Hypergraph categories

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ABSTRACT

Hypergraph categories have been rediscovered at least five times, under various names, including well-supported compact closed categories, dgs-monoidal categories, and dungeon categories. Perhaps the reason they keep being reinvented is two-fold: there are many applications—including to automata, databases, circuits, linear relations, graph rewriting, and belief propagation—and yet the standard definition is so involved and ornate as to be difficult to find in the literature. Indeed, a hypergraph category is, roughly speaking, a "symmetric monoidal category in which each object is equipped with the structure of a special commutative Frobenius monoid, satisfying certain coherence conditions".

Fortunately, this description can be simplified a great deal: a hypergraph category is simply a "cospan-algebra," roughly a lax monoidal functor from cospans to sets. The goal of this paper is to remove the scare-quotes and make the previous statement precise. We prove two main theorems. First is a coherence theorem for hypergraph categories, which says that every hypergraph category is equivalent to an *objectwise-free* hypergraph category. Second, we prove that the category of objectwise-free hypergraph categories is equivalent to the category of cospan-algebras.

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1. Introduction

Suppose you wish to specify the following picture:
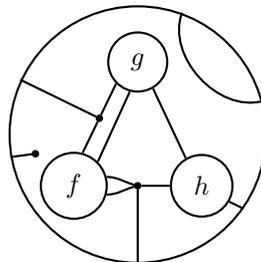
(1)

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This picture might represent, for example, an electrical circuit, a tensor network, or a pattern of shared variables between logical formulas.

One way to specify the picture in Eq. (1)—given the symbols f, g, h and their arities—is to define primitives that represent merging, initializing, splitting, and terminating wires. The picture can then be constructed piece by piece,


(2)
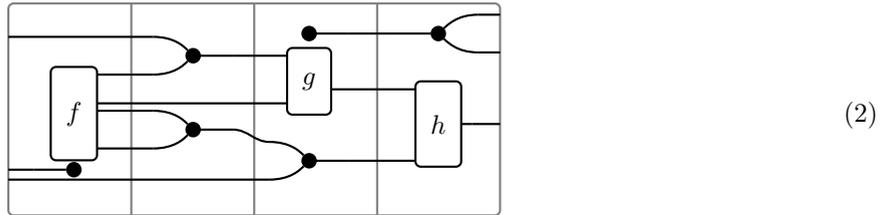
and described as a text-string, as follows:

$$(1 \otimes f \otimes \multimap \otimes 1) \,\fatsemi\, (\succ \otimes 1 \otimes \succ \otimes 1) \,\fatsemi\, (\bullet\!\!- \otimes g \otimes \succ) \,\fatsemi\, (\multimapinv \otimes h).$$

This system of notation provides a way of describing patterns of interconnection between f, g and h. The subtlety—a nontrivial one—lies in understanding when the morphisms defined by two different constructions should be considered the same. Indeed, we might equally we have chosen to represent the above picture as


(3)
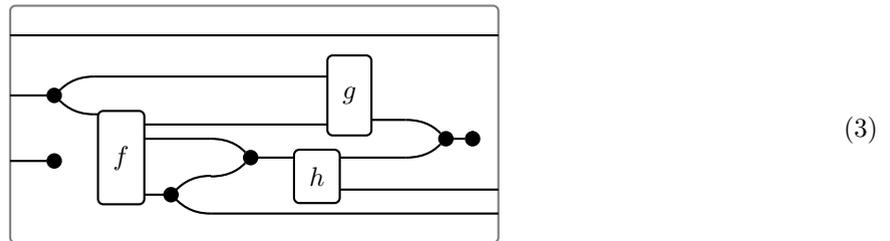
Similarly, but more simply, the diagrams
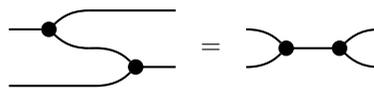
both represent the same pattern of interconnection.

Nonetheless, despite this subtlety, the method we have just outlined has been independently rediscovered, with various names and motivations, many times over the past few decades. The first were Carboni and Walters, who called this structure a well-support compact closed category, and used it to study categories of relations, as well as labeled transition systems and automata [3]. Bruni and Gadducci called it a dgs-monoidal category when studying Petri nets [9]. Morton called it a dungeon category when studying belief propagation [16]. Finally, converging on the name *hypergraph category*, Kissinger used it to study quantum systems and graph rewriting, and Fong, Baez, and Pollard used it to study electric circuits and chemical reaction networks [1,2,6,11].

Simultaneously, Spivak defined essentially the same structure in his work on databases [19], but preferred a more uniform, combinatorial approach. Instead of thinking about how to generate such pictures piece by piece, Spivak focused on writing down the connection patterns. For example, the picture in Eq. (1) can be described as follows. First, we define three sets, corresponding to the ports of the all three inner boxes (A, the white circles), the intermediate nodes (N, the black circles), and the ports of the outer box (B, the gray circles).

$$A \longrightarrow N \longleftarrow B \qquad (4)$$
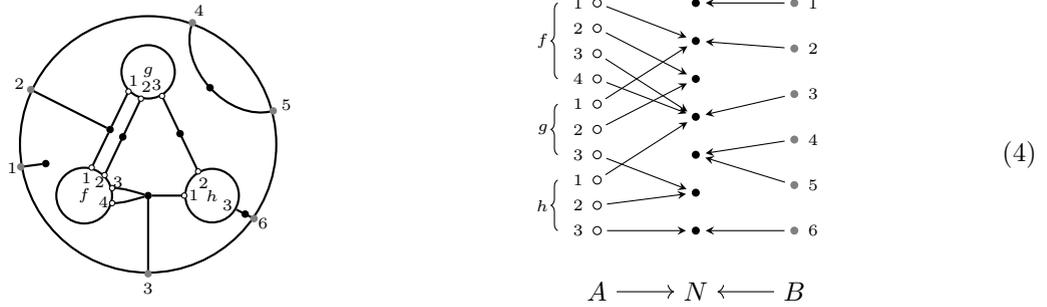
The picture is then described by a pair of functions $A \to N$, $B \to N$, that say how the wires on the boxes connect to the intermediate nodes. Writing this pair as $A \to N \leftarrow B$, we call this a cospan.

It was already noticed by Carboni and Walters that these two approaches should be similar, and aspects of this cospan idea have appeared in almost all the references above. In this paper we pin down the exact relationship. To do so requires a thorough investigation of hypergraph categories and their functors, including discussion of self-dual compact closed structure, free hypergraph categories, a factorization system on hypergraph functors, and a coherence theorem for hypergraph categories. Let us be a bit more precise.

1.1. Composition, wiring diagrams, and cospans

What is most relevant about the above diagrams is that they can be *composed*: new diagrams can be built from old. Let's explore how composition works for both hypergraph categories and cospan-algebras.

We refer to the primitives that represent wires merging, initializing, splitting, and terminating as *Frobenius generators*, and when their composites obey laws reflecting the above intuition about interconnection, we call the resulting structure a *special commutative Frobenius monoid*. A hypergraph category is a symmetric monoidal category in which every object is equipped with the structure of a special commutative Frobenius monoid in a way compatible with the monoidal product.

The monoidal structure gives notions of composition that come from concatenation: we may build new diagrams by placing them end to end—the categorical composition—or side by side—the monoidal product. The Frobenius generators, as special morphisms, take care of the network structure.

There is another perspective, however: that of substitution. Below is a pictorial representation of the sort of composition that makes sense in categories, monoidal categories, traced monoidal categories, and hypergraph categories.

category	monoidal category	traced monoidal category	hypergraph category

$$(5)$$
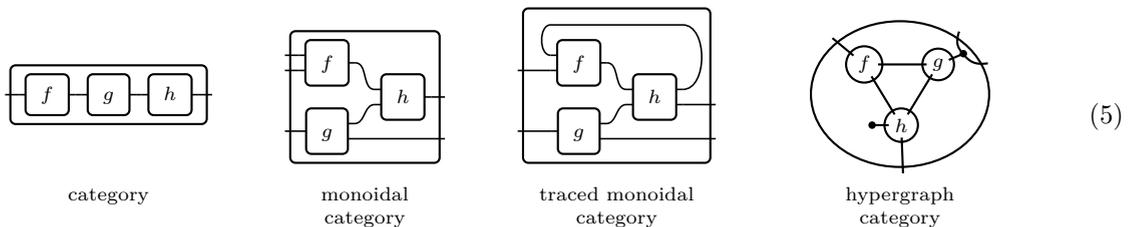
The above pictures are known as wiring diagrams. Here we think of the outer box as of the same nature as the inner boxes, which allows *substitution* of one wiring diagram into another.


$$(6)$$
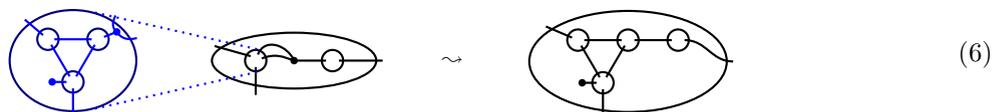
More formally, boxes, wiring diagrams, and substitution can be represented as objects, morphisms, and composition in an operad. The rules of this substitution—e.g. whether or not the "special rule" ⊸∘⊸ = —

holds, a question one might ask themselves if checking the details of Eq. (6)—are controlled by this operad. The above operadic viewpoint on wiring diagrams was put forth by Spivak and collaborators [17,19,21]. In particular it was shown in [20] that the operad governing traced monoidal categories is **Cob**, the operad of oriented 1-dimensional cobordisms.

In this paper we prove a similar result: the operad governing hypergraph categories is **Cospan**. Informally, what this means is that there is a one-to-one correspondence between the wiring diagrams that can be interpreted in a hypergraph category $\mathcal{H}$—or more precisely, equivalence classes thereof—and cospans labeled by the objects of $\mathcal{H}$.

Strictly speaking, every morphism in a category—including in a hypergraph category—has a domain and codomain, and thus should be represented as a two-sided figure, say a box with left and right sides, just like in the first three cases of Eq. (5). However, "morally speaking" (in the sense of [4]), a morphism $f \in \mathcal{H}$ in a hypergraph category is indexed not by a pair of objects $x_1, x_2 \in \mathrm{Ob}\,\mathcal{H}$, serving as the domain and codomain of f, but instead by a finite set $\{x_1, \ldots, x_n\} \subseteq \mathrm{Ob}\,\mathcal{H}$ of objects, which one can visualize as an "o-mane" for f, e.g.:
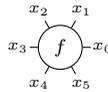
The reason not to distinguish between inputs and outputs is that the structures and axioms of hypergraph categories allow us to "bend arrows" arbitrarily, as we see in the difference between Eq. (2) and Eq. (3). The axioms of hypergraph categories ensure that these two diagrams denote the same composite morphism: directionality is irrelevant.

Thus we think of the cospan representation as an *unbiased* viewpoint on hypergraph categories. As an analogy, consider the case of ordinary monoids. A monoid is usually presented as a set M together with a binary operation $*: M \times M \to M$ and a constant, or 0-ary operation, $e \in M$, satisfying three equations. Once this structure is in place, one can uniquely define an n-ary operation, for any other n, by iterating the 2-ary operation.

An unbiased viewpoint on monoids is one in which all the n-ary operations are put on equal footing, rather than having 0- and 2-ary morphisms be special. One such approach is to say that a monoid is an algebra on the List monad: it is a set X equipped with a function $h: \mathsf{List}(X) \to X$ satisfying the usual monad-algebra equations. The 0-ary and 2-ary case are embedded in this structure as h applied to lists of length 0 and 2, respectively. Another unbiased approach is to use operads, which gives a very simple description: a monoid is an algebra on the terminal operad.

We similarly use cospan-algebras in this article to provide an unbiased viewpoint on hypergraph categories. However, doing so has a cost: while hypergraph categories and the functors between them are roughly cospan-algebras, the corresponding statement does not hold when one considers 2-categorical aspects. In other words, the natural transformations between hypergraph functors are not visible in the cospan formulation. Indeed, one can consider the category of cospan-algebras as a decategorification of the 2-category *Hyp* of hypergraph categories.

1.2. Statement of main theorems

Our first theorem is a strictification theorem. If $\mathcal{H}$ is a strict hypergraph category, an *objectwise-free* (OF) structure on $\mathcal{H}$ is a set Λ and a monoid isomorphism $i: \mathsf{List}(\Lambda) \cong \mathrm{Ob}\,\mathcal{H}$; in this case we say that $\mathcal{H}$ is OF or *objectwise-free* on Λ. Let $\mathcal{H}\!yp_{\mathsf{OF}}$ denote the 2-category that has OF-hypergraph categories as objects and for which 1- and 2-morphisms are those between underlying hypergraph categories. In other words, we have a full and faithful functor $U: \mathcal{H}\!yp_{\mathsf{OF}} \to \mathcal{H}\!yp$. The strictification theorem says that every hypergraph category is equivalent to an OF-hypergraph category.

Theorem 1.1. *The 2-functor $U \colon \mathcal{Hyp}_{\mathrm{OF}} \to \mathcal{Hyp}$ is a 2-equivalence.*

Our main theorem says that the category of cospan-algebras is a decategorification of $\mathcal{Hyp}_{\mathrm{OF}}$; in particular that it is isomorphic to the underlying 1-category $\mathbf{Hyp}_{\mathrm{OF}}$ of OF-hypergraph categories and all hypergraph functors between them. Before we can state this theorem, we need to say exactly what we mean by the category of cospan-algebras.

Let Λ be a set; we think of this as a set of wire labels. By a Λ-*labeled finite set*, we mean a nonnegative integer $m \in \mathbb{N}$ and a function $x \colon m \to \Lambda$, where $m \coloneqq \{1, \ldots, m\}$; in other words, just a list of elements in Λ. Let $\mathbf{Cospan}_\Lambda$ denote the category whose objects are Λ-labeled finite sets (m, x) and whose morphisms $f \colon (m, x) \to (n, y)$ are labeled cospans, i.e. isomorphism classes of commutative diagrams[1]

$$
\begin{array}{ccccc}
m & \xrightarrow{\ f_1\ } & p & \xleftarrow{\ f_2\ } & n \\
& \searrow_{x} & \downarrow_{z} & \swarrow_{y} & \\
& & \Lambda & &
\end{array}
\tag{7}
$$

Let $\mathbf{Hyp}_{\mathrm{OF}(\Lambda)} \subseteq \mathbf{Hyp}$ denote the subcategory of identity-on-objects functors between hypergraph categories that are objectwise-free on Λ. We will prove that this category is isomorphic to that of lax monoidal functors $a \colon \mathbf{Cospan}_\Lambda \to \mathbf{Set}$:

$$
\mathbf{Hyp}_{\mathrm{OF}(\Lambda)} \cong \mathbf{Lax}(\mathbf{Cospan}_\Lambda, \mathbf{Set}).
\tag{8}
$$

Maps in $\mathbf{Hyp}_{\mathrm{OF}}$ are just hypergraph functors $F \colon \mathcal{H} \to \mathcal{H}'$ between hypergraph categories that happen to be OF. In particular they need not send generators to generators; instead they send each generator in $\mathcal{H}$ to an arbitrary object, which is identified with a *list of generators* in $\mathcal{H}'$. Let $\mathbf{Set}_{\mathsf{List}}$ denote the Kleisli category of the list-monad, i.e. the category whose objects are sets, e.g. Λ, and for which a morphism from Λ to Λ' is a function $\Lambda \to \mathsf{List}(\Lambda')$. We will explain that the on-objects part of F induces a functor $\mathbf{Cospan}_{\mathrm{Ob}(F)} \colon \mathbf{Cospan}_\Lambda \to \mathbf{Cospan}_{\Lambda'}$, and that the on-morphisms part of F induces a monoidal natural transformation α

$$
\begin{array}{ccc}
\mathbf{Cospan}_\Lambda & \xrightarrow{\ a\ } & \\
\mathbf{Cospan}_{\mathrm{Ob}(F)} \Big\downarrow \quad \Downarrow \alpha & & \mathbf{Set} \\
\mathbf{Cospan}_{\Lambda'} & \xrightarrow[\ a'\ ]{} &
\end{array}
$$

Moreover, we will show that every morphism of cospan-algebras arises in this way.

Using the Grothendieck construction, we can package the above isomorphisms (8) into a single one; this is our second main theorem.

Theorem 1.2. *There is an isomorphism of 1-categories,*

$$
\mathbf{Hyp}_{\mathrm{OF}} \quad \cong \quad \int^{\Lambda \in \mathbf{Set}_{\mathsf{List}}} \mathbf{Lax}(\mathbf{Cospan}_\Lambda, \mathbf{Set}).
$$

1.3. Plan of paper

This paper has three remaining sections. In Section 2 we formally introduce the key concepts, cospan-algebras (§2.1), Frobenius monoids (§2.2), and hypergraph categories (§2.3), giving a few basic examples,

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[1] Two labeled cospans (f_1, p, z, f_2) and (f_1', p', z', f_2') as in Eq. (7) are considered equivalent if there is a bijection $i \colon p \cong p'$ with $f_1' = f_1 \,\mathring{,}\, i$, $f_2' = f_2 \,\mathring{,}\, i$, and $z = i \,\mathring{,}\, z'$.

and pointing out some basic facts. In particular, in Section 2.4 we note that hypergraph categories do not obey the principle of equivalence, which we argue motivates the cospan-algebra perspective, and in Section 2.5 we remark on the interaction between the operadic and the monoidal categorical perspectives.

Section 3 develops the theory of hypergraph categories. We discuss four key properties. In Section 3.1, we see that hypergraph categories have a natural self-dual compact closed structure, so morphisms may be described by their so-called names. In Section 3.2, we show that $\mathbf{Cospan}_\Lambda$ is the free hypergraph category over Λ. In Section 3.3, we see that there is an (identity-on-objects, fully faithful) factorization of any hypergraph functor, and this implies that the category of hypergraph categories is fibred over $\mathbf{Set}_{\mathsf{List}}$. In Section 3.4, we prove a coherence theorem, showing that the 2-category of hypergraph categories is 2-equivalent to the 2-category of those that are objectwise-free.

The final section, Section 4, is devoted to proving that the 1-categories of cospan-algebras and objectwise-free hypergraph categories are equivalent. We do this by first showing how cospan-algebras may be constructed from hypergraph categories (§4.1), then how hypergraph categories may be constructed from cospan-algebras (§4.2), and finally that these two constructions define an equivalence of categories (§4.3).

Acknowledgements

1.4. Notation and terminology

- We generally denote composition in 1-categories using diagrammatic order, writing $f \,\mathring{,}\, g$ rather than $g \circ f$.
- If $\mathcal{C}$ is a category, we may write $c \in \mathcal{C}$ to mean c is an object in $\mathcal{C}$.
- Unless otherwise indicated, we use $\otimes$ to denote the monoidal product in a monoidal category, and I to denote the monoidal unit.
- By $\mathbf{Set}$ we always mean the symmetric monoidal category of (small) sets and functions, with the Cartesian product monoidal structure $(\{*\}, \times)$.
- By $\mathbf{FinSet}$ we always mean the (strict skeleton of the) symmetric monoidal category of finite sets and functions, with the coproduct monoidal structure $(\varnothing, +)$.
- Following John Baez and his students [15], we refer to the coherence maps for lax monoidal functors as *laxators*; this is in keeping with widely-used terms like unitor, associator, etc.
- For any natural number $m \in \mathbb{N}$, we abuse notation to also let $m := \{1, \ldots, m\}$; in particular $0 = \varnothing$. If the coproduct of X and Y exists, we denote its coproduct by $X + Y$; given morphisms $f: X \to Z$ and $g: Y \to Z$, we denote the universal morphism from the coproduct by $[f, g] : X + Y \to Z$.
- If Λ is a set, we denote by $\mathsf{List}(\Lambda)$ the set of pairs (m, x), where $m \in \mathbb{N}$ and $x: m \to \Lambda$; we may also denote (m, x) by $[x_1, \ldots, x_m]$. We may denote the list simply as x, in which case it will be convenient to denote the indexing set m as $|x|$. It is well-known that List is a functor $\mathsf{List}: \mathbf{Set} \to \mathbf{Set}$, and extends to a monad when equipped with the singleton list transformation $\mathsf{sing}: \mathrm{id}_{\mathbf{Set}} \to \mathsf{List}$ and the flatten transformation $\mathsf{flat}: \mathsf{List} \,\mathring{,}\, \mathsf{List} \to \mathsf{List}$. We denote the concatenation of lists x and y by $x \oplus y$, and we often denote the empty list by $\varnothing$.
- Given a functor $F: \mathcal{C} \to \mathbf{Cat}^{\mathrm{op}}$, we write $\int^{\mathcal{C}} F \to \mathcal{C}$ for the Grothendieck construction on F; this is the category over $\mathcal{C}$ that has objects given by pairs (X, a), where $X \in \mathcal{C}$ and $a \in FX$, and morphisms $(X, a) \to (Y, b)$ given by pairs (f, g), where $f: X \to Y$ is a morphism in $\mathcal{C}$ and $g: a \to F(f)(b)$ is a morphism in FX.

2. Basic definitions: cospan-algebras and hypergraph categories

In this section we review the definitions of the basic concepts we will use: cospan-algebras (§2.1), Frobenius structures (§2.2), and hypergraph categories (§2.3). We then discuss some perhaps undesirable ways in which hypergraph categories do not behave well with respect to equivalence of categories, hence motivating the cospan-algebra viewpoint (§2.4), and also briefly touch on the (disappearing) role of operads in this paper (§2.5).

2.1. Cospans and cospan-algebras

The main character in our story is $\mathbf{Cospan}_\Lambda$, where Λ is an arbitrary set. We already defined its objects, Λ-labeled finite sets (m, x), and its morphisms, which we call *labeled cospans*, in Eq. (7). The composition formula is given by pushout; see [8]. The monoidal unit is denoted $\varnothing$ and defined to be $(0, !)$, where $!\colon \emptyset \to \Lambda$ is the unique function. The monoidal product is denoted $\oplus$ and defined to be $(m_1, x_1) \oplus (m_2, x_2) :=$ $(m_1 + m_2, [x_1, x_2])$. Note that $\mathbf{Cospan}_\Lambda$ is the category of cospans in the category $\mathbf{FinSet}_\Lambda$, in which objects are Λ-labeled finite sets (m, x), and morphisms are simply *functions* between them that respect the typing x.

When Λ is a one-element set, we can suppress it from the notation and simply write $\mathbf{Cospan}$. It is the usual category whose objects are finite sets and whose morphisms are isomorphism classes of cospans, as discussed in Footnote 1.

Note that flattening lists is a coproduct operation. Given $x\colon m \to \mathsf{List}(\Lambda)$, we have m-many indexing sets $|x_1|, \ldots, |x_m|$ and maps $|x_i| \xrightarrow{x_i} \Lambda$. The flattened list $\mathsf{flat}(x)$ is indexed by the coproduct of the indexing sets $|x_i|$, and its content is given by the universal map:

$$[x_1, \ldots, x_m]\colon \sum_{i \in m} |x_i| \to \Lambda.$$

Given a function $f\colon \Lambda \to \mathsf{List}(\Lambda')$, we define a functor $\mathbf{Cospan}_f\colon \mathbf{Cospan}_\Lambda \to \mathbf{Cospan}_{\Lambda'}$ as follows. For an object $(m, x) \in \mathrm{Ob}(\mathbf{Cospan}_\Lambda)$, we obtain a function $m \xrightarrow{x} \Lambda \xrightarrow{f} \mathsf{List}(\Lambda')$, and hence $\mathsf{flat}(x \mathbin{\fatsemi} f) \in \mathsf{List}(\Lambda') = \mathrm{Ob}(\mathbf{Cospan}_{\Lambda'})$ by applying the monad multiplication. Since List is functorial, composing a Λ-labeled cospan with f, as shown on the left, induces a Λ'-labeled cospan as shown on the right, by flattening:

$$
\begin{array}{ccc}
|x| \longrightarrow |z| \longleftarrow |y| & & \sum_m |x \mathbin{\fatsemi} f| \longrightarrow \sum_p |z \mathbin{\fatsemi} f| \longleftarrow \sum_n |y \mathbin{\fatsemi} f| \\
\end{array}
$$

(9)

Proposition 2.1. *The above defines a functor* $\mathbf{Cospan}_-\colon \mathbf{Set}_{\mathsf{List}} \to \mathbf{Cat}$.

Proof. We gave the data for the functor on objects $\Lambda \in \mathbf{Set}_{\mathsf{List}}$, namely $\Lambda \mapsto \mathbf{Cospan}_\Lambda$, and on morphisms $\Lambda \to \mathsf{List}(\Lambda')$ in Eq. (9). To check that $\mathbf{Cospan}_f$ is a functor, first note that it sends identity morphisms in $\mathbf{Cospan}_\Lambda$ to those in $\mathbf{Cospan}_{\Lambda'}$. Then, observe that showing it preserves composition reduces to checking that, for any $a, b, c \in \mathbb{N}$ and pushout diagram as to the left below

$$
\begin{array}{ccc}
a \longrightarrow c & & \sum_{i \in a} |w(i)| \longrightarrow \sum_{k \in c} |y(k)| \\
\downarrow \quad \quad \downarrow \quad y & & \downarrow \quad\quad\quad\quad \downarrow \\
b \longrightarrow b \sqcup_a c \quad z & & \sum_{j \in b} |x(j)| \longrightarrow \sum_{l \in b \sqcup_a c} |z(l)| \\
\quad\quad\quad\quad x \quad \mathsf{List}(\Lambda') & &
\end{array}
$$

where $w\colon a \to \mathsf{List}(\Lambda')$ is the composite map, the diagram to the right is also a pushout. This is an easy calculation.

It is also straightforward to observe that $\mathbf{Cospan}\_$ is itself functorial: $\mathbf{Cospan}_{\mathrm{id}_\Lambda} = \mathrm{id}_{\mathbf{Cospan}_\Lambda}$, and if f, g are composable morphisms in $\mathbf{Set}_{\mathsf{List}}$, then $\mathbf{Cospan}_f \, \mathring{\,}\, \mathbf{Cospan}_g = \mathbf{Cospan}_{f \mathring{\,} g}$. $\quad\square$

Definition 2.2. A cospan-algebra consists of a set Λ, called the *label set*, and a lax symmetric monoidal functor

$$a\colon (\mathbf{Cospan}_\Lambda, \oplus) \longrightarrow (\mathbf{Set}, \times).$$

Let (Λ, a) and (Λ', a') be cospan-algebras. A *morphism* between them consists of a function $f\colon \Lambda \to \mathsf{List}(\Lambda')$ and a monoidal natural transformation α as shown here:

We write **Cospan-Alg** for the category of cospan-algebras and cospan-algebra morphisms.

Example 2.3. For a set Λ, we can define the cospan-algebra $\mathsf{Part}_\Lambda\colon \mathbf{Cospan}_\Lambda \to \mathbf{Set}$ represented by the monoidal unit 0 of $\mathbf{Cospan}_\Lambda$. Explicitly, it is defined on $x\colon m \to \Lambda$ by

$$\mathsf{Part}_\Lambda(x) \coloneqq \mathbf{Cospan}_\Lambda(0, x) = \left\{ (p, f) \;\middle|\; \begin{array}{c} p \xleftarrow{\;f\;} m \\ {\scriptstyle z}\downarrow \quad \nearrow {\scriptstyle x} \\ \Lambda \leftarrow \end{array} \right\}$$

the set of ways to partition m into some number of (possibly empty) parts, respecting $x\colon m \to \Lambda$. This is clearly lax monoidal: given functions $m \to m'$ and $n \to n'$, one obtains a function $m + n \to m' + n'$.

We shall see in Section 4 that any hypergraph category gives rise to a cospan-algebra.

The following observation is immediate from Definition 2.2.

Proposition 2.4. *We have an isomorphism of categories*

$$\mathbf{Cospan}\text{-}\mathbf{Alg} \cong \int^{\Lambda \in \mathbf{Set}_{\mathsf{List}}} \mathbf{Lax}(\mathbf{Cospan}_\Lambda, \mathbf{Set}).$$

2.2. Special commutative Frobenius monoids

In a hypergraph category, every object is equipped with the structure of a special commutative Frobenius monoid, which we call a Frobenius structure. In this section we recall the definition and give important examples.

We will represent morphisms in monoidal categories using the string diagrams introduced by Joyal and Street [10]. We draw $\times\colon X \otimes Y \to Y \otimes X$ for the braiding in a symmetric monoidal category. Diagrams are to be read left to right; we shall suppress the labels, since we deal with a unique generating object and a unique generator of each type.

Definition 2.5. A special commutative Frobenius monoid $(X, \mu, \eta, \delta, \epsilon)$ in a symmetric monoidal category $(\mathcal{C}, \otimes, I)$ is an object X of $\mathcal{C}$ together with maps

$$\mu\colon X \otimes X \to X \qquad \eta\colon I \to X \qquad \delta\colon X \to X \otimes X \qquad \epsilon\colon X \to I$$

obeying the commutative monoid axioms

(associativity)   (unitality)   (commutativity)
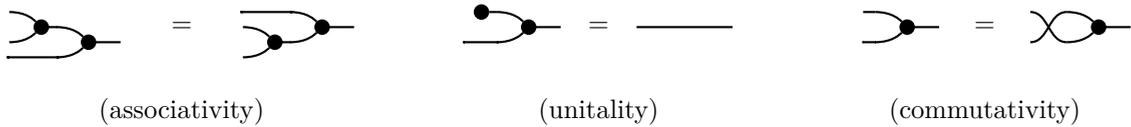
the cocommutative comonoid axioms

(coassociativity)   (counitality)   (cocommutativity)
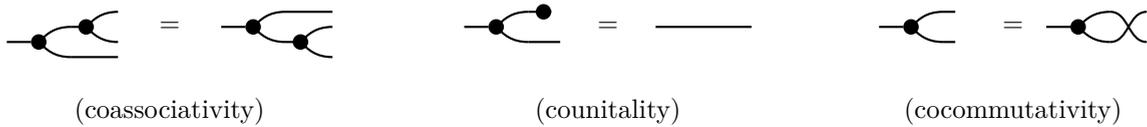
and the Frobenius and special axioms

(Frobenius)   (special)
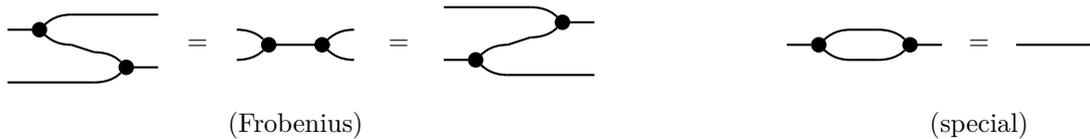
We say that $(\mu, \eta, \delta, \epsilon)$ is a Frobenius structure on X, and call these four morphisms Frobenius generators. We further refer to any morphism generated from these maps using composition, monoidal product, identity maps, and braiding maps as a Frobenius map.

Example 2.6. In any symmetric monoidal category, there is a canonical Frobenius structure on the monoidal unit I. Indeed, the left and right unitors give (equal) isomorphisms $\lambda_I = \rho_I\colon I \otimes I \cong I$, so define $\mu_I \coloneqq \rho_I$ and $\delta_I \coloneqq \rho_I^{-1}$, and define $\eta_I = \mathrm{id}_I = \epsilon_I$. It is easy to see that this is the only possible Frobenius structure for which $\eta_I = \mathrm{id}_I = \epsilon_I$.

Example 2.7. Let $\mathcal{C}$ be a symmetric monoidal category such that $I \otimes I = I$. For example, $\mathcal{C}$ could be a symmetric monoidal category with one object—that is, a commutative monoid considered as a one object category.

A Frobenius structure on I then consists of maps $\mu, \eta, \delta, \epsilon$, all of type $I \to I$. The morphisms $I \to I$ in a monoidal category always form a commutative monoid $(M, *, e)$; this follows from an Eckmann-Hilton argument [12]. The axioms of Frobenius structures then say that $\mu, \eta, \delta, \epsilon$, as elements of M, satisfy $\mu * \eta = e$, $\mu * \delta = e$, and $\delta * \epsilon = e$. This implies that a Frobenius structure on the unit of a symmetric monoidal category can be identified with an invertible element μ in the monoid of scalars.

Example 2.8. Consider the symmetric monoidal category **Cospan**. We will construct a Frobenius structure on the object 1. To do so, we need to define morphisms $\mu\colon 1 \oplus 1 \to 1$, $\eta\colon 0 \to 1$, $\delta\colon 1 \to 1 \oplus 1$, and $\epsilon\colon 1 \to 0$ in **Cospan**, and then check that they satisfy the required equations.

Recall that $1 \oplus 1 = 2$, the two element set $\{1, 2\}$. Each Frobenius generator will be the unique cospan of the required domain and codomain with apex 1—this is well-defined because 1 is terminal in **FinSet**. For example, we take $\mu\colon 2 \to 1$ to be the cospan
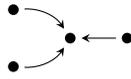
and $\epsilon\colon 1 \to 0$ to be the cospan

where the dotted square represents the empty set.

One can then check that the nine equations in Definition 2.5 hold: in each case both sides of the equation represent the unique cospan with apex 1. For example, the associativity axiom says that the composite cospans $(\mu \otimes 1)\,\mathring{,}\, \mu$ and $(1 \otimes \mu)\,\mathring{,}\, \mu$ are equal, namely to the cospan
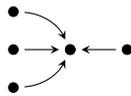
**Example 2.9.** Example 2.8 generalizes to any object in any category with finite colimits. Indeed, let $\mathcal{C}$ be a category with finite colimits. Write $\mathbf{Cospan}(\mathcal{C})$ for symmetric monoidal category with the objects of $\mathcal{C}$ as its objects, isomorphism classes of cospans in $\mathcal{C}$ as its morphisms, and coproduct $+$ as its monoidal product. Then each object X of $\mathbf{Cospan}(\mathcal{C})$ has a canonical Frobenius structure, with Frobenius maps exactly those cospans built from coproducts and copairings of id_X. In particular, each object (m, x) of $\mathbf{Cospan}_\Lambda$ has a canonical Frobenius structure, which can be thought of as m parallel copies of the Frobenius structure on $1 \in \mathbf{Cospan}$.

Next in Examples 2.10 and 2.11 we give two different Frobenius structures on the same object. Let $(\mathbf{LinRel}, \oplus)$ denote the symmetric monoidal category of finite-dimensional real vector spaces V and linear relations between them, with direct sum as the monoidal product. Recall that a linear relation between V and W—i.e. a morphism in $\mathbf{LinRel}$—is a linear subspace $R \subseteq V \oplus W$ of their direct sum. The composite of $R \subseteq V \oplus W$ and $S \subseteq W \oplus X$ is the relation

$$R\,\mathring{,}\,S := \big\{(v, x) \in V \oplus X \mid \exists (w \in W).\, (v, w) \in R \text{ and } (w, x) \in S.\big\} \tag{10}$$

The identity morphism on V is represented by the bare reflexive relation $\{(v, v') \mid v = v'\}$.

Example 2.10. We now define a Frobenius structure on the object $\mathbb{R} \in \mathbf{LinRel}$. Consider the relation $E \subseteq \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$ given by $(a, b, c) \in E$ iff $a = b = c$. This is a linear relation because it is closed under addition and scalar multiplication; hence we can take E to represent $\mu\colon (\mathbb{R} \oplus \mathbb{R}) \to \mathbb{R}$. We can also take E to represent $\delta\colon \mathbb{R} \to (\mathbb{R} \oplus \mathbb{R})$. We can take the maximal relation $\mathbb{R} \subseteq \mathbb{R}$ to represent $\eta\colon \mathbb{R}^0 \to \mathbb{R}$ and $\epsilon\colon \mathbb{R} \to \mathbb{R}^0$.

It is easy to check that the nine equations required by Definition 2.5 are satisfied. For example, unitality says that the composite of $\{(a, (b, c)) \mid c = a\}$ and $\{((b, c), d) \mid b = c = d\}$ should be the identity map $\{(a, d) \mid a = d\}$, and one checks that it is by working through Eq. (10).

Example 2.11. Here we define a different Frobenius structure on the object $\mathbb{R} \in \mathbf{LinRel}$. Let μ be represented by the relation $\{(a, b, c) \mid a + b = c\} \subseteq \mathbb{R}^3$, let η be represented by the relation $\{0\} \subseteq \mathbb{R}$. Similarly, let δ be represented by the relation $\{(a, b, c) \mid a = b + c\}$ and ϵ be represented by the relation $\{0\} \subseteq \mathbb{R}$.

Again, it is easy to check that the equations required by Definition 2.5 are satisfied. For example, the Frobenius law requires that for any $a_1, a_2, b_1, b_2 \in \mathbb{R}$, the equation $a_1 + a_2 = b_1 + b_2$ holds iff there exists an $x \in \mathbb{R}$ such that the equations $a_1 = b_1 + x$ and $x + a_2 = b_2$ hold; this is easily checked.

2.3. Hypergraph categories

In a hypergraph category, every object has a chosen Frobenius structure, chosen compatibly with the monoidal structure.

Definition 2.12. A hypergraph category is a symmetric monoidal category $(\mathcal{H}, \otimes, I)$ in which each object X is equipped with a Frobenius structure $(\mu_X, \eta_X, \delta_X, \epsilon_X)$, satisfying

(11)

as well as the *unit coherence axiom*, that $\eta_I = \mathrm{id}_I = \epsilon_I$.

A functor (F, φ) of hypergraph categories, or hypergraph functor, is a strong symmetric monoidal functor (F, φ) that preserves the hypergraph structure. More precisely, the latter condition means that if the Frobenius structure on X is $(\mu_X, \eta_X, \delta_X, \epsilon_X)$ then that on FX must be

$$\left(\varphi_{X,X} \, \mathbin{\mathring{,}} \, F\mu_X, \quad \varphi_I \, \mathbin{\mathring{,}} \, F\eta_X, \quad F\delta_X \, \mathbin{\mathring{,}} \, \varphi_{X,X}^{-1}, \quad F\epsilon_X \, \mathbin{\mathring{,}} \, \varphi_I^{-1} \right). \tag{12}$$

We write **Hyp** for the category with hypergraph categories as objects and hypergraph functors as morphisms, and $\mathcal{Hyp}$ for the 2-category with, in addition to these objects and morphisms, monoidal natural transformations as 2-morphisms.

Remark 2.13. Note that every natural transformation between hypergraph functors is invertible, i.e. a natural isomorphism. This follows from Proposition 3.1 and the fact that every natural transformation between compact closed categories is invertible.

Example 2.14. Following Example 2.9, the category of $\mathbf{Cospan}(\mathcal{C})$ of cospans in any category with finite colimits is canonically a hypergraph category. Note in particular that $\mathbf{Cospan}_\Lambda = \mathbf{Cospan}(\mathbf{FinSet}_\Lambda)$ is a hypergraph category.

Example 2.15. If we let $\mathcal{C}$ be the opposite of a category with finite limits in Example 2.14, we see that the category of $\mathbf{Span}(\mathcal{C})$ of *spans* in any category with finite limits is canonically a hypergraph category. In fact, if $\mathcal{C}$ is regular, then the category of relations in $\mathcal{C}$ is also a hypergraph category. In particular, the category $\mathbf{Rel}$ of sets and binary relations can be equipped with the structure of a hypergraph category [7].

Example 2.16. The category of finite-dimensional vector spaces where each object each has a chosen basis is a hypergraph category [11].

Remark 2.17. Note that the condition that the Frobenius structure on the monoidal unit be the structure $(\rho_I^{-1}, \mathrm{id}_I, \rho_I, \mathrm{id}_I)$ of Example 2.6 has been omitted from some, but not all, previous definitions. We shall see

in Theorem 3.22 that the unit coherence axiom is crucial for the strictification of hypergraph categories, and hence for the equivalence with cospan-algebras.

One reason that this additional unit coherence axiom may have been overlooked is that, in the strict case, this additional axiom does not alter the definition; we will prove this in Proposition 2.18. In Example 2.19 we will give an example which shows that the unit coherence axiom does not follow from the old ones; it really is a new addition.

Proposition 2.18. *Suppose $\mathcal{H}$ is a strict symmetric monoidal category in which each object is equipped with a Frobenius structure such that Eq. (11) is satisfied. Then $\mathcal{H}$ is a hypergraph category.*

Proof. We must show that the Frobenius structure on the monoidal unit is $(\rho_I^{-1}, \mathrm{id}_I, \rho_I, \mathrm{id}_I)$.

First, note that in any strict monoidal category, we have $I = I \otimes I$ and $\rho_I = \mathrm{id}_I$, so the Frobenius structure constructed from the unitors on I, as detailed in Example 2.6, is simply equal to $(\mathrm{id}_I, \mathrm{id}_I, \mathrm{id}_I, \mathrm{id}_I)$. This is the unique Frobenius structure on I obeying the equations of Eq. (11). To see this, recall that by Example 2.7, a Frobenius structure on I simply amounts to a choice of invertible map $\mu \colon I \to I$. The first equation of Eq. (11) requires further that $\mu = \mu * \mu$. But the only monoid element that is both idempotent and invertible is the identity, $\mathrm{id}_I = \mu * \mu^{-1} = \mu * \mu * \mu^{-1} = \mu$. $\square$

Example 2.19. To show that the unit coherence axiom indeed alters the definition of hypergraph category in general, here we provide a example of a (necessarily non-strict) symmetric monoidal category $\mathcal{X}$, equipped with a Frobenius structure on each object, that fails only this additional axiom.

Let $(\mathcal{X}, \oplus, I)$ be the symmetric monoidal category with two objects, I and O, such that every homset is equal to $\{0, 1\}$, such that $I \oplus I = O \oplus O = O$ and $I \oplus O = O \oplus I = I$, and such that composition and monoidal product of morphisms are all given by addition modulo 2. Note that the identity maps on I and O are both 0. The coherence maps for the monoidal product are also given by the maps 0; from this naturality and all coherence conditions are immediate.

We may choose Frobenius structures $(1, 1, 1, 1)$ on I and $(0, 0, 0, 0)$ on O. These structures obey the equations in Eq. (11), but do not obey the condition that the Frobenius structure on I is $(\rho_I^{-1}, \mathrm{id}_I, \rho_I, \mathrm{id}_I) = (0, 0, 0, 0)$.

2.4. Critiques of hypergraph categories as structured categories

In this brief subsection we sketch two examples to promote the idea that hypergraph categories should not be thought of as structured categories. First we show in Example 2.20 that hypergraph structures do not extend along equivalences of categories. Second we show in Example 2.21 that an essentially surjective and fully faithful hypergraph functor may fail to be a hypergraph equivalence. These critiques motivate the upcoming cospan-algebra perspective.

Example 2.20. Here we will produce a category $\mathbf{LinRel}_2$, an equivalence of categories $F \colon \mathbf{LinRel}_2 \to \mathbf{LinRel}$, and a hypergraph structure on $\mathbf{LinRel}_2$ for which there is no extension along F, i.e. there is no hypergraph structure on $\mathbf{LinRel}$ under which F is a hypergraph functor. The idea is to let $\mathbf{LinRel}_2$ house two copies of $\mathbf{LinRel}$ and to cause a problem by equipping them with the two different Frobenius structures from Examples 2.10 and 2.11.

Let $\mathbf{LinRel}_2$ be the hypergraph category with two isomorphic copies of every object in $\mathbf{LinRel}$, but the same maps.[2] By definition there is a functor $F \colon \mathbf{LinRel}_2 \to \mathbf{LinRel}$, which is both fully faithful

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[2] The definition of $\mathbf{LinRel}_2$ can be made more precise, once we have defined the (io, ff)-factorization of hypergraph categories (§3.3). Namely, consider the unique hypergraph functor $\mathbf{Cospan}_{\{1,2\}} \to \mathbf{LinRel}$ sending $1, 2 \mapsto \mathbb{R}$, let $\mathbf{Cospan}_{\{1,2\}} \xrightarrow{\text{io}} \mathcal{H} \xrightarrow{\text{ff}} \mathbf{LinRel}$ be its (io, ff)-factorization, and let $\mathbf{LinRel}_2 := \mathcal{H}$.

and essentially surjective, so it is an equivalence. As we shall see in detail in Lemma 3.10, we can put a hypergraph structure on $\mathbf{LinRel}_2$ by declaring a Frobenius structure on the two copies of $\mathbb{R}$; we use the two different such structures from Examples 2.10 and 2.11. Then no Frobenius structure on $\mathbb{R} \in \mathbf{LinRel}$ will satisfy Eq. (12) in Definition 2.12.

Example 2.21. To show that a fully faithful, essentially surjective hypergraph functor need not be a hypergraph equivalence, we simply run Example 2.20 the other way. Namely, let $\mathbf{LinRel}_2$ be the hypergraph category constructed in Example 2.20, give $\mathbf{LinRel}$ the additive hypergraph structure from Example 2.11, and consider the hypergraph functor $\mathbf{LinRel} \to \mathbf{LinRel}_2$ sending the generator to the appropriate generator. This is essentially surjective and fully faithful, but it is not an equivalence of hypergraph categories because, as we saw in Example 2.20, there is no hypergraph functor to serve as its inverse.

The critique leveled by Examples 2.20 and 2.21 is important, because it says that in an important sense hypergraph categories do not behave like structured categories. This critique dissolves—i.e. the above problems become impossible to state—when we treat hypergraph categories as cospan-algebras.

Thus thinking of hypergraph categories as cospan-algebras has distinct advantages. However, it also comes with a couple of costs. The first is that cospan-algebras do not take into account 2-morphisms, i.e. the natural transformations between hypergraph functors; the question of whether and/or how this can be rectified, and indeed if it needs to rectified, remains open. The second cost is that cospan-algebras correspond to hypergraph categories that are objectwise-free (`OF`). Luckily, this second issue is not very important: in Section 3.4, we will show that every hypergraph category is naturally equivalent to one that is `OF`.

2.5. A word on operads

In the introduction, we spoke of operads. Operads are generalizations of categories in which each morphism has a finite number of inputs and one output, e.g. $\varphi \colon x_1, \ldots, x_n \to y$. In the context of this paper, operads govern the structure of wiring diagrams like the ones in Eq. (6), and one should imagine the x's as the interior cells or boxes and the y as the exterior cell or box of a wiring diagram φ.

The reader who is unfamiliar with operads need not worry: the only operads we use are those that underlie symmetric monoidal categories $\mathcal{M}$, where operad morphisms φ as above come from morphisms $\varphi \colon (x_1 \otimes \cdots \otimes x_n) \to y$ in $\mathcal{M}$.

In fact, throughout this paper we work exclusively in the monoidal setting, so operads will disappear from the discussion. There are two reasons we bring up operads at all. First, they are a bit more general, so further work in this area will sometimes require one to use operads rather than monoidal categories. More relevant, however, is the fact that the wiring diagram pictures we want to draw more naturally fit with operads. For example, here we draw the "same morphism" in two ways: operadic style $\phi \colon f, g, h \to i$ on the left and monoidal style $\phi \colon f \otimes g \otimes h \to i$ on the right:
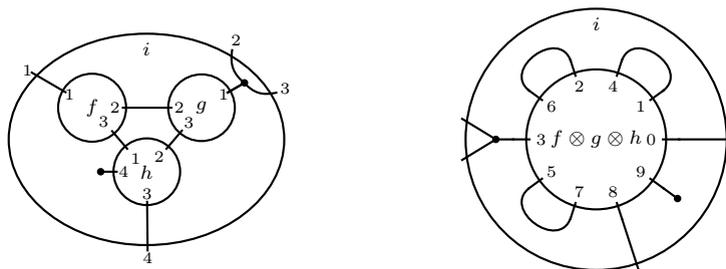
Although monoidal-style wiring diagrams are often more difficult to visually parse than operad-style, the symbolic notation for monoidal categories is often easier to parse than that of operads. Thus the only place operads will appear from now on is in visualizing wiring diagrams.

3. Properties of hypergraph categories

In this section, we discuss some basic properties of hypergraph categories. In Section 3.1, we show that they are self-dual compact closed. In Section 3.2.2, we show that $\mathbf{Cospan}_\Lambda$ is both the free hypergraph category and the free OF-hypergraph category on a set Λ. In Section 3.3 we show that any hypergraph functor can be factored as an identity-on-objects (io) hypergraph functor followed by a fully faithful (ff) hypergraph functor, and use this factorization to construct a Grothendieck fibration $U \colon \mathbf{Hyp}_{\mathsf{OF}} \to \mathbf{Set}_{\mathsf{List}}$. Finally in Section 3.4 we prove that every hypergraph category can be strictified to an equivalent hypergraph category that is objectwise-free.

3.1. Hypergraph categories are self-dual compact closed

A compact closed category is a symmetric monoidal category $(\mathcal{C}, \otimes, I)$ such that every object X is dualizable—i.e. there exists an object X^* and morphisms $\mathtt{cup}_X \colon I \to X \otimes X^*$, depicted $\subset$, and $\mathtt{cap}_X \colon X^* \otimes X \to I$, depicted $\supset$, which satisfy the zigzag identities[3]:

$$ \supset = \overline{} \qquad\qquad \subset = \overline{} \tag{13} $$

This notion generalizes duals in finite-dimensional vector spaces. A compact closed category is called *self-dual* if every object serves as its own dual, $X^* := X$; the category of finite-dimensional based vector spaces (where each vector space is equipped with basis) is self-dual compact closed.

A basic property of hypergraph categories is that they are self-dual compact closed. Indeed, a self-duality for each object can be constructed using the Frobenius maps for each object.

Proposition 3.1. *Every hypergraph category $\mathcal{H}$ is self-dual compact closed. Moreover, each object X is equipped with a canonical self-duality defined by $\mathtt{cup}_X := \eta_X \,\mathring{,}\, \delta_X \colon I \to X \otimes X$ and $\mathtt{cap}_X := \mu_X \,\mathring{,}\, \epsilon_X \colon X \otimes X \to I$.*

Proof. This result is well known; see for example [18]. The proof is straightforward: the zigzag identities (13) are an immediate consequence of the Frobenius and co/unitality axioms. For example:

This means that in any hypergraph category, we have a bijection between morphisms $X \to Y$, and morphisms $I \to X \otimes Y$.

Proposition 3.2. *For any two objects X, Y in a self-dual compact closed category $\mathcal{C}$, there is a bijection $\mathcal{C}(X, Y) \cong \mathcal{C}(I, X \otimes Y)$.*

Proof. For any $f \colon X \to Y$, and any $g \colon I \to X \otimes Y$ define

$$ \mathtt{cup}_X \,\mathring{,}\, (\mathrm{id}_X \otimes f) \qquad\qquad (\mathrm{id}_X \otimes g) \,\mathring{,}\, (\mathtt{cap}_X \otimes \mathrm{id}_Y) $$

It is easy to prove that $\widehat{\cdot}$ and $\widecheck{\cdot}$ are mutually inverse. $\square$

---

[3] These are often also called the triangle identities, as one can think of X^* as a left adjoint to X (see [5]), or the snake identities (see Eq. (13)).

We will refer to $\widehat{f}$ as the *name* of f. This notion will be critical for the equivalence between hypergraph categories and cospan-algebras: given a hypergraph category $\mathcal{H}$, the corresponding cospan-algebra $A_{\mathcal{H}}$ will record the *names* of the morphisms $\mathcal{H}$, rather than the morphisms themselves. But note that homsets $\mathcal{H}(X, Y)$ are indexed by *two* objects, X and Y, while $A_{\mathcal{H}}(X)$ just depends on one. It is the self-dual compact closed structure that allows us to switch between these two viewpoints.

For any three objects X, Y, Z in a self-dual compact closed category, we may define a morphism $\mathrm{comp}^Y_{X,Z} :=$ $\mathrm{id}_X \otimes \mathrm{cap}_Y \otimes \mathrm{id}_Z : X \otimes Y \otimes Y \otimes Z \longrightarrow X \otimes Z$:

$$\mathrm{comp}^Y_{X,Z} := \begin{array}{l} X \;\rule[0.5ex]{1.5em}{0.4pt} \\ Y \;\rule[0.5ex]{0.8em}{0.4pt}\!\!\supset \\ Y \;\rule[0.5ex]{0.8em}{0.4pt}\!\!\supset \\ Z \;\rule[0.5ex]{1.5em}{0.4pt} \end{array} \tag{14}$$

Below in Propositions 3.3 and 3.4, we show that the morphism comp acts like composition on names and that comp can be used to recover a morphism from its name; both propositions are immediate from the zigzag identities.

Proposition 3.3. *For any morphisms $f\colon X \to Y$, $g\colon Y \to Z$ in a self-dual compact closed category, we have* $(\widehat{f} \otimes \widehat{g}) \,\mathring{,}\, \mathrm{comp}^Y_{X,Z} = \widehat{f \,\mathring{,}\, g}$ *:*
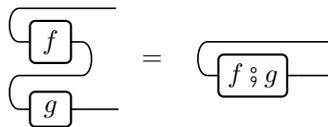
Proposition 3.4. *For any morphism $f\colon X \to Y$ in a self-dual compact closed category, we have $(\mathrm{id}_X \oplus \widehat{f}) \,\mathring{,}\,$* $\mathrm{comp}^X_{\varnothing, Y} = f$*:*
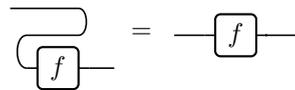
Example 3.5. In $\mathbf{Cospan}_\Lambda$ the morphism $\mathrm{comp}^Y_{X,Z}$ is given by the cospan

$$X \oplus Y \oplus Y \oplus Z \xrightarrow{\mathrm{id}_X \oplus [\mathrm{id}_Y, \mathrm{id}_Y] \oplus \mathrm{id}_Z} X \oplus Y \oplus Z \xleftarrow{\mathrm{id}_X \oplus !_Y \oplus \mathrm{id}_Z} X \oplus Z.$$
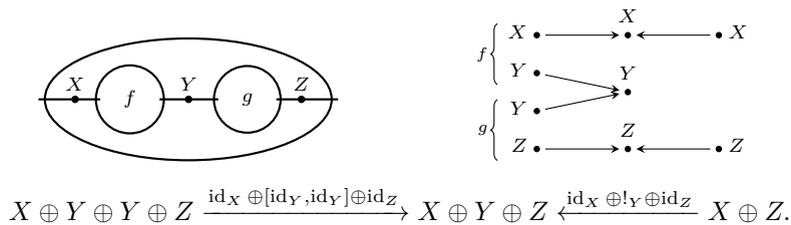
Example 3.5 will be useful later, when we see that not only are the Frobenius structures of hypergraph categories controlled by cospans, but so are the identities and the composition law!

3.2. Free hypergraph categories

In this section we show that $\mathbf{Cospan}_\Lambda$ is both the free hypergraph category and the free OF-hypergraph category on a set Λ. We first discuss the relationship between $\mathbf{Cospan}$ and Frobenius monoids.

3.2.1. Cospans and Frobenius monoids

Example 2.8, where we define a certain Frobenius structure on the object 1 in $\mathbf{Cospan}$, is central to the interplay between cospan-algebras and hypergraph categories. This is because $\mathbf{Cospan}$ is free special commutative Frobenius monoid on one generator.

Write $\sigma\colon 2 \to 2$ for the cospan
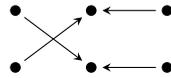
and id$\colon 1 \to 1$ for the cospan

$$(15)$$
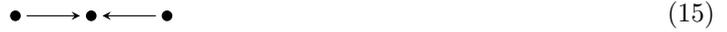
Lemma 3.6. *The category* $(\mathbf{Cospan}, \oplus)$ *is generated, as a symmetric monoidal category, by the morphisms* μ, η, δ, ϵ. *That is, given any cospan* $c = (m \to p \leftarrow n)$, *we may write down an expression that is equal to* c, *using only the cospans* μ, η, δ, ϵ; *the cospans* σ, id; *composition; and monoidal product.*

Proof. Note that any function can be factored as a permutation, followed by an order-preserving surjection, followed by an order-preserving injection. Applying this to each leg of a cospan $c = (m \to p \leftarrow n)$ gives a factorization

$$m \xrightarrow{\cong} m \xrightarrow{\text{ord.}} m' \rightarrowtail^{\text{ord.}} p \xleftarrow{\text{ord.}} n' \xleftarrow{\text{ord.}} n \xleftarrow{\cong} n. \qquad (16)$$

Each of these six functions may be viewed as a cospan, with the other leg supplied by the identity map on the codomain of the function. Since the pushout of a morphism a along an identity map is just a again, this gives a factorization of c into the composite of six cospans. It remains to show each of these cospans can be built from the required generators.

The key idea is that the permutations may be constructed by (composites of) braidings σ, the order-preserving surjection $m \twoheadrightarrow m'$ by multiplications, $m' \rightarrowtail p$ by units, $n' \leftarrowtail n$ by counits, and $p \leftarrow n'$ by comultiplications.

To elaborate, for the permutations, observe that any transposition of adjacent elements may be constructed as the monoidal product of σ with some number of identities id. It is then well known from group theory that transpositions of adjacent elements generate all permutations. For order-preserving surjections $s\colon m \twoheadrightarrow m'$, observe that this s is the monoidal product of surjections $s_i\colon s^{-1}(i) \twoheadrightarrow 1$ for each $i \in m'$. Thus without loss of generality we need only consider surjections $m \twoheadrightarrow 1$. Any such surjection may be written as $(\text{id} \oplus \cdots \oplus \text{id} \oplus \mu) \, \mathring{,} \cdots \mathring{,} \, (\text{id} \oplus \mu) \, \mathring{,} \, \mu$, where this is the composite of $m - 1$ terms, each of which is a sum of $m - 2$ identities and μ. For order-preserving injections $j\colon m' \rightarrowtail p$, we may take the monoidal product $j_1 \oplus \cdots \oplus j_p$, where $j_i = \eta$ if $j^{-1}(i) = \varnothing$, and $j_i = \text{id}$ otherwise. The remaining cases are analogous. $\quad\square$

The above argument is perhaps clearest through a detailed example.

Example 3.7. We now build the cospan Eq. (4)—shown to the left below—from the generators in Lemma 3.6.
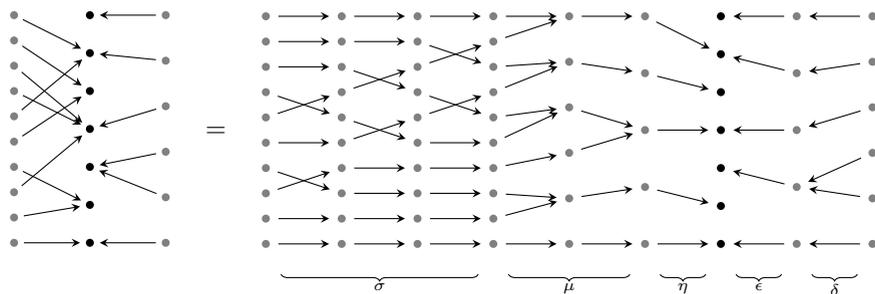
Lemma 3.6 implies that to define a symmetric monoidal functor $\mathbf{Cospan} \to \mathcal{C}$ we simply need to say where to send the generators of $\mathbf{Cospan}$, and check that the relevant equations between these generators

hold. The following proposition says that these equations are exactly the axioms of special commutative Frobenius monoids.

Proposition 3.8 *(Cospan is the theory of special commutative Frobenius monoids). Let $(\mathcal{C}, \otimes)$ be a symmetric monoidal category. There is a one-to-one correspondence:*

$$\left\{ \begin{matrix} \text{special commutative} \\ \text{Frobenius monoids in } \mathcal{C} \end{matrix} \right\} \leftrightarrow \left\{ \begin{matrix} \text{strict symmetric monoidal functors} \\ (F, \varphi) \colon (\mathbf{Cospan}, \oplus) \to (\mathcal{C}, \otimes) \end{matrix} \right\}.$$

Proof (sketch). Suppose we have a symmetric monoidal functor $(F, \varphi) \colon (\mathbf{Cospan}, \oplus) \to (\mathcal{C}, \otimes)$. Let $\mu, \eta, \delta, \epsilon$ be the Frobenius generators in **Cospan** as defined in Example 2.8. It is straightforward to verify that

$$(F1, \quad \varphi_{1,1} \, \fatsemi \, F\mu, \quad \varphi_\varnothing \, \fatsemi \, F\eta, \quad F\delta \, \fatsemi \, \varphi_{1,1}^{-1}, \quad F\epsilon \, \fatsemi \, \varphi_\varnothing^{-1})$$

is a special commutative Frobenius monoid in $\mathcal{C}$.

The converse is trickier. Suppose $(X, \mu_X, \eta_X, \delta_X, \epsilon_X)$ is a special commutative Frobenius monoid in $\mathcal{C}$; we wish to define a strict symmetric monoidal functor $\mathsf{Frob} \colon \mathbf{Cospan} \to \mathcal{C}$. We send the object 1 to $\mathsf{Frob}(1) := X$; this implies $m \in \mathbf{Cospan}$ maps to $\mathsf{Frob}(m) := X^{\otimes m}$. Using Lemma 3.6, to define a candidate strict symmetric monoidal functor we only need to say where to map the cospans μ, η, δ, and ϵ. This is easy: we map them to the corresponding Frobenius generator on X.

Verifying functoriality, however, amounts to a technical exercise verifying that the axioms of special commutative Frobenius monoids exactly describe pushouts of finite sets. This is treated at a high level, using distributive laws for props, in [13], and also remarked upon in [18]; we are not aware of any more detailed treatment in writing. Once functoriality is verified, it is straightforward to also check that Frob defines a strict symmetric monoidal functor.

These constructions are evidently inverses, and so we have the stated one-to-one correspondence. □

3.2.2. $\mathbf{Cospan}_\Lambda$ as free hypergraph category

We now wish to show that $\mathbf{Cospan}_\Lambda$ is a free hypergraph category on Λ. We begin with a lemma that provides an easy way to equip an $\mathsf{OF}(\Lambda)$ symmetric monoidal category (see Definition 3.9) with a hypergraph structure: assign a Frobenius structure to each element of Λ.

Definition 3.9. An *objectwise-free structure* on a strict symmetric monoidal category $(\mathcal{C}, \otimes)$ consists of a set Λ and an isomorphism of monoids $\mathsf{List}(\Lambda) \cong \mathrm{Ob}(\mathcal{C})$. In this case we say that $(\mathcal{C}, \otimes)$ is OF or $\mathsf{OF}(\Lambda)$.

Lemma 3.10. *Suppose that $\mathcal{C}$ is an $\mathsf{OF}(\Lambda)$ symmetric monoidal category. Assigning a Frobenius structure to each object $[l]$, for each $l \in \Lambda$, induces a unique hypergraph structure on $\mathcal{C}$.*

Furthermore, if $\mathcal{C}$ is as above, $\mathcal{D}$ is a hypergraph category, and $F \colon \mathcal{C} \to \mathcal{D}$ is a symmetric monoidal functor, then F is a hypergraph functor iff F preserves the Frobenius structure on each $l \in \Lambda$.

Proof. Suppose that for each $l \in \Lambda$ we are given a Frobenius structure $(\mu_l, \eta_l, \delta_l, \epsilon_l)$. We need to show that this uniquely determines a Frobenius structure on every object, satisfying (11) and restricting to the chosen one on each $l \in \Lambda$. Any object in $\mathcal{C}$ can be uniquely written as a list $[l_1, \cdots, l_n]$ for some $n \in \mathbb{N}$. By induction, we may assume $n = 0$ or $n = 2$. When $n = 0$ the Frobenius structure is by definition given by the unitors, while when $n = 2$ the Frobenius structure on the monoidal product $[l, m] = l \oplus m$ is forced to be that given by Eq. (11).

The second claim is similar and straightforward. □

Remark 3.11. It will be useful to give a more explicit description of the construction from Lemma 3.10, at least in the case of μ, in order to fix ideas. Given an object $l = l_1 \oplus \cdots \oplus l_n$ and a multiplication map $\mu_i \colon l_i \oplus l_i \to l_i$ for each i, the multiplication map μ_l is given by

$$\mu_l \colon (l_1 \oplus \cdots \oplus l_n) \oplus (l_1 \oplus \cdots \oplus l_n) \cong l_1 \oplus l_1 \oplus \cdots \oplus l_n \oplus l_n \xrightarrow{\mu_1 \oplus \ldots \oplus \mu_n} l_1 \oplus \cdots \oplus l_n.$$

Example 3.12. The category $\mathbf{Cospan}_\Lambda$ can be given the structure of a hypergraph category. Indeed, it is enough by Lemma 3.10 to give a Frobenius structure on each $l \in \Lambda$. We assign them all the same structure, namely the one given in Example 2.8.

Similarly, since $\mathbf{LinRel}$ is objectwise-free on $\mathbb{R}$, the Frobenius structures on $\mathbb{R}$ given in Examples 2.10 and 2.11 induce two different hypergraph structures on $\mathbf{LinRel}$.

Corollary 3.13. *The functor* $\mathbf{Cospan}_\_ \colon \mathbf{Set}_{\mathsf{List}} \to \mathbf{Cat}$ *from Proposition 2.1 factors through the inclusion* $\mathbf{Hyp}_{\mathsf{OF}} \subseteq \mathbf{Cat}$, *giving a functor*

$$\mathbf{Cospan}_\_ \colon \mathbf{Set}_{\mathsf{List}} \to \mathbf{Hyp}_{\mathsf{OF}}.$$

Proof. In Example 3.12, we showed that $\mathbf{Cospan}_\Lambda$ is a hypergraph category for each Λ, and it is objectwise-free because $\mathrm{Ob}(\mathbf{Cospan}_\Lambda) = \mathsf{List}(\Lambda)$. If $f \colon \Lambda \to \mathsf{List}(\Lambda')$ is a function, we need to check that $\mathbf{Cospan}_f$ preserves the Frobenius structure $(\mu, \eta, \delta, \epsilon)$ on every object. This is a simple calculation; we carry it out for μ and leave the others to the reader.

By Lemma 3.10, it suffices to check that μ is preserved for an arbitrary $l \in \Lambda$. The cospan μ_l is shown on the left of the diagram below, and if $f(l)$ is a list of length n, then by Eq. (9), $\mathbf{Cospan}_f(\mu_l)$ is shown on the right
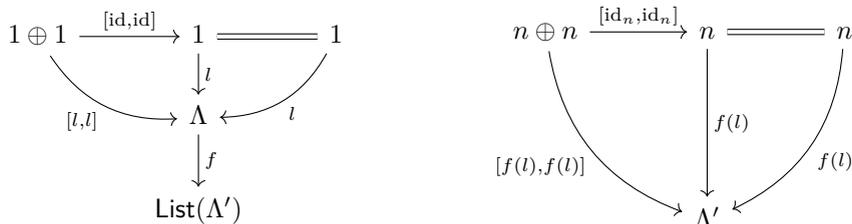
But this cospan is exactly the one from Lemma 3.10; see also Eq. (17). □

We also denote by $\mathbf{Cospan}_\_$ the composite of the functor from Corollary 3.13 with the faithful inclusion $\mathbf{Set} \to \mathbf{Set}_{\mathsf{List}}$ and the fully faithful inclusion $\mathbf{Hyp}_{\mathsf{OF}} \to \mathbf{Hyp}$:

The following theorem states that $\mathbf{Cospan}_\Lambda$ is the free hypergraph category on Λ. In particular, this theorem produces a hypergraph functor

$$\mathsf{Frob}_{\mathcal{H}} \colon \mathbf{Cospan}_{\mathrm{Ob}(\mathcal{H})} \to \mathcal{H}$$

which is so-named because its image provides all the Frobenius morphisms on all the objects of $\mathcal{H}$. In fact, Frob arises as the counit of an adjunction.

Theorem 3.14. $\mathbf{Cospan}_\Lambda$ *is the free hypergraph category on the set Λ. That is, there is an adjunction*

$$\mathbf{Set} \; \underset{\mathrm{Ob}}{\overset{\mathbf{Cospan}_-}{\rightleftarrows}} \; \mathbf{Hyp}.$$

Proof. We want to show Ob is right adjoint to $\mathbf{Cospan}_-$, so we provide a unit transformation and counit transformation and check the triangle identities.

For any $\Lambda \in \mathbf{Set}$, we have $\mathrm{Ob}(\mathbf{Cospan}_\Lambda) = \mathsf{List}(\Lambda)$, so we take the unit map $\Lambda \to \mathrm{Ob}(\mathbf{Cospan}_\Lambda)$ to be the unit natural transformation sing from the List monad.

Suppose $\mathcal{H}$ is a hypergraph category; for the counit of the adjunction, we need a hypergraph functor $\mathbf{Cospan}_{\mathrm{Ob}(\mathcal{H})} \to \mathcal{H}$. Note that for each object of $\mathcal{H}$, Proposition 3.8 gives a strong symmetric monoidal functor $\mathbf{Cospan} \to \mathcal{H}$. Observing that $\mathbf{Cospan}_{\mathrm{Ob}(\mathcal{H})}$ is the coproduct, in the category of symmetric monoidal categories and strong symmetric monoidal functors, of $\mathrm{Ob}(\mathcal{H})$-many copies of $\mathbf{Cospan}$, the copairing of all these functors thus gives a strong symmetric monoidal functor $\mathsf{Frob}_{\mathcal{H}} \colon \mathbf{Cospan}_{\mathrm{Ob}(\mathcal{H})} \to \mathcal{H}$. It is straightforward to observe that this functor is hypergraph.

It remains to check that Frob is natural (as its subscripts $\mathcal{H}$ vary), and that the triangle identities hold. The map Frob is natural because hypergraph functors $\mathcal{H} \to \mathcal{H}'$ are required to preserve Frobenius structures. Finally, for the triangle identities, we need to check that the following diagrams commute:

Both are straightforward calculations. $\square$

3.2.3. $\mathbf{Cospan}_\Lambda$ *as the free hypergraph category over Λ*

Corollary 3.15. *The functor* $\mathbf{Cospan}_- \colon \mathbf{Set}_{\mathsf{List}} \to \mathbf{Hyp}_{\mathsf{OF}}$*, constructed in Corollary 3.13, is fully faithful and has a right adjoint:*

$$\mathbf{Set}_{\mathsf{List}} \; \underset{\mathsf{Gens}}{\overset{\mathbf{Cospan}_-}{\rightleftarrows}} \; \mathbf{Hyp}_{\mathsf{OF}}.$$

Moreover, the components $\mathsf{Frob}_{\mathcal{H}} \colon \mathbf{Cospan}_{\mathsf{Gens}(\mathcal{H})} \to \mathcal{H}$ *of the counit transformation are* io *hypergraph functors.*

Proof (sketch). As a right adjoint to $\mathbf{Cospan}_-$, we propose the functor Gens given by sending an OF-hypergraph category $(\Lambda, \mathcal{H}, i)$ to the set Λ of generators. It is clearly functorial.

The proof that Gens is right adjoint to $\mathbf{Cospan}_-$ is analogous to, though a bit easier than, that of Theorem 3.14. Rather than the unit map being sing, the unit of the List monad, here it is simply the

identity map $\Lambda \to \Lambda$ in $\mathbf{Set}_{\mathsf{List}}$, so the triangle identities become trivial. For the $\mathcal{H}$-component $\mathsf{Frob}_{\mathcal{H}}$ of the counit transformation, simply replace $\mathrm{Ob}(\mathcal{H})$ with Λ throughout the proof. For any $x \in \mathcal{H}$ we have a list $(x_1, \ldots, x_n) \in \mathsf{List}(\Lambda)$ with $x = x_1 \oplus \cdots \oplus x_n$, so the hypergraph functor $\mathsf{Frob}_{\mathcal{H}}$ is indeed identity-on-objects.

Finally it is well-known that a left adjoint is fully faithful iff the corresponding unit map is a natural isomorphism, and indeed for any Λ, the unit map $\Lambda \to \mathsf{Gens}(\mathbf{Cospan}_{\Lambda}) = \Lambda$ is the identity. $\square$

Proposition 3.16. *In the case $\mathcal{H} = \mathbf{Cospan}_{\Lambda}$, the counit map $\mathsf{Frob}_{\mathbf{Cospan}_{\Lambda}} : \mathbf{Cospan}_{\Lambda} \to \mathbf{Cospan}_{\Lambda}$ is the identity.*

Proof. The main idea here is that the counit selects out the Frobenius morphisms of $\mathbf{Cospan}_{\Lambda}$, and since $\mathbf{Cospan}_{\Lambda}$ is free, these are *all* the morphisms.

More precisely, observe that since left adjoints preserve coproducts, and $\mathbf{Cospan}_{-}$ is a left adjoint (Corollary 3.15), we now know $\mathbf{Cospan}_{\Lambda} \cong \coprod_{l \in \Lambda} \mathbf{Cospan}$ in $\mathbf{Hyp}$. Fix $l \in \Lambda$. The Frobenius structure on the object $[l] \in \mathbf{Cospan}_{\Lambda}$ is given by Example 2.8, and thus the corresponding map $\mathsf{Frob}_l : \mathbf{Cospan} \to \mathbf{Cospan}_{\Lambda}$ given by Proposition 3.8 is precisely the inclusion into the lth summand. Moreover, by the proof of Theorem 3.14, $\mathsf{Frob}_{\mathbf{Cospan}_{\Lambda}}$ is built as the copairing over Λ of these maps Frob_l. Thus $\mathsf{Frob}_{\mathbf{Cospan}_{\Lambda}} = [\mathsf{Frob}_l]_{l \in \Lambda} = \mathrm{id}_{\mathbf{Cospan}_{\Lambda}}$, as required. $\square$

Remark 3.17. To summarize, consider the following diagram of categories and functors:

$$
\begin{array}{ccc}
& \overset{\mathbf{Cospan}_{-}}{\longrightarrow} & \\
\mathbf{Set} & \underset{\mathrm{Ob}}{\overset{\perp}{\longleftarrow}} & \mathbf{Hyp} \\
{\scriptstyle \mathsf{List}} \Big\Vdash \Big\downarrow {\scriptstyle i} & & {\scriptstyle U} \Big\uparrow \Big\downarrow {\scriptstyle \mathsf{Str}} \\
\mathbf{Set}_{\mathsf{List}} & \underset{\mathsf{Gens}}{\overset{\mathbf{Cospan}_{-}}{\underset{\perp}{\longleftarrow}}} & \mathbf{Hyp}_{\mathsf{OF}}
\end{array}
\tag{18}
$$

The left-hand adjunction is the usual one between $\mathbf{Set}$ and the Kleisli category of the List monad. The top adjunction was proved in Theorem 3.14, while the bottom adjunction was proved in Corollary 3.15. The right-hand map U just sends an OF-hypergraph category $(\Lambda, \mathcal{H}, i)$ to the underlying hypergraph category $\mathcal{H}$, and the strictification functor Str will be constructed in Theorem 3.22, namely as the underlying 1-functor of Eq. (23). But beware that U and Str are not adjoint: there are faux unit and counit maps $\mathcal{H} \to U\mathsf{Str}(\mathcal{H}) \to \mathcal{H}$, but they do not satisfy the triangle identities. In some sense this right-hand part of the diagram is stronger than the rest—it is the shadow of the 2-equivalence $\mathcal{H}yp_{\mathsf{OF}} \cong \mathcal{H}yp$ from Theorem 3.22—but in another sense it is weaker in that there is no adjunction between the underlying 1-categories.

3.3. Factoring hypergraph functors

Hypergraph functors naturally factor into two sorts: those that are identity-on-objects (io) and those that are fully faithful (ff)—roughly speaking, "identity on morphisms". Indeed, given a hypergraph functor $F : \mathcal{H}_1 \to \mathcal{H}_2$, define a new hypergraph category $\mathcal{H}_F$ as follows: its objects are the same as those of the domain, and for every two objects $x, y \in \mathrm{Ob}(\mathcal{H}_F)$, the hom-set is that of their images under F,

$$
\mathrm{Ob}(\mathcal{H}_F) \coloneqq \mathrm{Ob}(\mathcal{H}_1) \qquad \text{and} \qquad \mathcal{H}_F(x, y) \coloneqq \mathcal{H}_2(Fx, Fy).
\tag{19}
$$

The monoidal unit object and the monoidal product on objects in $\mathcal{H}$ are inherited from $\mathcal{H}_1$, and the monoidal product on morphisms together with all the Frobenius structures are inherited from $\mathcal{H}_2$. One easily constructs an identity-on-objects functor $\mathcal{H}_1 \xrightarrow{\text{io}} \mathcal{H}_F$ and a fully faithful functor $\mathcal{H}_F \xrightarrow{\text{ff}} \mathcal{H}_2$, of which the composite is F,

$$\mathcal{H}_1 \xrightarrow{\quad F \quad} \mathcal{H}_2$$
$$\mathcal{H}_1 \xrightarrow{\text{io}} \mathcal{H}_F \xrightarrow{\text{ff}} \mathcal{H}_2$$

Of course, there are details to check, but we leave them to the reader.

Remark 3.18. The above forms an *orthogonal factorization system* (io, ff) on the 2-category *Hyp*. See [20] for a definition and a similar result in the case of traced and compact closed categories. However, we will not need to use this fact, so we omit the proof.

In fact the (ff, io) factorization is special in that it leads to a fibration of categories, as we will show in Proposition 3.20; it will help to first prove a lemma.

Lemma 3.19. *Let* $g\colon \Lambda_1 \to \Lambda_2$ *be a morphism in* $\mathbf{Set}_{\mathsf{List}}$*, let* $\mathcal{H}_2$ *be a hypergraph category such that* $\mathsf{Gens}(\mathcal{H}_2) = \Lambda_2$*, and let* $\mathsf{Frob}_2\colon \mathbf{Cospan}_{\Lambda_2} \to \mathcal{H}_2$ *be the counit map on* $\mathcal{H}_2$ *of the adjunction* $\mathbf{Cospan}_{\_} \dashv \mathsf{Gens}$ *from Corollary 3.15.*

Consider the (io, ff) *factorization* $\mathbf{Cospan}_{\Lambda} \xrightarrow{i_1} \mathcal{H}_1 \xrightarrow{G} \mathcal{H}_2$ *of the composite* $\mathbf{Cospan}_g \,\text{\fontfamily{cmr}\selectfont\textsemicolon}\, \mathsf{Frob}_2$*:*

$$
\begin{array}{ccc}
\mathbf{Cospan}_{\Lambda_1} & \xrightarrow{\quad \mathbf{Cospan}_g \quad} & \mathbf{Cospan}_{\Lambda_2} \\
{\scriptstyle i_1\ (\text{io})}\big\downarrow & & \big\downarrow{\scriptstyle \mathsf{Frob}_2} \\
\mathcal{H}_1 & \xrightarrow[\quad G\ (\text{ff}) \quad]{} & \mathcal{H}_2
\end{array}
\tag{20}
$$

Then we have $i_1 = \mathsf{Frob}_1$*, the counit map on* $\mathcal{H}_1$*.*

Proof. Corollary 3.15 gives the bijection $\mathbf{Hyp}(\mathbf{Cospan}_{\Lambda_1}, \mathcal{H}_\infty) \cong \mathbf{Set}_{\mathsf{List}}(\Lambda_1, \mathsf{Gens}(\mathcal{H}_1))$. Note that $\Lambda_1 = \mathsf{Gens}(\mathcal{H}_1)$. The functor i_1 induces the identity map $\Lambda_1 \to \Lambda_1$ on generators and hence maps to the identity map on Λ_1 under this bijection. Since Frob_1 does the same, the two functors must be equal. □

Proposition 3.20. *The functor* $\mathsf{Gens}\colon \mathbf{Hyp}_{\mathsf{OF}} \to \mathbf{Set}_{\mathsf{List}}$ *from Corollary 3.15 is a split Grothendieck fibration.*

Proof. We first want to show Gens is a fibration, so suppose given a diagram

$$
\begin{array}{c}
\mathcal{H}_2 \\
\big\downarrow{\scriptstyle \mathsf{Gens}} \\
\Lambda_1 \xrightarrow{\quad g \quad} \Lambda_2
\end{array}
$$

We want to find a Cartesian morphism G over $g \in \mathbf{Set}_{\mathsf{List}}$. Since $\mathsf{Gens}(\mathcal{H}_2) = \Lambda_2$, we can factor $\mathbf{Cospan}_g \,\text{\fontfamily{cmr}\selectfont\textsemicolon}\,$ Frob_2 as in Lemma 3.19 to obtain the commutative square Eq. (20). We claim that the map $G\colon \mathcal{H}_1 \to \mathcal{H}_2$ is Cartesian. So suppose given a solid-arrow diagram as to the left below; it is equivalently described by the solid-arrow diagram to the right:

$$
\begin{array}{ccccc}
& \xrightarrow{\quad H \quad} & & & \\
\mathcal{H}_0 & \dashrightarrow & \mathcal{H}_1 & \xrightarrow{G} & \mathcal{H}_2 \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
\Lambda_0 & \xrightarrow{f} & \Lambda_1 & \xrightarrow{g} & \Lambda_2
\end{array}
\qquad
\begin{array}{ccccc}
\mathbf{Cospan}_{\Lambda_0} & \xrightarrow{\mathbf{Cospan}_f} & \mathbf{Cospan}_{\Lambda_1} & \xrightarrow{\mathbf{Cospan}_g} & \mathbf{Cospan}_{\Lambda_2} \\
{\scriptstyle(\text{io})}\big\downarrow & & {\scriptstyle(\text{io})}\big\downarrow & & \big\downarrow{\scriptstyle(\text{io})} \\
\mathcal{H}_0 & \dashrightarrow{\ \ F\ \ } & \mathcal{H}_1 & \xrightarrow{G\ (\text{ff})} & \mathcal{H}_2 \\
& & & \xrightarrow[\ \ H\ \ ]{} &
\end{array}
$$

We need to show there is a unique dashed map $F \colon \mathcal{H}_0 \to \mathcal{H}_1$ making the bottom triangle on the right-hand diagram commute. But because the vertical maps are io, we take F on objects to agree with $\mathbf{Cospan}_f$, and because G is fully faithful, we take F on morphisms to agree with H. This is the only possible choice to make the diagrams commute, and it will be a hypergraph functor because G and H are.

We have proved that Gens is a Grothendieck fibration. It is split, meaning that our choices of Cartesian maps are closed under composition, because Eq. (19) defines the factorization system up to equality. □

In general, split Grothendieck fibrations $p \colon E \to B$ can be identified with functors $\ulcorner p \urcorner \colon B \to \mathbf{Cat}^{\mathrm{op}}$. In the case of Proposition 3.20, the functor $\ulcorner \mathsf{Gens} \urcorner \colon \mathbf{Set}_{\mathsf{List}} \to \mathbf{Cat}^{\mathrm{op}}$ shall be denoted $\mathbf{Hyp}_{\mathsf{OF}(-)}$. It sends an object Λ to the category $\mathbf{Hyp}_{\mathsf{OF}(\Lambda)}$ of hypergraph categories that are objectwise-free on Λ and the io hypergraph functors between them. It sends a morphism $f \colon \Lambda_1 \to \mathsf{List}(\Lambda_2)$ to the functor $\mathbf{Hyp}_{\mathsf{OF}(f)} \colon \mathbf{Hyp}_{\mathsf{OF}(\Lambda_2)} \to \mathbf{Hyp}_{\mathsf{OF}(\Lambda_1)}$ defined by factorization as in Eq. (20); in other words, on objects $\mathbf{Hyp}_{\mathsf{OF}(f)}$ maps $\mathcal{H}_2$ to the hypergraph category $\mathcal{H}_1$ given by the factorization

$$
\begin{array}{ccc}
\mathbf{Cospan}_{\Lambda_1} & \xrightarrow{\;\mathbf{Cospan}_f\;} & \mathbf{Cospan}_{\Lambda_2} \\
{\scriptstyle \mathsf{Frob}_1}\Big\downarrow{\scriptstyle (\mathtt{io})} & & \Big\downarrow{\scriptstyle \mathsf{Frob}_2} \\
\mathbf{Hyp}_{\mathsf{OF}(f)}(\mathcal{H}_2) := \mathcal{H}_1 & \xrightarrow[\;F\;(\mathtt{ff})\;]{} & \mathcal{H}_2
\end{array}
\tag{21}
$$

The opposite direction, taking a functor $B \to \mathbf{Cat}^{\mathrm{op}}$ and returning a fibration over B is called the Grothendieck construction. We immediately have the following.

Corollary 3.21. *There is an equivalence of categories*

$$
\mathbf{Hyp}_{\mathsf{OF}} \;\overset{\cong}{\Rightarrow}\; \int^{\Lambda \in \mathbf{Set}_{\mathsf{List}}} \mathbf{Hyp}_{\mathsf{OF}(\Lambda)}.
$$

3.4. Strictification of hypergraph categories

In this subsection we prove that every hypergraph category is hypergraph equivalent to a strict hypergraph category. In fact, there is a 2-equivalence $\mathcal{H}yp \cong \mathcal{H}yp_{\mathsf{OF}}$. This coherence result will be the first step in formalizing the relationship between hypergraph categories and cospan-algebras.

Define the 2-category $\mathcal{H}yp_{\mathsf{OF}}$ as the full sub-2-category of objectwise-free hypergraph categories. That is, the objects of $\mathcal{H}yp_{\mathsf{OF}}$ are hypergraph categories $\mathcal{H}$ such that there exists a set Λ and a bijection $i \colon \mathsf{List}(\Lambda) \to \mathrm{Ob}(\mathcal{H})$, and given two OF-hypergraph categories $(\mathcal{H}, \Lambda, i)$ and $(\mathcal{H}', \Lambda', i')$, the hom-category between them is simply

$$
\mathcal{H}yp_{\mathsf{OF}}\big((\mathcal{H}, \Lambda, i), (\mathcal{H}', \Lambda', i')\big) := \mathcal{H}yp(\mathcal{H}, \mathcal{H}').
$$

There is an obvious forgetful functor $U \colon \mathcal{H}yp_{\mathsf{OF}} \to \mathcal{H}yp$, and by construction it is fully faithful.

Theorem 3.22. *The functor $U \colon \mathcal{H}yp_{\mathsf{OF}} \to \mathcal{H}yp$ is a 2-equivalence. In particular it is essentially surjective, i.e. every hypergraph category is hypergraph equivalent to an objectwise-free hypergraph category.*

Proof. Since U is fully faithful by definition, it suffices to show that it is essentially surjective.

Let $(\mathcal{H}, \otimes)$ be a hypergraph category. As $\mathcal{H}$ is, in particular, a symmetric monoidal category, a standard construction (see Mac Lane [14, Theorem XI.3.1]) gives an equivalent strict symmetric monoidal category $\mathcal{H}_{\mathsf{str}}$, the *strictification of* $\mathcal{H}$, whose construction we detail here.

Let $\Lambda := \mathrm{Ob}(\mathcal{H})$. The set of objects in the strictification is $\mathrm{Ob}(\mathcal{H}_{\mathtt{str}}) := \mathsf{List}(\Lambda)$, i.e., finite lists $[x_1, \ldots, x_m]$ of objects in $\mathcal{H}$. For each such list, let $Px := (((x_1 \otimes x_2) \otimes \ldots) \otimes x_m) \otimes I$ denote the "pre-parenthesized product of x" in $\mathcal{H}$ with all open parentheses at the front. Note that P applied to the empty list is the monoidal unit I, and that for any pair of lists x, y there is a canonical isomorphism $P([x, y]) \cong [Px, Py]$,

$$(((((x_1 \otimes x_2) \otimes \cdots \otimes x_m) \otimes y_1) \otimes y_2) \otimes \cdots \otimes y_n) \otimes I$$
$$\cong (((x_1 \otimes x_2) \otimes \cdots \otimes x_m) \otimes I) \otimes (((y_1 \otimes y_2) \otimes \cdots \otimes y_n) \otimes I). \quad (22)$$

The morphisms $[x_1, \ldots, x_m] \to [y_1, \ldots, y_n]$ in $\mathcal{H}_{\mathtt{str}}$ are the morphisms $Px \to Py$ in $\mathcal{H}$, and composition is inherited from $\mathcal{H}$. The monoidal structure on objects in $\mathcal{H}_{\mathtt{str}}$ is given by concatenation of lists; the monoidal unit is the empty list. The monoidal product of two morphisms in $\mathcal{H}_{\mathtt{str}}$ is given by their monoidal product in $\mathcal{H}$ pre- (and post-) composed with the canonical isomorphism (and its inverse) from Eq. (22).

By design, the associators and unitors of $\mathcal{H}_{\mathtt{str}}$ are simply identity maps, and the braiding $[x, y] \to [y, x]$ is given by the braiding $Px \otimes Py \to Py \otimes Px$ in $\mathcal{H}$, similarly pre- and post-composed with the isomorphisms from Eq. (22). This defines a strict symmetric monoidal category [14], and it is objectwise-free on Λ by construction. This construction is 2-functorial: given a strong monoidal functor between monoidal categories (resp. a monoidal natural transformation between monoidal functors), there is an evident strict monoidal functor (resp. a monoidal natural transformation) between strictifications.

To make $\mathcal{H}_{\mathtt{str}}$ into a hypergraph category, we equip each object $x = [x_1, \ldots, x_n]$ with an Frobenius structure $(\mu, \eta, \delta, \epsilon)$ using the monoidal product, over $i = 1, \ldots, n$, of corresponding Frobenius structures $(\mu_i, \eta_i, \delta_i, \epsilon_i)$ from $\mathcal{H}$, and pre- or post-composition with canonical isomorphisms from Eq. (22). For example, the multiplication μ on $x \in \mathrm{Ob}(\mathcal{H}_{\mathtt{str}})$ is given by

$$P([x, x]) \cong ((((x_1 \otimes x_1) \otimes (x_2 \otimes x_2)) \otimes \ldots) \otimes (x_n \otimes x_n)) \otimes I \xrightarrow{((\mu_1 \otimes \mu_2) \otimes \cdots \otimes \mu_n) \otimes \mathrm{id}_I} P(x).$$

As the coherence maps are natural, each special commutative Frobenius monoid axiom for this data on $[x_1, \ldots, x_n]$ reduces to a list of the corresponding axioms for the objects x_i in $\mathcal{H}$. Similarly, the coherence axioms and naturality of the coherence maps imply the Frobenius structure on the monoidal product of objects is given by the Frobenius structures on the factors in the required way.

Thus we have upgraded $\mathcal{H}_{\mathtt{str}}$ to a hypergraph category. Moreover, this construction is 2-functorial; all that needs to be checked is that the usual strictification of a hypergraph functor $\mathcal{H} \to \mathcal{H}'$ preserves the hypergraph structure on $\mathcal{H}_{\mathtt{str}}$ and $\mathcal{H}'_{\mathtt{str}}$ as defined above, which is easy to see. Thus we have a 2-functor

$$\mathsf{Str} \colon \mathcal{Hyp} \to \mathcal{Hyp}_{\mathsf{OF}}, \quad (23)$$

and it remains to prove the equivalence of $\mathcal{H}$ and $\mathcal{H}_{\mathtt{str}}$.

Mac Lane's standard construction further gives strong symmetric monoidal functors $P \colon \mathcal{H}_{\mathtt{str}} \to \mathcal{H}$, extending the map P above, and $S \colon \mathcal{H} \to \mathcal{H}_{\mathtt{str}}$ sending $x \in \mathcal{H}$ to the length-1 list $[x] \in \mathcal{H}_{\mathtt{str}}$, and P and S form an equivalence of symmetric monoidal categories.

Moreover, it is straightforward to check that P and S preserve the hypergraph structure defined above, and thus form an equivalence of hypergraph categories. The fact that P preserves the hypergraph structure follows from the compatibility of the Frobenius structures with the monoidal product required in the definition of hypergraph category.

Note in particular that $\mathcal{H}$ must obey the unit coherence axiom (see Definition 2.12) in order for the Frobenius structure on the monoidal unit $\varnothing$ of $\mathcal{H}_{\mathtt{str}}$ to map to the Frobenius structure on its image $P(\varnothing) = I$ of $\mathcal{H}$. By construction, the Frobenius structure on $\varnothing$ just comprises identity maps $\varnothing \to \varnothing$; indeed, since $\mathcal{H}_{\mathtt{str}}$ is strict, Proposition 2.18 shows this is the only Frobenius structure it could have. The image of this

Frobenius structure under P is then defined by P's monoidal coherence maps, and these coherence maps define precisely the canonical Frobenius structure on I detailed in Example 2.6. $\square$

4. Cospan-algebras and hypergraph categories are equivalent

In Theorem 3.22 we showed that there is a 2-equivalence between the bicategories $\mathcal{Hyp}$ and $\mathcal{Hyp}_{\mathsf{OF}}$. Our remaining goal is to show there is an equivalence between the (1-) categories $\mathbf{Hyp}_{\mathsf{OF}}$ and $\mathbf{Cospan\text{-}Alg}$. We will build this equivalence in parts.

In Section 4.1 we produce a functor $A_- \colon \mathbf{Hyp}_{\mathsf{OF}(\Lambda)} \to \mathbf{Lax}(\mathbf{Cospan}_\Lambda, \mathbf{Set})$ natural in Λ, and in Section 4.2 we produce a functor $\mathcal{H}_-$ in the opposite direction. In Section 4.3 we prove that A_- and $\mathcal{H}_-$ are mutually inverse, giving an equivalence of categories

$$\mathbf{Hyp}_{\mathsf{OF}(\Lambda)} \cong \mathbf{Lax}(\mathbf{Cospan}_\Lambda, \mathbf{Set}). \tag{24}$$

These equivalences will again be natural in $\Lambda \in \mathbf{Set}_{\mathsf{List}}$, so we will be able to gather them together into a single equivalence, $\mathbf{Hyp}_{\mathsf{OF}} \cong \mathbf{Cospan\text{-}Alg}$.

4.1. From hypergraph categories to cospan-algebras

Our aim in this subsection is to provide one half of the equivalence (24), converting any hypergraph category $\mathcal{H}$ into a cospan-algebra $A_{\mathcal{H}}$. This is given by the following.

Proposition 4.1. *For any $\Lambda \in \mathbf{Set}_{\mathsf{List}}$, we can naturally construct a functor*

$$A_- \colon \mathbf{Hyp}_{\mathsf{OF}(\Lambda)} \to \mathbf{Lax}(\mathbf{Cospan}_\Lambda, \mathbf{Set}).$$

This will be proved on page 4770. First we prove two lemmas, which we use to define A_- on objects and on morphisms of $\mathbf{Hyp}_{\mathsf{OF}(\Lambda)}$ respectively.

Lemma 4.2. *Let $\mathcal{H}$ be an OF hypergraph category with $\Lambda = \mathsf{Gens}(\mathcal{H})$; by Corollary 3.15 we have an identity-on-objects hypergraph functor $\mathsf{Frob} \colon \mathbf{Cospan}_\Lambda \to \mathcal{H}$. The set of maps out of the monoidal unit $I \in \mathcal{H}$ defines a lax symmetric monoidal functor*

$$A_{\mathcal{H}} \colon \mathbf{Cospan}_\Lambda \to \mathbf{Set}$$
$$A_{\mathcal{H}}(-) \coloneqq \mathcal{H}\big(I, \mathsf{Frob}(-)\big) \tag{25}$$

Proof. The formula (25) makes sense not only for objects in $\mathbf{Cospan}_\Lambda$ but also for morphisms, and it makes clear how to endow $A_{\mathcal{H}}$ with a lax structure. Indeed, given a morphism $f \colon X \to Y$ in $\mathbf{Cospan}_\Lambda$, composing with $\mathsf{Frob}(f)$ induces a function $\mathcal{H}(I, \mathsf{Frob}(X)) \to \mathcal{H}(I, \mathsf{Frob}(Y))$, and this defines $A_{\mathcal{H}}$ on morphisms. For the laxators, we need a function $\gamma \colon \{1\} \to A_{\mathcal{H}}(0)$ and a function $\gamma_{X,Y} \colon A_{\mathcal{H}}(X) \times A_{\mathcal{H}}(Y) \to A_{\mathcal{H}}(X \oplus Y)$ for any $X, Y \in \mathbf{Cospan}_\Lambda$. Since Frob is io (Corollary 3.15), we can define the functions γ and $\gamma_{X,Y}$ as follows:

$$\{1\} \xrightarrow{\mathrm{id}_I} \mathcal{H}(I, I) = A_{\mathcal{H}}(\varnothing), \tag{26}$$

$$\mathcal{H}(I, \mathsf{Frob}X) \times \mathcal{H}(I, \mathsf{Frob}Y) \xrightarrow{\otimes} \mathcal{H}(I, \mathsf{Frob}X \otimes \mathsf{Frob}Y) = A_{\mathcal{H}}(X \oplus Y).$$

It is easy to check that these satisfy the necessary coherence conditions. $\square$

We next want to define A_- on morphisms, so suppose that $\mathcal{H}$ and $\mathcal{H}'$ are hypergraph categories. Morphisms between them in $\mathbf{Hyp}_{\mathsf{OF}(\Lambda)}$ are $\mathtt{io}$ hypergraph functors $F\colon \mathcal{H} \to \mathcal{H}'$, and morphisms between their images in $\mathbf{Lax}(\mathbf{Cospan}_\Lambda, \mathbf{Set})$ are natural transformations $\alpha\colon A_\mathcal{H} \to A_{\mathcal{H}'}$

$$\mathbf{Cospan}_\Lambda \overset{\overset{A_\mathcal{H}}{\longrightarrow}}{\underset{\underset{A_{\mathcal{H}'}}{\longrightarrow}}{\Downarrow \alpha}} \mathbf{Set}.$$

So given F, in order to define $\alpha := A_F$, we first note that $F\colon \mathcal{H} \to \mathcal{H}'$ induces a commutative diagram of $\mathtt{io}$ hypergraph functors

$$
\begin{array}{ccc}
 & \mathbf{Cospan}_\Lambda & \\
{}^{\mathsf{Frob}}\nearrow & & \searrow^{\mathsf{Frob}'} \\
\mathcal{H} & \xrightarrow[\hspace{2em} F \hspace{2em}]{} & \mathcal{H}'.
\end{array}
\tag{27}
$$

Lemma 4.3. *Let $F\colon \mathcal{H} \to \mathcal{H}'$ be an $\mathtt{io}$ hypergraph functor between hypergraph categories over Λ. For any $X \in \mathbf{Cospan}_\Lambda$, define $\alpha_X \colon A_\mathcal{H}(X) \to A_{\mathcal{H}'}(X)$ as the composite*

$$A_\mathcal{H}(X) := \mathcal{H}(I, \mathsf{Frob}(X)) \xrightarrow{F} \mathcal{H}'(FI, F \circ \mathsf{Frob}(X)) = \mathcal{H}'(I, \mathsf{Frob}'(X)) =: A_{\mathcal{H}'}(X). \tag{28}$$

This defines a natural transformation $\alpha\colon \mathcal{H} \to \mathcal{H}'$.

Proof. The naturality and monoidality of α, i.e. the commutativity of the following diagrams for any X, Y and $f\colon X \to Y$ in $\mathbf{Cospan}_\Lambda$

$$
\begin{array}{ccc}
A_\mathcal{H}(X) & \xrightarrow{\alpha_X} & A_{\mathcal{H}'}(X) \\
{\scriptstyle A_\mathcal{H}(f)}\downarrow & & \downarrow{\scriptstyle A_\mathcal{H}(f)} \\
A_\mathcal{H}(Y) & \xrightarrow[\alpha_Y]{} & A_{\mathcal{H}'}(X)
\end{array}
\qquad
\begin{array}{c}
\{1\} \\
{}^{\gamma}\swarrow \quad \searrow^{\gamma'} \\
A_\mathcal{H}(\varnothing) \underset{\alpha_\varnothing}{\rightrightarrows} A_{\mathcal{H}'}(\varnothing)
\end{array}
\qquad
\begin{array}{ccc}
A_\mathcal{H}(X) \times A_\mathcal{H}(Y) & \xrightarrow{\gamma_{X,Y}} & A_\mathcal{H}(X \oplus Y) \\
{\scriptstyle \alpha_X \times \alpha_Y}\downarrow & & \downarrow{\scriptstyle \alpha_{X \oplus Y}} \\
A_{\mathcal{H}'}(X) \times A_{\mathcal{H}'}(Y) & \xrightarrow[\gamma'_{X,Y}]{} & A_{\mathcal{H}'}(X \oplus Y)
\end{array}
$$

arise from the functoriality and strong monoidality of F. $\square$

Proof of Proposition 4.1. Choose any set Λ.[4] We define A_- on objects by Lemma 4.2 and on morphisms by Lemma 4.3. It remains to check this is functorial, and that it is natural as Λ varies in $\mathbf{Set}_{\mathsf{List}}$.

If $F = \mathrm{id}_\mathcal{H}$ is the identity then each component $\alpha_X = A_F(X)$ defined in Eq. (28) is also the identity. Similarly, given two composable $\mathtt{io}$ hypergraph functors $\mathcal{H} \xrightarrow{F} \mathcal{H}' \xrightarrow{F'} \mathcal{H}''$, the associated commutative diagrams Eq. (27) compose, and again by Eq. (28), we have $A_F(X) \mathbin{\mathring{,}} A_{F'}(X) = A_{F \mathbin{\mathring{,}} F'}(X)$ for any $X \in \mathbf{Cospan}_\Lambda$.

We now show that the above is natural in $\Lambda \in \mathbf{Set}_{\mathsf{List}}$. Let $f\colon \Lambda \to \mathsf{List}(\Lambda')$ be a function; we want to show that the following square commutes:

$$
\begin{array}{ccc}
\mathbf{Hyp}_{\mathsf{OF}(\Lambda')} & \xrightarrow{A_-} & \mathbf{Lax}(\mathbf{Cospan}_{\Lambda'}, \mathbf{Set}) \\
{\scriptstyle \mathbf{Hyp}_{\mathsf{OF}(f)}}\downarrow & & \downarrow{\scriptstyle \mathbf{Cospan}_f} \\
\mathbf{Hyp}_{\mathsf{OF}(\Lambda)} & \xrightarrow[A_-]{} & \mathbf{Lax}(\mathbf{Cospan}_\Lambda, \mathbf{Set})
\end{array}
\tag{29}
$$

---

[4] To be more precise we might use Λ to annotate the functor A_- as A_-^Λ, but we leave off the superscript for typographical reasons.

That is, for any $\mathcal{H}' \in \mathbf{Hyp}_{\mathsf{OF}(\Lambda')}$, we want to show $\mathbf{Cospan}_f \, \mathring{,} \, A_{\mathcal{H}'} = A_{\mathbf{Hyp}_{\mathsf{OF}(f)}(\mathcal{H}')}$ as functors $\mathbf{Cospan}_\Lambda \to \mathbf{Set}$. Recalling from Eq. (25) the definition $A_{\mathcal{H}} = \mathsf{Frob} \, \mathring{,} \, \mathcal{H}(I, -)$, this is simply the commutativity of the diagram

$$
\begin{array}{ccc}
\mathbf{Cospan}_\Lambda & \xrightarrow{\ \mathbf{Cospan}_f\ } & \mathbf{Cospan}_{\Lambda'} \\
{\scriptstyle \mathsf{Frob}}\big\downarrow & & \big\downarrow{\scriptstyle \mathsf{Frob}'} \\
\mathbf{Hyp}_{\mathsf{OF}(f)}(\mathcal{H}') & \xrightarrow{\ \ F\ \ } & \mathcal{H}'
\end{array}
$$

$$
\mathbf{Hyp}_{\mathsf{OF}(f)}(\mathcal{H}')(I,-) \searrow \quad \swarrow \mathcal{H}'(I,-)
$$
$$
\mathbf{Set}
$$

which commutes by Eq. (21) and the fact that F is a fully faithful hypergraph functor. $\quad\square$

Remark 4.4. An analogous construction can be used to obtain a cospan-algebra from any hypergraph category $\mathcal{H}$, even if it is not objectwise-free. Rather than use the counit map $\mathsf{Frob}_{\mathcal{H}} \colon \mathbf{Cospan}_{\mathsf{Gens}(\mathcal{H})} \to \mathcal{H}$ from Corollary 3.15, we use the counit map $\mathsf{Frob}'_{\mathcal{H}} \colon \mathbf{Cospan}_{\mathrm{Ob}(\mathcal{H})} \to \mathcal{H}$ from Theorem 3.14. Then in place of the lax monoidal functor $\mathcal{H}(I, \mathsf{Frob}_{\mathcal{H}}(-))$ from Eq. (25), one defines a lax monoidal functor $\mathbf{Cospan}_{\mathrm{Ob}(\mathcal{H})} \to \mathbf{Set}$ by $\mathcal{H}(I, \mathsf{Frob}'_{\mathcal{H}}(-))$.

Although we will not give full details here, it is easy to show this construction extends to a functor $\mathbf{Hyp} \longrightarrow \mathbf{Cospan}\text{-}\mathbf{Alg}$.

To conclude this subsection, we use the functor A_- to construct the initial cospan-algebra over Λ; this will be useful in what follows. Recall from Corollary 3.15 that $\mathbf{Cospan}_\Lambda$ is the free hypergraph category over Λ.

Remark 4.5. Applying the functor A_- from Proposition 4.1 to the hypergraph category $\mathbf{Cospan}_\Lambda$ yields Part_Λ from Example 2.3:

$$
A_{\mathbf{Cospan}_\Lambda} = \mathsf{Part}_\Lambda.
$$

Indeed, Proposition 3.16 says that the map $\mathsf{Frob}_{\mathbf{Cospan}_\Lambda}$ used in Eq. (25) to define $A_{\mathbf{Cospan}_\Lambda}$ is in fact the identity. Thus $A_{\mathbf{Cospan}_\Lambda}(X) = \mathbf{Cospan}_\Lambda(\varnothing, X)$, which is exactly $\mathsf{Part}_\Lambda(X)$.

We often denote Part_Λ simply by Part if Λ is clear from context.

Proposition 4.6. $\mathsf{Part} \colon \mathbf{Cospan}_\Lambda \to \mathbf{Set}$ *is the initial cospan-algebra over* Λ.

Proof. Let $(A, \gamma) \colon \mathbf{Cospan}_\Lambda \to \mathbf{Set}$ be a cospan-algebra over Λ. We need to show there is a unique monoidal natural transformation $\alpha \colon \mathsf{Part} \to A$. Given $X \in \mathbf{Cospan}_\Lambda$, define

$$
\alpha_X \colon \mathsf{Part}(X) = \mathbf{Cospan}_\Lambda(\varnothing, X) \longrightarrow A(X);
$$
$$
f \longmapsto \left(1 \xrightarrow{\gamma} A\varnothing \xrightarrow{Af} AX \right). \tag{30}
$$

This is natural because A is a functor: $A(g)(A(f)(\gamma)) = A(f \, \mathring{,} \, g)(\gamma)$. To prove that α is monoidal, we must show that for any $f \colon \varnothing \to X$ and $g \colon \varnothing \to Y$ in $\mathbf{Cospan}_\Lambda$:

$$
\begin{array}{ccccccc}
1 & \xrightarrow{\ \gamma\ } & A(\varnothing) & \xrightarrow{\ \Delta\ } & A(\varnothing) \times A(\varnothing) & \xrightarrow{A(f) \times A(g)} & A(X) \times A(Y) \\
{\scriptstyle \gamma}\big\downarrow & & & & \big\downarrow{\scriptstyle \gamma_{\varnothing,\varnothing}} & & \big\downarrow{\scriptstyle \gamma_{X,Y}} \\
A(\varnothing) & & \xrightarrow[A(\lambda^{-1})]{} & & A(\varnothing \otimes \varnothing) & \xrightarrow[A(f \otimes g)]{} & A(X \otimes Y)
\end{array}
$$

where Δ is the diagonal map. This follows from the monoidality of A and the fact that $\Delta = \lambda^{-1} \colon 1 \to 1 \times 1$.

Finally, we must show that the definition of α in Eq. (30) is the only possible one. To see this, first note that by Eq. (26), the laxator $\gamma^{\mathsf{Part}} \colon 1 \to \mathsf{Part}(\varnothing)$ sends $1 \mapsto \mathrm{id}_\varnothing$. Then since α is assumed to be a monoidal natural transformation, the following diagram commutes for any $X \in \mathbf{Cospan}_\Lambda$:

$$
\begin{array}{ccc}
1 \xrightarrow{\quad\quad\quad f \quad\quad\quad} & & \\
\searrow^{\gamma^{\mathsf{Part}}} \quad \mathsf{Part}(\varnothing) \xrightarrow{\;\mathsf{Part}(f)\;} & \mathsf{Part}(X) & \\
\searrow^{\gamma} \qquad \downarrow^{\alpha_\varnothing} & \downarrow^{\alpha_X} & \\
A(\varnothing) \xrightarrow{\;A(f)\;} & A(X) &
\end{array}
$$

and this forces $\alpha_X(f) = \gamma \,\mathring{,}\, A(f)$ as in Eq. (30). $\square$

4.2. From cospan-algebras to hypergraph categories

Our aim in this subsection is to provide the other half of the equivalence (24), converting any cospan-algebra A into a hypergraph category $\mathcal{H}_A$. The aim of this subsection is detail the following construction.

Proposition 4.7. *For any* $\Lambda \in \mathbf{Set}_{\mathsf{List}}$*, we can naturally construct a functor*

$$\mathcal{H}_- \colon \mathbf{Lax}(\mathbf{Cospan}_\Lambda, \mathbf{Set}) \to \mathbf{Hyp}_{\mathsf{OF}(\Lambda)}.$$

This will be proved on page 4774. As in the previous subsection, we first prove two lemmas that we will use to define this functor on objects (Lemma 4.8) and then on morphisms (Lemma 4.9). We will then again conclude the subsection with some observations on what this implies about the interaction between cospans and composition in hypergraph categories.

Given a cospan-algebra (A, γ), we construct a hypergraph category $\mathcal{H}_A$ with objects and Frobenius structure coming from $\mathbf{Cospan}_\Lambda$, and the homsets coming from the image of objects under A.

Lemma 4.8. *Let* $A \colon \mathbf{Cospan}_\Lambda \to \mathbf{Set}$ *be a lax monoidal functor. We may define a strict hypergraph category* $\mathcal{H}_A \in \mathbf{Hyp}_{\mathsf{OF}(\Lambda)}$ *with:*

- *objects given by lists in* Λ,
- *morphisms* $X \to Y$ *given by the set* $A(X \oplus Y)$,
- *monoidal structure arising from the monoidal structure on* A, *and*
- *composition, identity, and hypergraph structure arising from the images of a certain cospans under* A.

Proof. We first detail the structure of $\mathcal{H}_A$, outlined above. For this we will need to give explicit names to the laxator maps, say $\gamma \colon \{1\} \to A(\varnothing)$ and $\gamma_{X,Y} \colon A(X) \times A(Y) \to A(X \oplus Y)$ for any $X, Y \in \mathbf{Cospan}_\Lambda$.

We define $\mathrm{Ob}(\mathcal{H}_A) := \mathsf{List}(\Lambda)$ and for any lists X, Y in Λ, we define the homset

$$\mathcal{H}_A(X, Y) := A(X \oplus Y). \tag{31}$$

The monoidal unit is the empty list $\varnothing$, the monoidal product on objects is given by concatenation of lists. The monoidal product on morphisms is given by the lax structure on A:

$$
A(W \oplus X) \times A(Y \oplus Z) \xrightarrow{\;\gamma_{W \oplus X, Y \oplus Z}\;} A(W \oplus X \oplus Y \oplus Z) \xrightarrow{\;A(\mathrm{id}_W \oplus \sigma_{X,Y} \oplus \mathrm{id}_Z)\;} A(W \oplus Y \oplus X \oplus Z).
$$

This is strict, so we need not define unitors and associators.

The structure maps in $\mathcal{H}_A$—the composition, identity, braiding, and Frobenius maps—are constructed using the image under A of particular cospans, as we now explain.

Let Λ be a set, and $X, Y, Z \in \Lambda$, and recall the morphism $\mathtt{comp}_{X,Z}^Y$ from Example 3.5. We define composition $\mathcal{H}_A(X,Y) \times \mathcal{H}_A(Y,Z) \to \mathcal{H}_A(X,Z)$ by the formula

$$A(X \oplus Y) \times A(Y \oplus Z) \xrightarrow{\gamma_{X \oplus Y, Y \oplus Z}} A(X \oplus Y \oplus Y \oplus Z) \xrightarrow{A(\mathtt{comp}_{X,Z}^Y)} A(X \oplus Z). \tag{32}$$

All of the remaining structure maps in $\mathcal{H}_A$ arise in similar ways, from cospans of the form $s \colon \varnothing \to X \oplus Y$, where X and Y are the domain and codomain of the map being constructed. Indeed, given such an s, the composite $\{1\} \xrightarrow{\gamma} A(\varnothing) \xrightarrow{A(s)} A(X \oplus Y)$ gives an element of $A(X \oplus Y) = \mathcal{H}_A(X,Y)$. The six required cospans s are given as follows[5]:


$$\tag{33}$$

| identity | braiding | (co)multiplication | (co)unit |
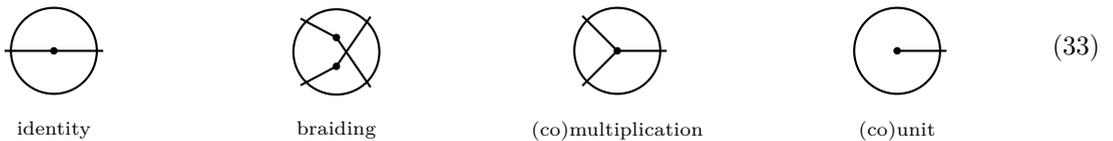
It remains to check the above data obeys the hypergraph category axioms, but this follows from routine calculation. In particular, the associativity, identity, symmetric, and hypergraph laws reduce to facts about the composition operation in $\mathbf{Cospan}_\Lambda$ (this is easy to prove; see Example 2.8 for intuition) although we must also use the naturality of γ and the functoriality of A to make this reduction. The interchange law additionally uses the fact that A is a *symmetric* monoidal functor. $\square$

A morphism $A \to B$ of lax monoidal functors $\mathbf{Cospan}_\Lambda \to \mathbf{Set}$ consists of a collection of functions $\alpha_X \colon A(X) \to B(X)$, one for each $X \in \mathrm{List}(\Lambda)$. Thus by Eq. (31), $\alpha_{X \oplus Y}$ provides a map $\mathcal{H}_A(X,Y) \to \mathcal{H}_B(X,Y)$ for all $X, Y \in \mathrm{Ob}(\mathcal{H}_A) = \mathrm{Ob}(\mathcal{H}_B)$. This is exactly the data of an $\mathtt{io}$ hypergraph functor $F_\alpha \colon \mathcal{H}_A \to \mathcal{H}_B$, though it remains to check that F_α is well-defined; we do that next.

Lemma 4.9. *Let $\alpha \colon A \to B$ be a morphism of cospan-algebras. Setting $F_\alpha(f) := \alpha(f)$ for each $f \in \mathcal{H}_A(X,Y)$ defines an $\mathtt{io}$ hypergraph functor $F_\alpha \colon \mathcal{H}_A \to \mathcal{H}_B$.*

Proof. The relevant axioms are routine consequences of the fact that α is a monoidal natural transformation. For example, F_α preserves composition when the following diagram commutes:

$$
\begin{array}{ccccc}
A(X \oplus Y) \times A(Y \oplus Z) & \xrightarrow{\gamma_A} & A(X \oplus Y \oplus Y \oplus Z) & \xrightarrow{Ac} & A(X \oplus Z) \\
\downarrow{\scriptstyle \alpha \times \alpha} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha} \\
B(X \oplus Y) \times B(Y \oplus Z) & \xrightarrow{\gamma_B} & B(X \oplus Y \oplus Y \oplus Z) & \xrightarrow{Bc} & B(X \oplus Z)
\end{array}
$$

where c is the cospan defining composition (see (32)). This is always true: the first square is the monoidality of α, and the second is the naturality of α with respect to c. Similarly, F_α preserves identities and all hypergraph structure as by the unit monoidality law for α and by the naturality of α with respect to the cospans that define the identity, braiding and Frobenius maps.

Note that also that F_α strict monoidal: $\mathcal{H}_A$ and $\mathcal{H}_B$ have the same objects and monoidal product on objects, and F_α is identity on objects. $\square$

---

[5] As in Eq. (33), the wiring diagram for a cospan of the form $\varnothing \to X \oplus Y$ will have no interior cells, and the ports of the exterior cell are partitioned into two disjoint sets.

Proof of Proposition 4.7. Choose any set Λ. For any lax monoidal functor $A\colon \mathbf{Cospan}_\Lambda \to \mathbf{Set}$, Lemma 4.8 produces a hypergraph category $\mathcal{H}_A \in \mathbf{Hyp}_{\mathsf{OF}(\Lambda)}$, and for any morphism $\alpha\colon A \to B$, Lemma 4.9 produces a hypergraph functor between them. Functoriality is straightforward: given composable cospan-algebra morphisms α and β, $(F_\alpha \,\mathring{\,}\, F_\beta)(-) = \beta(\alpha(-)) = F_{\alpha \mathring{\,} \beta}(-)$.

Now suppose that $f\colon \Lambda \to \Lambda'$ is a morphism in $\mathbf{Set}_{\mathsf{List}}$. For naturality, we need to check the commutativity of a diagram much like Eq. (29), which comes down to checking that $\mathcal{H}_{(\mathbf{Cospan}_f \mathring{\,} A')} = \mathbf{Hyp}_{\mathsf{OF}(f)} \mathcal{H}_{A'}$ holds for any $A'\colon \mathbf{Cospan}_{\Lambda'} \to \mathbf{Set}$. While an equality of categories may seem strange, these two categories are defined to have the same objects—namely both are $\mathrm{Ob}(\mathbf{Cospan}_\Lambda)$—as well as the same morphisms. Indeed, by the definition of $\mathbf{Hyp}_{\mathsf{OF}}$ in Eqs. (19) and (21), the definition of $\mathcal{H}_-$ in Eq. (31), and the fact that Frob is $\mathtt{io}$, we have

$$\mathcal{H}_{(\mathbf{Cospan}_f \mathring{\,} A')}(X,Y) := A'(\mathbf{Cospan}_f(X) \oplus \mathbf{Cospan}_f(Y)) =: \mathbf{Hyp}_{\mathsf{OF}(f)} \mathcal{H}_{A'}(X,Y)$$

for any $X, Y \in \mathbf{Cospan}_\Lambda$. This completes the proof. $\square$

Remark 4.10. The main difference in perspective between hypergraph categories and cospan-algebras is that the structure of hypergraph categories involves both operations and special morphisms, whereas the structure of cospan-algebras involves just operations. Indeed, a hypergraph category $\mathcal{H} \in \mathbf{Hyp}_{\mathsf{OF}(\Lambda)}$ has the 2-ary operations of composition and monoidal product, as well as the identity morphism id_X and four Frobenius morphisms $\mu_X, \eta_X, \delta_X, \epsilon_X$ for every $X \in \mathcal{H}$. We saw in Eqs. (32) and (33) that both the operations and the special morphisms can be encoded in various cospans—morphisms in $\mathbf{Cospan}_\Lambda$—and that a cospan-algebra A turns them all into operations.

We can now put several different ideas together. Recall the initial cospan-algebra $\mathsf{Part}\colon \mathbf{Cospan}_\Lambda \to \mathbf{Set}$ from Proposition 4.6 and the name bijection $\widehat{\cdot}\colon \mathcal{H}(X,Y) \to \mathcal{H}(\varnothing, X \oplus Y)$ from Proposition 3.2. The above construction (Lemma 4.8) constructs from the initial cospan-algebra over Λ a hypergraph category $\mathcal{H}_{\mathsf{Part}}$ over Λ, which comes equipped with the universal map $\mathsf{Frob}\colon \mathbf{Cospan}_\Lambda \to \mathcal{H}_{\mathsf{Part}}$ that selects its Frobenius morphisms. The following proposition tells us these Frobenius morphisms are simply the names of the corresponding cospans.

Lemma 4.11. *Let* Part *be the initial cospan-algebra, let* $\mathcal{H}_{\mathsf{Part}}$ *be the corresponding hypergraph category, and let* $\mathsf{Frob}\colon \mathbf{Cospan}_\Lambda \to \mathcal{H}_{\mathsf{Part}}$ *be the universal map. Then for any* $f\colon X \to Y$ *in* $\mathbf{Cospan}_\Lambda$, *we have* $\mathsf{Frob}(f) = \widehat{f}$.

Proof. Note that by definition we have $\mathcal{H}_{\mathsf{Part}}(X,Y) = \mathsf{Part}(X * Y) = \mathbf{Cospan}_\Lambda(\varnothing, X \oplus Y)$, so the above identity is well typed. Since $\mathbf{Cospan}_\Lambda$ is the coproduct of Λ-many copies of $\mathbf{Cospan}$, and since $\mathbf{Cospan}$ is generated by the Frobenius generators (Lemma 3.6) it is enough to check that these two maps Frob and $\widehat{\cdot}$ agree on the Frobenius generators. This is true by construction as detailed in Eq. (33) of Lemma 4.8. $\square$

Corollary 4.12. *Let* $A\colon \mathbf{Cospan}_\Lambda \to \mathbf{Set}$ *be a cospan-algebra, let* $c\colon X \to Y$ *be a cospan, and consider the counit map* $\mathsf{Frob}_A\colon \mathbf{Cospan}_\Lambda \to \mathcal{H}_A$ *from Corollary 3.15. Then*

$$\mathsf{Frob}_A(c) = A(\widehat{c})(\gamma).$$

Proof. Let $\mathsf{Part}\colon \mathbf{Cospan}_\Lambda \to \mathbf{Set}$ be the initial cospan-algebra, let $\alpha_!\colon \mathsf{Part} \to A$ be the unique map, and let $F_!\colon \mathcal{H}_{\mathsf{Part}} \to \mathcal{H}_A$ be the associated $\mathtt{io}$ hypergraph functor given by Lemma 4.9. By naturality of the counit Frob, we have $\mathsf{Frob}_A = \mathsf{Frob}_{\mathsf{Part}} \mathring{\,} F_!$.

Take any $c\colon X \to Y$ in $\mathbf{Cospan}_\Lambda$. By Lemma 4.11, $\mathsf{Frob}_{\mathsf{Part}}(c) = \widehat{c}$, and by definition of $F_!$ we have $\mathsf{Frob}_A(c) = F_!(\mathsf{Frob}_{\mathsf{Part}}(c)) = \alpha_!(\widehat{c})$. The result then follows from Proposition 4.6, specifically Eq. (30), which says $\alpha_!(\widehat{c}) = A(\widehat{c})(\gamma)$. $\square$

4.3. The equivalence between OF-hypergraph categories and cospan-algebras

We are now ready to prove the equivalence.

Theorem 4.13. *The functors from Propositions 4.1 and 4.7 define an equivalence of categories:*

$$\mathbf{Hyp}_{\mathtt{OF}(\Lambda)} \xrightarrow[\;\mathcal{H}_-\;]{\;A_-\;} \mathbf{Lax}(\mathbf{Cospan}_\Lambda, \mathbf{Set}).$$

Moreover, this equivalence is natural in $\Lambda \in \mathbf{Set}_{\mathsf{List}}$.

Proof. Choose a set Λ. We shall show that $\mathcal{H}_{A_-}$ is naturally isomorphic to the identity functor on $\mathbf{Hyp}_{\mathtt{OF}(\Lambda)}$, and that $A_{\mathcal{H}_-}$ is in fact equal to the identity functor on $\mathbf{Lax}(\mathbf{Cospan}_\Lambda, \mathbf{Set})$. We will then be done because both $\mathcal{H}_-$ and A_- were shown to be natural in $\Lambda \in \mathbf{Set}_{\mathsf{List}}$.

Let's first consider the case of $\mathcal{H}_{A_-}$. The hypergraph category $\mathcal{H}_{A_\mathcal{H}}$ has the same objects as $\mathcal{H}$ and has homsets $\mathcal{H}_{A_\mathcal{H}}(X, Y) = \mathcal{H}(I, X \oplus Y)$. Define an io hypergraph functor $\nu_\mathcal{H} \colon \mathcal{H} \to \mathcal{H}_{A_\mathcal{H}}$ by sending $f \in \mathcal{H}(X, Y)$ to its name $\nu_\mathcal{H}(f) := \widehat{f}$, as in Proposition 3.2.

This map $\nu_\mathcal{H}$ is a well-defined functor due to the compact closure axioms. Indeed, let $f \colon X \to Y$ and $g \colon Y \to Z$ be morphisms in $\mathcal{H}$. By Eq. (32), the composite of names $\widehat{f} \,\mathring{,}\, \widehat{g}$ and the name of the composite are equal in $\mathcal{H}_{A_\mathcal{H}}$, by Proposition 3.3.

Preservation of the hypergraph structure follows from the Frobenius axioms. For example, given the multiplication μ_X on some object X in $\mathcal{H}$, its image under ν is given by

which is exactly the morphism specified by the cospan $\varnothing \to X \oplus X \oplus X$ in $\mathcal{H}_{A_\mathcal{H}}$; see Eq. (33).

We thus see that $\nu_\mathcal{H}$ defines an identity-on-objects hypergraph functor. Moreover, compact closure (Proposition 3.1) implies ν is fully faithful, and hence $\nu_\mathcal{H}$ has an inverse hypergraph functor. We must check these functors $\nu_\mathcal{H}$ are natural in $\mathcal{H}$, i.e. that for any hypergraph functor $F \colon \mathcal{H} \to \mathcal{H}'$ the following naturality square commutes:

$$
\begin{array}{ccc}
\mathcal{H} & \xrightarrow{\;\nu_\mathcal{H}\;} & \mathcal{H}_{A_\mathcal{H}} \\
{\scriptstyle F}\big\downarrow & & \big\downarrow{\scriptstyle \mathcal{H}_{A_F}} \\
\mathcal{H}' & \xrightarrow{\;\nu_{\mathcal{H}'}\;} & \mathcal{H}_{A_{\mathcal{H}'}}
\end{array}
$$

This is so because all the maps in the square are io and for any morphism $f \in \mathcal{H}$ we have $\widehat{F(f)} = F(\widehat{f})$. Thus $\nu \colon \mathrm{id}_{\mathbf{Hyp}_{\mathtt{OF}(\Lambda)}} \to \mathcal{H}_{A_-}$ is a natural isomorphism, as desired.

Next we consider the case of $A_{\mathcal{H}_-}$; we want to show that for any cospan-algebra A, there is an equality $A_{\mathcal{H}_A} = A$ of lax symmetric monoidal functors $\mathbf{Cospan}_\Lambda \to \mathbf{Set}$. On the objects of $\mathbf{Cospan}_\Lambda$ this is straightforward: the cospan-algebra $A_{\mathcal{H}_A}$ maps an object $X \in \mathbf{Cospan}_\Lambda$ to $\mathcal{H}_A(\varnothing, X) = A(X)$, so by definition $A_{\mathcal{H}_A}(X) = A(X)$. For morphisms, let $c \colon X \to Y$ be a cospan over Λ. Then $A_{\mathcal{H}_A}(c)$ is the function $A(X) = \mathcal{H}_A(\varnothing, X) \to \mathcal{H}_A(\varnothing, Y) = A(Y)$ given by composition with $\mathsf{Frob}(c) \in \mathcal{H}_A(X, Y)$, where $\mathsf{Frob} \colon \mathbf{Cospan}_\Lambda \to \mathcal{H}_A$ is the functor from Corollary 3.15. By Corollary 4.12, however, $\mathsf{Frob}(c) = A(\hat{c})(\gamma)$. By definition of composition in $\mathcal{H}_A$ (32) and Proposition 3.4, this implies that $A_{\mathcal{H}_A}(c)$ is exactly the function $A(c)$. Thus $A_{\mathcal{H}_-}$ is identity-on-objects.

Next, we consider the action of $A_{\mathcal{H}_-}$ on morphisms. Suppose that $\alpha\colon A \to B$ is a morphism of cospan-algebras, i.e. a monoidal natural transformation. We shall show that $A_{\mathcal{H}_\alpha} = \alpha$. Indeed, $\mathcal{H}_\alpha\colon \mathcal{H}_A \to \mathcal{H}_B$ maps each $f \in \mathcal{H}_A(\varnothing, X)$ to $\alpha_X(f)$, and hence $A_{\mathcal{H}_\alpha}(X)\colon A(X) \to B(X)$ maps each $f \in A(X) = \mathcal{H}_A(\varnothing, X)$ to $\alpha(f) \in B(X) = \mathcal{H}_B(\varnothing, X)$. This is what we wanted to show.

We have shown $A_{\mathcal{H}_-} = \mathrm{id}_{\mathbf{Lax}(\mathbf{Cospan}_\Lambda, \mathbf{Set})}$, completing the proof.   □

Example 4.14. Recall the hypergraph category **LinRel** with the addition Frobenius structure given in Example 2.11, where for example the linear relation corresponding to μ is $\{(a, b, c) \mid a + b = c\} \subseteq \mathbb{R}^3$. We shall construct $\mathbf{LinRel}' \coloneqq \mathcal{H}_{A_{\mathbf{LinRel}}}$, and observe that it is hypergraph equivalent to **LinRel**. However, one may notice that the definition of μ is not symmetric with respect to a, b, c, e.g. in contrast with what one might call the symmetric version, $\{(a, b, c) \mid a + b + c = 0\}$. Since cospan-algebras have no notion of domain and codomain, we will see that the isomorphism $\nu\colon \mathbf{LinRel} \to \mathbf{LinRel}'$ must rectify the asymmetry with a minus-sign.

By Eq. (26), the lax symmetric monoidal functor $A_{\mathbf{LinRel}}\colon \mathbf{Cospan} \to \mathbf{Set}$ sends each natural number n to $\mathbf{LinRel}(\mathbb{R}^0, \mathbb{R}^n)$, which we identify with the set of linear subspaces of $\mathbb{R}^n$, and it sends each cospan $m \to n$ to the unique corresponding linear relation $\mathbb{R}^m \to \mathbb{R}^n$ defined by the Frobenius maps.

At first blush, the homsets $\mathbf{LinRel}'(m, n)$ appear to be the same as those of **LinRel**, but this is not quite so:

$$\mathbf{LinRel}(m, n) = \{R \subseteq \mathbb{R}^m \oplus \mathbb{R}^n\} \qquad \text{whereas} \qquad \mathbf{LinRel}'(m, n) = \{R \subseteq \mathbb{R}^{m+n}\}.$$

These are certainly isomorphic, but in more than one way. The particular isomorphism ν constructed in Theorem 4.13 is given by sending R to its name $\nu(R) = \widehat{R}$, which itself arises *via the Frobenius structures in* **LinRel**; see Section 3.1. Unwinding the definitions, we have

$$\nu(R) = \{(a, b) \in \mathbb{R}^{m+n} \mid \exists (a' \in \mathbb{R}^m).\, a + a' = 0 \wedge (a', b) \in R\}$$
$$= \{(a, b) \in \mathbb{R}^{m+n} \mid (-a, b) \in R\}.$$

Remark 4.15. The isomorphism between the categories of cospan-algebras and hypergraph categories is a special case of the fact that there exists an isomorphism

$$\mathbf{Hyp}^{\mathrm{io}}_{\mathcal{H}/} \cong \mathbf{Lax}(\mathcal{H}, \mathbf{Set})$$

between the category $\mathbf{Lax}(\mathcal{H}, \mathbf{Set})$ of lax symmetric monoidal functors $\mathcal{H} \to \mathbf{Set}$, and the coslice category $\mathbf{Hyp}^{\mathrm{io}}_{\mathcal{H}/}$ over $\mathcal{H}$ of hypergraph categories and identity-on-objects hypergraph functors between them. Above we have just taken $\mathcal{H}$ to be the free hypergraph category $\mathbf{Cospan}_\Lambda$ over Λ. Nonetheless, the proof above generalizes to the case where $\mathcal{H}$ is any hypergraph category.

We can now easily prove the main theorem.

Theorem 4.16. *We have an equivalence of categories*

$$\mathbf{Hyp}_{\mathsf{OF}} \cong \mathbf{Cospan}\text{-}\mathbf{Alg}.$$

Proof. Theorem 4.13 provides an equivalence $\mathbf{Hyp}_{\mathsf{OF}(\Lambda)} \cong \mathbf{Lax}(\mathbf{Cospan}_\Lambda, \mathbf{Set})$, natural in $\Lambda \in \mathbf{Set}_{\mathsf{List}}$. Since the Grothendieck construction is functorial, the middle map below is also an equivalence

$$\mathbf{Hyp}_{\mathsf{OF}} \cong \int^{\Lambda \in \mathbf{Set}_{\mathsf{List}}} \mathbf{Hyp}_{\mathsf{OF}(\Lambda)} \cong \int^{\Lambda \in \mathbf{Set}_{\mathsf{List}}} \mathbf{Lax}(\mathbf{Cospan}_\Lambda, \mathbf{Set}) \cong \mathbf{Cospan}\text{-}\mathbf{Alg}$$

and the first and third equivalence follow from Corollary 3.21 and Proposition 2.4.   □

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