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The multiplicative group action on singular varieties and Chow varieties

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ABSTRACT

We answer two questions of Carrell on a singular complex projective variety admitting the multiplicative group action, one positively and the other negatively. The results are applied to Chow varieties and we obtain Chow groups of 0-cycles and Lawson homology groups of 1-cycles for Chow varieties. A brief survey on the structure of Chow varieties is included for comparison and completeness. Moreover, we give counterexamples to Shafarevich's problem on the rationality of the irreducible components of Chow varieties.

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1. Introduction

Let V be a holomorphic vector field defined on a projective algebraic variety X. The zero subscheme Z is the subspace of X defined by the ideal generated by $V\mathcal{O}_X$ and we denote it by X^V.

The existence of a holomorphic vector field with zeroes on a smooth projective variety imposes restrictions on the topology of the manifold. For examples, the Hodge numbers $h^{p,q}(X) = 0$ if $|p - q| > \dim Z$ (see [5]). For a smooth complex projective variety X admitting a $\mathbb{C}^*$-action, Bialynicki-Birula' structure theorem describes the relation between the structure of X and that of the fixed points set ([3]).

According to Lieberman ([37]), the existence of a holomorphic vector field V on a complex algebraic projective variety X with nonempty zeroes is equivalent to the existence of the action a 1-parameter group G, which is isomorphic to a product of $\mathbb{C}^*$'s and at most one $\mathbb{C}$'s. This encourages us to study the structure of projective varieties admitting a $\mathbb{C}^*$-action or a $\mathbb{C}$-action.

In this paper, we consider the case that X is a singular projective variety admitting a $\mathbb{C}$-action (resp. $\mathbb{C}^*$-action). In this case, the relation between the structure of X and that of the fixed point set is subtle. A general result of Bialynicki-Birula says that X and the fixed point set are $\mathbb{C}$-equivalent (resp. $\mathbb{C}^*$-equivalent). In [27], we got the Chow group of zero cycles and Lawson homology group of 1-cycles for X admitting a $\mathbb{C}$-action.

When X is singular and admitting a $\mathbb{C}^*$-action, the Bialynicki-Birula type structure theorem also holds for singular homology groups if the action is "good" in sense of [4]. In general, it does not hold for a singular X admitting a $\mathbb{C}^*$-action without additional conditions. For a singular variety X admitting the certain $\mathbb{C}^*$-action with isolated fixed points, Carrell asked if the odd Betti numbers of X vanish, etc. In section 3, we ask parallel questions to those of Carrell and give answers to all of them. We give negative answers to some of these questions. We compute the Chow groups of 0-cycles for singular varieties admitting a $\mathbb{C}^*$-action with isolated fixed points. As a contrast to projective varieties admitting a $\mathbb{C}$-action, the parallel result for Lawson homology group of 1-cycles does not hold any more (see Example 3.29).

In section 4, we briefly review and summarize some known algebraic and topological invariants for Chow varieties $C_{p,d}(\mathbb{P}^n)$ parameterizing effective p-cycles of degree d in the complex projective space $\mathbb{P}^n$. We give a counterexample to the problem of Shafarevich on the rationality of the irreducible components of Chow varieties, based on the work of Eisenbud, Harris, Mumford, etc.

As applications of section 2 and 3, we compute the Chow group of zero cycles and Lawson homology groups of 1-cycles for Chow varieties.

2. Invariants under the additive group action

Let X be a possible singular complex projective algebraic variety which admits an additive group action. Our main purpose is to compare certain algebraic and topological invariants (such as the Chow group of zero cycles, Lawson homology, the singular homology, etc.) of X to those of the fixed point set $X^{\mathbb{C}}$. If X is a smooth projective variety, then most of its topological invariants are studied and computed in details. However, some of algebro-geometric invariants are still hard to identify. Some of those invariants have been investigated even if X is singular. In this section, we will identify some of these invariants including the Chow groups of zero cycles, Lawson homology for 1-cycles, the singular homology with integer coefficients, etc.

2.1. A-equivalence

Let A be a fixed complex quasi-projective algebraic variety. Recall that an algebraic scheme X_1 is **simply A-equivalent** to an algebraic variety X_2 if X_1 is isomorphic to a closed subvariety X_2' of X_2 and there exists an isomorphism $f : X_2 - X_2' \to Y \times A$, where Y is an algebraic variety. The smallest equivalence relation containing the relation of simple A-equivalences is called the **A-equivalence** and we denote it by $\sim$ (see [2]). A result of Bialynicki-Birula says that $X \sim X^{\mathbb{C}}$ if X is a quasi-projective variety admitting a $\mathbb{C}$-action. A similar statement holds for X admitting a $\mathbb{C}^*$-action. From this, Bialynicki-Birula showed that $H^0(X,\mathbb{Z}) \cong H^0(X^{\mathbb{C}},\mathbb{Z})$ and $H^1(X,\mathbb{Z}) \cong H^1(X^{\mathbb{C}},\mathbb{Z})$ in the case that X admits a $\mathbb{C}$-action, where $\chi(X) = \chi(X^{\mathbb{C}^*})$ in the case that X admits a $\mathbb{C}$-action (see [2]). Along this route, more additive invariants have been calculated for varieties admits a $\mathbb{C}$ or $\mathbb{C}^*$-action (see [28]).

2.2. Chow groups and Lawson homology

Let X be any complex projective variety or scheme of dimension n and let $\mathcal{Z}_p(X)$ be the group of algebraic p-cycles on X. Let $\mathrm{Ch}_p(X)$ be the Chow group of p-cycles on X, i.e.

$$\mathrm{Ch}_p(X) = \mathcal{Z}_p(X)/\{\text{rational equivalence}\}.$$

Set $\mathrm{Ch}_p(X)_{\mathbb{Q}} := \mathrm{Ch}_p(X) \otimes \mathbb{Q}$, $\mathrm{Ch}_p(X) = \bigoplus_{p \geq 0} \mathrm{Ch}_p(X)$. For more details on the Chow theory, the reader is referred to Fulton ([21]).

Proposition 2.1. *[27] Let X be a (possible singular) connected complex projective variety. If X admits a $\mathbb{C}$-action with isolated fixed points, then* $\mathrm{Ch}_0(X) \cong \mathbb{Z}$.

Remark 2.2. More generally, by using the same method, we can show that if X admits a $\mathbb{C}$-action with a fixed points set $X^{\mathbb{C}}$, then $\mathrm{Ch}_0(X) \cong \mathrm{Ch}_0(X^{\mathbb{C}})$.

The *Lawson homology* $L_pH_k(X)$ of p-cycles for a projective variety is defined by

$$L_pH_k(X) := \pi_{k-2p}(\mathcal{Z}_p(X)) \quad for \quad k \geq 2p \geq 0,$$

where $\mathcal{Z}_p(X)$ is provided with a natural topology (cf. [18], [34]).

In [19], Friedlander and Mazur showed that there are natural transformations, called *Friedlander-Mazur cycle class maps*

$$\Phi_{p,k} : L_pH_k(X) \to H_k(X,\mathbb{Z}) \tag{2.3}$$

for all $k \geq 2p \geq 0$.

Set

$$\begin{aligned} L_pH_k(X)_{hom} &:= \ker\{\Phi_{p,k} : L_pH_k(X) \to H_k(X)\}; \\ L_pH_k(X)_{\mathbb{Q}} &:= L_pH_k(X) \otimes \mathbb{Q}. \end{aligned}$$

Denoted by $\Phi_{p,k,\mathbb{Q}}$ the map $\Phi_{p,k} \otimes \mathbb{Q} : L_pH_k(X)_{\mathbb{Q}} \to H_k(X,\mathbb{Q})$. The *Griffiths group* of dimension p-cycles is defined to be

$$\mathrm{Griff}_p(X) := \mathcal{Z}_p(X)_{hom}/\mathcal{Z}_p(X)_{alg}.$$

Set

$$\mathrm{Griff}_p(X)_{\mathbb{Q}} := \mathrm{Griff}_p(X) \otimes \mathbb{Q};$$

It was proved by Friedlander [18] that, for any smooth projective variety X,

$$L_pH_{2p}(X) \cong \mathcal{Z}_p(X)/\mathcal{Z}_p(X)_{alg}.$$

Therefore

$$L_pH_{2p}(X)_{hom} \cong \mathrm{Griff}_p(X).$$

Proposition 2.4. *[27] Under the same assumption as Proposition 2.1, we have*

$$L_1H_k(X) \cong H_k(X, \mathbb{Z})$$

for all $k \geq 2$. In particular, $\mathrm{Griff}_1(X) = 0$.

Remark 2.5. The isomorphism $L_0H_k(X) \cong H_k(X, \mathbb{Z})$ holds for any integer $k \geq 0$, which is the special case of the Dold-Thom Theorem.

Remark 2.6. The assumption of "connectedness" in Proposition 2.4 is not necessary. By the same reason, we can remove the connectedness in Proposition 2.1, while the conclusion "$\mathrm{Ch}_0(X) \cong \mathbb{Z}$" would be replaced by "$\mathrm{Ch}_0(X) \cong H_0(X, \mathbb{Z})$".

2.3. The virtual Betti and Hodge numbers

Recall that the *virtual Hodge polynomial* $H : Var_{\mathbb{C}} \to \mathbb{Z}[u, v]$ is defined by the following properties:

(1) $H_X(u, v) := \sum_{p,q}(-1)^{p+q} \dim H^q(X, \Omega_X^p)u^pv^q$ if X is nonsingular and projective (or complete).
(2) $H_X(u, v) = H_U(u, v) + H_Y(u, v)$ if Y is a closed algebraic subset of X and $U = X - Y$.
(3) If $X = Y \times Z$, then $H_X(u, v) = H_Y(u, v) \cdot H_Z(u, v)$.

The existence and uniqueness of such a polynomial follow from Deligne's Mixed Hodge theory (see [12,13]). The coefficient of u^pv^q of $H_X(u, v)$ is called the *virtual Hodge* (p, q)*-number* of X and we denote it by $\tilde{h}^{p,q}(X)$. Note that from the definition, $\tilde{h}^{p,q}(X)$ coincides with the usual Hodge number (p, q)-number $h^{p,q}(X)$ if X is a smooth projective variety. The sum $\tilde{\beta}^k(X) := \sum_{i+j=k} \tilde{h}^{p,q}(X)$ is called the k-th *virtual Betti number* of X. The *virtual Poincaré polynomial* of X is defined to be

$$\widetilde{P}_X(t) := \sum_{k=0}^{2\dim_{\mathbb{C}} X} \beta^k(X)t^k,$$

which coincides to the usual Poincaré polynomial defined through the corresponding usual Betti numbers.

3. Results related to the multiplicative group action

In this section we will give all kinds of relations between a complex variety (not necessarily smooth, irreducible) and the fixed point set of a multiplicative group action or an additive group action.

Let X be a smooth complex projective variety which admits a $\mathbb{C}^*$-action with fixed point set $X^{\mathbb{C}^*}$. Denote by $F_1, \cdots, F_r$ the connected components. It was shown by Bialynicki-Birula that there is a homology basis formula ([3]):

$$H_k(X, \mathbb{Z}) \cong \bigoplus_{j=1}^{r} H_{k-2\lambda_j}(F_j, \mathbb{Z}), \tag{3.1}$$

where λ_j is the fiber dimension of the bundle in $P_j : X_j^+ \to F_j$ and $X_j^+ := \{x \in X : \lim_{t\to 0} t \cdot x \in F_j\}$. This result has been generalized to compact Kähler manifolds without change by Carrell-Sommese [8] and Fujiki independently. In fact, when X is a compact Kähler manifold, the Hodge structure on X is completely determined by that on the fixed point set in an obvious way.

Furthermore, there are similar basis formulas for Chow groups (see [10] for $X^{\mathbb{C}^*}$ finite and [40] for the general case) and Lawson homology (see [39] for $X^{\mathbb{C}^*}$ finite and [33] for the general case), as applications of Bialynicki-Birula' structure theorem ([3]).

However, if X is a singular projective algebraic variety, Equation (3.1) would be failed in general. Under some additional conditions, Equation (3.1) may still hold. For example, if the $\mathbb{C}^*$-action on X is "good" in the sense of Carrell and Goresky, Equation (3.1) has been shown to hold (cf. [4]).

There are several questions related to the structure of X and $X^{\mathbb{C}^*}$. J. Carrell asked the question how does the mixed Hodge structure on X relate to the mixed Hodge structure on the fixed point set in the case of good action.

Question 3.2. ([6, p.21]) In the case of a good action, how does the mixed Hodge structure on X relate to the mixed Hodge structure on $X^{\mathbb{C}^*}$?

We will give an explicit relation on the mixed Hodge structure between X and $X^{\mathbb{C}^*}$, especially the relation of their virtual Hodge numbers (see Proposition 3.10).

When X is a possibly singular complex projective variety with a $\mathbb{C}^*$-action, where a "variety" means a reduced, not necessary irreducible scheme, Carrell and Goresky showed that there still exists an integral homology basis formula under the assumption that the $\mathbb{C}^*$-action is "good" ([4]).

Carrell asked the following question.

Question 3.3. ([6, p.22]) If an irreducible complex projective variety X admits not necessarily good $\mathbb{C}^*$-action with isolated fixed points, do the odd homology groups of X vanish?

The following example gives a negative answer to his question.

Example 3.4. Let C be a cubic plane curve with a node singular point p, e.g. $(zy^2 = x^3 + x^2z) \subset \mathbb{P}^2$. The normalization $\sigma : \tilde{C} \to C$ of C is isomorphic to $\mathbb{P}^1$. Let $\mathbb{C}^* \times \mathbb{P}^1 \to \mathbb{P}^1$ be the holomorphic $\mathbb{C}^*$-action given by $(t, [x : y]) \mapsto [tx : y]$. The fixed point set of this action contains two points, $[1 : 0]$ and $[0 : 1]$. We can always assume $\sigma([1, 0]) = \sigma([0 : 1]) = p_0$ by composing a suitable automorphism of $\mathbb{P}^1$, where $p_0 = [0 : 0 : 1]$ is the singular point of C. The holomorphic $\mathbb{C}^*$-action on $\mathbb{P}^1$ descends to a holomorphic $\mathbb{C}^*$-action on C whose fixed point set is the single point p. More explicitly, such a map σ can be given by the formula: $\sigma : \mathbb{P}^1 \to C$, $[s : t] \mapsto [st(s + t) : st(s - t) : (s + t)^3]$.

However, the fundamental group of C is isomorphic to $\mathbb{Z}$, so $H_1(C, \mathbb{Z}) \cong \mathbb{Z}$ and $\beta_1(C) = 1 \neq 0$.

In each dimension $n \geq 1$, there exists a projective variety X satisfying the assumption in Question 3.3 such that $\beta_1(X) \neq 0$. To see this, note that $\mathbb{P}^{n-1}$ admits a $\mathbb{C}^*$-action with isolated fixed points for each integer $n \geq 1$. Hence $X := C \times \mathbb{P}^{n-1}$ admits an induced $\mathbb{C}^*$-action from each component with isolated fixed points. Therefore, we get $\beta_1(X) = \beta_1(C)$ by the Künneth formula and the later is nonzero from Example 3.4.

In Example 3.4, X admits a $\mathbb{C}^*$-action but the odd homology group $H_1(X, \mathbb{Z})$ is nonzero. However, the odd *virtual Betti numbers* and the *virtual Hodge numbers* $\tilde{h}^{p,q}(X)$ are zero, where $p \neq q$. To see this, we can write $C = \mathbb{C}^* \cup p_0$ and so $H_C(u, v) = (uv - 1) + 1 = uv$. Hence $\tilde{h}^{1,0}(C) = \tilde{h}^{0,1}(C) = 0$ and $\tilde{\beta}^1(C) = 0$.

In certain sense, the virtual Betti numbers are more suitable to reveal the topology of a singular variety. A natural question would be the following modified version of Carrell's Question in virtual Betti numbers.

Question 3.5 *(Modified version of Carrell).* If an irreducible complex projective variety X admits not necessarily good $\mathbb{C}^*$-action with isolated fixed points, do the odd virtual Betti numbers of X vanish?

If X is irreducible and $\dim X = 1$, the answer to the question is positive. In this case, $X = \mathbb{C}^* \cup Y$ and Y is a set of finite points. Then $H_X(u,v) = (uv-1)+k = uv+k-1$ and the odd virtual Betti numbers of X are zero, where k is the number of points of Y.

If X is smooth projective, then the answer to the question is positive ([3]). Moreover, if the $\mathbb{C}^*$-action on X is "good" in the sense of Carrell and Goresky, the answer is also positive (see Corollary 3.16 for a weaker condition such that the answer is positive).

The following example of a projective variety admits a not "good" $\mathbb{C}^*$-action, but the answer to Question 3.5 is positive.

Example 3.6. Let $X := \mathrm{SP}^d(\mathbb{P}^n)$ be the d-th symmetric product of the complex projective space $\mathbb{P}^n$. The standard $(\mathbb{C}^*)^n$-action on $\mathbb{P}^n$ induces a $(\mathbb{C}^*)^n$-action on $\mathrm{SP}^d(\mathbb{P}^n)$ with isolated fixed points. It follows from Cheah [9] that the k-th virtual Betti number of $\mathrm{SP}^d(\mathbb{P}^n)$ is the coefficient of $t^d x^k$ in the power series of $\prod_{j=0}^n (1-tx^{2j})^{-1}$. Hence $\tilde{\beta}_k(\mathrm{SP}^d(\mathbb{P}^n)) = 0$ for and all d and all odd k.

Under a weaker condition than Carrell and Goresky's "good" condition, the answer to Question 3.5 is positive (see Corollary 3.16).

However, in general, the answer to Question 3.5 is negative. There is an irreducible projective algebraic surface S admitting $\mathbb{C}^*$-action with isolated zeroes such that the first virtual betti number $\tilde{\beta}_1(S) \neq 0$. Such a surface was constructed by Lieberman ([38, p.111]) as a nonrational surface admitting a holomorphic vector field with isolated zeroes. A suitable modification fulfills our purpose. The following example gives a negative answer to Question 3.5.

Example 3.7. Let $Y = \mathbb{P}^1 \times C$, where C is a smooth projective curve with genus $g(C) \geq 1$. Let us consider the $\mathbb{C}^*$-action $\phi : \mathbb{C}^* \times Y \to Y$ given by $(t, ([u:v], z)) \to ([u:tv], z)$, where $[u:v]$ denotes the homogeneous coordinates for $\mathbb{P}^1$ and z denotes the coordinate for the curve C. The fixed points of the action ϕ are $C_1 := [1:0] \times C$ and $C_2 := [0:1] \times C$. Each of these curves has self-intersection zero. Let $\sigma : \widetilde{S} \to Y$ be obtained from Y by blowing up one point p_i on each C_i $(i = 1, 2)$, and let $\tilde{\phi} : \mathbb{C} \times \widetilde{S} \to \widetilde{S}$ be the equivariant lifting action. The fixed points of $\tilde{\phi}$ are the proper transforms $\widetilde{C}_i$ of C_i and two other isolated points. Since the self-intersection number of $\widetilde{C}_i$ on $\widetilde{S}$ is -1. One can blow down $\tilde{\sigma} : \widetilde{S} \to S$ the $\widetilde{C}_i$ to obtain a projective surface S, which admits the induced $\mathbb{C}^*$-action. Moreover $S^{\mathbb{C}^*}$ are four isolated points. In explicitly, we have the following relations

$$\begin{tikzcd} \widetilde{S} \arrow[r, "\tilde{\sigma}"] \arrow[d, "\sigma"] & S \\ Y \arrow[r, equal] & \mathbb{P}^1 \times C. \end{tikzcd} \tag{3.8}$$

Now we can compute the virtual Betti numbers from the construction. Since $\widetilde{S} - \widetilde{C}_1 - \widetilde{C}_2 \cong S - \tilde{\sigma}(\widetilde{C}_1) - \tilde{\sigma}(\widetilde{C}_2)$ and $\widetilde{S} - E_1 - E_2 \cong Y - p_1 - p_2$, where $E_1 \cong E_2 \cong \mathbb{P}^1$, we have by using the additive property of the virtual Poincaré polynomial

$$\begin{aligned}
\widetilde{P}_S(t) &= \widetilde{P}_{\widetilde{S}}(t) - \widetilde{P}_{\widetilde{C}_1}(t) - \widetilde{P}_{\widetilde{C}_2}(t) + \widetilde{P}_{\tilde{\sigma}(\widetilde{C}_1)}(t) + \widetilde{P}_{\tilde{\sigma}(\widetilde{C}_2)}(t) \\
&= \widetilde{P}_{\widetilde{S}}(t) - 2\widetilde{P}_C(t) + 2 \\
&= \widetilde{P}_Y(t) + 2\widetilde{P}_{\mathbb{P}^1}(t) - 2 - 2\widetilde{P}_C(t) + 2 \\
&= \widetilde{P}_{\mathbb{P}^1 \times C}(t) + 2\widetilde{P}_{\mathbb{P}^1}(t) - 2 - 2\widetilde{P}_C(t) + 2 \\
&= \widetilde{P}_{\mathbb{P}^1}(t)\widetilde{P}_C(t) + 2\widetilde{P}_{\mathbb{P}^1}(t) - 2 - 2\widetilde{P}_C(t) + 2 \\
&= (t^2+1)(t^2+2g(C)t+1) + 2(t^2+1) - 2(t^2+2g(C)t+1) \\
&= t^4 + 2g(C)t^3 + 2t^2 - 2g(C)t + 1.
\end{aligned}$$

Since $g(C) \geq 1$, $\tilde{\beta}_1(S) = -2g(C) \neq 0$.

Remark 3.9. We can also construct examples of projective varieties in any dimension greater than or equals to 2 such that the answer to Question 3.5 is negative. Since $\mathbb{P}^n$ admits a $\mathbb{C}^*$-action with isolated fixed points, so $S \times \mathbb{P}^n$ admits a $\mathbb{C}^*$-action with isolated points, where S is the projective surface constructed in Example 3.7. By using the product property of the virtual Poincaré polynomial, it is easy to compute that $\tilde{\beta}_1(S \times \mathbb{P}^n) = -2g(C)$.

Now we shall show that the answer to Question 3.5 is positive under certain not "good" condition. For a singular variety X with a $\mathbb{C}^*$-action, one can always find an analytic Whitney stratification whose strata are $\mathbb{C}^*$-invariant. Recall that the $\mathbb{C}^*$-action on X is *singularity preserving* as $t \to 0$ if there exists an equivariant Whiteny stratification of X such that for every stratum A, and for every $x \in A$, the limit $x_0 = \lim_{t\to 0} t \cdot x$ is also in A (cf. [4]). In this case, $X = \bigcup_{j=1}^r X_j^+$, and $X_j^+ \to F_j$ is a topologically locally trivial affine space bundle (cf. [4, Lemma 1]). Denote m_j be the dimension of the fiber of the bundle $X_j^+ \to F_j$.

Then we have the following relation on virtual Hodge polynomials between X and the fixed point set.

Proposition 3.10. *Suppose X admits a Whitney stratification which is singularity preserving as $t \to 0$. Then*

$$H_X(u,v) = \sum_{j=1}^{r} H_{F_j}(u,v) u^{m_j} v^{m_j},$$

where F_j and m_j are given as before.

Proof. Suppose X has a Whitney stratification that is singularity preserving as $t \to 0$ and let F_j denote a fixed point component. For a stratum A, the map $F_j \cap A$, $p_j^{-1}(F_j \cap A) := \{x \in X : \lim_{t\to 0}(t \cdot x) \in F_j \cap A\}$ is Zariski locally trivial affine space bundle (cf. [3], [7]). Since $F_j = \cup_{A\in\mathbb{S}}(F_j \cap A)$, where $\mathbb{S}$ is the set of all strata of X in the given Whitney stratification. Hence the total space of the topological locally trivial affine space bundle $X_j^+ \to F_j$ can be written the disjoint union of subvarieties $p_j^{-1}(F_j \cap A)$.

Therefore, we have

$$\begin{aligned} H_X(u,v) &= \textstyle\sum_{j=1}^r H_{X_j^+}(u,v) \\ &= \textstyle\sum_{j=1}^r \sum_{A\in\mathbb{S}} H_{p_j^{-1}(F_j\cap A)}(u,v) \\ &= \textstyle\sum_{j=1}^r \sum_{A\in\mathbb{S}} H_{F_j\cap A}(u,v) \cdot H_{\mathbb{C}^{m_j}}(u,v) \\ &= \textstyle\sum_{j=1}^r H_{F_j}(u,v) \cdot H_{\mathbb{C}^{m_j}}(u,v) \\ &= \textstyle\sum_{j=1}^r H_{F_j}(u,v)(uv)^{m_j}. \quad \square \end{aligned}$$

Remark 3.11. Proposition 3.10 does not have to hold if the singularity preserving property fails. For example, X is the cone in $\mathbb{P}^{n+1}$ over a smooth projective variety $V \subset \mathbb{P}^n = (z_{n+1} = 0)$ with vertex $\mathbb{P}^0 = [0 : \cdots : 0 : 1]$, the $\mathbb{C}^*$-action on X induced by the action $(t, [z_0 : \cdots : z_n : z_{n+1}]) \mapsto [tz_0 : \cdots : tz_n : z_{n+1}]$ on $\mathbb{P}^{n+1}$. The fixed point set is V and $\mathbb{P}^0$, and the action is not singularity preserving as $t \to 0$. In this case we observe that $H_X(u,v) \neq H_V(u,v) + h_{\mathbb{P}^0}(u,v)u^n v^n$. However, if the action is given as $(t, [z_0 : \cdots : z_n : z_{n+1}]) \mapsto [z_0 : \cdots : z_n : tz_{n+1}]$ on $\mathbb{P}^{n+1}$, it is singularity preserving as $t \to 0$. So $H_X(u,v) = H_V(u,v)uv + H_{\mathbb{P}^0}(u,v) = H_V(u,v)uv + 1$.

From the proof of the above theorem, we see that if X can be decomposed as the disjoint union of locally closed subvarieties (not necessarily irreducible) W_j for $j = 1, \cdots, r$, where W_i is a locally trivial affine space bundle over F_j with fiber $\mathbb{C}^{m_j}$ in Zariski topology, then $H_X(u,v) = \sum_{j=1}^r H_{F_j}(u,v)u^{m_j}v^{m_j}$.

From Proposition 3.10, we see that the mixed Hodge structure of X is partially determined by the mixed Hodge structures of the fixed point set. One also obtains from Proposition 3.10 that the virtual Hodge numbers of X are nonnegative if all F_j are smooth projective varieties.

Corollary 3.12. *Suppose X admits a Whitney stratification which is singularity preserving as $t \to 0$. Then*

$$\tilde{h}^{p,q}(X) = 0, \quad \forall |p-q| > \dim X^{\mathbb{C}^*}.$$

In particular, if $X^{\mathbb{C}^*}$ contains only isolated points, then $\tilde{h}^{p,q}(X) = 0$ for all $p \neq q$.

One obtains the relations between virtual Betti numbers of X and those of the fixed point set immediately from Proposition 3.10.

Corollary 3.13. *Suppose X admits a Whitney stratification which is singularity preserving as $t \to 0$. Then*

$$\tilde{P}_X(t) = \sum_{j=1}^{r} \tilde{P}_{F_j}(t) t^{2m_j}. \tag{3.14}$$

If the $\mathbb{C}^*$-action on a projective variety X is "good" in the sense of Carrell and Goresky (cf. [4]), then the usual Poincaré polynomial $P_X(t)$ of X can be expressed in terms of that of the fixed point set as follows:

$$P_X(t) = \sum_{j=1}^{r} P_{F_j}(t) t^{2m_j}. \tag{3.15}$$

Furthermore, if F_j are smooth projective varieties, then $\tilde{P}_X(t) = P_X(t)$ since $\tilde{P}_{F_j}(t) = P_{F_j}(t)$ for each F_j and Equation (3.14)-(3.15). In other words, the virtual Betti numbers and the usual Betti numbers coincide for such projective varieties. This gives us the following corollary.

Since the answer to Question 3.5 is negative in general, the following corollary gives a sufficient condition for the $\mathbb{C}^*$-action such that the odd virtual Betti numbers vanish. This condition is much weaker than Carrell and Goresky's "good" condition.

Corollary 3.16. *Under the assumption in Question 3.5 and suppose X admits a Whitney stratification which is singularity preserving as $t \to 0$. Then*

$$\tilde{\beta}_{2k-1}(X) = 0, \quad \forall k > \dim X^{\mathbb{C}^*}.$$

In particular, if $X^{\mathbb{C}^*}$ contains only isolated points, then $\tilde{\beta}_k(X) = 0$ for all odd k.

For a $\mathbb{C}^*$-action on algebraic varieties, there is a relation between virtual Hodge numbers between X and $X^{\mathbb{C}^*}$ (see [28]), i.e.,

$$\sum_{p-q=i} \tilde{h}^{p,q}(X) = \sum_{p-q=i} \tilde{h}^{p,q}(X^{\mathbb{C}^*}), \forall i. \tag{3.17}$$

If we set $\tilde{b}_{even}(X) := \sum_i \tilde{b}_{2i}(X)$ and $\tilde{b}_{odd}(X) := \sum_i \tilde{b}_{2i-1}(X)$, then we get from equation (3.17)

$$\begin{aligned} \tilde{b}_{even}(X) &= \tilde{b}_{even}(X^{\mathbb{C}^*}) \\ \tilde{b}_{odd}(X) &= \tilde{b}_{odd}(X^{\mathbb{C}^*}). \end{aligned} \tag{3.18}$$

In particular, X admits a $\mathbb{C}^*$-action with isolated zeroes, then $\tilde{b}_{odd}(X) = 0$, i.e., the sum of all odd virtual Betti numbers is zero.

Note that the Euler characteristic $\chi(X)$ of X is equal to $\tilde{b}_{even}(X) - \tilde{b}_{odd}(X)$ and Equation (3.18) implies the fixed point formula for the Euler characteristic: $\chi(X) = \chi(X^{\mathbb{C}^*})$.

When X admits $\mathbb{C}$-action with isolated fixed point, it was shown that $\mathrm{Ch}_0(X) \cong \mathbb{Z}$ (see Proposition 2.1). Inspired by this result, it is natural to ask if $\mathrm{Ch}_0(X) \cong \mathbb{Z}$ holds for a $\mathbb{C}^*$-action. Amazingly, such a statement still holds.

Proposition 3.19. *If X is a connected projective variety admitting a $\mathbb{C}^*$-action with isolated fixed points, then we have $\mathrm{Ch}_0(X) \cong \mathbb{Z}$.*

Proof. Since X admits a $\mathbb{C}^*$-action with isolated fixed points, there exists a $\mathbb{C}^*$-invariant Zariski open set $U \subset X$ such that $U \cong U' \times \mathbb{C}^*$ (see [2]). Such U and U' can be assumed to be non-singular if necessary. Set $Z = X - U$. By the localization sequence of higher chow groups and homotopy invariance, we get $\mathrm{Ch}_0(U' \times \mathbb{C}^*, 1) \cong \mathrm{Ch}_0(U')$. From the Poicaré duality, homotopy invariance of cohomology and the Künneth formula for the Borel-Moore homology, we obtain that $H_1^{BM}(U' \times \mathbb{C}^*) \cong H^{2n-1}(U' \times \mathbb{C}^*) \cong H^{2n-1}(U' \times S^1) \cong H_0^{BM}(U' \times S^1) \cong H_0^{BM}(U')$. Note that the cycle class map $\mathrm{Ch}_0(U') \to H_0^{BM}(U', \mathbb{Z})$ is always surjective. Hence the higher cycle class map $\phi_0(U, 1) : \mathrm{Ch}_0(U, 1) \to H_1^{BM}(U, \mathbb{Z})$ is surjective.

By applying the localization sequence to (X, Z) and using the natural transform for the higher chow group to the singular homology group, we get

$$\begin{array}{ccccccccc}
\mathrm{Ch}_0(U,1) & \longrightarrow & \mathrm{Ch}_0(Z) & \longrightarrow & \mathrm{Ch}_0(X) & \longrightarrow & \mathrm{Ch}_0(U) & \longrightarrow & 0 \\
\downarrow & & \downarrow\cong & & \downarrow & & \downarrow\cong & & \\
H_1^{BM}(U,\mathbb{Z}) & \longrightarrow & H_0(Z,\mathbb{Z}) & \longrightarrow & H_0(X,\mathbb{Z}) & \longrightarrow & H_0^{BM}(U,\mathbb{Z}) & \longrightarrow & 0.
\end{array} \tag{3.20}$$

By induction hypothesis, we have the isomorphism $\mathrm{Ch}_0(Z) \xrightarrow{\cong} H_0(Z, \mathbb{Z})$. Note that $\mathrm{Ch}_0(U) \cong \mathrm{Ch}_0(U' \times \mathbb{C}^*) = 0$ since a point moves in a $\mathbb{C}^*$ direction to infinite, which is not on U. Therefore $\mathrm{Ch}_0(U) = 0 = H_0^{BM}(U, \mathbb{Z})$. Now we get the isomorphism $\mathrm{Ch}_0(X) \xrightarrow{\cong} H_0(X, \mathbb{Z})$ by the Five Lemma. Hence $\mathrm{Ch}_0(X) \xrightarrow{\cong} \mathbb{Z}$ since X is connected. This completes the proof of the proposition. $\square$

Remark 3.21. In fact, from the proof of Proposition 3.19, we have shown the following result: If X is a connected projective variety admitting a $\mathbb{C}^*$-action with a nonempty fixed point set $X^{\mathbb{C}^*}$, then the inclusion $i : X^{\mathbb{C}^*} \to X$ induces a surjective $\mathrm{Ch}_0(X^{\mathbb{C}^*}) \to \mathrm{Ch}_0(X)$.

Remark 3.22. If X is smooth projective variety admitting a $\mathbb{C}^*$-action with isolated fixed points, then X admits a cellular decomposition (see [3]) and $\mathrm{Ch}_p(X) \cong H_{2p}(X, \mathbb{Z})$ for all $p \geq 0$. However, in the case that X is singular, $\mathrm{Ch}_p(X) \cong H_{2p}(X, \mathbb{Z})$ can be wrong for $p > 0$ by the following example.

Example 3.23. Let S be the surface construction in Example 3.7, $\mathrm{Ch}_1(S) \ncong H_2(S, \mathbb{Z})$. Moreover, $\mathrm{Ch}_1(S)_{hom} \neq 0$. Recall that the relations among $\widetilde{S}, S$ and Y were given in diagram (3.8). By using $\tilde{\sigma} : \widetilde{S} \to S$ and the localization sequence for Chow group of 1-cycles, we get the difference between $\mathrm{Ch}_1(\widetilde{S})$ and $\mathrm{Ch}_1(S)$ is at most rank 2 (generated by the cycle classes of $\widetilde{C}_1$ and $\widetilde{C}_2$) since the sequence $\mathrm{Ch}_1(\widetilde{C}_1 \cup \widetilde{C}_2) \to \mathrm{Ch}_1(\widetilde{S}) \to \mathrm{Ch}_1(S) \to 0$ is exact and $\mathrm{Ch}_1(\widetilde{C}_1 \cup \widetilde{C}_2) \cong \mathbb{Z} \oplus \mathbb{Z}$. So $\mathrm{Ch}_1(\widetilde{S})_{hom} \cong \mathrm{Ch}_1(S)_{hom}$. On the other hand, $\mathrm{Ch}_1(\widetilde{S}) \cong \mathrm{Ch}_1(Y) \oplus \mathbb{Z}^2 \cong (\mathrm{Ch}_0(C) \oplus \mathbb{Z}) \oplus \mathbb{Z}^2$. Hence $\mathrm{Ch}_1(\widetilde{S})_{hom} \cong \mathrm{Ch}_0(C) \cong J(C) \neq 0$, where $J(C)$ is the Jacobi of C of genus $g(C) \geq 1$.

By applying to a possible singular projective variety carrying a holomorphic vector field with isolated zeroes, we have the following result.

Corollary 3.24. *Let X be a (possible singular) complex projective algebraic variety which admits a holomorphic vector field V whose zero set Z is isolated and nonempty. Then we have* $\mathrm{Ch}_0(X) \cong \mathbb{Z}$.

Proof. Recall that a holomorphic vector field generates a G-action on X, where $G \cong (\mathbb{C}^*)^k \times \mathbb{C}$ or $G \cong (\mathbb{C}^*)^k$. Write $G \cong G_1 \times \mathbb{C}^*$ and $X_1 := X^{\mathbb{C}^*}$. From Remark 3.21, the inclusion $X_1 \to X$ induces a surjection $\mathrm{Ch}_0(X_1) \to \mathrm{Ch}_0(X)$. If $G \cong (\mathbb{C}^*)^k$, we get the surjection $\mathrm{Ch}_0(V) \to \mathrm{Ch}_0(X)$ by induction. If $G \cong (\mathbb{C}^*)^k \times \mathbb{C}$, we get the surjection $\mathrm{Ch}_0(V_1) \to \mathrm{Ch}_0(X)$ by induction, where $V_1 := X^{(\mathbb{C}^*)^k}$. Note that V_1 admits a $\mathbb{C}$-action whose fixed point is V. By Proposition 2.1, we have $\mathrm{Ch}_0(V) \cong \mathrm{Ch}_0(V_1)$. Therefore, the inclusion $V \hookrightarrow X$ induces a surjection $\mathrm{Ch}_0(V) \to \mathrm{Ch}_0(X)$. By assumption, V is a set of finite points. Hence $\mathrm{Ch}_0(X)$ is of finite rank and so $\mathrm{Ch}_0(X) \to H_0(X, \mathbb{Z}) \cong \mathbb{Z}$ is injective. Clearly, $\mathrm{Ch}_0(X) \neq 0$ and we get $\mathrm{Ch}_0(X) \cong \mathbb{Z}$. □

Applying to Lawson homology, we get the structure for 1-cycles.

Lemma 3.25. *For any projective variety X and any integer $k \geq 2r \geq 0$ and $n \neq 0$, we have the following formula*

$$L_rH_k(X \times \mathbb{C}^*) \cong L_{r-1}H_{k-2}(X) \oplus L_rH_{k-1}(X). \tag{3.26}$$

Proof. First, we note that the pair $(X \times \mathbb{C}, X \times \{0\})$, we have the long exact sequence of Lawson homology:

$$\ldots \xrightarrow{\partial} L_rH_k(X) \xrightarrow{i_*} L_rH_k(X \times \mathbb{C}) \xrightarrow{Res} L_rH_k(X \times \mathbb{C}^*) \xrightarrow{\partial} L_rH_{k-1}(X) \to \ldots \tag{3.27}$$

where $i : X = X \times \{0\} \to X \times \mathbb{C}$ is the inclusion, Res is the restriction map and ∂ is the boundary map.

The long exact sequence of Lawson homology for the pair $(X \times \mathbb{P}^1, X \times \{0\})$ is

$$\ldots \xrightarrow{\partial} L_rH_k(X) \xrightarrow{i_{\infty *}} L_rH_k(X \times \mathbb{P}^1) \xrightarrow{Res} L_rH_k(X \times \mathbb{C}) \xrightarrow{\partial} L_rH_{k-1}(X) \to \ldots$$

where $i_\infty : X = X \times \{\infty\} \to X \times \mathbb{P}^1$ is the inclusion.

Then, from the $\mathbb{C}^1$-homotopy invariance of Lawson homology, we get $i_{0*} = i_{\infty *} : L_pH_k(X) \to L_pH_k(X \times \mathbb{P}^1)$, where $i_0 : X = X \times \{0\} \to X \times \mathbb{P}^1$ is the inclusion. From the definition of i and i_0, we have $i_* = Res \circ i_{0*}$, where $Res : L_rH_k(X \times \mathbb{P}^1) \to L_rH_k(X \times \mathbb{C})$ is the restriction map. Hence we obtain

$$i_* = Res \circ i_{0*} = Res \circ i_{\infty *} = 0.$$

Therefore, Equation (3.27) is broken into short exact sequences

$$0 \to L_rH_k(X \times \mathbb{C}) \xrightarrow{Res} L_rH_k(X \times \mathbb{C}^*) \xrightarrow{\partial} L_rH_{k-1}(X) \to 0.$$

This sequence splits since the map $\mathcal{Z}_r(X \times \mathbb{C}^*) = \mathcal{Z}_r(X \times \mathbb{C})/\mathcal{Z}_r(X \times \{0\}) \to \mathcal{Z}_{r-1}(X) \simeq \mathcal{Z}_r(X \times \mathbb{C})$ given by $c \mapsto c \cap (X \times \{0\})$ gives a section of the projection $\mathcal{Z}_r(X \times \mathbb{C}) \to \mathcal{Z}_r(X \times \mathbb{C})/\mathcal{Z}_r(X \times \{0\})$. So we get Equation (3.26). This completes the proof of the lemma. □

Now we study the structure of Lawson homology under a $\mathbb{C}^*$-action. When X admits $\mathbb{C}$-action with isolated fixed points, it was shown that $L_1H_k(X) \cong H_k(X, \mathbb{Z})$ (see Proposition 2.4). Inspired by this result, it is natural to ask the following question.

Question 3.28. Let X be a complex projective variety admitting a $\mathbb{C}^*$-action with isolated fixed point. Does $L_1H_k(X) \cong H_k(X, \mathbb{Z})$ hold for $k \geq 2$?

The positive answer to this question would be an analogue of Proposition 2.4. Contrary to the analogue between Proposition 2.1 and 3.19, it is surprising to a certain degree that the answer to Question 3.28 is negative in the sense that for each $k \geq 2$, we can find X (depending on k) satisfying conditions in the question such that $L_1H_k(X) \ncong H_k(X,\mathbb{Z})$.

Example 3.29. Let S be the variety given in Example 3.7, then $S \times S$ admits a $\mathbb{C}^*$-action with isolated fixed points induced by the $\mathbb{C}^*$-action on S. We have

$$L_1H_2(S \times S) \cong H_2(S \times S, \mathbb{Z}),$$

and

$$L_1H_3(S \times S) \ncong H_3(S \times S, \mathbb{Z}).$$

Proof. The $\mathbb{C}^*$-action $\phi : \mathbb{C}^* \times S \to S$, $(t,x) \mapsto \phi(t,x)$ induces a $\mathbb{C}^*$-action $(t,(x,y)) \mapsto (tx,ty)$ on $S \times S$. The fixed point set $(S \times S)^{\mathbb{C}^*} \subset S^{\mathbb{C}^*} \times S^{\mathbb{C}^*}$ is finite since $S^{\mathbb{C}^*}$ is.

By the construction of S, we have $H_1(S,\mathbb{Z}) = 0$. By the Künneth formula, $H_2(S \times S,\mathbb{Z}) \cong H_2(S,\mathbb{Z}) \oplus H_2(S,\mathbb{Z})$. Note $H_2(S,\mathbb{Z}) \cong \mathbb{Z}^3$ is generated by the homological classes of algebraic cycles $\tilde{\sigma}(\sigma^{-1}(\mathbb{P}^1 \times c_0))$, $\tilde{\sigma}(\sigma^{-1}(p_i))$, where c_0 is a point of C different from p_i for $i = 1,2$. Hence $H_2(S \times S,\mathbb{Z})$ is generated by algebraic cycles and so the cycle class map $L_1H_2(S \times S) \to H_2(S \times S,\mathbb{Z})$ is surjective.

From the construction in Example 3.7, $\sigma : \widetilde{S} \to Y = C \times \mathbb{P}^1$ is the blow up of two point $p_1 \in C_1, p_2 \in C_2$. Set $U := Y - C_1 - C_2 \cong \widetilde{S} - \sigma^{-1}(C_1) - \sigma^{-1}(C_2)$, where $\sigma^{-1}(C_i) = \widetilde{C}_i \cup E_i$ and $E_i \cong \mathbb{P}^1$. One gets $U \cong C \times \mathbb{C}^*$. Since $\tilde{\sigma} : \widetilde{S} \to S$ is the blow down and each $\widetilde{C}_i$ collapses to a point, $S - \tilde{\sigma}(E_1) - \tilde{\sigma}(E_2) \cong U$. Since only $\widetilde{C}_i$ collapses under $\tilde{\sigma}$, $\tilde{\sigma}(E_i) \cong E_i \cong \mathbb{P}^1$. Set $Z := S \times S - U \times U$ and $\widetilde{E}_i := \tilde{\sigma}(E_i)$, then Z is the union $\big((\widetilde{E}_1 \cup \widetilde{E}_2) \times S\big) \bigcup \big(S \times (\widetilde{E}_1 \cup \widetilde{E}_2)\big)$. Set $\widetilde{Z} := \widetilde{S} \times \widetilde{S} - U \times U$ and then $\widetilde{Z}$ is the union $\big((\sigma^{-1}(C_1) \cup \sigma^{-1}(C_2)) \times \widetilde{S}\big) \bigcup \big(\widetilde{S} \times (\sigma^{-1}(C_1) \cup \sigma^{-1}(C_2))\big)$. From the long localization exact sequence of Lawson homology for $(\widetilde{S},\widetilde{Z})$ and (S,Z), we have the following commutative diagram

$$\begin{tikzcd}
\ldots \arrow[r] & L_1H_3(\widetilde{U}) \arrow[r] \arrow[d, "="] & L_1H_2(\widetilde{Z}) \arrow[r] \arrow[d] & L_1H_2(\widetilde{S} \times \widetilde{S}) \arrow[r] \arrow[d, "(\sigma\times\sigma)_*"] & L_1H_2(U) \arrow[d] \\
\ldots \arrow[r] & L_1H_3(\widetilde{U}) \arrow[r] & L_1H_2(Z) \arrow[r] & L_1H_2(S \times S) \arrow[r] & L_1H_2(U)
\end{tikzcd}$$

By the homotopy invariance and localization sequences of Lawson homology, one gets $L_1H_k(Z) \cong H_k(Z,\mathbb{Z})$ and $L_1H_k(\widetilde{Z}) \cong H_k(\widetilde{Z},\mathbb{Z})$ for $k \geq 2$. From the construction, the collapse $\widetilde{Z} \to Z$ induces a surjective map $H_2(\widetilde{Z},\mathbb{Z}) \to H_2(Z,\mathbb{Z})$.

From $U \cong C \times \mathbb{C}^*$ and Lemma 3.25, we get isomorphisms

$$\begin{aligned}
L_1H_2(U \times U) &\cong L_1H_2(C \times C \times \mathbb{C}^* \times \mathbb{C}^*) \\
&\cong L_0H_0(C \times C \times \mathbb{C}^*) \\
&\cong H_0^{BM}(C \times C \times \mathbb{C}^*, \mathbb{Z}) \\
&= 0.
\end{aligned}$$

Therefore, $(\sigma\times\sigma)_*$ is a surjective map. Note that $\widetilde{S}\times\widetilde{S}$ is nonsingular and projective, a direct computation by the localization sequence and the blowup formula for Lawson homology (see [29]) yields $L_1H_2(\widetilde{S}\times\widetilde{S})_{hom} = 0$. Hence $L_1H_2(S \times S)_{hom} = 0$ and $L_1H_2(S \times S) \to H_2(S \times S,\mathbb{Z})$ is injective.

We need to identify $L_1H_3(U \times U)$ and $H_3^{BM}(U \times U,\mathbb{Z})$ so that one can compare the relation between $L_1H_3(S \times S)$ and $H_3(S \times S,\mathbb{Z})$.

By Lemma 3.25, we get

$$\begin{aligned} L_1H_3(U\times U) &\cong L_1H_3(C\times C\times \mathbb{C}^*\times \mathbb{C}^*) \\ &\cong L_0H_1(C\times C\times \mathbb{C}^*)\oplus L_1H_2(C\times C\times \mathbb{C}^*) \\ &\cong L_0H_0(C\times C)\oplus L_0H_0(C\times C) \\ &\cong \mathbb{Z}\oplus\mathbb{Z}. \end{aligned}$$

Similarly, we get

$$\begin{aligned} H_3^{BM}(U\times U,\mathbb{Z}) &\cong H_3^{BM}(C\times C\times \mathbb{C}^*\times \mathbb{C}^*,\mathbb{Z}) \\ &\cong H_1^{BM}(C\times C\times \mathbb{C}^*,\mathbb{Z})\oplus H_2^{BM}(C\times C\times \mathbb{C}^*,\mathbb{Z}) \\ &\cong H_0^{BM}(C\times C)\oplus H_0^{BM}(C\times C)\oplus H_1^{BM}(C\times C) \\ &\cong \mathbb{Z}\oplus\mathbb{Z}\oplus H_1(C\times C). \end{aligned}$$

Hence the cycle class map

$$\Phi_{1,3}(U\times U): L_1H_3(U\times U)\to H_3^{BM}(U\times U,\mathbb{Z})$$

is not surjective if $g(C)>0$, as we chose. In particular, $\Phi_{1,3}(U)$ is not an isomorphism.

For simplicity in diagram $X:=S\times S$, $\widetilde{U}:=U\times U$. From the following commutative diagram

$$\begin{tikzcd} L_1H_3(Z) \arrow[r] \arrow[d, "\cong"] & L_1H_3(X) \arrow[r] \arrow[d, "\cong?"] & L_1H_3(\widetilde{U}) \arrow[r] \arrow[d, "\Phi_{1,3}(\widetilde{U})"] & L_1H_2(Z) \arrow[r] \arrow[d, "\cong"] & L_1H_2(X) \arrow[d, "\cong"] \\ H_3(Z,\mathbb{Z}) \arrow[r] & H_3(X,\mathbb{Z}) \arrow[r] & H_3^{BM}(\widetilde{U},\mathbb{Z}) \arrow[r] & H_2(Z,\mathbb{Z}) \arrow[r] & H_2(X,\mathbb{Z}) \end{tikzcd}$$

and the Five lemma, we obtain that $\Phi_{1,3}(U)$ is an isomorphism if $\Phi_{1,3}(X): L_1H_3(X)\to H_3(X,\mathbb{Z})$ is. Therefore, $\Phi_{1,3}(X)$ is not an isomorphism. □

Remark 3.30. From Lemma 3.25 and Example 3.29, for each $k\geq 3$, one can construct projective varieties X admitting $\mathbb{C}^*$-action with isolated fixed points such that $L_1H_k(X)\ncong H_k(X,\mathbb{Z})$. Such a X can be chosen as $X:=S\times S\times C^{k-3}$, where C is the curve in Example 3.4. For $k=2$, a direct calculation shows that $L_1H_2(C\times C)\ncong H_2(C\times C,\mathbb{Z})$. The detail is left to the interested reader.

4. Applications to Chow varieties

In this section, we shall first very briefly review some known facts about Chow varieties, especially in algebraic and topological aspects and then give some new results. Unless otherwise specified, Chow varieties defined over the complex numbers.

One of our purpose is to understand the algebraic and topological structure on the complex Chow variety $C_{p,d}(\mathbb{P}^n_{\mathbb{C}})$ (or simply $C_{p,d}(\mathbb{P}^n)$ if there is no confusion) parameterizing effective p-cycles of degree d in the complex projective space $\mathbb{P}^n$.

In degree 1 case, $C_{p,1}(\mathbb{P}^n)$ is exactly the Grassmannian of $(p+1)$-planes in $\mathbb{C}^{n+1}$, which is a space of fundamental importance in geometry and topology. In dimension 0 case, $C_{0,d}(\mathbb{P}^n)$ is the d-th symmetric product of $\mathbb{P}^n$, a “correct” object to realize homology when d tends to infinity. It is needless to explain here the importance of Chow varieties in the algebraic cycle theory. Until recent years, it is surprising that not many topological and algebraic invariants were known about $C_{p,d}(\mathbb{P}^n)$ for $d>1$.

4.1. The origin of Chow variety

Let $X \subset \mathbb{P}^n$ be a complex projective variety and let $C_{p,d}(X) \subset C_{p,d}(\mathbb{P}^n)$ be the subset containing those cycles $c = \sum a_i V_i \in C_{p,d}(\mathbb{P}^n)$ whose support $\mathrm{supp}(c) = \cup V_i$ lies in X, where V_i is an irreducible projective variety of dimension $\dim V_i = p$, $a_i \in \mathbb{Z}^+$ and $\sum a_i = d$. It has been established by Chow and Van der Waerden in 1937 that each $C_{p,d}(X)$ canonically carries the structure of a projective algebraic set (see [11]). More intrinsically, the space of all effective p-cycles can be written as a countable disjoint union $\coprod_{\alpha \in H_{2p}(X,\mathbb{Z})} C_{p,\alpha}(X)$, where each $C_{p,\alpha}(X)$ carries the structure of a projective algebraic set.

4.2. The dimension and number of irreducible components

In general, $C_{p,d}(\mathbb{P}^n)$ is not irreducible. The simplest non-irreducible Chow variety is $C_{1,2}(\mathbb{P}^3)$, which has two irreducible components. Moreover, the different irreducible components may have different dimension. Examples of Chow varieties including those parametrizing curves of low degrees (less than or equals to 4) in $\mathbb{P}^3$ can be found in [22].

The exact number of irreducible components for $C_{p,d}(\mathbb{P}^n)$ is not known in general, even for $C_{1,d}(\mathbb{P}^3)$. An upper bound of the number of irreducible components of $C_{p,d}(\mathbb{P}^n)$ was given by $N_{p,d,n} := \binom{nd+d}{n}^{m_{p,d}}$, where $m_{p,d} := d\binom{d+p-1}{p} + \binom{d+p-1}{p-1}$ (see Kollar [41, Exer.3.28]). We should mention that Kollar's book contains an excellent exposition on families of cycles over arbitrary schemes. Of course, this upper bound is usually much higher than the actual number of irreducible components for $C_{p,d}(\mathbb{P}^n)$ in many known cases. For example, there is exactly one component for $C_{0,d}(\mathbb{P}^n)$ for any d and n. For $d = 1$ and arbitrary $n, p \geq 0$, $C_{p,1}(\mathbb{P}^n)$ is the Grassmannian parametering $(p+1)$-vector spaces in $\mathbb{C}^{n+1}$, which is irreducible. For $d = 2$ and arbitrary $n, p \geq 0$, there are at most two irreducible components for $C_{p,2}(\mathbb{P}^n)$. By checking the possible genus of an irreducible curve with a given degree in $\mathbb{P}^3$ (see [25, Ch. IV]), one can obtain that the irreducible components of $C_{1,d}(\mathbb{P}^3)$ are 1,2,4,8,14,27,46 corresponding to d from 1 to 7. These numbers are really much smaller than the corresponding numbers $N_{p,d,n}$.

The dimension of $C_{p,d}(\mathbb{P}^n)$ we mean the maximal number of the dimension of its irreducible components. Eisenbud and Harris in 1992 showed that the dimension of the space of effective 1-cycles of degree d in $\mathbb{P}^n$ is

$$\dim C_{1,d}(\mathbb{P}^n) = \max\{2d(n-1), 3(n-2) + d(d+3)/2\}$$

(see [14]).

The dimension of $C_{p,d}(\mathbb{P}^n)$ was computed by Azcue in 1992 in his Ph.D. thesis under the direct of Harris (see [1]). The explicit formula for $\dim C_{p,d}(\mathbb{P}^n)$ can be found in a paper by Lehmann in 2017 (see [36]), that is,

$$\dim C_{p,d}(\mathbb{P}^n) = \max\left\{ d(p+1)(n-p), \binom{d+p+1}{p+1} - 1 + (p+2)(n-p-1) \right\}.$$

4.3. Homotopy and homology groups

It is not hard to show that $C_{p,d}(\mathbb{P}^n)$ is connected as a topological space since every element c is path-connected to $d \cdot L$, where L is any fixed p-plane in $\mathbb{P}^n$. By comparing connectedness of the morphism between a variety and the fixed point set under the additive group action, Horrocks showed in 1969 that the algebraic fundamental group of the Chow variety $C_{p,d}(\mathbb{P}^n)_K$ defined over an algebraically closed field K is trivial (see [26]). By using the similar method to complex varieties, A. Fujiki showed in 1995 that the topological fundamental group of $C_{p,d}(\mathbb{P}^n)$ is trivial, i.e., $C_{p,d}(\mathbb{P}^n)$ is simply connected (see [20]).

In a complete different way, Lawson in 1989 gave a very short proof of the simply connectedness of $C_{p,d}(\mathbb{P}^n)$ by using Sard theorem for families (see [34]). More important, in that paper, Lawson has established the Lawson homology theory and showed the famous Complex Suspension Theorem. The author has observed that the methods in proving the Complex Suspension Theorem can be used to compute the higher homotopy group of $C_{p,d}(\mathbb{P}^n)$. The author showed in 2010 that $\pi_2(C_{p,d}(\mathbb{P}^n)) \cong \mathbb{Z}$ for all $d \geq 1$ and $0 \leq p < n$. This statement $\pi_2(C_{p,d}(\mathbb{P}^n)) \cong \mathbb{Z}$ was conjectured by Lawson in 1995 in [35, p.141]. For $p = n$, $C_{p,d}(\mathbb{P}^n)$ is a single point and so $\pi_2(C_{p,d}(\mathbb{P}^n))$ is trivial. More results can be found in [30] on the stability of the homotopy group of $C_{p,d}(\mathbb{P}^n)$ when p or n increases.

For higher homotopy groups, a slightly weaker version of Lawson's open question is that whether there is an isomorphism $\pi_k(C_{p,d}(\mathbb{P}^n)) \cong \widetilde{H}_{k+2p}(\mathbb{P}^n, \mathbb{Z})$ for $k \leq 2d$, where $\widetilde{H}_{\cdot}(-, \mathbb{Z})$ denotes the reduced singular homology with integer coefficients (see [34, p.256]). Lawson showed that there is a natural surjective map from $\pi_k(C_{p,d}(\mathbb{P}^n))$ to $\widetilde{H}_{k+2p}(\mathbb{P}^n, \mathbb{Z})$. The author showed in 2015 that the surjective map is actually an isomorphism. Moreover, as its corollary, the homology group of $C_{p,d}(\mathbb{P}^n)$ has been computed up to $2d$ (see [31]).

4.4. Euler characteristic

By establishing a fixed point formula for compact complex spaces under a weakly analytic S^1-action, Lawson and Yau showed in 1987 that the Euler characteristic $\chi(C_{p,d}(\mathbb{P}^n))$ of the complex Chow variety is given by a beautiful formula

$$\chi(C_{p,d}(\mathbb{P}^n)) = \binom{v_{p,n} + d - 1}{d},$$

where $v_{p,n} = \binom{n+1}{p+1}$.

In 2013, the author gave a direct and elementary proof of this formula (see [32]). One of the main techniques is "pulling of normal cone" established by Fulton, which was used by Lawson in proving his Complex Suspension Theorem (see [34]). The key observation was that one can write $C_{p+1,d}(\mathbb{P}^{n+1})$ as a disjoint union of quasi-projective varieties

$$C_{p+1,d}(\mathbb{P}^{n+1})_i = \coprod_{i=0}^{d} C_{p+1,i}(\mathbb{P}^n) \times T_{p+1,d-i}(\mathbb{P}^{n+1}),$$

where $T_{p+1,d-i}(\mathbb{P}^{n+1})$ is homotopic to $C_{p,d-i}(\mathbb{P}^n)$ by the technique "pulling of normal cone". Hence one obtains by the additive property of the Euler characteristic a recursive formula

$$\chi(C_{p+1,d}(\mathbb{P}^{n+1})) = \chi(C_{p,d}(\mathbb{P}^n)) + \sum_{i=1}^{d} \chi(C_{p+1,i}(\mathbb{P}^n)) \cdot \chi(C_{p,d-i}(\mathbb{P}^n))$$

and give the short proof of Lawson and Yau's formula.

The techniques above are also able to use the compute the l-adic Euler-Poincaré characteristic of the Chow varieties $C_{p,d}(\mathbb{P}^n)_K$ defined over an algebraically closed field K. As an analogue in the complex case, Friedlander showed in 1991 that there is an algebraic homotopy from $T_{p+1,d-i}(\mathbb{P}^{n+1})$ to $C_{p,d-i}(\mathbb{P}^n)$. One got the generalization of Lawson-Yau's formula directly to Chow varieties over an algebraically closed field K:

$$\chi(C_{p,d}(\mathbb{P}^n)_K, l) = \binom{v_{p,n}+d-1}{d}, \quad \text{where } v_{p,n} = \binom{n+1}{p+1},$$

where $\chi(X_K, l)$ denotes the l-adic Euler-Poincaré Characteristic of an algebraic variety X_K over K. The Euler Characteristic for the space of right-quaternionic cycles was also given with an explicit formula (see [32]).

It seems that there is no way to compute the Euler characteristic $C_{p,\alpha}(X)$ for X a generic projective variety. However, for special varieties, such as toric varieties, Elizondo gave a beautiful formula for their Euler characteristic in terms of the fans of the variety [16].

If one denotes the **p-th Euler series** of a toric variety X is defined by the following formal power series

$$E_p(X) := \sum_{\alpha \in H_{2p}(X,\mathbb{Z})} \chi(C_{p,\alpha}(X))\alpha.$$

A toric variety X is a projective variety containing the algebraic group $T = (\mathbb{C}^*)^{\times n}$ as a Zariski open subset such that the action of $(\mathbb{C}^*)^{\times n}$ on itself extends to an action on X. The action of T on X induces action on $C_{p,\alpha}(X)$.

Denote by $V_1, ..., V_N$ the p-dimensional invariant irreducible subvarieties of X. Let $e_{[V_i]}$ be the characteristic function of the subset $\{[V_i], i = 1, 2, ..., N\}$ of the homology group $H_{2p}(X, \mathbb{Z})$, where $[V]$ denotes its class in $H_{2p}(X, \mathbb{Z})$. Elizondo showed in 1994 that there is a beautiful formula for $E_p(X)$:

$$E_p(X) = \prod_{1 \le i \le N} \left(\frac{1}{1 - e_{[V_i]}} \right).$$

Elizondo and Lima-Filho showed 1998 that the Euler-Chow series of the projectivization of the direct sum of two algebraic vector bundles can be computed in terms of that of the projectivization of each of the vector bundles and their fiber product (see [17]). More specifically, let E_1 and E_2 be two algebraic vector bundles over a projective variety X. Let $\mathbb{P}(E_1)$ (resp. $\mathbb{P}(E_2)$) be the projectivization of E_1 (resp. E_2). Then the Euler-Chow series $E_p(\mathbb{P}(E_1 \oplus E_2))$ can be computed in terms of that of $\mathbb{P}(E_1)$, $\mathbb{P}(E_2)$ and $\mathbb{P}(E_1) \times_X \mathbb{P}(E_2)$, where the last one is the fiber product of $\mathbb{P}(E_1)$ and $\mathbb{P}(E_2)$ over X. This result can be used to compute the Chow series of Grassmannian and certain flag varieties.

4.5. Virtual Betti and Hodge numbers

For integers $n \ge p \ge 0$ and $d \ge 0$, the author showed in 2013 that the virtual Hodge (r, s)-number of the Chow variety $C_{p,d}(\mathbb{P}^n)$ satisfies the following equations:

$$\sum_{r-s=i} \tilde{h}^{r,s}(C_{p,d}(\mathbb{P}^n)) = 0$$

for all $i \neq 0$,

$$\sum_{r \ge 0} \tilde{h}^{r,r}(C_{p,d}(\mathbb{P}^n)) = \chi(C_{p,d}(\mathbb{P}^n)),$$
$$\tilde{h}^{0,0}(C_{p,d}(\mathbb{P}^n)) = 1,$$
$$\tilde{h}^{r,0}(C_{p,d}(\mathbb{P}^n)) = 0,$$

and

$$\tilde{h}^{0,r}(C_{p,d}(\mathbb{P}^n)) = 0$$

for $r > 0$ (see [28]).

This also implies that $\tilde{\beta}^0(C_{p,d}(\mathbb{P}^n)) = 1$ and $\tilde{\beta}^1(C_{p,d}(\mathbb{P}^n)) = 0$. It is worth to remark that for a complex singular projective variety X, $\tilde{\beta}^0(X) = 1$ is independent of the connectedness of X, while $\tilde{\beta}^1(X) = 0$ is independent of the simply connectedness of X.

Due to the lack understanding of the structure of $C_{p,d}(\mathbb{P}^n)$, we post the following wild conjecture on their virtual Hodge numbers and virtual Betti numbers.

Conjecture 4.1. $\tilde{h}^{r,s}(C_{p,d}(\mathbb{P}^n)) = 0$ for all $r \neq s$. In particular, we conjecture that $\tilde{\beta}^i(C_{p,d}(\mathbb{P}^n)) = 0$ for i odd.

There are several examples supporting this conjecture. When $p = 0$, $C_{p,d}(\mathbb{P}^n) = \mathrm{SP}^d(\mathbb{P}^n)$, its virtual Betti numbers and virtual Hodge numbers have been computed in [9] and all their odd virtual Betti and virtual Hodge numbers vanish. When $p = n - 1$, $C_{p,d}(\mathbb{P}^n) = C_{n-1,d}(\mathbb{P}^n) = \mathbb{P}^{\binom{n+d}{d}-1}$ and its virtual Betti (resp. virtual Hodge numbers) are the same as its usual Betti numbers (resp. usual Hodge numbers), which are zeroes. When $d = 1$, $C_{p,d}(\mathbb{P}^n)$ is the Grassmannian $G(p+1, \mathbb{C}^{n+1})$, then one has $\tilde{h}^{r,s}(G(p+1, \mathbb{C}^{n+1})) = h^{r,s}(G(p+1, \mathbb{C}^{n+1})) = 0$ for all $r \neq s$, where $h^{r,s}(G(p+1, \mathbb{C}^{n+1}))$ denotes the Hodge (r, s)-number of $G(p+1, \mathbb{C}^{n+1})$.

Example 4.2. For $d = 2$ and all p, n, one also has $h^{r,s}(C_{p,2}(\mathbb{P}^n)) = 0$ for $r \neq s$ and $\tilde{\beta}^{2i-1}(C_{p,2}(\mathbb{P}^n)) = 0$ for $i > 0$.

Proof. Note that $C_{p,2}(\mathbb{P}^n)$ can be written as the union

$$C_{p,2}(\mathbb{P}^n) = \mathrm{SP}^2(G(p+1, \mathbb{C}^{n+1})) \cup Q_{p,n},$$

where $Q_{p,n}$ consists of effective irreducible p-cycles of degree 2 in $\mathbb{P}^n$ and $Q_{p,n}$ is a fiber bundle over the Grassmannian $G(p+2, n+1)$ with fiber the space S of all smooth quadrics in $\mathbb{P}^{p+1}$. Note that S is isomorphic to $\mathbb{P}^{\binom{p+3}{2}-1} - \mathrm{SP}^2(\mathbb{P}^{p+1})$ (see [30]). Therefore,

$$\begin{aligned}
\widetilde{P}_{C_{p,2}(\mathbb{P}^n)}(t) &= \widetilde{P}_{\mathrm{SP}^2(G(p+1,\mathbb{C}^{n+1}))}(t) + \widetilde{P}_{Q_{p,n}}(t) \\
&= \widetilde{P}_{\mathrm{SP}^2(G(p+1,\mathbb{C}^{n+1}))}(t) + \widetilde{P}_{G(p+2,n+1)} \cdot \widetilde{P}_{\mathbb{P}^{\binom{p+3}{2}-1} - \mathrm{SP}^2(\mathbb{P}^{p+1})}(t) \\
&= \widetilde{P}_{\mathrm{SP}^2(G(p+1,\mathbb{C}^{n+1}))}(t) + \widetilde{P}_{G(p+2,n+1)} \cdot (\widetilde{P}_{\mathbb{P}^{\binom{p+3}{2}-1}}(t) - \widetilde{P}_{\mathrm{SP}^2(\mathbb{P}^{p+1})}(t)).
\end{aligned}$$

This implies that the odd betti numbers of $C_{p,2}(\mathbb{P}^n)$ are zeroes since those of Grassmannians and the symmetric product of Grassmannians are zeroes. Similar computations work for the virtual Hodge numbers. $\square$

4.6. Ruledness and rationality of irreducible components

Since $C_{p,d}(\mathbb{P}^n)$ admits a $\mathbb{C}$-action with an isolated fixed point ([26]), each of its irreducible component is preserved under the action. Hence each irreducible component of $C_{p,d}(\mathbb{P}^n)$ admits a $\mathbb{C}$-action with an isolated fixed point. From Lieberman's result ([37, Th.1]), we obtain that each component of $C_{p,d}(\mathbb{P}^n)$ is a *ruled* variety.

In general, the rationality of irreducible components of $C_{p,d}(\mathbb{P}^n)$ is an open problem, which can be found in Shafarevich's book (see [42]). As a remark, Shafarevich said "*Whether every irreducible component of them is rational, in general, is 'an apparently very difficult but very fundamental problem'.*"

Question 4.3 *(Shafarevich).* Is each irreducible component of $C_{p,d}(\mathbb{P}^n)$ is rational for all $0 \leq p \leq n$ and $d \geq 1$?

Surely, $C_{p,1}(\mathbb{P}^n)$ is rational for all $0 \le p \le n$ since $C_{p,1}(\mathbb{P}^n)$ is just the complex Grassmannian manfold $G(p+1, \mathbb{C}^{n+1})$, which is rational. When $p = 1, n = 3$, the irreducible components of $C_{1,d}(\mathbb{P}^3)$ have been shown to be rational for d small ([42]). However, even if the proof of rationality for $C_{0,d}(\mathbb{P}^n)$ is nontrivial (see [22, Ch.4,Th2.8] and references cited there).

For $d = 2$ and $0 \le p \le n$, the explicit structure of each irreducible component has been studied in details in [30]. From that, one obtains that each irreducible component is rational since the symmetric products of complex Grassmannian manfolds are rational.

It is not hard to show that an irreducible component of the maximal dimension in $C_{p,d}(\mathbb{P}^n)$ is rational. This follows from the fact that the symmetric product of a rational variety is rational and at least one irreducible component of the maximal dimension either consists of all d-tuples p-dimensional linear spaces in $\mathbb{P}^n$ or irreducible p-dimensional hypersurfaces degree d in $\mathbb{P}^{p+1} \subset \mathbb{P}^n$ (see [36]).

However, the answer to Question 4.3 is negative, as explained in the following counterexample, which should be known earlier but it cannot be found in the literature.

Example 4.4. Let M_g ($g \ge 2$) be the moduli space of smooth complex algebraic curves of genus g. Now we recall the construction of M_g from the geometric invariant theory (cf. [23]). Let $\mathcal{H}_{d,g,r}$ be the Hilbert scheme of curves of degree d and (arithmetic) genus g in $\mathbb{P}^r$. For any integer $n \ge 3$, a smooth curve C can be embedded as a curve of degree $2(g-1)n$ in $\mathbb{P}^N$ by the complete linear series $|nK_C|$, where $N = (2n-1)(g-1) - 1$. Let us consider pairs $(C, \varphi : C \to \mathbb{P}^N)$, where C is a curve and $\varphi : C \to \mathbb{P}^N$ is an n-canonical embedding. The family of all such pairs corresponds to a locally closed subset $\mathcal{K}$ of the Hilbert scheme $\mathcal{H}_{d,g,N}$ of smooth curves of degree d and genus g in $\mathbb{P}^N$, where $d = 2(g-1)n$. The projective general linear group $PGL(N+1, \mathbb{C})$ acts on $\mathcal{K}$ with quotient is M_g. The locally closed subset $\mathcal{K}$ is just a Zariski open set of an irreducible component of $C_{1,d}(\mathbb{P}^N)$. Therefore, there exists an irreducible component of $C_{1,d}(\mathbb{P}^N)$, denoted by $I_{1,d}(\mathbb{P}^N)$, and a dominant rational map $I_{1,d}(\mathbb{P}^N) \dashrightarrow M_g$ for each $g \ge 2$.

Note that it was shown in [15] and [24] that M_g is a quasi-projective variety of the general type for $g \ge 24$. This together with the dominant rational map $I_{1,d}(\mathbb{P}^N) \dashrightarrow M_g$ implies that $I_{1,d}(\mathbb{P}^N)$ is not rational since a variety dominated by an rational variety is a unirational variety. This completes the construction of the example.

One can go further to construct counterexamples to Shafarevich's question for cycles in arbitrary dimensions.

Fix a hyperplane $\mathbb{P}^n \subset \mathbb{P}^{n+1}$ and a point $P = [0 : \cdots : 0 : 1] \in \mathbb{P}^{n+1} - \mathbb{P}^n$. Let $V \subset \mathbb{P}^n$ be any closed algebraic subset. The algebraic suspension of V with vertex P (i.e., cone over P) is the set

$$\Sigma_P V := \cup\{l \mid l \text{ is a projective line through } P \text{ and intersects } V\}.$$

Set

$$T_{p+1,d}(\mathbb{P}^{n+1}) := \left\{ c = \sum n_i V_i \in C_{p+1,d}(\mathbb{P}^{n+1}) | \dim(V_i \cap \mathbb{P}^n) = p, \forall i \right\}.$$

It has been shown in [34] that $T_{p+1,d}(\mathbb{P}^{n+1}) \subset C_{p+1,d}(\mathbb{P}^{n+1})$ is a Zariski open set and there is a continuous algebraic surjective map $T_{p+1,d}(\mathbb{P}^{n+1}) \to C_{p,d}(\mathbb{P}^n)$ (cf. [18] for the case over arbitrary algebraically closed field). A continuous algebraic map is a rational map which can be extended to a continuous map in the complex topology. Hence, for each irreducible component $I_{p,d,n}$ of $C_{p,d}(\mathbb{P}^n)$, there exists an irreducible component $J_{p+1,d,n+1}$ of $T_{p+1,d}(\mathbb{P}^{n+1})$ such that $J_{p+1,d,n+1} \to I_{p,d,n}$ is a continuous algebraic surjective map. In particular, it is a dominant rational map. Let $\overline{J}_{p+1,d,n+1}$ be the closure of $J_{p+1,d,n+1}$ in $C_{p+1,d}(\mathbb{P}^{n+1})$. Then we get a dominant rational map $\overline{J}_{p+1,d,n+1} \dashrightarrow I_{p,d,n}$ from $J_{p+1,d,n+1} \to I_{p,d,n}$. Since $T_{p+1,d}(\mathbb{P}^{n+1}) \subset C_{p+1,d}(\mathbb{P}^{n+1})$ is a Zariski open set, $\overline{J}_{p+1,d,n+1}$ is an irreducible component $I_{p+1,d,n+1}$ of $C_{p+1,d}(\mathbb{P}^{n+1})$. So if $I_{p,d,n} \dashrightarrow M_g$ is a dominant rational map, then $\overline{J}_{p+1,d,n+1} \dashrightarrow M_g$ is also a dominant rational map. Therefore there is a dominant rational map is $I_{p+1,d,n+1} \dashrightarrow M_g$ from the irreducible component $I_{p+1,d,n+1}$ of $C_{p+1,d,n+1}$ to the moduli space of curve of genus g. From the construction of Example 4.4, there exist d, n such that $I_{1,d,n} \dashrightarrow M_g$ is a dominant rational map for $g \geq 2$. Moreover, M_g is of general type if $g \geq 24$ by results in [15] and [24]. Hence $I_{p+1,d,n+1}$ is not a rational variety since it dominates a variety of general type.

In summary, the above argument provides a proof to the following theorem by induction.

Theorem 4.5. *For any $p \geq 1$, there exists an irreducible component $I_{p,d,n}$ of $C_{p,d}(\mathbb{P}^n)$ such that $I_{p,d,n}$ is not rational if d, n large.*

Remark 4.6. The $I_{p,d,n}$ in Theorem 4.5 admits a $\mathbb{C}^*$-action with isolated fixed points but it is not rational.

4.7. Chow groups and Lawson homology

By using the results in sections above, we shall compute Chow groups of 0-cycles and Lawson homology of 1-cycles for Chow varieties $C_{p,d}(\mathbb{P}^n)$.

We consider the action of $\mathbb{C}^*$ on $\mathbb{P}^n$ given by setting

$$\Phi_t([z_0 : \dots : z_n]) = [t_0 z_0 : \dots : t_n z_n],$$

where $t = (t_0 : \dots : t_n) \in (\mathbb{C}^*)^{n+1}$ and $[z_0 : \dots : z_n]$ are homogeneous coordinates for $\mathbb{P}^{n+1}$.

This action on $\mathbb{P}^n$ induces an action of $(\mathbb{C}^*)^n$ on $C_{p,d}(\mathbb{P}^n)$. From the definition of the action $(\mathbb{C}^*)^n$ on $\mathbb{P}^n$, it is clear that any irreducible subvariety V of $\dim V = p$ is invariant under the action $(\mathbb{C}^*)^n$ if and only if V is spanned by $(p+1)$-coordinate points in $\mathbb{P}^n$ and hence the fixed point set is finite.

Proposition 4.7. *For all $d > 0, 0 \leq p \leq n$, we have*

$$\mathrm{Ch}_0(C_{p,d}(\mathbb{P}^n)) \cong \mathbb{Z}.$$

Proof. Since $C_{p,d}(\mathbb{P}^n)$ admits a $(\mathbb{C}^*)^n$-action with isolated fixed points, one obtains from the diagonal embedding $\Delta = \{(t^{a_0}, \dots, t^{a_n}) | t \in \mathbb{C}^*\} \subset (\mathbb{C}^*)^n$ to get a $\mathbb{C}^*$-action on $C_{p,d}(\mathbb{P}^n)$ with isolated fixed points, where $a_i \in \mathbb{Z}$ are different to each other. Now by Proposition 3.19, we get $\mathrm{Ch}_0(C_{p,d}(\mathbb{P}^n)) \cong \mathbb{Z}$ since $C_{p,d}(\mathbb{P}^n)$ is connected. An alternative method is to use the induction on the number of $\mathbb{C}^*$-action (a $(\mathbb{C}^*)^n$-action is a sequence of $\mathbb{C}^*$-actions) and Remark 3.21 to obtain that $\mathrm{Ch}_0(C_{p,d}(\mathbb{P}^n))^{(\mathbb{C}^*)^n}) \to \mathrm{Ch}_0(C_{p,d}(\mathbb{P}^n))$ is surjective and hence is of finite rank. Then one gets $\mathrm{Ch}_0(C_{p,d}(\mathbb{P}^n)) \cong H_0(C_{p,d}(\mathbb{P}^n), \mathbb{Z}) \cong \mathbb{Z}$ by the connectedness of $C_{p,d}(\mathbb{P}^n)$. $\square$

Proposition 4.8. *For all $d > 0, 0 \leq p \leq n$, we have*

$$L_1 H_k(C_{p,d}(\mathbb{P}^n)) \cong H_k(C_{p,d}(\mathbb{P}^n), \mathbb{Z})$$

for all $k \geq 2$. In particular, $L_1 H_2(C_{p,d}(\mathbb{P}^n)) \cong \mathbb{Z}$. Equivalently, the homotopy groups of the space of 1-cycles of the Chow variety $C_{p,d}(\mathbb{P}^n)$ coincide with the corresponding singular homology groups with integer coefficients, i.e.,

$$\pi_{k-2} \mathcal{Z}_1(C_{p,d}(\mathbb{P}^n)) \cong H_k(C_{p,d}(\mathbb{P}^n), \mathbb{Z})$$

for all $k \geq 2$.

Proof. By [26], we know $C_{p,d}(\mathbb{P}^n)$ admits an action of a solvable group G with a single fixed point, where $G = G_r \supset G_{r-1} \supset \cdots \supset G_1 \supset G_0 = \{0\}$ is a normal series with quotients G_i/G_{i-1} isomorphic to the additive group scheme $\mathbb{C}$.

By Proposition 2.4, we can show that if X admits an action of a solvable group G with a single fixed pint, then $L_1H_k(X) \cong H_k(X,\mathbb{Z})$. In fact, we have the following inclusion $X^G = X^{G_r} \subset X^{G_{r-1}} \subset \cdots\cdots X^{G_2} \subset X^{G_1} \subset X^{G_0} = X$. Since $G_r/G_{r-1} \cong \mathbb{C}$ and X^{G_r} is a single point, we get by Proposition 2.4 that $L_1H_k(X^{G_{r-1}}) \cong H_k(X^{G_{r-1}},\mathbb{Z})$ from the fact $L_1H_k(X^{G_r}) \cong H_k(X^{G_r},\mathbb{Z})$. Since $G_i/G_{i-1} \cong \mathbb{C}$ for all $i \geq 1$ and by induction and Proposition 2.4, we have

$$L_1H_k(X^{G_0}) \cong H_k(X^{G_0},\mathbb{Z}),$$

that is, $L_1H_k(X) \cong H_k(X,\mathbb{Z})$.

By applying this to $X = C_{p,d}(\mathbb{P}^n)$, we have $L_1H_k(C_{p,d}(\mathbb{P}^n)) \cong H_k(C_{p,d}(\mathbb{P}^n),\mathbb{Z})$ for all $k \geq 2$. This completes the proof of the proposition. □

Similar to Conjecture 4.1, we post another wild conjecture on their Chow groups and Lawson homology groups.

Conjecture 4.9. For $d \geq 0$ and $0 \leq p \leq n$, one has

$$\mathrm{Ch}_q(C_{p,d}(\mathbb{P}^n)) \cong H_{2q}(C_{p,d}(\mathbb{P}^n),\mathbb{Z})$$

for all $q \geq 0$ and $L_qH_k(C_{p,d}(\mathbb{P}^n)) \cong H_k(C_{p,d}(\mathbb{P}^n),\mathbb{Z})$ for all $k \geq 2q \geq 0$.

Remark 4.10. For $p = 0$, $C_{p,d}(\mathbb{P}^n) \cong \mathrm{SP}^d(\mathbb{P}^n)$, one can show that these conjectures are true in rational coefficients. For $1 \leq p \leq n-2$ and d large, we have no idea to show or disprove $\mathrm{Ch}_q(C_{p,d}(\mathbb{P}^n)) \cong H_{2q}(C_{p,d}(\mathbb{P}^n),\mathbb{Z})$ even for $q = 1$.

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