



Hilbert-Kunz density function for graded domains

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ARTICLE INFO

Article history:

Received 14 October 2020

Received in revised form 11 June 2021

Available online 13 July 2021

Communicated by C.A. Weibel

MSC:

13A02; 13A35; 14C20; 13A50

Keywords:

 \mathbb{N} -graded domainsChar p methods

Hilbert-Kunz density functions

Reflexive sheaves

 \mathbb{Q} -divisors F -thresholds

ABSTRACT

We prove the existence of HK density function for a graded pair (R, I) , where R is an \mathbb{N} -graded domain of finite type over a perfect field and $I \subset R$ is a graded ideal of finite colength. This generalizes our earlier result where one proves the existence of such a function for a pair (R, I) , where, in addition R is standard graded.

Other properties of the HK density functions also hold for the graded pairs: for example, it is a multiplicative function for Segre products, its maximum support is the F -threshold of an \mathfrak{m} -primary ideal provided $\text{Proj } R$ is smooth, it has a closed formula when either I is generated by a system of parameters or R is of dimension two.

As one of the consequences we show that if G is a finite group scheme acting linearly on a polynomial ring R of dimension d then the HK density function f_{R^G, \mathfrak{m}_G} , of the pair (R^G, \mathfrak{m}_G) , is a piecewise polynomial function of degree $d - 1$.

We also compute the HK density functions for (R^G, \mathfrak{m}_G) , where $G \subset SL_2(k)$ is a finite group acting linearly on the ring $k[X, Y]$.

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1. Introduction

In this paper a pair (R, I) is a *graded pair* if R is an \mathbb{N} -graded domain of dimension $d \geq 2$ and finite type over a perfect field k of characteristic $p > 0$, and I is a graded ideal of finite colength. The main aim here is to prove the existence of the Hilbert-Kunz (HK) density function for such a pair.

The notion of HK density function was introduced by the first author in [19] for the purpose of studying the Hilbert-Kunz multiplicity (or HK multiplicity) $e_{HK}(R, I)$. The well known notion of HK multiplicity $e_{HK}(R, I)$ was introduced by P. Monsky [14] for an arbitrary Noetherian ring R (in characteristic $p > 0$)

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and an ideal $I \subset R$ of finite colength. In the same paper he showed that it is a positive real number given by

$$e_{HK}(R, I) = \lim_{n \rightarrow \infty} \frac{\ell(R/I^{[q]})}{q^d}.$$

Though this char p invariant has many interesting applications it is difficult to compute and hence difficult to guess and prove its various properties.

The advantage of dealing with the HK density function is that it behaves well (for standard graded pairs) for various operations like tensor products, Segre products etc. Moreover it is a limit of a uniformly converging sequence (which could be suitably renormalized to study a given specific property). Other than realizing HK multiplicity as its integral, it realizes the F -threshold $c^I(\mathbf{m})$ as its maximum support provided R is strongly F -regular on its punctured spectrum.

In [19], we proved the existence of HK density function for a *standard graded pair* (R, I) , where by a standard graded pair we mean R is a standard graded ring (but not necessarily a domain). The result was as follows (see Theorem 1.1 in [19]).

Theorem ([19]). *Let (R, I) be a standard graded pair. Then for a finitely generated graded module M over R there is a sequence $\{g_n(M, I) : [0, \infty) \rightarrow [0, \infty)\}_{n \in \mathbb{N}}$ of compactly supported continuous and piecewise linear functions such that*

- (1) *the sequence $\{g_n(M, I)\}_n$ is uniformly convergent. Moreover*
- (2) *the HK density function $f_{M, I} : [0, \infty) \rightarrow [0, \infty)$ given by $x \rightarrow \lim_{n \rightarrow \infty} g_n(M, I)(x)$ is a compactly supported continuous function, and*

$$e_{HK}(M, I) = \int_0^\infty f_{M, I}(x) dx.$$

Here, for a finitely generated graded R -module M , the functions $\{g_n(M, I) : [0, \infty) \rightarrow [0, \infty)\}_n$ are given as follows:

For $x \geq 0$, if $q = p^n$ and $x = (1 - t)\frac{\lfloor xq \rfloor}{q} + (t)\frac{\lfloor xq + 1 \rfloor}{q}$, for some $t \in [0, 1)$ then

$$g_n(M, I)(x) = \frac{1}{q^{d-1}} \left((1 - t)\ell(M/I^{[q]}M)_{\lfloor xq \rfloor} + (t)\ell(M/I^{[q]}M)_{\lfloor xq + 1 \rfloor} \right).$$

In this paper we generalize the above result to the case of *graded pair* (R, I) , where R need not be standard graded. There are many interesting \mathbb{N} -graded rings which are not standard graded, for examples the ring of invariants and the positive affine semi group rings, in particular the affine toric rings.

To do this we generalize the notion of $g_n(M, I)$ (see Definition 2.2) which coincides with the above notion of $g_n(M, I)$ whenever $\gcd\{n \mid R_n \neq 0\} = 1$.

More precisely we prove the following in this paper (see Section 4).

Theorem 1.1 (Main Theorem). *If M is a finitely generated graded R -module, where (R, I) is a graded pair then there is a sequence $\{g_n(M, I) : [0, \infty) \rightarrow [0, \infty)\}_n$ of compactly supported continuous and piecewise linear functions such that*

- (1) *$\{g_n(M, I)\}_{n \in \mathbb{N}}$ is a uniformly convergent sequence of compactly supported functions.*

(2) If $f_{M,I} : [0, \infty) \rightarrow [0, \infty)$ given by $x \rightarrow \lim_{n \rightarrow \infty} g_n(M, I)(x)$ then $f_{M,I}$ is a compactly supported continuous function such that

$$(a) \quad f_{M,I} = (\text{rank } M) f_{R,I} \quad \text{and} \quad (b) \quad e_{HK}(M, I) = \int_0^\infty f_{M,I}(x) dx.$$

Here we discuss the obstacles in applying the proof of [19] to graded rings which are not standard graded rings.

If R is a standard graded domain (not necessarily normal) and I is an ideal of finite colength generated by homogeneous generators f_1, \dots, f_s of degrees d_1, \dots, d_s then there exists a very ample invertible sheaf $\mathcal{O}_X(D)$ on $X = \text{Proj } R$ (associated to a Cartier divisor D) with an injective graded ring homomorphism $R \rightarrow \bigoplus_{m \geq 0} H^0(X, \mathcal{O}_X(mD))$ which is an isomorphism in all graded degrees $m \gg 0$. This gives us a short exact sequence of \mathcal{O}_X -modules

$$0 \rightarrow V \rightarrow \bigoplus_i \mathcal{O}_X((1 - d_i)D) \xrightarrow{\phi} \mathcal{O}_X(D) \rightarrow 0, \quad (1.1)$$

$\phi(\sum_i a_i) = \sum_i a_i f_i$. Since $\mathcal{O}_X(D)$ (in fact every $\mathcal{O}_X(mD)$) is invertible the sequence (1.1) is locally split exact and hence taking the pull backs along the absolute Frobenius gives the exact sequence (here $q = p^n$)

$$0 \rightarrow F^{n*}V \rightarrow \bigoplus_i \mathcal{O}_X((q - qd_i)D) \xrightarrow{\phi_{0,q}} \mathcal{O}_X(qD) \rightarrow 0. \quad (1.2)$$

Since $\mathcal{O}_X(mD) \otimes \mathcal{O}_X(nD) \simeq \mathcal{O}_X((m+n)D)$ tensoring (1.2) by $\mathcal{O}_X(mD)$ we get the exact sequence

$$0 \rightarrow F^{n*}V \otimes \mathcal{O}_X(mD) \rightarrow \bigoplus_i \mathcal{O}_X((m+q - qd_i)D) \xrightarrow{\phi_{m,q}} \mathcal{O}_X((m+q)D) \rightarrow 0. \quad (1.3)$$

Now, for given $q = p^n$ and for $x \geq 0$, let $[xq] = m + q$, where $m \in \mathbb{Z}_{\geq 0}$. Let us define the step function

$$\begin{aligned} f_n(R, I)(x) &:= f_n(R, I)\left(\frac{m+q}{q}\right) = \frac{1}{q^{d-1}} \ell(R/I^{[q]})_{m+q} \\ &= \frac{1}{q^{d-1}} [h^0(X, \mathcal{O}_X(m+q)D) - \bigoplus_i h^0(X, \mathcal{O}_X(m+q - qd_i)D) + h^0(X, F^{n*}V \otimes \mathcal{O}_X(mD))]. \end{aligned} \quad (1.4)$$

The sequence $\{g_n(R, I)\}_n$ in the theorem is obtained from $\{f_n(R, I)\}_n$ by a simple modification. In particular the computations depend only on the cohomologies of the Frobenius pullbacks of the locally free sheaves V and $\mathcal{O}_X(D)$ and their twists (by the line bundles $\mathcal{O}_X(mD)$).

On the other hand, let us consider the case when $R = \bigoplus_{m \geq 0} R_m$ is an arbitrary normal graded domain. By the theorem of Demazure (see Theorem 3.1), there is a \mathbb{Q} -divisor D such that $R_m = H^0(X, \mathcal{O}_m)$, for all m , where we denote $\mathcal{O}_n = \mathcal{O}_X(nD)$. But \mathcal{O}_1 need not be an invertible sheaf. Moreover the multiplication map $\mathcal{O}_m \otimes \mathcal{O}_n \rightarrow \mathcal{O}_{m+n}$ need not be an isomorphism, in general. In particular the sequence (the map ϕ defined as in (1.1))

$$0 \rightarrow \text{Ker } \phi \rightarrow \bigoplus_i \mathcal{O}_{1-d_i} \xrightarrow{\phi} \mathcal{O}_1 \rightarrow 0$$

need not be locally split exact and $\text{Ker } \phi$ may not be locally free. Hence a version of (1.3) cannot be derived from a single exact sequence like (1.1), and therefore $\text{Ker } \phi_{m,q}$, where $\phi_{m,q} : \bigoplus_i \mathcal{O}_{m+q-qd_i} \rightarrow \mathcal{O}_{m+q}$ is the canonical map, does not come from ‘twists of’ a single sheaf (unlike in the standard graded situation, where $\text{Ker } \phi_{m,q} = F^{n*}V \otimes \mathcal{O}_X(mD)$, for all m and $q = p^n$).

However the sheaf \mathcal{O}_m , associated to such a Weil \mathbb{Q} -divisor does have some special properties which we exploit, for example \mathcal{O}_m is a rank one reflexive sheaf of \mathcal{O}_X -modules, hence invertible outside the singular

locus of X . As a result though one does not have a direct relation between the sequences (1.3) (as m and q vary), we are able to relate their cohomologies by estimates

$$|h^0(X, \text{Ker } \phi_{mp+n_1, qp}) - p^{d-1}h^0(X, \text{Ker } \phi_{m, q})| = O((m+q)^{d-2}), \quad \text{for } 0 \leq n_1 < p.$$

In particular the structure theorem of Demazure, which realizes each graded component R_m of R as the space of sections of the \mathbb{Q} -divisor mD , allows us to give a simpler proof (than in [19]) for this more general setting of the graded pairs.

However in [19] the existence of the HK density function $f_{M,I}$ was proved directly (and without the assumption that R is a domain). Here we prove the Main Theorem with the assumption that R is a domain. The proof is in three steps: We prove the theorem when $(M, I) = (R, I)$ and where R is a normal domain such that $\gcd\{m > 0 \mid R_m \neq 0\} = 1$. This is the main part. Then we extend the result for the pair (R, I) , where R is a graded domain. Then we further extend this to graded modules over such pairs.

We can extend the notion of the HK density function by removing the domain condition on the ring, by defining

$$f_{R,I} := \sum_{P \in \Lambda} \lambda(R_P) f_{R/P, (I+P)/P},$$

where $\Lambda = \{P \in \text{Spec } R \mid \dim R = \dim R/P\}$. This is clearly an additive function and hence can be extended canonically to the notion of $f_{M,I}$. Since $e_{HK}(-)$ is an additive function we get $\int f_{M,I}(x)dx = e_{HK}(M, I)$, with such a definition.

However, this definition may not reconcile (see Remark 4.3) with the following definition given for a normal domain or for a standard graded domain, which is

$$f_{R,I}(x) = \lim_{n \rightarrow \infty} 1/q^{d-1} \ell(R/I^{[q]})_{[xq]n_0},$$

where $n_0 = \gcd\{m > 0 \mid R_m \neq 0\}$, unless $\gcd\{m > 0 \mid (R/P)_m \neq 0\} = n_0$, for all $P \in \Lambda$.

As a consequence of our Main Theorem (Theorem 1.1) we have the following corollary (see subsection 7.3 for a proof).

Corollary 1.2. *Let (S, I) be a graded pair. Suppose there is a graded ring R with a degree preserving finite map $S \rightarrow R$ of k -algebras such that $\text{proj dim}_R(R/IR) < \infty$. Then*

- (1) *the HK density function $f_{S,I}$ is a piecewise polynomial function of degree $\dim S - 1$, explicitly given in terms of the graded Betti numbers of the resolution of IR as an R -module. Moreover,*
- (2) *if IR has pure resolution as an R -module then the graded Betti numbers of the resolution can be recovered from the function $f_{S,I}$.*

In particular, if $R = k[X_1, \dots, X_d]$ is a polynomial ring and G is a finite group (scheme) acting linearly on R then for any graded pair (R^G, I) , where R^G is the ring of invariants, the function $f_{R^G, I}$ is a piecewise polynomial of degree $d - 1$.

Thus for a faithful linear representation $G \rightarrow GL_d(k)$, where G is a finite group scheme, we have a piecewise polynomial function $f_{R^G, I}$ for any graded pair (R^G, I) . It would be interesting to see that beside the HK multiplicities of R^G , what other information this new invariant can provide for the ring R^G .

As a result of the above corollary, we explicitly write down the HK density function $f_{R, \mathbf{m}}$, where $R = k[[X_1, X_2, X_3]]/(f)$ is a rational double point.

As in the case of standard graded pairs, the HK density function is multiplicative (Theorem 6.1) for the graded pairs too.

We also generalize the other results of [20] to the graded pairs. We prove that, if R is strongly F -regular on the punctured spectrum then the maximum support $\alpha(R, I)$ of the function $f_{R, I}$ can be realized as the F -threshold of an explicit \mathbf{m} -primary ideals. We give the formula for $f_{R, I}$ when either the dimension of R is two, or I is generated by a system of parameters.

2. Preliminaries

Notation 2.1. By a *graded pair* (R, I) we mean that $R = \bigoplus_{m \geq 0} R_m$ is a Noetherian graded domain of dimension $d \geq 2$, and of finite type over a perfect field $k = R_0$ of characteristic $p > 0$, and $I \subset R$ is a graded ideal such that $\ell(R/I) < \infty$.

Definition 2.2. Let (R, I) be a graded pair, and let M be a finitely generated graded R -module. Let $n_0 = \gcd \{n > 0 \mid R_n \neq 0\}$. Then the sequence of density functions $\{f_n(M, I) : [0, \infty) \rightarrow [0, \infty)\}_n$ for the pair (M, I) is a sequence of step functions given by

$$f_n(M, I)(x) = \left(\ell(M/I^{[q]}M)_{\lfloor xq \rfloor n_0} + \cdots + \ell(M/I^{[q]}M)_{\lfloor xq \rfloor n_0 + n_0 - 1} \right) / q^{d-1}.$$

Alternatively we consider a sequence $\{g_n(M, I) : [0, \infty) \rightarrow [0, \infty)\}_n$ of continuous functions given by

$$g_n(M, I)(x) = (1 - t)f_n(M, I)(x) + (t)f_n(M, I)(x + (1/q)),$$

where $x = (1 - t)\lfloor xq \rfloor / q + (t)\lfloor xq + 1 \rfloor / q$, for some $t \in [0, 1)$.

In particular, each $g_n(R, I)$ is continuous, and the uniform convergence of the sequence $\{g_n(M, I)\}_n$ is equivalent to the uniform convergence of $\{f_n(M, I)\}_n$. Moreover the functions $g_n(M, I)$ and $f_n(M, I)$ are compactly supported with a bound on the support which is independent of n .

Remark 2.3. Unlike in the case of standard graded pairs, here if $R \rightarrow R'$ is a degree preserving finite graded morphism of rings and M is a graded R' -module then $f_n(M_R, I)$ and $f_n(M_{R'}, IR')$ may differ (though recoverable from each other as shown in Lemma 4.4).

In such case, to avoid the ambiguity, we will write $f_n(M_R, I)$ and $g_n(M_R, I)$ instead of $f_n(M, I)$ and $g_n(M, I)$ to emphasize the fact that we are considering M as an R -module.

Lemma 2.4. Each $g_n(M, I)$ is a compactly supported continuous function. Moreover, for a given pair (M, I) there is a constant \tilde{m} (independent of n) such that

$$\text{supp } g_n(M, I) \subseteq [0, \tilde{m}], \quad \text{for all } n \geq 1.$$

In particular $\text{supp } f_n(M, I) \subseteq [0, \tilde{m}]$, for all $n \geq 1$.

Proof. We choose the integers s, l, m_μ and n_ν as follows: Let $\mu(I) = s$. Let $J = \bigoplus_{m > 0} R_m$ with a set of homogeneous generators h_1, \dots, h_μ of degrees, say, $m_1 \leq \cdots \leq m_\mu$ respectively. Let l be an integer such that $J^l \subseteq I$. Let M be generated by homogeneous elements g_1, \dots, g_ν of degrees $n_1 \leq \cdots \leq n_\nu$.

Since $R_m = h_1 R_{m-m_1} + \cdots + h_\mu R_{m-m_\mu}$ and $M_m = g_1 R_{m-n_1} + \cdots + g_\nu R_{m-n_\nu}$,

$$m - n_\nu \geq (m_\mu)lsq \implies M_m \subseteq J^{lsq}M \subseteq I^{sq}M \subseteq I^{[q]}M.$$

Hence $(M/I^{[q]}M)_m = 0$, for all $m \geq n_\nu + (m_\mu)lsq$. \square

The following is a well known result.

Lemma 2.5. *Let $R = \bigoplus_{n \geq 0} R_n$ be a Noetherian graded domain such that R_0 is a field. Then the following three conditions are equivalent:*

- (1) $\gcd \{n > 0 \mid R_n \neq 0\} = n_0$.
- (2) $n_0 > 0$ is the least integer with the property: there is $m_1 > 0$ such that $R_{mn_0} \neq 0$, for all $m \geq m_1$.
- (3) $n_0 > 0$ is the least integer such that the quotient field of R has a homogeneous element of degree n_0 .

Proof. Left as an exercise for the reader. \square

2.1. Some general facts about \mathbb{Q} -divisors and reflexive sheaves

Here we refer the reader to Hartshorne [7] (for more consolidated information one can also look at the notes by Schewde [17]).

Let X be a normal projective variety over a perfect field k . The *prime divisors* of X are the codimension 1 integral subschemes of X . The set $W\text{Div}(X, \mathbb{Z})$, of *Weil divisors* on X , is the group $W\text{Div}(X, \mathbb{Z}) = \{D = \sum_i n_i D_i \mid n_i \in \mathbb{Z} \text{ and } D_i \text{ prime divisors of } X\}$. The set $W\text{Div}(X, \mathbb{Q}) = W\text{Div}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ of *Weil \mathbb{Q} -divisor* is the group

$$W\text{Div}(X, \mathbb{Q}) = \{D = \sum_i a_i D_i \mid a_i \in \mathbb{Q} \text{ and } D_i \text{ prime divisors of } X\}.$$

Let $K(X)$ denote the function field of X and $\mathcal{K}(X)$ the constant sheaf on X given by $K(X)$. Then given $D = \sum_i a_i D_i$ in $W\text{Div}(X, \mathbb{Q})$ we can associate a coherent subsheaf $\mathcal{O}_X(D) \subset \mathcal{K}(X)$ as follows: Consider a group homomorphism $K(X) \rightarrow W\text{Div}(X, \mathbb{Z})$ given by $f \rightarrow \text{div}(f) := \sum_i v_{D_i}(f) D_i$, where a prime divisor D_i of X gives a canonical discrete valuation $v_{D_i} : K(X) \rightarrow \mathbb{Z} \cup \{\infty\}$. Then, for an open set $U \subset X$, the sheaf $\mathcal{O}_X(D)|_U$ is defined as the space of sections

$$H^0(U, \mathcal{O}_X(D)) = \{f \in K(X) \mid \text{div}(f)|_U + D|_U \geq 0\},$$

where, we say a divisor $D = \sum_i a_i D_i \geq 0$ if each $a_i \geq 0$.

The set $\text{C-div}(X)$ of Cartier divisors is

$$\text{C-div}(X) = \{D \in W\text{Div}(X, \mathbb{Z}) \mid \mathcal{O}_X(D) \text{ is an invertible subsheaf of } \mathcal{K}(X)\}.$$

If $D = \sum_i a_i D_i \in W\text{Div}(X, \mathbb{Q})$ then it follows from the above definition that $\mathcal{O}_X(D) = \mathcal{O}_X(\lfloor D \rfloor)$, where $\lfloor D \rfloor := \sum_i \lfloor a_i \rfloor D_i$. If D is a Cartier divisor then $D = \lfloor D \rfloor$ as $D = \sum a_i D_i$, where $a_i \in \mathbb{Z}$.

The sheaves $\mathcal{O}_X(D)$ associated to a divisor D form a special class of coherent sheaves, known as reflexive sheaves.

Definition 2.6. If \mathcal{F} is a coherent sheaf on X and $\mathcal{F}^\vee = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$ then \mathcal{F} is *reflexive* if the natural map of \mathcal{O}_X -modules $\alpha : \mathcal{F} \rightarrow (\mathcal{F}^\vee)^\vee$ is an isomorphism.

It is known that

- (1) $\{\mathcal{O}_X(D) \mid D \in W\text{Div}(X, \mathbb{Q})\} =$ rank one reflexive subsheaves of $\mathcal{K}(X)$ and
- (2) $\{\mathcal{O}_X(D) \mid D \in \text{C-div}(X)\} =$ invertible reflexive subsheaves of $\mathcal{K}(X)$.
- (3) The rank one reflexive sheaves are invertible on the regular locus of X .

3. The HK density functions for normal graded domains

In this section we prove the existence of the HK density function for a graded pair (R, I) , where R is a normal domain over a perfect field k with $\gcd\{n > 0 \mid R_n \neq 0\} = 1$.

For such a ring R the following result of Demazure [5] gives a description of each graded component R_m of R in terms of a single Weil \mathbb{Q} -divisor in $X = \text{Proj } R$.

Theorem 3.1 (Demazure). *Let $R = \bigoplus_{n \geq 0} R_n$ be a normal graded domain of finite type over a field k . Suppose there is a homogeneous element T of degree 1 in the quotient field of R . Then for $X = \text{Proj } R$, there exists a unique Weil \mathbb{Q} -divisor D in $W\text{Div}(X, \mathbb{Q})$ such that $R_n = H^0(X, \mathcal{O}_X(nD)) \cdot T^n$, for every $n \geq 0$.*

If R is standard graded ring then there is the canonical closed embedding $X \hookrightarrow \mathbb{P}^n$, which gives an ample line bundle $\mathcal{O}_X(1) = \mathcal{O}_X(D)$ for some Cartier divisor D . Hence $R_m = H^0(X, \mathcal{O}_X(mD))$, for $m \gg 0$.

On the other hand, by the above theorem of Demazure, if R is a graded normal domain then there exists a Weil \mathbb{Q} -divisor D such that $R_m = H^0(X, \mathcal{O}_X(mD))$, for all $m \geq 0$. However, in this case D need not be Cartier. But it is known that, for some $l_1 \in \mathbb{N}$, $l_1 D$ is Cartier and, moreover, for some $r \in \mathbb{N}$, $\mathcal{O}_X(rD)$ is very ample line bundle on X .

In the following lemma we show that the integers l_1 and r can be given in terms of the generators of R as R_0 -algebra (can also refer to Bourbaki [2], Chap III § Proposition 3).

Lemma 3.2. *For R and D as in Theorem 3.1, let h_1, \dots, h_μ denote a set of homogeneous generators of R as R_0 -algebra, of degrees m_1, \dots, m_μ respectively, and let $l_1 = \text{lcm}(m_1, \dots, m_\mu)$. Then*

(a) *for $n \in l_1 \mathbb{N}$, the sheaf $\mathcal{O}_X(nD)$ is a line bundle on X . In particular the canonical multiplication map*

$$\mathcal{O}_X(nD) \otimes \mathcal{O}_X(aD) \longrightarrow \mathcal{O}_X((n+a)D) \quad \text{is an isomorphism, for all } a.$$

(b) *If R is \mathbb{N} -graded ring as above but not necessarily normal, then, for $r = l_1 \mu$ the ring $R^{(r)} = \bigoplus_{m \geq 0} R_{rm}$, where $R_m^{(r)} := R_{rm}$, is a standard graded ring.*

(c) *For $r = l_1 \mu$ the line bundle $\mathcal{O}_X(rD)$ is very ample on X .*

Proof. (a): By the above hypothesis, the variety X has the affine open cover $\{D_+(h_i)\}_i = \text{Spec } R_{(h_i)}$, where $R_{(h_i)} = \{f/h_i^m \mid \deg(f) - m \deg(h_i) = 0 \text{ and } f \in R_{\deg(f)}\}$ and $\mathcal{O}_X|_{D_+(h_i)} = R_{(h_i)}$.

By Lemma 2.1 in [21], sheaves of \mathcal{O}_X -modules $\widetilde{R(n)} \simeq \mathcal{O}_X(nD)$, for every $n \in \mathbb{N}$. Therefore, for an integer $a \in \mathbb{Z}$, the sheaf $\mathcal{O}_X(aD)$ on the open set $D_+(h_i)$ is defined as

$$\mathcal{O}_X(aD)|_{D_+(h_i)} = \widetilde{R(n)}|_{(h_i)} = \{f/h_i^m \mid \deg(f) - m \deg(h_i) = a \text{ and } f \in R_{\deg(f)}\}.$$

This implies that for $n \in l_1 \mathbb{N}$

$$\mathcal{O}_X(nD)|_{D_+(h_i)} = h_i^{n/m_i} \mathcal{O}_X|_{D_+(h_i)}, \quad \text{where } h_i^{n/m_i} \in H^0(D_+(h_i), \mathcal{O}_X(nD)|_{D_+(h_i)}).$$

Therefore $\mathcal{O}_X(nD)$ is a line bundle and nD is a Cartier divisor.

Moreover this also implies that $\mathcal{O}_X(nD) \otimes \mathcal{O}_X(aD) = \mathcal{O}_X((n+a)D)$ as

$$\begin{aligned} \mathcal{O}_X((n+a)D)|_{D_+(h_i)} &= \{f/h_i^m \mid \deg(f) - m \deg(h_i) = n+a \text{ and } f \in R_{\deg(f)}\} \\ &= \{h^{n/m_i}(f/h_i^{m+n/m_i}) \mid \deg(f) - m \deg(h_i) = n+a \text{ and } f \in R_{\deg(f)}\} \\ &= \mathcal{O}_X(nD)|_{D_+(h_i)} \otimes \mathcal{O}_X(aD)|_{D_+(h_i)}. \end{aligned}$$

(b): First we prove that $n \geq l_1\mu$ implies $R_n \subseteq R_{l_1} \cdot R_{n-l_1}$. Consider $x \in R_n$ which we can write as $x = h_1^{i_1} \cdots h_\mu^{i_\mu}$ where for some j , we have $m_j i_j \geq l_1$; this allows us to rewrite $x = h_j^{l_1/m_j} (h_1^{i_1} \cdots h_j^{i_j-l_1/m_j} \cdots h_\mu^{i_\mu}) \in R_{l_1} \cdot R_{n-l_1}$.

Now this proves $R^{(r)}$ is a standard graded ring; because, for $m \geq 1$

$$R_m^{(r)} = R_{mr} \subseteq R_{l_1}^{\mu(m-1)} R_{l_1\mu} \subseteq R_{l_1\mu}^{m-1} R_{l_1\mu} = (R_1^{(r)})^m.$$

(c): Since $X = \text{Proj } R^{(r)}$, for $h = h^0(X, \mathcal{O}_X(rD)) - 1$, the sections of $\mathcal{O}_X(rD)$ give a canonical surjective map $k[Y_0, \dots, Y_h] \longrightarrow R^{(r)}$ and hence a closed immersion X into \mathbb{P}_k^h . This proves the last assertion. \square

In the rest of the section we have the following notation.

Notation 3.3. The pair (R, I) is a fixed graded pair, where R is a normal graded domain of dimension $d \geq 2$ and $\gcd \{n > 0 \mid R_n \neq 0\} = 1$. Then, by Theorem 3.1, $R = R(X, D) = \bigoplus_{n \geq 0} H^0(X, \mathcal{O}_X(nD)).T^n$, where $D \in W\text{Div}(X, \mathbb{Q})$ is a Weil \mathbb{Q} -divisor on X and T is a homogeneous element of degree 1 in the quotient field of R .

We also fix $r \in \mathbb{N}$, so that $\mathcal{O}_X(rD)$ is a very ample divisor on X and $R_m \neq 0$, for all $m \geq r$. We fix a set of homogeneous generators f_1, \dots, f_s of I of degrees d_1, \dots, d_s respectively. For the sake of abbreviation we adopt the following notation.

Let $\mathcal{O}_n = \mathcal{O}_X(nD)$.

Let $\mathcal{L} = \mathcal{O}_r$ be the very ample line bundle on X .

Let $m_{q,d_i} = m + q - qd_i$ where $q = p^n$, for some $n \geq 1$.

Let $(m_{q,d_i}) = \lfloor \frac{m+q}{r} \rfloor r - qd_i$.

Since $\gcd \{m > 0 \mid R_m \neq 0\} = 1$, the definition of the sequences $\{f_n(R, I)\}_n$ and $\{g_n(R, I)\}_n$, given as in Definition 2.2 coincides with the definition given for the case of standard graded pair in [19].

Definition 3.4. For the pair (R, I) , the sequence of density functions $f_n(R, I) : [0, \infty) \rightarrow [0, \infty)\}_{n \in \mathbb{N}}$ is given by

$$f_n(R, I)(x) = 1/q^{d-1} \ell(R/I^{[q]})_{\lfloor xq \rfloor}, \quad \text{where } q = p^n.$$

Henceforth in this section we will denote $f_n(R, I)$ by f_n . We also define a sequence of continuous functions $\{g_n(R, I)\}_n$ given by

$$g_n(R, I)(x) = (1-t)f_n(m/q) + (t)f_n(m+1/q),$$

where $x = (1-t)m/q + t(m+1)/q$, for some $t \in [0, 1)$.

In this section our aim is to prove the uniform convergence of $\{f_n\}_n$, which is equivalent to proving the uniform convergence of the sequence $\{g_n\}_n$. As a result the limiting function $f_{R,I}$ will be a compactly supported continuous function and $\int_0^\infty f_{R,I}(x) = e_{HK}(R, I)$.

For given $m \in N$ and $q = p^n$, we consider the following short exact sequence of \mathcal{O}_X -modules

$$0 \longrightarrow \mathcal{F}_{m,q} \longrightarrow \bigoplus_{i=1}^s \mathcal{O}_{m_{q,d_i}} \xrightarrow{\varphi_{m,q}} \mathcal{O}_{m+q} \longrightarrow 0, \quad (3.1)$$

where $\varphi_{m,q}(a_1, \dots, a_s) = \sum_i a_i f_i^q$ and $\mathcal{F}_{m,q} = \text{Ker } \varphi_{m,q}$. This implies

$$\begin{aligned}
f_n\left(\frac{m+q}{q}\right) &= \frac{1}{q^{d-1}} \ell(R/I^{[q]})_{m+q} = \frac{1}{q^{d-1}} [\ell(R_{m+q}) - \sum_{i=1}^s \ell(f_i^q R_{m+q-qd_i})] \\
&= \frac{1}{q^{d-1}} \left[h^0(X, \mathcal{O}_{m+q}) - \sum_i h^0(X, \mathcal{O}_{m_{q,d_i}}) + h^0(X, \mathcal{F}_{m,q}) \right].
\end{aligned}$$

The following technical result, which will be proved in Section 5, is crucial in proving the uniform convergence of the sequence $\{f_n\}_n$.

Lemma 3.5 (Main Lemma). *For $\mathcal{F}_{m,q}$ as in (5.1), there exists a constant C such that, for all $m \geq 0$ and $q = p^n$ and $0 \leq n_1 < p$,*

$$|h^0(X, \mathcal{F}_{mp+n_1,qp}) - p^{d-1}h^0(X, \mathcal{F}_{m,q})| \leq C(mp + qp)^{d-2}.$$

Moreover

$$|h^0(X, \mathcal{O}_{pm_q+n_1}) - p^{d-1}h^0(X, \mathcal{O}_{m_q})| \leq C(mp + qp)^{d-2},$$

where m_q denotes the integer m_{q,d_i} or (m_{q,d_i}) , for $1 \leq i \leq s$, or $m_q = m + q$.

Assuming the proof of Lemma 3.5 we prove the existence of the HK density function for normal graded domains.

Proposition 3.6. *If $R = \bigoplus_{n \geq 0} R_n$ is a normal graded domain and $\gcd\{n > 0 \mid R_n \neq 0\} = 1$ then for a graded pair (R, I) the sequence $\{f_n(R, I)\}_n$ is uniformly convergent.*

Proof. Let $x \geq 1$. For $q = p^n$ and $m + q \leq xq < m + q + 1$,

$$f_n(x) = 1/q^{d-1} \ell(R/I^{[q]})_{[xq]} = 1/q^{d-1} \ell(R/I^{[q]})_{m+q}.$$

Therefore there is n_1 such that $0 \leq n_1 < p$ and

$$f_{n+1}(x) = 1/(qp)^{d-1} \ell(R/I^{[qp]})_{mp+qp+n_1}.$$

For $\mathcal{O}_m = \mathcal{O}_X(mD)$ we have the short exact sequences of \mathcal{O}_X -modules (as in (5.1))

$$\begin{aligned}
0 \longrightarrow \mathcal{F}_{m,q} \longrightarrow \bigoplus_{i=1}^s \mathcal{O}_{m+q-qd_i} \xrightarrow{\varphi_{m,q}} \mathcal{O}_{m+q} \longrightarrow 0 \quad \text{and} \\
0 \longrightarrow \mathcal{F}_{mp+n_1,qp} \longrightarrow \bigoplus_{i=1}^s \mathcal{O}_{mp+qp+n_1-qp d_i} \xrightarrow{\varphi_{mp+n_1,qp}} \mathcal{O}_{mp+qp+n_1} \longrightarrow 0.
\end{aligned}$$

Therefore

$$\begin{aligned}
f_n(x) &= \frac{p^{d-1} [h^0(X, \mathcal{O}_{m+q}) - \sum_i h^0(X, \mathcal{O}_{m+q-qd_i}) + h^0(X, \mathcal{F}_{m,q})]}{(qp)^{d-1}} \quad \text{and} \\
f_{n+1}(x) &= \frac{[h^0(X, \mathcal{O}_{mp+qp+n_1}) - \sum_i h^0(X, \mathcal{O}_{mp+qp+n_1-qp d_i}) + h^0(X, \mathcal{F}_{mp+n_1,qp})]}{(qp)^{d-1}}.
\end{aligned}$$

By Lemma 3.5, there is a constant $C_0 > 0$ such that

$$|f_n(x) - f_{n+1}(x)| \leq C_0(mp + qp)^{d-2}/(qp)^{d-1}.$$

Since $\text{supp } f_n \subseteq [0, \tilde{m}]$, where \tilde{m} is as in Lemma 2.4, we can assume $(m + q)/q \leq \tilde{m}$ and hence there is a constant C_1 such that

$$|f_n(x) - f_{n+1}(x)| \leq C_1/qp, \quad \text{for all } x \geq 1.$$

We can further choose C_1 such that the above inequality also holds for all $0 \leq x \leq 1$: Because if $0 \leq x < 1$ then

$$f_n(x) = \ell(R)_{\lfloor xq \rfloor} / q^{d-1} = P(\lfloor xq \rfloor) / q^{d-1} \quad \text{for all } q = p^n \gg 0,$$

where $P(x) \in \mathbb{Q}[X]$ is the Hilbert polynomial of R hence of degree $d-1$. This proves the proposition. \square

4. The Main Theorem

In this section we prove the Main Theorem 1.1 modulo the proof of the Main Lemma 3.5.

Throughout this section (R, I) denotes a graded pair of dimension $d \geq 2$ and $\gcd \{m > 0 \mid R_m \neq 0\} = n_0$.

Also if $S \rightarrow R$ is a map of rings then $f_n(R_S, I)$ are the density functions as in Definition 2.2, where R is considered as a S -module. See Remark 2.3 for more details.

Remark 4.1. If $S \rightarrow R$ is a degree preserving finite map of graded domains, where (S, I) is a graded pair, then we can associate two sequences of density functions to (R, IR)

- (1) the sequence $\{f_n(R, IR)\}_n$ of density functions for (R, IR) , where R is considered as the module over itself, (here $q = p^n$)

$$f_n(R, IR)(x) = \frac{1}{q^{d-1}} \left(\ell(R/I^{[q]}R)_{\lfloor xq \rfloor n_0} \right), \quad \text{where } n_0 = \gcd \{n > 0 \mid R_n \neq 0\},$$

- (2) the sequence $\{f_n(R_S, I)\}_n$ of density functions of (R, IR) , where R is considered as the module over S ,

$$f_n(R_S, I)(x) = \frac{1}{q^{d-1}} \left(\ell(R/I^{[q]}R)_{\lfloor xq \rfloor m_0} + \cdots + \ell(R/I^{[q]}R)_{\lfloor xq \rfloor m_0 + m_0 - 1} \right),$$

where $m_0 = \gcd \{n > 0 \mid S_n \neq 0\}$.

The (uniform) convergence of both the sequences will follow from Theorem 1.1 and the limiting function will be denoted by $f_{R, IR}$ and $f_{R_S, I}$ respectively. Though both the functions are not the same unless $m_0 = n_0$, by Lemma 4.4 they can be recovered from each other. Moreover $\int f_{R, IR}(x)dx = \int f_{R_S, R}(x)dx = e_{HK}(R, IR)$.

We use the following lemma to reduce the problem of convergence of the sequence $\{f_n(M, I)\}_n$ to the problem of convergence of the sequence $\{f_n(S, J)\}_n$, where S is a normal graded domain with $\gcd \{n > 0 \mid S_n \neq 0\} = 1$.

Lemma 4.2. If $\gcd \{m > 0 \mid R_m \neq 0\} = 1$ and N, N' are finitely generated graded R -modules with the exact sequence of graded R -linear maps

$$0 \rightarrow N \xrightarrow{\phi} N' \rightarrow Q'' \rightarrow 0 \tag{4.1}$$

such that the $\text{supp dim } Q'' < d$ and the map ϕ is of degree 0.

Then the sequence $\{f_n(N, I)\}_n$ is uniformly convergent if and only if $\{f_n(N', I)\}_n$ is so. Moreover in that case

$$\lim_{n \rightarrow \infty} f_n(N, I) = \lim_{n \rightarrow \infty} f_n(N', I).$$

Proof. Here if M is a graded R -module then the function $f_n(M, I) : [0, \infty) \rightarrow [0, \infty)$ is given by $x \rightarrow \ell(M/I^{[q]}M)_{\lfloor xq \rfloor} / q^{d-1}$, where $q = p^n$.

Let I have homogeneous generators f_1, \dots, f_s of degree d_1, \dots, d_s respectively. Then for any graded R -module M , we define

$$\Phi_M^n : \oplus_i^s M(-qd_i) \longrightarrow M \quad \text{given by} \quad (m_1, \dots, m_s) \rightarrow \sum_i f_i^q m_i$$

This gives degree preserving maps of graded R -modules, which is functorial in M ,

$$0 \longrightarrow \text{Ker } \Phi_M^n \longrightarrow \oplus_1^s M(-qd_i) \xrightarrow{\Phi_M^n} M \longrightarrow \text{Coker } \Phi_M^n \longrightarrow 0.$$

Now the snake lemma applied to (4.1) gives the following exact sequence of graded R -modules

$$\longrightarrow \text{Ker } \Phi_{Q''}^n \longrightarrow \text{Coker } \Phi_N^n \longrightarrow \text{Coker } \Phi_{N'}^n \longrightarrow \text{Coker } \Phi_{Q''}^n \longrightarrow 0,$$

where

$$f_n(N', I)\left(\frac{m+q}{q}\right) = \ell(\text{Coker } \Phi_{N'}^n)_{m+q} \quad \text{and} \quad f_n(N, I)\left(\frac{m+q}{q}\right) = \ell(\text{Coker } \Phi_N^n)_{m+q}.$$

Let $C_{Q''}$ be constant such that, for all $m > 0$, $\ell(Q''_m) \leq C_{Q''} m^{d-2}$ (such a constant exists by the hypothesis on the support dimension of Q'').

$$|\ell(\text{Coker } \Phi_{N'}^n)_{m+q} - \ell(\text{Coker } \Phi_N^n)_{m+q}| \leq 2C_{Q''}(m+q)^{d-2}.$$

Now, for $x \geq 1$ we have $m+q \leq xq < m+q+1$ for some $m \geq 0$ and so we have

$$|f_n(N, I)(x) - f_n(N', I)(x)| \leq 2C_{Q''} x_0^{d-2} / q,$$

where by Lemma 2.4, we may fix an x_0 such that $\text{supp } f_n(N, I)$ and $\text{supp } f_n(N', I)$ are subsets of $[0, x_0]$, for all $n \geq 1$.

If $0 \leq x < 1$ then $m \leq xq < m+1$, for some $m < q$. It is easy to check that in this case

$$|f_n(N, I)(x) - f_n(N', I)(x)| = 2C_{Q''} m^{d-2} / q^{d-1} \leq 2C_{Q''} / q.$$

This proves the lemma. \square

Now we are ready to prove the Main Theorem.

Proof of the Main Theorem 1.1. It is enough to prove that the sequence $\{g_n(M, I)\}_n$ is uniformly convergent as part (2) of the theorem follows from this assertion. Let $\gcd \{n > 0 \mid R_n \neq 0\} = n_0$.

Let $S = R^{(n_0)}$, where the n^{th} degree component of $R^{(n_0)}$ is R_{nn_0} . Then S is a graded domain, where $\gcd \{n > 0 \mid S_n \neq 0\} = 1$. (Note that $S = R$ as rings, but the grading is changed.)

It is easy to see that the uniform convergence of $\{g_n(M, I)\}_n$ is equivalent to the uniform convergence of $\{f_n(M, I)\}_n$.

We first prove the theorem for $M = R$, where it is sufficient to prove the uniform convergence of $\{f_n(R, I)\}_n$. By definition, $f_n(R, I) = f_n(S, I)$, for all n .

Let $\tilde{S} = \oplus_n \tilde{S}_n$ denote the normalization of S in its quotient field then the inclusion map $S \rightarrow \tilde{S}$ is a module finite graded map of degree 0, and we have the short exact sequence of graded S -modules

$$0 \longrightarrow S \longrightarrow \tilde{S} \longrightarrow Q'' \longrightarrow 0,$$

where $\text{support dim } Q'' \leq d - 1$.

By Proposition 3.6, the sequence $\{f_n(\bar{S}, I\bar{S})\}_n$ is uniformly convergent. But, by Definition $f_n(\bar{S}_S, I) = f_n(\bar{S}, I\bar{S})$. Hence the uniform convergence of $\{f_n(R, I)\}_n$ follows by Lemma 4.2.

We now consider the general case of a finitely generated graded R -module M .

Let $\bar{M} = \bigoplus_n \bar{M}_n$, where $\bar{M}_n = M_{nn_0} + \cdots + M_{nn_0+n_0-1}$ denotes the degree n component of \bar{M} . If M is generated by homogeneous elements g_1, \dots, g_ν as an R -module then \bar{M} is generated by g_1, \dots, g_ν as an S -module. Hence \bar{M} is a finitely generated graded S -module. Also, for all $n \geq 1$, $f_n(M_R, I) = f_n(\bar{M}_S, IS)$ and $\text{rank}_R M = \text{rank}_S \bar{M}$.

Claim. *There exists an exact sequence of graded S -modules*

$$0 \longrightarrow \bigoplus^{n_1} S(-a) \xrightarrow{\phi} \bar{M} \longrightarrow Q'' \longrightarrow 0,$$

where ϕ is a graded map of degree 0 and $\dim (Q'') \leq d - 1$.

Proof of the claim. For the multiplicatively closed set $T = S \setminus \{0\}$, the $T^{-1}S$ -module $T^{-1}\bar{M}$ is free of finite rank, say n_1 and is generated by a finite set of homogeneous elements. Hence we can choose homogeneous elements m_1, \dots, m_{n_1} in \bar{M} of degrees d_1, \dots, d_{n_1} respectively such that the m'_i 's give a basis for $T^{-1}\bar{M}$.

Since $\gcd\{n > 0 \mid S_n \neq 0\} = 1$, we have m_0 such that $S_m \neq 0$, for all $m \geq m_0$. Let $a > 0$ such that $a \geq \max\{m_0 + d_i, m_0\}_i$ and let $s_i \in S_{a-d_i} \setminus \{0\}$. Then $s_1 m_1, \dots, s_{n_1} m_{n_1} \in \bar{M}$ are homogeneous elements (each of degree a) and generate $T^{-1}\bar{M}$ as $T^{-1}S$ -module. Hence we have a generically isomorphic map $\bigoplus^{n_1} S(-a) \longrightarrow \bar{M}$ of graded S -modules of degree 0. The map is injective as S is a domain. This proves the claim and hence the theorem. \square

Remark 4.3. Recall that for a standard graded pair (R, I) of dimension $d \geq 2$ and a graded R -module M if $\Lambda = \{p \in \text{Spec } R \mid \dim R = \dim R/P\}$ then we have

$$f_{M,I}(x) = \frac{1}{q^{d-1}} \ell(M/I^{[q]}M)_{[xq]} = \sum_{p \in \Lambda} \lambda(M_P) f_{R/P, (I+P)/P}.$$

When R is not a domain, we can still extend the notion of the HK density function of a module M over a graded pair (R, I) by such an additive formula, that means

$$f_{M,I}(x) := \sum_{p \in \Lambda} \lambda(M_P) f_{R/P, (I+P)/P} = \lim_{n \rightarrow \infty} \left[\sum_{p \in \Lambda} \lambda(M_P) f_n(R/P, (I+P)/P) \right].$$

This makes $f_{M,I}$ the limit of a uniformly converging sequence of continuous functions and hence is continuous. However

$$f_{R,I}(x) \neq g_{R,I}(x) := \lim_{n \rightarrow \infty} 1/q^{d-1} \ell(R/I^{[q]})_{[xq]n_0},$$

where $n_0 = \gcd\{m > 0 \mid R_m \neq 0\}$, unless $\gcd\{m > 0 \mid (R/P)_m \neq 0\} = n_0$, for all $P \in \Lambda$. For example let

$$R = k[X^2, Y^7, Z^{14}]/(X^2 Y^7) \quad \text{and} \quad \mathbf{m} = (X^2, Y^7, Z^{14}) \quad \text{and} \quad \text{char } k = p > 7.$$

Then R is a two dimensional ring and $\Lambda = \{X^2 R, Y^7 R\}$. One can check that the function $g_{R,\mathbf{m}}(x) = 0$, for $x \in \{(14m+9)/p^n \mid n, m \in \mathbb{N}\}$. In particular $g_{R,\mathbf{m}}$ vanishes on a dense set, hence is not continuous (not even almost everywhere continuous function).

Next we show that for a finite map $S \rightarrow R$ as in Notation 4.1, the two HK density functions, for the pair (R, IR) , namely $f_{R,IR}$ and $f_{R_S,I}$ need not be the same functions but can be recovered from each other.

Lemma 4.4. *Let $S \rightarrow R$ be a module-finite map as in Notation 4.1. Let $m_0 = \gcd\{n > 0 \mid S_n \neq 0\}$ and $n_0 = \gcd\{n > 0 \mid R_n \neq 0\}$ then*

$$(l_0)f_{R,IR}(xl_0) = f_{R_S,I}(x) = (\text{rank}_S R) f_{S,I}(x), \quad \text{for all } x \in \mathbb{R}_{\geq 0},$$

and $l_0 = m_0/n_0$ is an integer. In particular $f_{R,IR} \equiv f_{R_S,I}$ if $m_0 = n_0$.

Proof. We first prove that n_0 divides m_0 . Otherwise $m_0 = n_0 l_0 + n_1$, where $0 < n_1 < n_0$. Now if $x \in Q(S)$ is a homogeneous element of degree m_0 and $y \in Q(R)$ is a homogeneous element of degree n_0 then $(x)(y^{-l_0})$ is a homogeneous element of degree n_1 in $Q(R)$. By Lemma 2.5, this contradicts the hypothesis that $\gcd\{n > 0 \mid R_n \neq 0\} = n_0$.

Now, replacing R by $R^{(n_0)}$ and S by $S^{(n_0)}$ we can assume $n_0 = 1$ and $m_0 = l_0$.

By definition

$$f_{R_S,I}(x) = \lim_{n \rightarrow \infty} \frac{1}{q^{d-1}} \left(\ell(R/I^{[q]}R)_{\lfloor xq \rfloor l_0} + \cdots + \ell(R/I^{[q]}R)_{\lfloor xq \rfloor l_0 + l_0 - 1} \right),$$

and

$$f_{R,IR}(xl_0) = \lim_{n \rightarrow \infty} \frac{1}{q^{d-1}} \left(\ell(R/I^{[q]}R)_{\lfloor xl_0 q \rfloor} \right).$$

For all $x \geq 0$ and $q = p^n$, we have $|\lfloor xql_0 \rfloor - \lfloor xq \rfloor l_0| \leq l_0$. Let m_1 be such that $R_m \neq 0$, for $m \geq m_1$. Then for each $0 \leq l_i \leq 2l_0$, we have generically isomorphic graded maps $R(-l_i) \rightarrow R(m_1)$ and $R \rightarrow R(m_1)$ of degree 0. Now by Lemma 4.2, there is a constant C_{l_0} such that

$$\frac{1}{q^{d-1}} |\ell(R/I^{[q]})_{\lfloor xql_0 \rfloor} - \ell(R/I^{[q]})_{\lfloor xql_0 \rfloor + l_i}| \leq C_{l_0}, \quad \text{for all } x, q \text{ and } 0 \leq l_i \leq 2l_0$$

which implies

$$f_{R,I}(xl_0) = \lim_{n \rightarrow \infty} \frac{1}{q^{d-1}} \ell(R/I^{[q]})_{\lfloor xql_0 \rfloor} = \lim_{n \rightarrow \infty} \frac{1}{q^{d-1}} \ell(R/I^{[q]})_{\lfloor xq \rfloor l_0 + l_i}.$$

This proves the lemma. \square

5. Proof of the Main Lemma

Here we prove the technical Main Lemma 3.5. This will complete the proof of the Main Theorem 1.1. Throughout this section we follow the Notation 3.3.

As we mentioned earlier, the sequence

$$0 \rightarrow \mathcal{F}_{m,q} \rightarrow \bigoplus_{i=1}^s \mathcal{O}_{m_q, d_i} \xrightarrow{\varphi_{m,q}} \mathcal{O}_{m+q} \rightarrow 0, \quad (5.1)$$

need not be locally split exact as the sheaf \mathcal{O}_{m+q} is not invertible in general. Instead we consider the following exact sequence of \mathcal{O}_X -modules

$$0 \rightarrow \mathcal{G}_{m,q} \rightarrow \bigoplus_{i=1}^s \mathcal{O}_{(m_q, d_i)} \xrightarrow{\bar{\varphi}_{m,q}} \mathcal{L}^{\lfloor m+q/r \rfloor} \rightarrow 0, \quad (5.2)$$

where $\bar{\varphi}_{m,q}(a_1, \dots, a_s) = \sum_i a_i f_i^q$. In particular the sequence is locally split, but still the sheaf $\mathcal{G}_{m,q}$ may not be locally free since $\mathcal{O}_{(m_q, d_i)}$ may not be so. We note that $\mathcal{G}_{m,q} = \mathcal{F}_{\lfloor m+q/r \rfloor r - q, q}$ because $\mathcal{O}_{\lfloor m+q/r \rfloor r} = \mathcal{L}^{\lfloor m+q/r \rfloor}$. Moreover in case $m+q$ is divisible by r , the sequence (5.2) is same as the sequence (5.1).

In Lemma 5.3, we show that the length of the cohomology of $\mathcal{F}_{m,q}$ differs from the length of the cohomology of $\mathcal{G}_{m,q}$ by a function of order $(m+q)^{d-2}$, where $\dim X = d-1$. Similarly we compare \mathcal{O}_{m+q} with $\mathcal{L}^{\lfloor m+q/r \rfloor}$ and \mathcal{O}_{m_q, d_i} with $\mathcal{O}_{(m_q, d_i)}$. Hence it will be sufficient to prove the Main Lemma 3.5 for $\mathcal{G}_{m,q}$ instead of $\mathcal{F}_{m,q}$.

In the rest of the section we use the following

Terminology We fix a pair (R, I) , the line bundle \mathcal{L} , the integer r along with a choice of generators f_1, \dots, f_s of I as in Notation 3.3.

Here C_L denotes a constant which depends only on L and the above fixed data, where L might stand for a number, a set, a map or a coherent sheaf.

By $\text{supp dim } \mathcal{F}$, we mean the dimension of the support of \mathcal{F} .

Here $\text{supp dim } \mathcal{O}_n = \text{supp dim } X = d-1 \geq 1$.

Some version of the following two lemmas can be found in the literature. Since we will be repeatedly using them we state them and sketch their proofs.

Lemma 5.1. *For a given coherent sheaf \mathcal{N} of \mathcal{O}_X -modules with $\text{supp dim } \mathcal{N} < d-1$, there is a constant $C_{\mathcal{N}}$ such that for $j \geq 0$, $q = p^n$*

(1) *if $m \in \mathbb{Z}$ then*

$$h^j(X, \mathcal{L}^m \otimes \mathcal{N}) \leq C_{\mathcal{N}}(|m|)^{d-2} \quad \text{and} \quad h^j(X, \mathcal{O}_m \otimes \mathcal{N}) \leq C_{\mathcal{N}}(|m|)^{d-2},$$

(2) *In particular, if $m \geq 0$ then*

$$h^j(X, \mathcal{O}_{m_q} \otimes \mathcal{N}) \leq C_{\mathcal{N}}(m+q)^{d-2} \quad \text{and} \quad h^j(X, \mathcal{G}_{m,q} \otimes \mathcal{N}) \leq C_{\mathcal{N}}(m+q)^{d-2},$$

where $m_q \in \{m_q, d_i, (m_q, d_i), m+q \mid 1 \leq i \leq s\}$, where m_q, d_i and (m_q, d_i) are as in Notation 3.3.

Proof. (1) By the Serre vanishing theorem in [6] $h^j(X, \mathcal{N} \otimes \mathcal{L}^m) = 0$, for $j > 0$ and $m \gg 0$. Also, since \mathcal{L} is very ample, for $m \gg 0$, $h^0(X, \mathcal{N} \otimes \mathcal{L}^m)$ is a polynomial of degree equal to $\dim \mathcal{N} < d-1$.

By Lemma 3.2 (a), we have $\mathcal{O}_m = \mathcal{L}^{\lfloor m/r \rfloor} \otimes \mathcal{O}_{r_1}$, where $r_1 = m - \lfloor m/r \rfloor r < r$. Since the support of $\mathcal{O}_m \otimes \mathcal{N}$ = the support of \mathcal{N} , the second inequality of the assertion (1) follows from the fact that $\mathcal{O}_{r_1} \otimes \mathcal{N}$ belongs to the finite set $\{\mathcal{O}_0 \otimes \mathcal{N}, \mathcal{O}_1 \otimes \mathcal{N}, \dots, \mathcal{O}_{r-1} \otimes \mathcal{N}\}$ of coherent sheaves of \mathcal{O}_X -modules.

(2) The first inequality of assertion (2) follows from assertion (1) as $|m+q-d_i q| \leq d_i(m+q)$.

Since the sequence (5.2) is locally split exact, the induced sequence

$$0 \longrightarrow \mathcal{G}_{m,q} \otimes \mathcal{N} \longrightarrow \bigoplus_{i=1}^s \mathcal{O}_{(m_q, d_i)} \otimes \mathcal{N} \xrightarrow{\bar{\varphi}_{m,q} \otimes \mathcal{N}} \mathcal{L}^{\lfloor m+q/r \rfloor} \otimes \mathcal{N} \longrightarrow 0$$

is exact. Now the resulting long exact sequence of cohomologies and assertion (1) give the second inequality of the assertion (2). \square

Lemma 5.2. (1) *Let $S = \{\psi_j : \mathcal{E}_j \longrightarrow \mathcal{F}_j \mid 1 \leq j \leq s\}$ be a fixed finite set of \mathcal{O}_X -linear maps, where \mathcal{E}_j and \mathcal{F}_j are coherent sheaves of \mathcal{O}_X -modules. For $m \in \mathbb{Z}$, let*

$$\psi_j(m) := \text{Id}_{\mathcal{L}^m} \otimes \psi_j : \mathcal{L}^m \otimes \mathcal{E}_j \longrightarrow \mathcal{L}^m \otimes \mathcal{F}_j$$

be the canonically induced maps. Assume that $\text{supp dim } (\text{Ker } \psi_j)$ and $\text{supp dim } (\text{Coker } \psi_j)$ are each $< d - 1$. Then there exists a constant C_S such that

$$\begin{aligned} h^i(X, (\text{Ker } \psi_j)(m)) &\leq (C_S)m^{d-2} \\ h^i(X, (\text{Coker } \psi_j)(m)) &\leq (C_S)m^{d-2}, \quad \text{for all } i \geq 0. \end{aligned}$$

(2) Moreover if $\{0 \rightarrow \mathcal{N}'_m \rightarrow \mathcal{M}'_m \xrightarrow{\phi_m} \mathcal{M}_m \rightarrow \mathcal{N}_m \rightarrow 0\}_{m \in \mathbb{Z}}$ denote a family of exact sequences of \mathcal{O}_X -modules and C_1 and C_2 are constants such that

$$h^i(X, \mathcal{N}'_m) \leq C_1(n_m)^{d-2} \quad \text{and} \quad h^i(X, \mathcal{N}_m) \leq C_2(n_m)^{d-2}, \quad \text{for all } i \geq 0,$$

then

$$|h^0(X, \mathcal{M}'_m) - h^0(X, \mathcal{M}_m)| \leq (C_1 + C_2)(n_m)^{d-2}.$$

Proof. We note that, for any $m \in \mathbb{Z}$,

$$\text{Ker } \psi_j(m) \simeq \mathcal{L}^m \otimes \text{Ker } \psi_j \quad \text{and} \quad \text{Coker } \psi_j(m) \simeq \mathcal{L}^m \otimes \text{Coker } \psi_j,$$

where $\text{Ker } \psi_j$ and $\text{Coker } \psi_j$ are in a fixed family of finite number of coherent sheaves of \mathcal{O}_X -modules. Hence the first assertion follows by Lemma 5.1.

The second assertion follows by splitting the exact sequence into two canonical two short exact sequences

$$\begin{aligned} 0 \rightarrow \mathcal{N}'_m \rightarrow \mathcal{M}'_m \rightarrow \text{Im}(\phi_m) \rightarrow 0, \\ 0 \rightarrow \text{Im}(\phi_m) \rightarrow \mathcal{M}_m \rightarrow \mathcal{N}_m \rightarrow 0. \quad \square \end{aligned}$$

Lemma 5.3. For all m and $q = p^n$,

(1) there is a constant C such that

$$|h^0(X, \mathcal{G}_{m,q}) - h^0(X, \mathcal{F}_{m,q})| \leq C(m+q)^{d-2}.$$

(2) For given integer l_0 , there exists a constant C_{l_0} such that for every $0 \leq l \leq l_0$,

$$\begin{aligned} |h^0(X, \mathcal{G}_{m,q}) - h^0(X, \mathcal{G}_{m+l,q})| &\leq C_{l_0}(m+q)^{d-2}, \\ |h^0(X, \mathcal{O}_{m_q}) - h^0(X, \mathcal{O}_{m_q+l})| &\leq C_{l_0}(m+q)^{d-2}, \end{aligned}$$

where $m_q \in \{m_{q,d_i}, (m_{q,d_i}), m+q \mid 1 \leq i \leq s\}$.

Proof.

Claim (A). For a given $\tilde{r} \in \mathbb{Z}$, if $H^0(X, \mathcal{O}_{\tilde{r}}) \neq \{0\}$ then there exists a constant $C_{\tilde{r}}$ such that, for $i \geq 0$ and $m \geq 0$

$$|h^0(X, \mathcal{F}_{m,q}) - h^0(X, \mathcal{F}_{m+\tilde{r},q})| \leq C_{\tilde{r}}(m+q)^{d-2}.$$

Proof of the claim. An element $h \in H^0(X, \mathcal{O}_{\tilde{r}}) \setminus \{0\}$ gives an injective map $\Phi_m^h : \mathcal{O}_m \rightarrow \mathcal{O}_{m+\tilde{r}}$, for all m . In particular we have the following canonical diagram of sheaves of \mathcal{O}_X -modules:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathcal{F}_{m+\tilde{r},q} & \longrightarrow & \bigoplus_{i=1}^s \mathcal{O}_{m+q+\tilde{r}-qd_i} & \xrightarrow{\varphi_{m+\tilde{r},q}} & \mathcal{O}_{m+\tilde{r}+q} \longrightarrow 0 \\
& & \uparrow \Psi_{m,q}^h & & \uparrow \bigoplus_i \Phi_{m,q}^h & & \uparrow \Phi_{m,q}^h \\
0 & \longrightarrow & \mathcal{F}_{m,q} & \longrightarrow & \bigoplus_{i=1}^s \mathcal{O}_{m+q-qd_i} & \xrightarrow{\varphi_{m,q}} & \mathcal{O}_{m+q} \longrightarrow 0 \\
& & \uparrow & & \uparrow & & \uparrow \\
& & 0 & & 0 & & 0
\end{array}$$

The map $\bigoplus_i \Phi_{m,q}^h : \bigoplus_i \mathcal{O}_{m+q-qd_i} \longrightarrow \bigoplus_i \mathcal{O}_{m+\tilde{r}+q-qd_i}$ is same as

$$\bigoplus_i (\text{Id}_{\mathcal{L}} \otimes \phi_i^h) : \bigoplus_i (\mathcal{L}^{\lfloor m_{q,d_i}/r \rfloor} \otimes \mathcal{E}_i) \longrightarrow \bigoplus_i (\mathcal{L}^{\lfloor m_{q,d_i}/r \rfloor} \otimes \mathcal{F}_i)$$

and where

$$\mathcal{E}_i = \mathcal{O}_{m+q-qd_i - \lfloor m_{q,d_i}/r \rfloor r} \quad \text{and} \quad \mathcal{F}_i = \mathcal{O}_{m+\tilde{r}+q-qd_i - \lfloor m_{q,d_i}/r \rfloor r}$$

and the map $\phi_i^h : \mathcal{E}_i \longrightarrow \mathcal{F}_i$ is the multiplication map by h .

Also the map $\Phi_{m,q}^h : \mathcal{O}_{m+q} \longrightarrow \mathcal{O}_{m+\tilde{r}+q}$ is

$$\text{Id}_{\mathcal{L}} \otimes \phi_0^h : \mathcal{L}^{\lfloor m+q/r \rfloor} \otimes \mathcal{E}_0 \longrightarrow \mathcal{L}^{\lfloor m+q/r \rfloor} \otimes \mathcal{F}_0,$$

where

$$\mathcal{E}_0 = \mathcal{O}_{m+q - \lfloor m+q/r \rfloor r} \quad \text{and} \quad \mathcal{F}_0 = \mathcal{O}_{m+\tilde{r}+q - \lfloor m+q/r \rfloor r}$$

and the map $\phi_0^h : \mathcal{E}_0 \longrightarrow \mathcal{F}_0$ is given by the multiplication by h . Note that $\mathcal{E}_i \in \{\mathcal{O}_0, \dots, \mathcal{O}_{r-1}\}$ and $\mathcal{F}_i \in \{\mathcal{O}_{\tilde{r}}, \dots, \mathcal{O}_{\tilde{r}+r-1}\}$ and $\text{supp dim}(\text{Coker } \phi_i) < d-1$. Hence the claim follows by Lemmas 5.1 and 5.2 and the short exact sequence

$$0 \longrightarrow \text{Coker } \Psi_{m,q}^h \longrightarrow \text{Coker } (\bigoplus_i^s \Phi_{m,q}^h) \longrightarrow \text{Coker } \Phi_{m,q}^h \longrightarrow 0.$$

Assertion (2). It is enough to prove the Assertion (2) for $\mathcal{F}_{m,q}$ instead of $\mathcal{G}_{m,q}$. Since there exists $x_1 \in H^0(X, \mathcal{O}_{2r}) \setminus \{0\}$, the above claim implies that we have a constant C_{2r} such that

$$|h^0(X, \mathcal{F}_{m,q}) - h^0(X, \mathcal{F}_{m+2r,q})| \leq C_{2r}(m+q)^{d-2}, \quad \text{for } i \geq 0.$$

Case 1. If $l \leq r$ then there exists $x_2 \in H^0(X, \mathcal{O}_{2r-l}) \setminus \{0\}$, and therefore we have a constant C_{2r-l} such that, for $i \geq 0$,

$$|h^0(X, \mathcal{F}_{m+l,q}) - h^0(X, \mathcal{F}_{m+2r,q})| \leq C_{2r-l}(m+q)^{d-2}.$$

Case 2. If $l \geq r$ then we can choose $x_3 \in H^0(X, \mathcal{O}_l) \setminus \{0\}$ and therefore get a constant C_l such that

$$|h^0(X, \mathcal{F}_{m,q}) - h^0(X, \mathcal{F}_{m+l,q})| \leq C_l(m+q)^{d-2},$$

for $i \geq 0$. Since, for given $0 \leq l \leq l_0$, there are finitely many choices of such C_l , we get Assertion (2) of the lemma.

Similarly we prove the lemma for \mathcal{O}_{m_q} .

Assertion (1). It follows from the proof of Assertion (2). \square

5.1. The Main Lemma for $\mathcal{G}_{m,q}$

Here we compare $h^0(X, \mathcal{G}_{mp,qp})$ with $h^0(X, \oplus^{p^{d-1}} \mathcal{G}_{m,q})$, and $h^0(X, (\mathcal{O}_{m_qp})$ with $h^0(X, \oplus^{p^{d-1}} \mathcal{O}_{m_q})$. The key point is that, if \mathcal{M} is a sheaf of \mathcal{O}_X -modules then the sequence (5.2) is exact for the functor $(-) \otimes \mathcal{M}$ as it is locally split exact to begin with. Using this fact we construct below a generically isomorphic map $F^* \mathcal{G}_{m,q} \longrightarrow \mathcal{G}_{m',qp}$, provided $|mp - m'|$ is bounded by constant for all m and m' .

Lemma 5.4. *There is a constant C_0 such that for every $m \geq 0$ and $q = p^n$*

$$|h^0(X, (F^* \mathcal{G}_{m,q})) - h^0(X, \mathcal{G}_{mp,qp})| \leq C_0(mp + qp)^{d-2},$$

$$|h^0(X, (F^* \mathcal{O}_{m_q})) - h^0(X, \mathcal{O}_{m_qp})| \leq C_0(mp + qp)^{d-2},$$

where $m_q \in \{m_{q,d_j}, (m_{q,d_j}), m + q \mid 1 \leq j \leq s\}$.

Proof.

Claim. *For given n there is a generically isomorphic map $\psi_n : F^* \mathcal{O}_n \longrightarrow \mathcal{O}_{np}$.*

Proof of the claim. By notation $\mathcal{O}_n = \mathcal{O}_X(nD)$, where D is a Weil \mathbb{Q} -divisor. Moreover the sheaf \mathcal{O}_n on $X \setminus X_{\text{sing}}$ is invertible. Let $D = \sum a_i D_i$, where $a_i \in \mathbb{Q}$ and D_i are prime divisors. Now

$$[npD] = \sum_i [a_i n] p D_i + \sum_i m_i D_i = p[nD] + \sum_i m_i D_i,$$

where $0 \leq m_i \leq p$ are integers. Let $\mathcal{M} = \mathcal{O}_X(p[nD])$ then $\mathcal{M} \xrightarrow{\bar{f}_n} \mathcal{O}_{np}$ is an inclusion such that $\text{supp dim Coker } \bar{f}_n < d - 1$.

On the other hand, we can define the map $\phi_n : F^* \mathcal{O}_n \longrightarrow \mathcal{M}$ as follows: For the Frobenius map $F : X_1 \longrightarrow X$ let $F^* \mathcal{O}_n = F^{-1} \mathcal{O}_n \otimes_{F^{-1} \mathcal{O}_X} \mathcal{O}_{X_1}$ and let $\{D_+(f)\}_f$ denote the affine open cover of X , where $f \in R$ is a homogeneous element of R . Then the map $\phi_n|_{D_+(f)}$ is given by

$$v/f^j \otimes u/f^i \rightarrow (v/f^j)^p \cdot u/f^i, \quad \text{if } v/f^j \in F^{-1} \mathcal{O}_n \text{ and } u/f^i \in \mathcal{O}_{X_1}.$$

Since the map ϕ_n is an isomorphism on the regular locus X_{reg} of X the map $\psi_n = \bar{f}_n \cdot \phi_n$ is generically an isomorphism. This proves the claim.

The claim implies that the map

$$\psi_{m,q} := \oplus_i \psi_{(m_q, d_i)} : \oplus_i F^* \mathcal{O}_{(m_q, d_i)} \longrightarrow \oplus_i \mathcal{O}_{(m_q, d_i)p}$$

is generically an isomorphism. Also the similar map $\phi_{m,q} : F^* \mathcal{L}^{\lfloor m+q/r \rfloor} \longrightarrow \mathcal{L}^{\lfloor m+q/r \rfloor p}$ is an isomorphism such that $\phi_{m,q} \circ F^* \phi_{m,q} = \bar{\phi}_{m',qp} \circ \phi_{m,q}$, where $m' = \lfloor (m+q)/r \rfloor rp - qp$. This gives us a map $f_{m,q} : F^* \mathcal{G}_{m,q} \longrightarrow \mathcal{G}_{m',qp}$ with the following commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{G}_{m',qp} & \longrightarrow & \oplus_{i=1}^s \mathcal{O}_{(m_q, d_i)p} & \xrightarrow{\bar{\varphi}_{m',qp}} & \mathcal{L}^{\lfloor m+q/r \rfloor p} \longrightarrow 0 \\ & & \uparrow f_{m,q} & & \uparrow \psi_{m,q} & & \uparrow \phi_{m,q} \\ 0 & \longrightarrow & F^* \mathcal{G}_{m,q} & \longrightarrow & \oplus_{i=1}^s F^* \mathcal{O}_{(m_q, d_i)} & \xrightarrow{F^* \varphi_{m,q}} & F^* \mathcal{L}^{\lfloor m+q/r \rfloor} \longrightarrow 0, \end{array}$$

where the second horizontal sequence is also exact as the sequence (5.2) is locally split exact. Hence we have the short exact sequence

$$0 \longrightarrow \operatorname{Ker} \psi_{m,q} \longrightarrow F^* \mathcal{G}_{m,q} \longrightarrow \mathcal{G}_{m',qp} \longrightarrow \operatorname{Coker} \psi_{m,q} \longrightarrow 0.$$

Now the rest of the lemma follows from the following

Claim. *There is a constant C_0 such that, for all $m \geq 0$ and $q = p^n$*

$$|h^0(X, \operatorname{Coker} \phi_{m,q})| \leq C_0(m+q)^{d-2} \quad \text{and} \quad |h^0(X, \operatorname{Ker} \phi_{m,q})| \leq C_0(m+q)^{d-2},$$

where the map $\phi_{m,q} : F^* \mathcal{O}_{m_q} \longrightarrow \mathcal{O}_{m_q p}$ is defined as in the above claim.

Proof of the claim. Note that the map $\phi_{m,q} : F^* \mathcal{O}_{m_q} \longrightarrow \mathcal{O}_{m_q p}$ is same as the map

$$\phi_{\lfloor m_q/r \rfloor r} \otimes \psi_{i_j} : F^* \mathcal{L}^{\lfloor m_q/r \rfloor} \otimes F^* \mathcal{O}_{i_j} \longrightarrow \mathcal{L}^{\lfloor m_q/r \rfloor p} \otimes \mathcal{O}_{i_j p},$$

where $\phi_{\lfloor m_q/r \rfloor r} : F^* \mathcal{L}^{\lfloor m_q/r \rfloor} \longrightarrow \mathcal{L}^{\lfloor m_q/r \rfloor p}$ is an isomorphism. Since $m' = mp - r_1 p$, for some $0 \leq r_1 < r$, the maps ϕ_{i_j} belong to the finite set $\{\phi_j : F^* \mathcal{O}_j \longrightarrow \mathcal{O}_{jp} \mid -r \leq j \leq r\}$ of generically isomorphic maps. Now the claim follows by Lemma 5.2. \square

Lemma 5.5. *There is a constant C_1 such that, for every $m \geq 0$ and $q = p^n$,*

$$|p^{d-1} h^0(X, \mathcal{G}_{m,q}) - h^0(X, F^* \mathcal{G}_{m,q})| \leq C_1(mp + qp)^{d-2}$$

$$|p^{d-1} h^0(X, \mathcal{O}_{m_q}) - h^0(X, F^* \mathcal{O}_{m_q})| \leq C_1(mp + qp)^{d-2},$$

where $m_q \in \{m_{q,d_i}, (m_{q,d_i}), m+q \mid 1 \leq i \leq s\}$.

Proof. Recall $X = \operatorname{Proj} R = \operatorname{Proj} R^{(r)}$, where $R^{(r)}$ is a standard graded domain. Therefore by Lemma 2.9 in [19], there is an integer $m_2 \in \mathbb{N}$ (it will be a multiple of r) such that we have a short exact sequence of sheaves of \mathcal{O}_X -modules

$$0 \longrightarrow \oplus^{p^{d-1}} \mathcal{O}_X(-m_2 D) \xrightarrow{\eta} F_* \mathcal{O}_X \longrightarrow Q'' \longrightarrow 0, \quad (5.3)$$

where support dimension Q'' is $< d-1$.

Let $M_1 = \oplus^{p^{d-1}} \mathcal{O}_X(-m_2 D)$ and $M = F_* \mathcal{O}_X$. Then the short exact sequences $0 \longrightarrow M_1 \xrightarrow{\eta} M \longrightarrow Q'' \longrightarrow 0$ and (5.2) give the following commutative diagram of canonical maps

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & \mathcal{G}_{m,q} \otimes Q'' & \longrightarrow & \oplus_{i=1}^s \mathcal{O}_{(m_q, d_i)} \otimes Q'' & \longrightarrow & \mathcal{L}^{\lfloor m+q/r \rfloor} \otimes Q'' \longrightarrow 0 \\ & & \uparrow h_{\mathcal{G}_{m,q}} & & \uparrow h_{\mathcal{L}_{m,q}} & & \uparrow \\ 0 & \longrightarrow & \mathcal{G}_{m,q} \otimes M & \longrightarrow & \oplus_{i=1}^s \mathcal{O}_{(m_q, d_i)} \otimes M & \xrightarrow{\bar{\varphi}_{m,q}} & \mathcal{L}^{\lfloor m+q/r \rfloor} \otimes M \longrightarrow 0 \\ & & \uparrow f_{\mathcal{G}_{m,q}} & & \uparrow f_{\mathcal{L}_{m,q}} & & \uparrow \\ 0 & \longrightarrow & \mathcal{G}_{m,q} \otimes M_1 & \longrightarrow & \oplus_{i=1}^s \mathcal{O}_{(m_q, d_i)} \otimes M_1 & \xrightarrow{\varphi_{m,q}} & \mathcal{L}^{\lfloor m+q/r \rfloor} \otimes M_1 \longrightarrow 0 \\ & & \uparrow & & \uparrow & & \uparrow \\ & & \operatorname{Ker} (f_{\mathcal{G}_{m,q}}) & = & \operatorname{Ker} (f_{\mathcal{L}_{m,q}}) & & 0, \\ & & \uparrow & & \uparrow & & \\ & & 0 & & 0 & & \end{array}$$

Since $\mathcal{O}_{(m_q, d_i)} = \mathcal{L}^{\lfloor m_q, d_i/r \rfloor} \otimes \mathcal{E}_i$, where $\mathcal{E}_i = \mathcal{O}_{(m_q, d_i) - \lfloor m_q, d_i/r \rfloor r} \in \{\mathcal{O}_0, \mathcal{O}_1, \dots, \mathcal{O}_{2r}\}$ the map $f_{\mathcal{L}_{m,q}}$ is same as the map

$$\oplus_{i=1}^s \text{Id}_{\mathcal{L}} \otimes \psi_i : \oplus_{i=1}^s \mathcal{L}^{\lfloor m_q, d_i/r \rfloor} \otimes \mathcal{E}_i \otimes M_1 \longrightarrow \oplus_{i=1}^s \mathcal{L}^{\lfloor m_q, d_i/r \rfloor} \otimes \mathcal{E}_i \otimes M,$$

where the map $\psi_i = \text{Id}_{\mathcal{E}_i} \otimes \eta : \mathcal{E}_i \otimes M_1 \longrightarrow \mathcal{E}_i \otimes M$ is generically isomorphic. Therefore

$$\text{supp dim of } \left(\text{Ker } (f_{\mathcal{G}_{m,q}}) = \text{Ker } (f_{\mathcal{L}_{m,q}}) = \oplus_{i=1}^s (\mathcal{L}^{\lfloor m_q, d_i/r \rfloor} \otimes \text{Ker } \psi_i) \right) < d - 1.$$

Now the long exact sequence

$$0 \longrightarrow \text{Ker } (f_{\mathcal{G}_{m,q}}) \longrightarrow \mathcal{G}_{m,q} \otimes M_1 \longrightarrow \mathcal{G}_{m,q} \otimes M \longrightarrow \mathcal{G}_{m,q} \otimes Q'' \longrightarrow 0$$

gives (Lemma 5.2 (2))

$$|h^0(X, \mathcal{G}_{m,q} \otimes M_1) - h^0(\mathcal{G}_{m,q} \otimes F_* \mathcal{O}_X)| = C_\eta(m+q)^{d-2}, \quad (5.4)$$

for some constant C_η . On the other hand, as F is a finite map, for any coherent sheaf M of \mathcal{O}_X -modules the projection formula $F_*(F^* \mathcal{G}_{m,q} \otimes M) = \mathcal{G}_{m,q} \otimes F_* M$ holds.

This implies

$$h^0(X, \mathcal{G}_{m,q} \otimes F_* \mathcal{O}_X) = h^0(X, F_*(F^* \mathcal{G}_{m,q})) = h^i(X, F^* \mathcal{G}_{m,q}) \quad (5.5)$$

Now the lemma follows by (5.4) and (5.5).

The second assertion follows by the same line of arguments. \square

Proof of Main Lemma 3.5. It follows from Lemma 5.4, Lemma 5.5 and Lemma 5.3 (1). \square

6. Some properties of the HK density functions for graded rings

6.1. The multiplicative properties of HK density functions

Let (R, I) and (S, J) be two pairs, where $R = \oplus_{n \geq 0} R_n$ and $S = \oplus_{n \geq 0} S_n$ are graded domains of dimension $d_1 \geq 2$ and $d_2 \geq 2$ respectively, over a perfect field k , and $I \subset R$ and $J \subset S$ are graded ideals of finite colengths.

Moreover let $F_R : [0, \infty) \longrightarrow [0, \infty)$ and $F_S : [0, \infty) \longrightarrow [0, \infty)$ be the Hilbert-Samuel density functions given by

$$F_R(x) = e_0(R)x^{d_1-1}/(d_1-1)!, \quad \text{and} \quad F_S(x) = e_0(S)x^{d_2-1}/(d_2-1)!,$$

where, for a graded ring R , $e_0(R)$ is the Hilbert-Samuel multiplicity of R with respect to its graded maximal ideal.

In [19], we had proved that the HK density function is multiplicative for Segre products of standard graded rings. In Theorem 6.1 and Corollary 6.3, we show that this property extends to graded domains.

Theorem 6.1. *Let $\gcd \{m > 0 \mid R_m \neq 0\} = 1$ and $\gcd \{m > 0 \mid S_m \neq 0\} = 1$, where (R, I) and (S, J) are the pairs given as above. Then the HK density function of the pair $(R \# S, I \# J)$, where $R \# S = \oplus_{n \geq 0} R_n \otimes_k S_n$ is the Segre product of R and S , is given by*

$$F_{R \# S} - f_{R \# S, I \# J} = [F_R - f_{R, I}] [F_S - f_{S, J}]$$

and also

$$\begin{aligned} e_{HK}(R\#S, I\#J) &= \frac{e_0(R)}{(d_1-1)!} \int_0^\infty x^{d_1-1} f_{S,J}(x) dx + \frac{e_0(S)}{(d_2-1)!} \int_0^\infty x^{d_2-1} f_{R,I}(x) dx \\ &\quad - \int_0^\infty f_{R,I}(x) f_{S,J}(x) dx. \end{aligned}$$

Proof. Note that $R\#S$ is a graded integral domain with $\gcd \{n > 0 \mid (R\#S)_n \neq 0\} = 1$. Therefore

$$\begin{aligned} (q^{d_1+d_2-2}) f_n(R\#S, I\#J)(m/q) &= \ell(R\#S/(I\#J)^{[q]})_m \\ &= \ell(R_m) \ell(S_m) - \left[\ell(R_m) - \ell(R/I^{[q]})_m \right] \left[\ell(S_m) - \ell(S/J^{[q]})_m \right]. \end{aligned}$$

Hence

$$f_n(R\#S, I\#J)(x) = f_n(S, J) \frac{\ell(R)_{\lfloor xq \rfloor}}{q^{d_1-1}} + f_n(R, I) \frac{\ell(S)_{\lfloor xq \rfloor}}{q^{d_2-1}} - f_n(R, I) f_n(S, J).$$

Since $\{f_n(R\#S, I\#J)\}_n$, $\{f_n(R, I)\}_n$ and $\{f_n(S, J)\}_n$ are uniformly convergent sequences with bounded supports, taking limit as $n \rightarrow \infty$ we get,

$$f_{R\#S, I\#S}(x) = F_R(x) f_{S,J}(x) + F_S(x) f_{R,I}(x) - f_{R,I}(x) f_{S,J}(x) \quad \text{for all } x \geq 0.$$

The rest of the proof follows as $F_{R\#S}(x) = F_R(x) F_S(x)$. \square

Notation 6.2. For a graded domain $R = \bigoplus_{n \geq 0} R_n$ with a graded ideal $I = \bigoplus_{n \geq 1} I_n$, we denote $R^{(m)} = \bigoplus_n R_{nm}$ where the degree n component is R_{nm} and $I^{(m)} = I \cap R^{(m)} = \bigoplus_n I_{nm}$.

Corollary 6.3. Let $\gcd \{m > 0 \mid R_m \neq 0\} = n_1$ and $\gcd \{m > 0 \mid S_m \neq 0\} = n_2$. Let $l = \text{lcm}(n_1, n_2)$ and let $f_{R^{(l)}, I^{(l)}}$ denote the HK density function of the pair $(R^{(l)}, I^{(l)})$. Then

$$F_{R\#S} - f_{R\#S, I\#J} = [F_{R^{(l)}} - f_{R^{(l)}, I^{(l)}}] [F_{S^{(l)}} - f_{S^{(l)}, J^{(l)}}]$$

and

$$\begin{aligned} e_{HK}(R\#S, I\#J) &= \frac{e_0(R^{(l)})}{(d_1-1)!} \int_0^\infty x^{d_1-1} f_{S^{(l)}, J^{(l)}}(x) dx + \frac{e_0(S^{(l)})}{(d_2-1)!} \int_0^\infty x^{d_2-1} f_{R^{(l)}, I^{(l)}}(x) dx \\ &\quad - \int_0^\infty f_{R^{(l)}, I^{(l)}}(x) f_{S^{(l)}, J^{(l)}}(x) dx. \end{aligned}$$

Proof. Since $R\#S = R^{(l)}\#S^{(l)}$ and $I\#J = I^{(l)}\#J^{(l)}$ and $\gcd \{m > 0 \mid R_m^{(l)} \neq 0\} = \gcd \{m > 0 \mid S_m^{(l)} \neq 0\} = 1$, the corollary follows from the above theorem. \square

6.2. F -threshold and support of $f_{R,I}$

We recall that for a pair of ideals I and J , the F -threshold of J with respect to I is defined as

$$c^I(J) = \lim_{q \rightarrow \infty} \frac{\min \{r \mid J^{r+1} \subseteq I^{[q]}\}}{q}.$$

This notion was first introduced by Mustař-Takagi-Watanabe [15] for regular rings, and in full generality by Huneke-Mustař-Takagi-Watanabe [10] and Betancourt-Pérez-Stefani [4].

In this section we consider an invariant attached to the HK density function $f_{R,I}$, namely $\alpha(R, I) = \sup \{x \mid f_{R,I}(x) \neq 0\}$. When R is a standard graded ring and F -regular on the punctured spectrum $\text{Spec } R \setminus \mathfrak{m}$, we showed in [20] (Corollary 3.10) that $\alpha(R, I) = c^I(\mathfrak{m})$.

Here we generalize this result when R is an \mathbb{N} -graded domain.

Theorem 6.4. *Let (R, I) be a graded pair of dimension $d \geq 2$. If R is F -regular on the punctured spectrum then there exists r such that*

$$\alpha(R, I) = r \cdot c^I(\mathfrak{a}) \quad \text{where} \quad \mathfrak{a} := \bigoplus_{n \geq 0} R_{r+n}.$$

In fact if R as an R_0 -algebra is generated by μ homogeneous generators of degrees m_1, \dots, m_μ and $l_1 = \text{lcm}\{m_1, \dots, m_\mu\}$ then we can choose any $r \in l_1\mu\mathbb{N}$.

Proof. Let $n_0 = \gcd\{n > 0 \mid R_n \neq 0\}$. Since $\alpha(R, I) = \alpha(R^{(n_0)}, I)$ and $c^I(\mathfrak{a})$ is independent of the grading of R , we can assume without loss of generality that $n_0 = 1$.

We fix an integer $r \in l_1\mu\mathbb{N}$. Then, by Lemma 3.2, the ring $R^{(r)}$ is a standard graded ring. By Lemma 4.2, for a given integer $n_1 \in \mathbb{N}$ we have $f_{R,I} = f_{R(n_1)R,I}$, where $R(n_1)$ is an R -module with n^{th} degree component equal to R_{n+n_1} .

Since, for all $x > 0$ and $q = p^n$, we have $0 \leq \lfloor xrq \rfloor - \lfloor xq \rfloor r < r$,

$$f_{R,I}(rx) = \lim_{n \rightarrow \infty} (1/q^{d-1})\ell(R/I^{[q]})_{\lfloor xrq \rfloor} = \lim_{n \rightarrow \infty} (1/q^{d-1})\ell(R/I^{[q]})_{\lfloor xq \rfloor r}.$$

Let $c = c^I(\mathfrak{a})$. Then, for $x > c$, we have $\mathfrak{a}^{\lfloor xq \rfloor} \subseteq I^{[q]}$. Therefore $R_{r\lfloor xq \rfloor} = R_r^{\lfloor xq \rfloor} \subseteq I^{[q]}$. This implies $\ell(R/I^{[q]})_{\lfloor xq \rfloor r} = 0$ and hence $f_{R,I}(rx) = 0$, for all $x \geq c$. Therefore $\alpha(R, I) \leq rc$.

To prove the converse, it is enough to show that if $\beta \in \mathbb{N}[1/p]$ such that $\beta < c$ then $r\beta < \alpha(R, I)$. Now the F -regularity property of R on the punctured spectrum implies that there is n_0 such that $\mathfrak{a}^{n_0} \subseteq \tau(R)$ the test ideal of R . We choose $\epsilon > 0$ such that $\beta + 2\epsilon < c$. Let q_0 such that $\beta q \in \mathbb{N}$ and $\epsilon q \geq n_0$, for $q \geq q_0$. We can further choose $q_1 \geq q_0$ such that $\mathfrak{a}^{\beta q_1 + \lfloor \epsilon q_1 \rfloor} \not\subseteq I^{[q_1]}$ and therefore $\mathfrak{a}^{\beta q_1} \not\subseteq I^{[q_1]*}$.

Let $z \in \mathfrak{a}^{\beta q_1} \setminus I^{[q_1]*}$ and let $J = (z, I^{[q_1]})$. Now $\deg z^q \geq (r\beta q_1)q$ implies $(I^{[qq_1]})_{\lfloor xq \rfloor} = (z^q, I^{[qq_1]})_{\lfloor xq \rfloor}$, for $x < r\beta q_1$. Hence

$$\int_0^{r\beta q_1} f_{R,I^{[q_1]}}(x)dx = \int_0^{r\beta q_1} f_{R,J}(x)dx.$$

But, by Hochster-Huneke [8] and Aberbach [1], $e_{HK}(R, I^{[q_1]}) > e_{HK}(R, J)$. Therefore $f_{R,I^{[q_1]}}(x) \neq 0$, for some $x > r\beta q_1$. In particular $\alpha(R, I^{[q_1]}) > r\beta q_1$ which means $\alpha(R, I) > r\beta$. This proves the theorem. \square

Remark 6.5. (1) If R is a graded domain such that $X = \text{Proj } R$ is a smooth variety then on the punctured spectrum the ring R is smooth and hence F -regular by Theorem 5.10 of Hochster-Huneke [9].

(2) If R is a standard graded ring then $\mathfrak{a} = \bigoplus_{n \geq 0} R_{r+n} = \mathfrak{m}^r$ which implies $c^I(\mathfrak{a}) = c^I(\mathfrak{m})/r$. Therefore Theorem 6.4 is a generalization of Corollary 3.10 of [20].

7. HK density functions in some special cases

The HK density function for a graded pair (R, I) has a neat formula when (1) either I is generated by a system of parameters, (2) the ring R is of dimension two or (3) there is finite degree preserving map of graded rings $R \rightarrow R'$ such that in the extension ring the ideal I has finite projective dimension.

The first and second cases are a generalization of the Lemma 3.2 of [20] and Example 3.3 of [19], respectively. Using (3) we prove that the HK density function is a piecewise polynomial for ring of invariants where the underlying group is a finite group acting linearly.

7.1. The HK density function $f_{R,I}$ when $\mu(I) = \dim R$

Definition 7.1. Given nonnegative integers n_1, \dots, n_d , consider a d -parallelotope $P = [0, n_1] \times \dots \times [0, n_d]$. We define a volume function

$$\text{Vol}_{d-1}(n_1, \dots, n_d) : [0, \infty) \longrightarrow [0, \infty) \quad \text{given by} \quad x \rightarrow \text{Vol}_{d-1}(P \cap H_x),$$

where $H_x = \{(y_1, \dots, y_d) \in \mathbb{R}^d \mid \sum_i y_i = x\}$ is a $d-1$ -dimensional hyperplane in \mathbb{R}^d and $\text{Vol}_{d-1}(P \cap H_x)$ is the relative volume of the $d-1$ dimensional convex set $P \cap H_x$.

Lemma 7.2. If R is a graded domain of dimension $d \geq 2$ with $\gcd\{n > 0 \mid R_n \neq 0\} = 1$ and I is generated by homogeneous system of parameters of degrees n_1, \dots, n_d then

$$f_{R,I}(x) = \text{Vol}_{d-1}(n_1, \dots, n_d)(x) \left[\lim_{t \rightarrow 1} (1-t)^d P(R, t) \right],$$

where $P(R, t) = \sum_{n \geq 0} \ell(R_n) t^n$ denotes the Poincare series of R .

Proof. Let f_1, \dots, f_d be a set of homogeneous generators of I with degree n_1, \dots, n_d respectively. Let $S = k[f_1, \dots, f_d]$ with $m_0 = \gcd\{n > 0 \mid S_n \neq 0\} = \gcd\{n_1, \dots, n_d\}$.

Let $J = (f_1, \dots, f_d) \subset S$. Then

$$m_0 f_{R,I}(x m_0) = (\text{rank}_S R) f_{S,J}(x) = e_0(R, I) f_{S,J}(x).$$

On the other hand if we consider S as a subring of $\tilde{R} = k[X_1, \dots, X_d]$, sending $f_i \rightarrow X_i^{n_i}$ then

$$f_{S,J}(x) = \frac{m_0}{\text{rank}_S \tilde{R}} f_{\tilde{R}, J \tilde{R}}(x m_0) = \frac{m_0}{n_1 \dots n_d} \text{Vol}_{d-1}(n_1, \dots, n_d)(x m_0),$$

where the last equality follows, by Lemma 2.2 of [20]. Hence

$$f_{R,I}(x) = \frac{e_0(R, I)}{n_1 \dots n_d} \text{Vol}_{d-1}(n_1, \dots, n_d)(x).$$

But Proposition 2.10 of Huneke-Takagi-Watanabe [11] gives

$$e_0(R, I) = (n_1 \dots n_d) \left[\lim_{t \rightarrow 1} (1-t)^d P(R, t) \right],$$

which proves the lemma. \square

Remark 7.3. If R is a normal graded domain with $R = R(X, D)$, where the projective variety X and the Weil \mathbb{Q} -divisor D in X are as in Notation 3.3, then by Proposition 2.1 of Tomari [18]

$$\begin{aligned} \lim_{t \rightarrow 1} (1-t)^d P(R, t) &= \deg D \quad \text{for } d = 2 \\ &= D^{d-1} \quad \text{for } d \geq 3. \end{aligned}$$

7.2. The HK density function $f_{R,I}$ when $\dim R = 2$

If (R, I) is a two dimensional graded pair then its HK density function $f_{R,I}$ is an explicit piecewise linear polynomial with rational coefficients and rational break points (the proof follows from the same arguments as in [20]):

Let f_1, \dots, f_s be a set of homogeneous generators of I of degrees d_1, \dots, d_s . Let \tilde{S} denote the normalization of $R^{(n_0)} = \bigoplus_{n \geq 0} R_{nn_0}$, where $\gcd\{m > 0 \mid R_m \neq 0\} = n_0$. Let $X = \text{Proj}(\tilde{S})$. Then for the \mathbb{Q} -Weil divisor D (which is Cartier in this case) corresponding to the normal ring \tilde{S} (as in Theorem 3.1) the sheaf $\mathcal{O}_n = \mathcal{O}_X(D)$ is invertible. Hence the sequence (5.1) is

$$0 \longrightarrow F^{n*}V \otimes \mathcal{O}_m \longrightarrow \bigoplus_i \mathcal{O}_{m+q-qd_i} \xrightarrow{\phi_{m,q}} \mathcal{O}_{m+q} \longrightarrow 0,$$

where

$$0 \longrightarrow V \longrightarrow \bigoplus_i \mathcal{O}_{1-d_i} \xrightarrow{\phi} \mathcal{O}_1 \longrightarrow 0$$

and where $\phi(x_1, \dots, x_s) = \sum x_i f_i$. This gives

$$f_{R,I}(x) = f_{V, \mathcal{O}_1}(x) - f_{\bigoplus_i \mathcal{O}_{1-d_i}, \mathcal{O}_1}(x), \quad \text{for } x \geq 1,$$

where, for a vector bundle E on X with strong HN data $(\{a_1, \dots, a_{l+1}\}, \{r_1, \dots, r_{l+1}\})$ and $d = \deg \mathcal{O}_1$, the function f_{E, \mathcal{O}_1} denotes the HK density function of E with respect to \mathcal{O}_1 and is given by

$$f_{E, \mathcal{O}_1}(x) = \begin{cases} - \left[\sum_{i=1}^{l+1} a_i r_i + d(x-1)r_i \right] & \text{if } x < 1 - a_1/d \\ - \left[\sum_{k=i+1}^{l+1} a_k r_k + d(x-1)r_k \right] & \text{if } 1 - a_i/d \leq x < 1 - a_{i+1}/d. \end{cases}$$

7.3. The HK density function $f_{R,I}$ when $\text{proj dim}_R I < \infty$

We recall that the Hilbert-Kunz (HK) function of (R, I) is the function given by $HK(R, I) : \mathbb{N} \rightarrow \mathbb{N}$, where $n \rightarrow \ell(R/I^{[p^n]})$. Though in general, this function is not a polynomial function, it is a polynomial function and the coefficients have a nice geometric description, given by Kurano [12], provided $\text{proj dim}_R I < \infty$.

Here we prove that the HK density function too has a nice description for such graded pairs.

Proposition 7.4. *Let (R, I) be a graded pair such that $\text{proj dim}_R(R/I) < \infty$ then the HK density function $f_{R,I}$ is a piecewise polynomial function of degree $d-1$, where $f_{R,I}$ (and hence $e_{HK}(R, I)$) is given in terms of the graded Betti numbers of the minimal graded R -resolution of R/I .*

Proof. Consider the minimal graded resolution of R/I over the graded ring R

$$0 \longrightarrow \bigoplus_{j \in \mathbb{Z}} R(-j)^{\beta_{d,j}} \longrightarrow \dots \longrightarrow \bigoplus_{j \in \mathbb{Z}} R(-j)^{\beta_{1,j}} \longrightarrow R \longrightarrow R/I \longrightarrow 0.$$

Since the Frobenius functor is exact on the category of modules of finite type and finite projective dimension (a corollary of the acyclicity lemma by Peskine-Szpiro [16]), we have a long exact sequence

$$0 \longrightarrow \bigoplus_{j \in \mathbb{Z}} R(-qj)^{\beta_{d,j}} \longrightarrow \dots \longrightarrow \bigoplus_{j \in \mathbb{Z}} R(-qj)^{\beta_{1,j}} \longrightarrow R \longrightarrow R/I^{[q]} \longrightarrow 0.$$

Let $\tilde{e}_0 = e_0(R, \mathbf{m})/(d-1)!$ and let

$$\mathbb{B}(j) = \beta_{0j} - \beta_{1j} + \beta_{2j} + \dots + (-1)^d \beta_{dj}.$$

Note that $\beta_{00} = 1$ and $\beta_{0,j} = 0$ for $j \neq 0$. For $j < 0$, $\mathbb{B}(j) = 0$. If l be the largest integer such that $\beta_{il} \neq 0$ for some i . Then

$$\ell(R/I^{[q]})_m = \ell(R_m) + \mathbb{B}(1)\ell(R_{m-q}) + \mathbb{B}(2)\ell(R_{m-2q}) + \cdots + \mathbb{B}(l)\ell(R_{m-lq})$$

and therefore

$$f_{R,I}(x) = \begin{cases} \tilde{e}_0 [x^{d-1}] & 0 \leq x \leq 1 \\ \tilde{e}_0 [x^{d-1} + \mathbb{B}(1)(x-1)^{d-1}] & 1 \leq x \leq 2 \\ \vdots & \\ \tilde{e}_0 [x^{d-1} + \mathbb{B}(1)(x-1)^{d-1} + \cdots + \mathbb{B}(i)(x-i)^{d-1}] & i \leq x \leq (i+1) \\ \tilde{e}_0 [x^{d-1} + \mathbb{B}(1)(x-1)^{d-1} + \cdots + \mathbb{B}(l-1)(x-l+1)^{d-1}] & l-1 \leq x \leq l \\ 0 & l \leq x. \end{cases}$$

Note that $f_{R,I}$ is a compactly supported function which implies that the polynomial

$$x^{d-1} + \mathbb{B}(1)(x-1)^{d-1} + \cdots + \mathbb{B}(l)(x-l)^{d-1} = 0.$$

Hence $\text{supp}(f_{R,I}) \subseteq [0, l]$.

Moreover

$$e_{HK}(R, I) = \frac{e_0}{d!} [\mathbb{B}(0)l^d + \mathbb{B}(1)(l-1)^d + \cdots + \mathbb{B}(i)(l-i)^d + \cdots + \mathbb{B}(l-1)] \quad \square$$

Proof of Corollary 1.2. Suppose for a given (S, I) there is a finite degree preserving map $S \rightarrow R$ of \mathbb{N} -graded rings such that $\text{proj dim}_R I < \infty$. Let $m_0 = \gcd\{n > 0 \mid S_n \neq 0\}$ and $n_0 = \gcd\{n > 0 \mid R_n \neq 0\}$ and $l_0 = m_0/n_0$. Further let $f_{R,S,I}$ denote the HK density function of R with respect to I as the S -module and let $f_{R,IR}$ denote the HK density function of R with respect to IR as the module over itself. Then, by Theorem 1.1 and Lemma 4.4, for $x \geq 0$,

$$f_{S,I}(x) = f_{R,S,I}(x)/(\text{rank}_S R) = f_{R,IR}(xl_0)(l_0/\text{rank}_S R)$$

and $e_{HK}(S, I) = e_{HK}(R, IR)/(\text{rank}_S R)$. Now the corollary follows from Proposition 7.4. \square

Remark 7.5. If (R, I) is a graded pair such that R/I has the finite pure resolution

$$0 \rightarrow \oplus^{\beta_d} R(-j_d) \rightarrow \cdots \rightarrow \oplus^{\beta_2} R(-j_2) \rightarrow \oplus^{\beta_1} R(-j_1) \rightarrow R \rightarrow R/I \rightarrow 0$$

then $j_1 < j_2 < \cdots < j_d$ and

$$\begin{aligned} \mathbb{B}(1) &= \cdots = \mathbb{B}(j_1 - 1) = 0 \quad \text{and} \quad \mathbb{B}(j_1) = -\beta_1 \\ \mathbb{B}(j_{n-1} + 1) &= \cdots = \mathbb{B}(j_n - 1) = 0 \quad \text{and} \quad \mathbb{B}(j_n) = (-1)^n \beta_n. \end{aligned}$$

Hence

$$f_{R,I}(x) = \begin{cases} \tilde{e}_0 [x^{d-1}] & 0 \leq x \leq j_1 \\ \tilde{e}_0 [x^{d-1} - \beta_1(x-j_1)^{d-1}] & j_1 \leq x \leq j_2 \\ \tilde{e}_0 [x^{d-1} - \beta_1(x-j_1)^{d-1} + \cdots + (-1)^{d-1} \beta_{d-1}(x-j_{d-1})^{d-1}] & j_{d-1} \leq x \leq j_d \\ 0 & j_d \leq x. \end{cases}$$

Here the maximum support of $f_{R,I} = \alpha(R, I) = j_d$, as $\beta_d \neq 0$.

8. Some concrete examples

We recall the Hilbert-Burch theorem (see the proof of Theorem 1.4.16 in Bruns-Herzog [3]).

Theorem 8.1. *Let $\psi : R^n \rightarrow R^{n+1}$ be an R -linear map, where R is a Noetherian ring. Let $I = I_n(\psi)$ be the ideal generated by $n+1$ elements consisting of n minors of the $(n+1) \times n$ matrix U given by ψ . Let δ_i be the minor of U where the i^{th} row is deleted.*

Then grade $I_n(\psi) \geq 2$ implies that the sequence

$$0 \rightarrow R^n \xrightarrow{\psi} R^{n+1} \xrightarrow{\phi} I \rightarrow 0$$

is exact, where $\phi(e_i) = (-1)^i \delta_i$ and ψ is given by the matrix U .

In the following examples we compute the HK density function $f_{S,I}$, where $S = R^G$ is the ring of invariants in $R = k[x_1, x_2]$ and $G \in \{A_n, D_n, E_6, E_7, E_8\}$ and I is the graded maximal ideal of S . This in particular recovers the computations of $e_H(R^G, I)$ given in Theorem 5.1 by Watanabe-Yoshida [22].

By the proof of Corollary 1.2, it is enough to construct the minimal graded resolution of IR as an R -module. To do this we adopt the following common strategy for all such R^G .

Recall that for such group G we have $R^G = k[h_1, h_2, h_3] \subset k[x_1, x_2]$, where h_1, h_2, h_3 are explicit homogeneous polynomials in x_1, x_2 (for example see Chap X, page 225 of [13] by Miller-Blichfeldt-Dickson).

Now to construct a minimal R -graded resolution of IR it is enough to find an element $U \in M_{3 \times 2}(R)$ with homogeneous elements as entries such that $IR = (h_1, h_2, h_3) = (\delta_1, \delta_2, \delta_3)$, where δ_i is a 2-minor of U with i^{th} deleted. This is because IR being an \mathfrak{m} -primary ideal of $k[x_1, x_2]$ has grade $= 2$, hence by Theorem 8.1, we have a short exact sequence of graded R -linear maps

$$0 \rightarrow R(-l_1) \oplus R(-l_2) \xrightarrow{\psi} \bigoplus_{i=1}^3 R(-\deg \delta_i) \xrightarrow{\phi} IR \rightarrow 0.$$

In particular this a minimal graded R -resolution for IR .

Example 8.2. Let $G = A_n$ then $|G| = n \geq 2$ and $\text{char } k = p \geq 2$ and $(p, n) = 1$.

$$R^G = k[h_1, h_2, h_3] \cong \frac{k[x_1, x_2, x_3]}{(x_1^n + x_2 x_3)},$$

where $h_1 = x_1 x_2$, $h_2 = x_1^n$ and $h_3 = x_2^n$. The map ψ is given by the matrix

$$\begin{bmatrix} x_1^{n-1} & -x_2 & 0 \\ x_2^{n-1} & 0 & -x_1 \end{bmatrix}.$$

Then the sequence

$$0 \rightarrow R(-n-1) \oplus R(-n-1) \xrightarrow{\psi} R(-2) \oplus R(-n) \oplus R(-n) \rightarrow IR \xrightarrow{\phi} 0$$

is the minimal resolution for IR as $I_2(\psi) = IR$. Here $\mathbb{B}(2) = -1$, $\mathbb{B}(n) = -2$ and $\mathbb{B}(n+1) = 2$. If n is even then the HK density function $f_{R^G, I}$ is given by

$$f_{R^G, I}(x) = \begin{cases} 4x/(n+1) & \text{if } 0 \leq x \leq 1 \\ 4/(n+1) & \text{if } 1 \leq x \leq n/2 \\ 2(2-4x+2n)/(n+1) & \text{if } n/2 \leq x \leq (n+1)/2 \end{cases}$$

If n is odd then the HK density function $f_{R^G, I}$ is given by

$$f_{R^G, I}(x) = \begin{cases} x/(n+1) & \text{if } 0 \leq x \leq 2 \\ 2/(n+1) & \text{if } 2 \leq x \leq n \\ (2-2x+2n)/(n+1) & \text{if } n \leq x \leq n+1 \end{cases}$$

Example 8.3. Let $G = D_n$ the dihedral group then $|G| = 4n$ and $\text{char } k = p \geq 3$ and $(p, n) = 1$.

$$R^G = k[h_1, h_2, h_3] = \frac{k[x_1, x_2, x_3]}{(x_3^2 + x_1x_2^2 + x_1^{n+1})},$$

where

$$h_1 = -2x_1^2x_2^2, \quad h_2 = x_1^{2n} + (-1)^n x_2^{2n}, \quad h_3 = x_1x_2(x_1^{2n} - (-1)^n x_2^{2n}).$$

We assume n is even. Let map ψ is given by the matrix

$$\begin{bmatrix} -2x_1^{2n-1} & x_1x_2^2 & x_2 \\ -2x_2^{2n-1} & -x_1^2x_2 & x_1 \end{bmatrix}$$

Then the sequence

$$0 \longrightarrow R(-2n-3) \oplus R(-2n-3) \xrightarrow{\psi} R(-4) \oplus R(-2n) \oplus R(-2n-2) \xrightarrow{\phi} IR \longrightarrow 0$$

is the minimal resolution for IR as $I_2(\psi) = IR$.

Here $\mathbb{B}(4) = -1$, $\mathbb{B}(2n) = -1$ and $\mathbb{B}(2n+2) = -1$. If n is even then the HK density function $f_{R^G, I}$ is given by

$$f_{R^G, I}(x) = \begin{cases} x/n-2 & \text{if } 0 \leq x \leq 2 \\ 2/(n-2) & \text{if } 2 \leq x \leq n \\ (n+2-x)/(n-2) & \text{if } n \leq x \leq n+1 \\ (2n+3-2x)/(n-2) & \text{if } n+1 \leq x \leq n+3/2 \end{cases}$$

and hence $e_{HK}(R^G, I) = 2 - 1/4n$.

Example 8.4. Let $G = E_6$ the tetrahedral group then $|G| = 24$ and $\text{char } k = p \geq 5$.

$$R^G = k[h_1, h_2, h_3] = \frac{k[x_1, x_2, x_3]}{(6ax_1^2 - x_2^3 + x_3^3)}, \quad \text{where } a = 2\sqrt{-3}$$

$$h_1 = x_1^5x_2 - x_1x_2^5, \quad h_2 = x_1^4 + ax_1^2x_2^2 + x_2^4, \quad h_3 = x_1^4 - ax_1^2x_2^2 + x_2^4.$$

Let ψ be given by the matrix

$$\begin{bmatrix} x_1 & -(a/2)x_1^2x_2 - x_2^3 & (a/2)x_1^2x_2 - x_2^3 \\ x_2 & x_1^3 + (a/2)x_1x_2^2 & x_1^3 - (a/2)x_1x_2^2 \end{bmatrix}$$

Then $I_2(\psi) = (ah_1, h_2, h_3)R$. If $a \neq 0$ in k then the canonical sequence

$$0 \longrightarrow R(-7) \oplus R(-7) \xrightarrow{\psi} R(-6) \oplus R(-4) \oplus R(-4) \xrightarrow{\phi} IR \longrightarrow 0$$

is the minimal R -resolution of IR .

Here $\mathbb{B}(4) = -2$, $\mathbb{B}(6) = -1$ and $\mathbb{B}(7) = 2$ and the HK density function $f_{R^G, I}$ is given by

$$f_{R^G, I}(x) = \begin{cases} x/6 & \text{if } 0 \leq x \leq 2 \\ (4-x)/6 & \text{if } 2 \leq x \leq 3 \\ (7-2x)/6 & \text{if } 3 \leq x \leq 7/2 \\ 0 & \text{otherwise} \end{cases}$$

Example 8.5. Let $G = E_7$ octahedral group then $|G| = 24$ and $\text{char } k \geq 5$ and

$$R^G = k[h_1, h_2, h_3] = \frac{k[x_1, x_2, x_3]}{(108x_1^4 - x_2^3 + x_3^2)}, \quad \text{where} \\ h_1 = x_1^5 x_2 - x_1 x_2^5, \quad h_2 = x_1^8 + 14x_1^4 x_2^4 + x_2^8, \quad h_3 = x_1^{12} - 33(x_1^8 x_2^4) - 33(x_1^4 x_2^8) + x_2^{12}.$$

Let ψ be given by the matrix

$$\begin{bmatrix} -7x_1^4 x_2^3 - x_2^7 & x_1^5 & x_1 \\ 7x_1^3 x_2^4 + x_1^7 & x_2^5 & x_2 \end{bmatrix}$$

If $\text{char } k > 3$ then $(h_1, h_2, h_3)R = I_2(\psi)$ and hence the minimal R -resolution for IR is given by

$$0 \longrightarrow R(-13) \oplus R(-13) \xrightarrow{\psi} R(-6) \oplus R(-8) \oplus R(-12) \xrightarrow{\phi} IR \longrightarrow 0.$$

Here $\mathbb{B}(6) = -1$, $\mathbb{B}(8) = -1$, $\mathbb{B}(12) = -1$ and $\mathbb{B}(13) = 2$. Hence the HK density of (S, I) is given by

$$f_{R^G, I}(x) = \begin{cases} x/48 & \text{if } 0 \leq x \leq 6 \\ 6/48 & \text{if } 6 \leq x \leq 8 \\ (14-x)/48 & \text{if } 8 \leq x \leq 12 \\ (26-2x)/48 & \text{if } 12 \leq x \leq 13 \\ 0 & \text{otherwise,} \end{cases}$$

and hence $e_{HK}(R^G, I) = 2 - (1/24)$.

Example 8.6. Let $G = E_8$ the icosahedral group then $|G| = 120$ and $\text{char } k \geq 7$. Now

$$R^G = k[h_1, h_2, h_3] = \frac{k[x_1, x_2, x_3]}{(x_2^2 + x_3^3 - 1728x_1^5)},$$

where

$$h_1 = x_1 x_2 (x_1^{10} + 11x_1^5 x_2^5 - x_2^{10}) \\ h_2 = x_1^{30} + x_2^{30} + 522(x_1^{25} x_2^5 - x_2^{25} x_1^5) - 10005(x_1^{20} x_2^{10} + x_1^{10} x_2^{20}) \\ h_3 = -x_1^{20} - x_2^{20} + 228(x_1^{15} x_2^5 - x_2^{15} x_1^5) - 494(x_1^{10} x_2^{10}).$$

Let ψ be given by the matrix

$$\begin{bmatrix} x_1 & f_2 & f_3 \\ x_2 & g_2 & g_3 \end{bmatrix}$$

where

$$\begin{aligned} f_2 &= -x^{11} - (11/2)x_1^6x_2^5, & f_3 &= x_2^{19} + ax_1^5x_2^{14} + (b/2)x_1^{10}x_2^9 \\ g_2 &= -x_2^{11} + (11/2)x_1^5x_2^6, & g_3 &= -x_1^{19} + ax_1^{14}x_2^5 - (b/2)x_1^9x_2^{10} \end{aligned}$$

and where $a = 228$ and $b = 494$.

In particular $(h_1, h_2, h_3)R = I_2(\psi)$.

Hence the minimal R -resolution for IR is given by

$$0 \longrightarrow R(-31) \oplus R(-31) \xrightarrow{\psi} R(-12) \oplus R(-30) \oplus R(-20) \longrightarrow IR \longrightarrow 0.$$

$\mathbb{B}(12) = -1$, $\mathbb{B}(20) = -1$, $\mathbb{B}(30) = -1$ and $\mathbb{B}(31) = 2$.

Hence the HK density of (S, I) is given by

$$f_{R^G, I}(x) = \begin{cases} x/30 & \text{if } 0 \leq x \leq 6 \\ 6/30 & \text{if } 6 \leq x \leq 10 \\ (16-x)/30 & \text{if } 10 \leq x \leq 15 \\ (31-2x)/30 & \text{if } 15 \leq x \leq 31/2 \\ 0 & \text{otherwise} \end{cases}$$

and hence $e_{HK}(R^G, I) = 2 - (1/120)$.

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