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The Giry monad is not strong for the canonical symmetric monoidal closed structure on **Meas**

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ABSTRACT

We show that the Giry monad is not strong with respect to the canonical symmetric monoidal closed structure on the category **Meas** of all measurable spaces and measurable functions.

1. Introduction

A motivation of this research is to give a denotational semantics of Higher-order continuous probabilistic programming language. The denotational semantics of discrete probabilistic programming language is categorified by using the (sub-)distribution monad on the category **Set** of all sets and functions. The categorified semantics supports the Higher-order functions since the category **Set** is cartesian closed, and the (sub-)distribution monad is commutative strong with respect to the cartesian monoidal structure. On the other hand, denotational semantics of continuous first-order probabilistic programming language is categorified by using the (sub-probabilistic) *Giry monad*. The Giry monad is a monad on the category **Meas** of measurable spaces and functions, which introduced by Giry to give a categorical definition of continuous random processes such as (Labelled) Markov processes in the paper [1]. The Giry monad is commutative strong with respect to the cartesian monoidal structure of **Meas**, and hence it supports first-order seman-

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tics for continuous probabilistic language. However, it does not support Higher-order functions because the category **Meas** is not cartesian closed [2].

To give a categorical semantics of higher-order continuous probabilistic programming language, we have to find a monoidal closed structure which supports continuous probabilistic processes/calculations.

In fact, there is a canonical *symmetric monoidal closed structure* on **Meas** that is defined by the finest σ-algebra $\Sigma_{X\otimes Y}$ over product sets $|X|\times|Y|$ that makes all constant graph functions measurable (Section 2). If the Giry monad was strong with respect to it then we obtained a categorical semantics of higher-order continuous probabilistic programming language. However, unfortunately, the Giry monad *is not strong* with respect to the canonical symmetric monoidal closed structure.

In this paper, we prove that Giry monad is not strong with respect to the canonical symmetric monoidal closed structure as follows: We recall that a strength of a monad with respect to a symmetric monoidal closed category corresponds bijectively to a tensorial strength [3]. We show that a tensorial strength for any monad on **Meas** with respect to the canonical symmetric monoidal closed structure is uniquely determined if exists (Section 3). This implies that there is a unique candidate of the strength of Giry monad with respect to the canonical symmetric monoidal closed structure. We give a counterexample that the candidate associates a non-measurable function to some pair of measurable spaces (Section 4).

1.1. Preliminaries

We refer the definitions of monads, monoidal categories, and monoidal functors from [4], and refer the definition of strong monads on symmetric monoidal closed categories from [5,3]. The notion of tensorial strength can be relaxed to a monad on a symmetric monoidal category. We often call monads equipped with tensorial strengths strong monads (see [6, Definition 3.2] or [7, Section 7.1]).

Throughout this paper, we use the category **Meas** of all measurable spaces and measurable functions. The category **Meas** is complete and cocomplete. Hence we enjoy the cartesian monoidal structure $(\mathbf{Meas},\times,1)$. Moreover, it is a *topological category* [8,9]. We emphasise that the category **Meas** is *not cartesian closed* because there is no σ-algebra over $\mathbf{Meas}([0,1],2)$ satisfying the axioms of exponential object [2].

We also introduce the following notations on measure theory:

- For each measurable spaces, we denote by $|X|$ and Σ_X the underlying set and σ-algebra of X respectively.
- The indicator function $\chi_A\colon X\to\mathbb{R}$ of a subset A of X is defined by $\chi_A(x)=1$ $(x\in A)$ and $\chi_A(x)=0$ $(x\notin A)$. Note that χ_A is measurable if and only if the corresponding subset A is measurable (i.e. $A\in\Sigma_X$).

We recall that the Borel σ-algebra $\mathcal{B}(\mathbb{R})$ over the real line $\mathbb{R}$ is generated from the family of all half-open intervals $\{\,[\alpha,\beta)\mid\alpha,\beta\in\mathbb{R}\,\}$. We remark that each singleton $\{r\}$ is a Borel set because $\{r\}=\bigcap_{n\in\mathbb{N}}[r,r+1/(n+1))$, and hence any countable subset of $\mathbb{R}$ is a Borel set. By the Caratheodory's extension theorem, there is a unique measure on the Borel σ-algebra $\mathcal{B}(\mathbb{R})$ assigning $\beta-\alpha$ to each half-open interval $[\alpha,\beta)$. We denote it by m, and call it *the Lebesgue measure on* $\mathcal{B}(\mathbb{R})$. Strictly speaking, the measure m is the restriction of the Lebesgue measure m^* over $\mathbb{R}$ to the Borel σ-algebra $\mathcal{B}(\mathbb{R})$, and the Lebesgue measure space $(\mathbb{R},\mathcal{L}(\mathbb{R}),m^*)$ is indeed the completion of the measure space $(\mathbb{R},\mathcal{B}(\mathbb{R}),m)$. We remark that $m(\{r\})=0$ for any $r\in\mathbb{R}$, and hence $m(A)=0$ for any countable subset $A\subseteq\mathbb{R}$.

1.2. The Giry monad

The Giry monad [1] is a monad on the category **Meas** that is introduced by Giry, which captures continuous/non-discrete probabilistic computations such as Labelled Markov processes. For example, Markov processes are arrows in the Kleisli category of Giry monad, and the Chapman–Kolmogorov equation for Markov processes is characterised as associativity of multiplications of the Giry monad.

The structure of Giry monad $\mathcal{G}$ is defined as follows:

- For any measurable space X, the measurable space $\mathcal{G}X$ is defined by
  - the underlying set $|\mathcal{G}X|$ is *the set of probability measures* on X.
  - the σ-algebra $\Sigma_{\mathcal{G}X}$ is the *coarsest* one over $|\mathcal{G}X|$ that makes the evaluation function $\mathrm{ev}_A \colon \mathcal{G}X \to [0,1]$ defined by $\nu \mapsto \nu(A)$ measurable for any $A \in \Sigma_X$, where $\Sigma_{[0,1]}$ is the Borel σ-algebra over the unit interval, which is introduced in the same way as $\mathcal{B}(\mathbb{R})$.
- For each $f \colon X \to Y$ in **Meas**, $\mathcal{G}f \colon \mathcal{G}X \to \mathcal{G}Y$ is defined by $(\mathcal{G}f)(\mu) = \mu(f^{-1}(-))$.
- The unit η is defined by $\eta_X(x) = \delta_x$, where δ_x is the *Dirac measure* centred on x.
- The Kleisli lifting of $f \colon X \to \mathcal{G}Y$ is given by $f^\sharp(\mu)(A) = \int_X f(-)(A)\, d\mu$ $(\mu \in \mathcal{G}X)$.

We also consider the subprobabilistic variant $\mathcal{G}_{\mathrm{sub}}$ of the Giry monad; the underlying set $|\mathcal{G}_{\mathrm{sub}}X|$ is the set of *subprobability* measures on X.

Both the Giry monad $\mathcal{G}$ and its subprobabilistic variant $\mathcal{G}_{\mathrm{sub}}$ is strong and commutative with respect to the cartesian monoidal structure on **Meas** in the sense of [6]. The tensorial strength $\mathrm{st}^{\mathcal{G}\times}_{-,=} \colon (-) \times \mathcal{G}(=) \Rightarrow \mathcal{G}(- \times =)$ is given by the product measure $\mathrm{st}_{X,Y}(x,\nu) = \delta_x \times \nu$. The commutativity is shown from the Fubini theorem, and the double strength $\mathrm{dst}^{\mathcal{G}\times}_{-,=} \colon \mathcal{G}(-) \times \mathcal{G}(=) \Rightarrow \mathcal{G}(- \times =)$ is given by $\mathrm{dst}_{X,Y}(\nu_1,\nu_2) = \nu_1 \times \nu_2$.

2. The canonical symmetric monoidal closed structure on Meas

The category **Meas** is not cartesian closed, but there is the canonical symmetric monoidal closed structure $(\otimes, 1, \multimap)$ on the topological category **Meas** (see also [10,11]). We first consider the following two families of constant graph functions:

- $\Gamma_x \colon |Y| \to |X| \times |Y|$ defined by $\Gamma_x(y) = (x,y)$ for any $y \in |Y|$ $(x \in |X|)$.
- $\Gamma_y \colon |X| \to |X| \times |Y|$ defined by $\Gamma_y(x) = (x,y)$ for any $x \in |X|$ $(y \in |Y|)$.

Next, we introduce the following symmetric monoidal closed structure on **Meas**:

- The monoidal product functor $\otimes$ is defined by $X \otimes Y = (|X| \times |Y|, \Sigma_{X\otimes Y})$ where the σ-algebra $\Sigma_{X\otimes Y}$ is the finest σ-algebra Σ such that
  - Γ_x is a measurable function $Y \to (|X| \times |Y|, \Sigma)$ for any $x \in X$, and
  - Γ_y is a measurable function $X \to (|X| \times |Y|, \Sigma)$ for any $y \in Y$,
- The internal Hom functor $\multimap$ is defined by $(X \multimap Y) = (\mathbf{Meas}(X,Y), \Sigma_{X\multimap Y})$ where the σ-algebra $\Sigma_{X\multimap Y}$ is the coarsest one generated by

$$\langle\!\langle x, U \rangle\!\rangle = \{\, f \in \mathbf{Meas}(X,Y) \mid f(x) \in U \,\} \quad (x \in |X|, U \in \Sigma_Y).$$

We remark that the forgetful functor $|-| \colon \mathbf{Meas} \to \mathbf{Set}$ forms a strict symmetric monoidal functor from $(\mathbf{Meas}, \otimes, 1)$ to $(\mathbf{Set}, \times, 1)$.

Lemma 1. *The currying operation forms a natural isomorphism* $\mathbf{Meas}(X \otimes Y, Z) \simeq \mathbf{Meas}(X, Y \multimap Z)$ *for all measurable spaces* X, Y, Z.

Proof. Let f be an arbitrary function of type $|X| \times |Y| \to |Z|$. The curried function $\lceil f \rceil$ is then a function of type $|X| \to \mathbf{Set}(|Y|, |Z|)$. The currying operator $\lceil - \rceil$ is obviously natural and isomorphic as a transformation on just functions. Hence, it suffices to show that the original f is measurable if and only if the curried $\lceil f \rceil$ returns measurable functions, and is measurable itself.

$$\begin{aligned}
& f \in \mathbf{Meas}(X \otimes Y, Z) \\
\iff & \forall V \in \Sigma_Z. f^{-1}(V) \in \Sigma_{X\otimes Y} \\
\iff & \forall V \in \Sigma_Z. \left((\forall x \in X. {\Gamma_x}^{-1}(f^{-1}(V)) \in \Sigma_Y) \wedge (\forall y \in Y. {\Gamma_y}^{-1}(f^{-1}(V)) \in \Sigma_X)\right) \\
\iff & \forall V \in \Sigma_Z. \Big((\forall x \in X. (\lceil f \rceil (x))^{-1}(V) \in \Sigma_Y) \wedge (\forall y \in Y. \lceil f \rceil^{-1}\langle\!\langle y, V \rangle\!\rangle \in \Sigma_X)\Big) \\
\iff & (\forall x \in X. \lceil f \rceil (x) \in \mathbf{Meas}(Y, Z)) \wedge (\forall V \in \Sigma_Z. \forall y \in Y. \lceil f \rceil^{-1}\langle\!\langle y, V \rangle\!\rangle \in \Sigma_X) \\
\iff & \lceil f \rceil \in \mathbf{Meas}(X, Y \multimap Z) \quad \square
\end{aligned}$$

We remark that the uncurried mapping of the identity mapping $\mathrm{id}_{X\multimap Y} \colon X \multimap Y \to X \multimap Y$ on $X \multimap Y$ is called the evaluation mapping $\mathrm{ev}_{X,Y} \colon (X \multimap Y) \otimes X \to Y$.

2.1. Topological categories

The construction of the above canonical symmetric monoidal closed structure $(\otimes, 1, \multimap)$ on **Meas** is similar to the classical one on the category **Top** of topological spaces and continuous functions (see [12, Example 6.1.9.g], [13, Section 3], and [14, Remark 6.4]). Why these constructions are similar because they are given by the same categorical construction of the canonical symmetric monoidal closed structure on a *topological category* along its *topological functor* [8, Section 2].

A faithful functor $U \colon \mathbb{C} \to \mathbf{Set}$ is a *topological functor* if

1. Every family in the form $(f_j \colon B \to UA_j)_{j\in J}$ (called source) has an initial lift along U, that is, a family $(\overline{f_j} \colon \overline{B} \to A_j)_{j\in J}$ such that $U\overline{f_j} = f_j$, and there is a unique arrow $\overline{k} \colon C \to \overline{B}$ such that $\overline{f_j} \circ \overline{k} = g_j$ for any pair of $k \colon UC \to B$ and $(g_j \colon C \to A_j)_{j\in J}$ satisfying $f_j \circ k = Ug_j$.
2. Every family in the form $(f_j \colon UA_j \to B)_{j\in J}$ (called sink) has a final U-lift $(\overline{f_j} \colon A_j \to \overline{B})_{j\in J}$ (the dual condition of (1.)).

The category $\mathbb{C}$ is then called a *topological (concrete) category.*

Remark that the conditions (1.) and (2.) are equivalent. Hence, either of them is often omitted from the definition of topological functors.

Every topological category is complete and cocomplete, and its topological functor preserves limits and colimits. In addition, it is separatable whose separator is the terminal object 1.

Both the forgetful functors $|-| \colon \mathbf{Meas} \to \mathbf{Set}$ and $|-| \colon \mathbf{Top} \to \mathbf{Set}$ are topological. For instance, the topologicity of $|-| \colon \mathbf{Meas} \to \mathbf{Set}$ are given as follows:

1. For any source $(f_j \colon B \to |A_j|)_{j\in J}$, the initial lift is $(f_j \colon (B, \Sigma_B) \to A_j)_{j\in J}$ where Σ_B is the *coarsest* σ-algebra Σ_B over B that makes f_j measurable.
2. For any sink $(f_j \colon |A_j| \to B)_{j\in J}$, the final lift is $(f_j \colon A_j \to (B, \Sigma'_B))_{j\in J}$ where Σ_B is the *finest* σ-algebra Σ'_B over B that makes f_j measurable.

Here, by replacing σ-algebras and measurability respectively to topologies and continuity, we obtain the topologicity of $|-| \colon \mathbf{Top} \to \mathbf{Set}$.

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2.2. The canonical symmetric monoidal closed structure

Consider a topological category $\mathbb{C}$ and its topological functor $U\colon \mathbb{C} \to \mathbf{Set}$. The canonical symmetric monoidal closed structure $(\otimes, 1, \multimap)$ on the topological category $\mathbb{C}$ is introduced as follows:

- The tensor product $X \otimes Y$ is the codomain of the final lift $(X \xrightarrow{\overline{\Gamma_y}} X \otimes Y \xleftarrow{\overline{\Gamma_x}} Y)_{x \in UX, y \in UY}$ of $(UX \xrightarrow{\Gamma_y} UX \times UY \xleftarrow{\Gamma_x} UY)_{x \in UX, y \in UY}$ along U where Γ_y and Γ_x are the constant graph functions defined by $x \mapsto (x, y)$ and $y \mapsto (x, y)$ respectively.
- The tensor unit is a terminal object 1 in $\mathbb{C}$.
- The internal hom $X \multimap Y$ is the domain of the initial lift $(X \multimap Y \xrightarrow{\overline{\mathrm{ev}_x}} Y)_{x \in UX}$ of $(\mathbb{C}(X, Y) \xrightarrow{\dot{U}_{X,Y}} \mathbf{Set}(UX, UY) \xrightarrow{\mathrm{ev}_x} UY)_{x \in UX}$ along U where $\dot{U}_{X,Y}$ and ev_x are defined by $f \mapsto Uf$ (here, $\mathbb{C}$ is locally small) and $g \mapsto g(x)$ respectively.

It is straightforward to check that $(\mathbb{C}, \otimes, 1, \multimap)$ is indeed a symmetric monoidal closed category, and that $U\colon \mathbb{C} \to \mathbf{Set}$ forms a strict monoidal functor $(\mathbb{C}, \otimes, 1) \to (\mathbf{Set}, \times, 1)$ preserving curryings/uncurryings.

By instantiating this construction to $|-|\colon \mathbf{Meas} \to \mathbf{Set}$, we obtain the canonical symmetric monoidal closed structure on **Meas**. We have the classical symmetric monoidal closed structure on **Top** in [12, Example 6.1.9.g] by instantiating this construction to $|-|\colon \mathbf{Top} \to \mathbf{Set}$.

2.3. Comparison of symmetric monoidal (closed) structures on Meas

Any symmetric monoidal closed structure $(\overline{\otimes}, I, \overline{\multimap})$ on $\mathbb{C}$ has the following canonical form, where the above canonical symmetric monoidal closed structure $(\otimes, 1, \multimap)$ is already given in the canonical form.

Proposition 2 *([14, Proposition 3.1]). Any symmetric monoidal closed structure $(\overline{\otimes}, I, \overline{\multimap})$ on $\mathbb{C}$ is isomorphic to a unique one $(\dot{\otimes}, 1, \dot{\multimap})$ (we call it the canonical form of $(\overline{\otimes}, I, \overline{\multimap})$) that satisfies the following conditions:*

- *The functor U is strictly symmetric monoidal $(\mathbb{C}, \otimes, 1) \to (\mathbf{Set}, \times, 1)$.*
- *The isomorphism $\Phi_{X,Y,Z}\colon \mathbb{C}(X \dot{\otimes} Y, Z) \cong \mathbb{C}(X, Y \dot{\multimap} UZ)$ satisfies*

$$(\dot{U}_{X \dot{\otimes} Y, Z}(g))(x, y) = \dot{U}_{Y,Z}((\dot{U}_{X, Y \dot{\multimap} Z}(\Phi_{X,Y,Z}(g))(x))(y))$$

for any $x \in UX$, $y \in UY$, and $g \in \mathbb{C}(X \dot{\otimes} Y, Z)$.

Consider an arbitrary symmetric monoidal closed structure $(\dot{\otimes}, 1, \dot{\multimap})$ on **Meas** in the canonical form. Since the forgetful functor $|-|\colon \mathbf{Meas} \to \mathbf{Set}$ is strict monoidal, we have $|X \dot{\otimes} Y| = |X| \times |Y|$, $|f \dot{\otimes} g| = |f| \times |g|$, $|\lambda_X| = \pi_2$, and $|\rho_X| = \pi_1$ hold, and $|{\lambda_X}^{-1}|$ and $|{\rho_X}^{-1}|$ are the functions $x \mapsto (*, x)$ and $x \mapsto (x, *)$ respectively, where $\lambda_X\colon 1 \dot{\otimes} X \cong X$ and $\rho_X\colon X \dot{\otimes} 1 \cong X$ are respectively the left and right unitors.

Hence, the measurable functions $(\overline{x} \dot{\otimes} Y) \circ {\lambda_Y}^{-1}\colon Y \to X \dot{\otimes} Y$ and $(X \dot{\otimes} \overline{y}) \circ {\rho_X}^{-1}\colon X \to X \dot{\otimes} Y$ are exactly the constant graph functions Γ_x and Γ_y respectively. Here, $x \in |X|$ and $y \in |Y|$, and $\overline{x}\colon 1 \to X$ and $\overline{y}\colon 1 \to Y$ are the element functions defined by $* \mapsto x$ and $* \mapsto y$ respectively.

Since the σ-algebra $\Sigma_{X \otimes Y}$ is the finest one such that constant graph functions are measurable, the identity function on $|X| \times |Y|$ forms a measurable function $X \otimes Y \to X \dot{\otimes} Y$.

Also, the identity function on $|X| \times |Y|$ forms a measurable function $X \dot{\otimes} Y \to X \times Y$ because $\pi_1 = |\rho_X \circ (X \dot{\otimes} !_Y)|$ and $\pi_2 = |\lambda_Y \circ (!_X \dot{\otimes} Y)|$.

3. The uniqueness of tensorial strength

We show the uniqueness of tensorial strength for any monad on **Meas** with respect to the canonical symmetric monoidal closed structure $(\otimes, 1, \multimap)$ on **Meas**.

Moggi proved the uniqueness of tensorial strength of a monad on a well-pointed cartesian closed category [6, Proposition 3.4], but its situation is relaxed a separatable symmetric monoidal closed category, because the original Moggi's proof uses only two things: the separator equals the tensor unit of the symmetric monoidal closed structure, and all generalised elements of a tensor product are splitten into two generalised elements of its two components. We state the extended version of the uniqueness of tensorial strength in [6] below.

We recall that an object I in a category $\mathbb{C}$ is called a *separator* if for any pair of arrows $f, g \colon X \to Y$ in $\mathbb{C}$, the equality $f = g$ holds when $f \circ e = g \circ e$ for each $e \colon I \to X$.

Lemma 3 *([6, Proposition 3.4], Extended). Consider a symmetric monoidal category $(\mathbb{C}, \otimes, I)$ whose tensor unit I is a separator of $\mathbb{C}$ such that for any morphism $\overline{z} \colon I \to X \otimes Y$, there are $\overline{x} \colon I \to X$ and $\overline{y} \colon I \to Y$ satisfying $\overline{z} = (\overline{x} \otimes \overline{y}) \circ {\lambda_I}^{-1}$.*

If T is a strong monad with respect to $(\mathbb{C}, \otimes, I)$ then its tensorial strength $\mathrm{st}^T \colon (-) \otimes T(=) \Rightarrow T(- \otimes =)$ is determined uniquely by

$$\mathrm{st}^T_{X,Y} \circ (\overline{x} \otimes \overline{\xi}) \circ {\lambda_I}^{-1} = T((\overline{x} \otimes Y) \circ {\lambda_Y}^{-1}) \circ \overline{\xi}$$

where $\overline{x} \colon I \to X$ and $\overline{\xi} \colon I \to TY$.

Proof. From the naturality of st^T and λ and bifunctoriality of $\otimes$, we obtain,

$$\begin{aligned} \mathrm{st}^T_{X,Y} \circ (\overline{x} \otimes \overline{\xi}) \circ {\lambda_I}^{-1} &= \mathrm{st}^T_{X,Y} \circ (\overline{x} \otimes TY) \circ (I \otimes \overline{\xi}) \circ {\lambda_I}^{-1} \\ &= T(\overline{x} \otimes Y) \circ \mathrm{st}^T_{I,Y} \circ {\lambda_{TY}}^{-1} \circ \overline{\xi} \circ \lambda_I \circ {\lambda_I}^{-1} \\ &= T((\overline{x} \otimes Y) \circ {\lambda_Y}^{-1}) \circ \overline{\xi} \end{aligned}$$

for any pair $\overline{x} \colon I \to X$ and $\overline{\xi} \colon I \to TY$. Since any arrow $\overline{z} \colon I \to X \otimes TY$ is written as $\overline{z} = (\overline{x} \otimes \overline{\xi}) \circ {\lambda_I}^{-1}$ for some $\overline{x} \colon I \to X$ and $\overline{\xi} \colon I \to TY$, the arrow $\mathrm{st}^T_{X,Y}$ is determined uniquely for each X and Y. □

The uniqueness of tensorial strength [6, Proposition 3.4] is indeed a corollary of Lemma 3: any well-pointed cartesian monoidal category $(\mathbb{C}, \times, 1)$ and any strong monad T with respect to the cartesian products satisfy the assumption of Lemma 3 because the terminal object 1 is both a tensor unit and a generator, and the left unitor $\lambda_X \colon 1 \times X \cong X$ are given by $\lambda_X = \pi_2$ and ${\lambda_X}^{-1} = \langle !_X, \mathrm{id} \rangle$ respectively.

The symmetric monoidal closed categories $(\mathbf{Meas}, \otimes, 1, \multimap)$ and $(\mathbf{Top}, \otimes, 1, \multimap)$ discussed in Section 2 satisfy the assumption of the lemma because the terminal object 1 is both a tensor unit and a generator, and each element $\overline{z} \colon 1 \to X \otimes Y$ $(* \mapsto (x, y))$ is obviously decomposed into a pair of elements $\overline{x} \colon 1 \to X$ $(* \mapsto x)$ and $\overline{y} \colon 1 \to Y$ $(* \mapsto y)$.

4. The Giry monad is not strong

We show that the Giry monad $\mathcal{G}$ is not strong with respect to the canonical symmetric monoidal closed structure $(\otimes, 1, \multimap)$ on **Meas**. In the following discussion, we consider the Giry monad $\mathcal{G}$, but we are able to prove that the subprobabilistic variant $\mathcal{G}_{\mathrm{sub}}$ is not strong in the same way.

Theorem 4. *Giry monad $\mathcal{G}$ is not strong with respect to the canonical symmetric monoidal closed structure $(\otimes, 1, \multimap)$ on* **Meas**.

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Assume that the Giry monad $\mathcal{G}$ is strong with respect to the symmetric monoidal structure $(\mathbf{Meas}, \otimes, \multimap, 1)$. From [3], the strength $\mathrm{G}_{X,Y} \colon (X \multimap Y) \to (\mathcal{G}X \multimap \mathcal{G}Y)$ correspond bijectively to the tensorial strength $\mathrm{st}^{\mathcal{G}}_{X,Y} \colon X \otimes \mathcal{G}Y \to \mathcal{G}(X \otimes Y)$. From the construction of $(\otimes, 1, \multimap)$, we obtain $\mathrm{G}_{X,Y} = \left\lceil \mathcal{G}(\mathrm{ev}_{X,Y}) \circ \mathrm{st}^{\mathcal{G}}_{X \multimap Y, X} \right\rceil$. By Lemma 3, the tensorial strength $\mathrm{st}^{\mathcal{G}}$ of $\mathcal{G}$ is determined uniquely by for any $x \in X$ and $\mu \in \mathcal{G}Y$,

$$\mathrm{st}^{\mathcal{G}}_{X,Y}(x, \mu) = \mathcal{G}((\overline{x} \otimes Y) \circ {\lambda_Y}^{-1}) \circ \overline{\mu} = \mu(((\overline{x} \otimes Y) \circ {\lambda_Y}^{-1})^{-1}(-)) = \mu({\Gamma_x}^{-1}(-)).$$

Hence, the following calculation shows that the strength $\mathrm{G}_{X,Y}$ is uniquely determined by the mapping that takes $f \colon X \to Y$, and returns $\mathcal{G}f \colon \mathcal{G}X \to \mathcal{G}Y$:

$$\mathcal{G}(\mathrm{ev}_{X,Y}) \circ \mathrm{st}^{\mathcal{G}}_{X \multimap Y, X}(f, \mu) = \mu(\mathrm{ev}_{X,Y} \circ {\Gamma_f}^{-1}(-)) = \mu(f^{-1}(-)) = \mathcal{G}(f)(\mu).$$

However, as we show below, the component $\mathrm{G}_{X,Y}$ is not even a measurable function of type $(X \multimap Y) \to (\mathcal{G}X \multimap \mathcal{G}Y)$ for some X and Y. Hence, the Giry monad is not strong with respect to the canonical symmetric monoidal structure.

4.1. Non-measurability of $\mathrm{G}_{X,Y}$

We recall that $\Sigma_{\mathcal{G}X \multimap \mathcal{G}Y}$ is generated by $\langle\!\langle \mu, {\mathrm{ev}_U}^{-1}(A) \rangle\!\rangle$ for parameters $\mu \in \mathcal{G}X$, $U \in \Sigma_Y$, and $A \in \Sigma_{[0,1]}$. We thus have,

$${\mathrm{G}_{X,Y}}^{-1} \langle\!\langle \mu, {\mathrm{ev}_U}^{-1}(A) \rangle\!\rangle = \left\{ f \in \mathbf{Meas}(X,Y) \;\middle|\; \mu(f^{-1}(U)) \in A \right\}.$$

Hence, the σ-algebra $\left\{ {\mathrm{G}_{X,Y}}^{-1}(K) \;\middle|\; K \in \Sigma_{\mathcal{G}X \multimap \mathcal{G}Y} \right\}$ of the inverse images of measurable subsets of $\mathcal{G}X \multimap \mathcal{G}Y$ along $\mathrm{G}_{X,Y}$ is at least finer than or equal to $\Sigma_{X \multimap Y}$, because for any $x \in X$ and $U \in \Sigma Y$, we obtain

$$\langle\!\langle x, U \rangle\!\rangle = \left\{ f \in \mathbf{Meas}(X,Y) \;\middle|\; x \in f^{-1}(U) \right\} = {\mathrm{G}_{X,Y}}^{-1} \langle\!\langle \delta_x, {\mathrm{ev}_U}^{-1}(\{1\}) \rangle\!\rangle.$$

We prove that there are X and Y such that the σ-algebra induced by $\mathrm{G}_{X,Y}$ is strictly finer than $\Sigma_{X \multimap Y}$.

4.1.1. A σ-algebra $\Omega_{X,Y}$

Consider measurable spaces X and Y whose underlying sets are infinite. We define the family $\Omega_{X,Y}$ of all subsets of the form

$$\langle h, V \rangle = \left\{ f \in \mathbf{Meas}(X,Y) \;\middle|\; \langle f(h(n)) \rangle_{n \in \mathbb{N}} \in V \right\}$$

where $h \colon \mathbb{N} \to X$ is an arbitrary injection and $V \subseteq |Y|^{\mathbb{N}}$ is an arbitrary subset.

Lemma 5. *The collection $\Omega_{X,Y}$ forms a σ-algebra over $\mathbf{Meas}(X,Y)$ including $\Sigma_{X \multimap Y}$.*

Proof. We have $\emptyset = \langle h, \emptyset \rangle \in \Omega_{X,Y}$ where h is an arbitrary injection. For any $\langle h, V \rangle \in \Omega_{X,Y}$, we have $\mathbf{Meas}(X,Y) \setminus \langle h, V \rangle = \left\langle h, |Y|^{\mathbb{N}} \setminus V \right\rangle \in \Omega_{X,Y}$. For any countable family $\{\langle h_m, V_m \rangle\}_{m \in \mathbb{N}}$ with $\langle h_m, V_m \rangle \in \Omega_{X,Y}$, we obtain $\bigcup_{m \in \mathbb{N}} \langle h_m, V_m \rangle = \langle h, V \rangle$ in the following steps:

1. The image $I = \{ h_m(n) \mid m, n \in \mathbb{N} \}$ is countably infinite, hence there is a bijection $k \colon \mathbb{N} \to I$. Now we define $k_m = k^{-1} \circ h_m$ for each $m \in \mathbb{N}$ and $h = \iota \circ k$ where $\iota \colon I \rightharpoonup X$ is the inclusion. Since $(k \circ k_m)(n) = h_m(n)$ for all $m, n \in \mathbb{N}$, the injection h and the family $\{k_m\}_{m \in \mathbb{N}}$ satisfy $h \circ k_m = h_m$ for each $m \in \mathbb{N}$.

2. We take the projection $\pi_l \colon |Y|^{\mathbb{N}} \to Y$ ($\langle x_L \rangle_{L\in\mathbb{N}} \mapsto x_l$) for each $l \in \mathbb{N}$ and the tuple $\langle \pi_{k_m(n)} \rangle_{n\in\mathbb{N}} : |Y|^{\mathbb{N}} \to |Y|^{\mathbb{N}}$ indexed by $\{k_m(n)\}_{n\in\mathbb{N}}$ for each $m \in \mathbb{N}$. Then the inverse image $W_m = \langle \pi_{k_m(n)} \rangle_{n\in\mathbb{N}}^{-1}(V_m)$ satisfies $\langle h_m, V_m \rangle = \langle h, W_m \rangle$ for each $m \in \mathbb{N}$.
3. We have $\bigcup_{m\in\mathbb{N}} \langle h_m, V_m \rangle = \bigcup_{m\in\mathbb{N}} \langle h, W_m \rangle = \langle h, \bigcup_{m\in\mathbb{N}} W_m \rangle$ (thus $V = \bigcup_{m\in\mathbb{N}} W_m$).

Hence, $\Omega_{X,Y}$ is indeed a σ-algebra over $\mathbf{Meas}(X,Y)$.

For each $x \in X$ and $U \in \Sigma_Y$, we have $\langle\langle x, U \rangle\rangle = \langle h, {\pi_0}^{-1}(U) \rangle$ where h is an arbitrary injection such that $h(0) = x$. From the minimality of $\Sigma_{X\multimap Y}$, the σ-algebra $\Omega_{X,Y}$ is finer than $\Sigma_{X\multimap Y}$. □

The inclusion $\Sigma_{X\multimap Y} \subseteq \Omega_{X,Y}$ implies that for each measurable set $K \in \Sigma_{X\multimap Y}$, the membership $f \in K$ is determined by checking outputs $f(x_0), f(x_1), \dots$ for some *countable sequence* $x_0, x_1, \dots$ of inputs.

4.1.2. A counterexample

Theorem 6. *Let $X = Y = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$, and let $\mu \in \mathcal{G}X$ be absolutely continuous with respect to the Lebesgue measure m on $\mathcal{B}(\mathbb{R})$ (i.e. $m(A) = 0 \implies \mu(A) = 0$ for any $A \in \Sigma_X$). We then obtain ${\mathrm{G}_{X,Y}}^{-1}\langle\langle \mu, {\mathrm{ev}_{\{0\}}}^{-1}(\{1\}) \rangle\rangle \notin \Sigma_{X\multimap Y}$.*

Proof. We write $K = {\mathrm{G}_{X,Y}}^{-1}\langle\langle \mu, {\mathrm{ev}_{\{0\}}}^{-1}(\{1\}) \rangle\rangle$. We assume $K = \langle h, U \rangle \in \Omega_{X,Y}$ holds for some injection $h \colon \mathbb{N} \to X$ and subset $U \subseteq \mathbb{R}^{\mathbb{N}}$. We then have $U \neq \mathbb{R}^{\mathbb{N}}, \emptyset$ because K is neither the whole space $\mathbf{Meas}(X,Y)$ nor the empty function space. Hence, there is a pair of sequences $\langle s_n \rangle_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ and $\langle t_n \rangle_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ such that $\langle s_n \rangle_{n\in\mathbb{N}} \in U$ and $\langle t_n \rangle_{n\in\mathbb{N}} \notin U$.

We consider a measurable function $f \in \mathbf{Meas}(X,Y)$. We give the functions f_1 and f_2 by replacing the output of f at each $h(n)$ to s_n and t_n ($n \in \mathbb{N}$) respectively, that is,

$$f_1 = f + \sum_{n\in\mathbb{N}} (s_n - f(h(n))) \cdot \chi_{\{h(n)\}}, \qquad f_2 = f + \sum_{n\in\mathbb{N}} (t_n - f(h(n))) \cdot \chi_{\{h(n)\}}.$$

Here, for each $n \in \mathbb{N}$, $\chi_{\{h(n)\}}$ is the indicator function of the closed subset $\{h(n)\}$, and hence it is a measurable function. Since $\mathbf{Meas}(X,Y)$ (i.e. the set of Borel measurable functions on $\mathbb{R}$) is closed under scalar multiplication and countable addition, the functions f_1 and f_2 are measurable. We obtain $f_1 \in K$ and $f_2 \notin K$ since $\langle f_1(h(n)) \rangle_{n\in\mathbb{N}} = \langle s_n \rangle_{n\in\mathbb{N}} \in U$ and $\langle f_2(h(n)) \rangle_{n\in\mathbb{N}} = \langle t_n \rangle_{n\in\mathbb{N}} \notin U$.

However, $f_1 \in K \iff f_2 \in K$ must hold. From the definition of f_1 and f_2, $\{\, x \in X \mid f_1(x) \neq f_2(x) \,\}$ is a subset of $\{h(n)\}_{n\in\mathbb{N}}$, and hence it is countable. We then have $\mu(\{\, x \in X \mid f_1(x) \neq f_2(x) \,\}) = 0$ since μ is absolutely continuous with respect to m. Since $K = \left\{\, g \in \mathbf{Meas}(X,Y) \,\middle|\, \mu(g^{-1}(\{0\})) = 1 \,\right\}$, we obtain,

$$\begin{aligned} f_1 \in K &\iff \mu({f_1}^{-1}(\{0\})) = 1 \\ &\iff \mu({f_1}^{-1}(\{0\}) \setminus \{\, x \in X \mid f_1(x) \neq f_2(x) \,\}) = 1 \\ &\iff \mu({f_2}^{-1}(\{0\})) = 1 \iff f_2 \in K. \end{aligned}$$

This is a contradiction. Hence, there is no h and U such that $K = \langle h, U \rangle \in \Omega_{X,Y}$. From the definition of $\Omega_{X,Y}$, we have $K \notin \Omega_{X,Y}$. Thus, we conclude $K \notin \Sigma_{X\multimap Y}$. □

5. Concluding remarks

The proof that the Giry monad is strong with respect to the canonical symmetric monoidal closed structure $(\otimes, 1, \multimap)$ in the preprint [11] has the following error: The statement of [11, Theorem 3.1] is just the naturality of $\mathrm{st}^{\mathcal{G}\times}_{X,Y} \circ \mathrm{id}_{|X|\times|Y|} \colon X \otimes \mathcal{G}Y \to \mathcal{G}(X \times Y)$. Here we remark $\mathrm{st}^{\mathcal{G}\times}_{X,Y}$ is the tensorial strength for $\mathcal{G}$ with respect to the cartesian product on **Meas**, and $\mathrm{id}_{|X|\times|Y|}$ obviously forms a symmetric monoidal natural transformation $X \otimes Y \to X \times Y$. However, the above statement is mistaken for the existence of the natural transformation of the type $X \otimes \mathcal{G}Y \to \mathcal{G}(X \otimes Y)$ in the proof of existence of the tensorial strength for the Giry monad $\mathcal{G}$ with respect to the canonical symmetric monoidal closed structure $(\otimes, 1, \multimap)$.

If there is a symmetric monoidal closed structure $(\dot{\otimes}, 1, \dot{\multimap})$ on **Meas** with respect to which makes the Giry monad strong, then there is a strong symmetric monoidal functor U from $(\dot{\otimes}, 1, \dot{\multimap})$ to the canonical symmetric monoidal closed structure $(\otimes, 1, \multimap)$. Moreover, by converting to the "normal form" discussed in the last two paragraphs of Section 2, the σ-algebra $\Sigma_{X\dot{\otimes}Y}$ of the space $X\dot{\otimes}Y$ satisfies $\Sigma_{X\times Y} \subsetneq \Sigma_{X\dot{\otimes}Y} \subsetneq \Sigma_{X\otimes Y}$. We have not found yet an intermediate symmetric monoidal closed structure on **Meas** which is intermediate between the cartesian monoidal structure and the canonical symmetric monoidal closed structure.

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