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# Quantum Borchers-Bozec algebras and their integrable representations

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## ABSTRACT

We investigate the fundamental properties of quantum Borchers-Bozec algebras and their representations. Among others, we prove that the quantum Borchers-Bozec algebras have a triangular decomposition and the category of integrable representations is semi-simple.

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## 0. Introduction

The *quantum Borchers-Bozec algebras* were introduced by T. Bozec in his geometric investigation of the representation theory of quivers with loops [1,2]. He gave a construction of Lusztig's canonical basis for the positive half of a quantum Borchers-Bozec algebra in terms of simple perverse sheaves on the representation variety of quivers with loops (cf. [12]).

On algebraic side, he developed the essential part of crystal basis theory for quantum Borchers-Bozec algebras. First of all, he defined the Kashiwara operators on the integrable representations and on the neg-

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ative half of a quantum Borcherds-Bozec algebra, which provides an important framework for Kashiwara's grand-loop argument (cf. [10]).

He went on to define the notion of abstract crystals for quantum Borcherds-Bozec algebras and gave a geometric construction of the crystal for the negative half of a quantum Borcherds-Bozec algebra based on the theory of Lusztig's quiver varieties (cf. [11,8]). Moreover, using Nakajima's quiver varieties, he also gave a geometric construction of crystals for the integrable highest weight representations (cf. [14,9]).

The purpose of this paper is to provide a rigorous foundation for the theory of quantum Borcherds-Bozec algebras and their representations. Among others, we prove that the quantum Borcherds-Bozec algebras have a triangular decomposition and the category of integrable representations is semi-simple; i.e., all the integrable representations are completely reducible.

Compared with the theory of quantum Kac-Moody algebras and quantum Borcherds algebras, one of the main difficulties lies in the fact that the commutation relations between positive part and negative part are much more complicated for quantum Borcherds-Bozec algebras due to the Drinfeld-type defining relations. Also the co-multiplication formulas need a lot more careful treatment when we show that the quantum Borcherds-Bozec algebras have a triangular decomposition (Theorem 3.2). In fact, we first need to verify that we have a well-defined co-multiplication on the quantum Borcherds-Bozec algebras (Proposition 2.5). By a detailed analysis of Drinfeld-type commutation relations, we prove one of the key ingredients for our main results (Proposition 4.2), which will lead to a characterization of irreducible highest weight representations with dominant integral highest weights. Thanks to the character formula for integrable highest weight representations [3], we can follow the outline given in [5,6] to prove that all the integrable representations are completely reducible (Theorem 5.10).

This paper is organized as follows. In Section 1, we recall Bozec's construction of quantum Borcherds-Bozec algebras. Section 2 is devoted to a detailed analysis of Drinfeld-type commutation relations. We investigate the structure of quantum string algebras and prove that there exists a well-defined co-multiplication on the quantum Borcherds-Bozec algebras. In Section 3, we show that the quantum Borcherds-Bozec algebras have a triangular decomposition. In Section 4, using the detailed analysis of Drinfeld-type commutation relations, we prove Proposition 4.2, a key ingredient for our main results. Finally, in Section 5, we prove that all the integrable representations are completely reducible.

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## 1. Quantum Borcherds-Bozec algebras

We first review Bozec's construction of quantum Borcherds-Bozec algebras [2].

Let  $I$  be an index set which can be countably infinite. An integer-valued matrix  $A = (a_{ij})_{i,j \in I}$  is called an *even symmetrizable Borcherds-Cartan matrix* if it satisfies the following conditions:

- (i)  $a_{ii} = 2, 0, -2, -4, \dots$ ,
- (ii)  $a_{ij} \leq 0$  for  $i \neq j$ ,
- (iii) there exists a diagonal matrix  $D = \text{diag}(s_i \in \mathbf{Z}_{>0} \mid i \in I)$  such that  $DA$  is symmetric.

Set  $I^{\text{re}} = \{i \in I \mid a_{ii} = 2\}$ ,  $I^{\text{im}} = \{i \in I \mid a_{ii} \leq 0\}$  and  $I^{\text{iso}} = \{i \in I \mid a_{ii} = 0\}$ .

A *Borcherds-Cartan datum* consists of:

- (a) an even symmetrizable Borcherds-Cartan matrix  $A = (a_{ij})_{i,j \in I}$ ,
- (b) a free abelian group  $P$ , the *weight lattice*,
- (c)  $\Pi = \{\alpha_i \in P \mid i \in I\}$ , the set of *simple roots*,

- (d)  $P^\vee := \text{Hom}(P, \mathbf{Z})$ , the *dual weight lattice*,
- (e)  $\Pi^\vee = \{h_i \in P^\vee \mid i \in I\}$ , the set of *simple coroots*

satisfying the following conditions

- (i)  $\langle h_i, \alpha_j \rangle = a_{ij}$  for all  $i, j \in I$ ,
- (ii)  $\Pi$  is linearly independent over  $\mathbf{Q}$ ,
- (iii) for each  $i \in I$ , there exists an element  $\Lambda_i \in P$  such that

$$\langle h_j, \Lambda_i \rangle = \delta_{ij} \quad \text{for all } i, j \in I.$$

Given an even symmetrizable Borchers-Cartan matrix, it can be shown that such a Borchers-Cartan datum always exists, which is not necessarily unique. The  $\Lambda_i$  ( $i \in I$ ) are called the *fundamental weights*.

We denote by

$$P^+ := \{\lambda \in P \mid \langle h_i, \lambda \rangle \geq 0 \text{ for all } i \in I\}$$

the set of *dominant integral weights*. The free abelian group  $Q := \bigoplus_{i \in I} \mathbf{Z} \alpha_i$  is called the *root lattice*. Set  $Q_+ := \sum_{i \in I} \mathbf{Z}_{\geq 0} \alpha_i$  and  $Q_- := -Q_+$ . For  $\beta = \sum k_i \alpha_i \in Q_+$ , we define its *height* to be  $|\beta| := \sum k_i$ .

Let  $\mathfrak{h} := \mathbf{Q} \otimes_{\mathbf{Z}} P^\vee$  be the *Cartan subalgebra*. We define a partial ordering on  $\mathfrak{h}^*$  by setting  $\lambda \geq \mu$  if and only if  $\lambda - \mu \in Q_+$  for  $\lambda, \mu \in \mathfrak{h}^*$ .

Since  $A$  is symmetrizable and  $\Pi$  is linearly independent over  $\mathbf{Q}$ , there exists a non-degenerate symmetric bilinear form  $(\ , \ )$  on  $\mathfrak{h}^*$  satisfying

$$(\alpha_i, \lambda) = s_i \langle h_i, \lambda \rangle \quad \text{for all } \lambda \in \mathfrak{h}^*.$$

For each  $i \in I^{\text{re}}$ , we define the *simple reflection*  $r_i \in \mathfrak{h}^*$  by

$$r_i(\lambda) = \lambda - \langle h_i, \lambda \rangle \alpha_i \quad \text{for } \lambda \in \mathfrak{h}^*.$$

The subgroup  $W$  of  $GL(\mathfrak{h}^*)$  generated by the simple reflections  $r_i$  ( $i \in I^{\text{re}}$ ) is called the *Weyl group* of the Borchers-Cartan datum given above. It is easy to check that  $(\ , \ )$  is  $W$ -invariant.

We now proceed to define the notion of *quantum Borchers-Bozec algebras*. Let  $q$  be an indeterminate and set

$$q_i = q^{s_i}, \quad q_{(i)} = q^{\frac{(\alpha_i, \alpha_i)}{2}} = q_i^{\frac{\alpha_i i}{2}}.$$

For each  $i \in I^{\text{re}}$  and  $n \in \mathbf{Z}_{>0}$ , we define

$$[n]_i = \frac{q_i - q_i^{-n}}{q_i - q_i^{-1}}, \quad [n]_i! = [n]_i [n-1]_i \cdots [1]_i, \quad \begin{bmatrix} n \\ k \end{bmatrix}_i = \frac{[n]_i!}{[k]_i! [n-k]_i!}.$$

Set  $I^\infty := (I^{\text{re}} \times \{1\}) \cup (I^{\text{im}} \times \mathbf{Z}_{>0})$ . For simplicity, we will often write  $i$  for  $(i, 1)$  ( $i \in I^{\text{re}}$ ).

Let  $\mathcal{F} = \mathbf{Q}(q)\langle f_{il} \mid (i, l) \in I^\infty \rangle$  be the free associative algebra defined on the set of alphabet  $\{f_{il} \mid (i, l) \in I^\infty\}$ . By setting  $\deg f_{il} = -l\alpha_i$ ,  $\mathcal{F}$  becomes a  $Q_-$ -graded algebra.

We define a *twisted* multiplication on  $\mathcal{F} \otimes \mathcal{F}$  by

$$(a_1 \otimes a_2)(b_1 \otimes b_2) = q^{-(\deg a_2, \deg b_1)} a_1 b_1 \otimes a_2 b_2 \quad \text{for } a_1, a_2, b_1, b_2 \in \mathcal{F}. \quad (1.1)$$

It can be shown that there is an algebra homomorphism (called the *co-multiplication*)  $\delta : \mathcal{F} \rightarrow \mathcal{F} \otimes \mathcal{F}$  given by

$$\delta(f_{il}) = \sum_{m+n=l} q_{(i)}^{-mn} f_{im} \otimes f_{in}. \quad (1.2)$$

Here, we understand  $f_{i0} = 1$ ,  $f_{il} = 0$  for  $l < 0$ .

**Proposition 1.1.** [13] *For each  $\tau = (\tau_{il})_{(i,l) \in I^\infty}$  with  $\tau_{il} \in \mathbf{Q}(q)$ , there exists a symmetric bilinear form  $(\ , \ )_L$  on  $\mathcal{F}$  satisfying the following conditions:*

- (a)  $(x, y)_L = 0$  unless  $\deg x = \deg y$ ,
- (b)  $(f_{il}, f_{il})_L = \tau_{il}$  for all  $(i, l) \in I^\infty$ ,
- (c)  $(x, yz)_L = (\delta(x), y \otimes z)_L$  for all  $x, y, z \in \mathcal{F}$ .

From now on, we assume that

$$\tau_{il} \in 1 + q\mathbf{Z}_{\geq 0}[[q]] \quad \text{for all } (i, l) \in I^\infty. \quad (1.3)$$

We define  $\widehat{U}$  to be the  $\mathbf{Q}(q)$ -algebra with  $\mathbf{1}$  generated by the elements  $q^h$  ( $h \in P^\vee$ ) and  $e_{il}, f_{il}$  ( $(i, l) \in I^\infty$ ) with the defining relations

$$\begin{aligned} q^0 &= \mathbf{1}, \quad q^h q^{h'} = q^{h+h'} \quad \text{for } h, h' \in P^\vee, \\ q^h e_{jl} q^{-h} &= q^{l\langle h, \alpha_j \rangle} e_{jl}, \quad q^h f_{jl} q^{-h} = q^{-l\langle h, \alpha_j \rangle} f_{jl} \quad \text{for } h \in P^\vee, (j, l) \in I^\infty, \\ \sum_{k=0}^{1-la_{ij}} (-1)^k \begin{bmatrix} 1-la_{ij} \\ k \end{bmatrix}_i e_i^{1-la_{ij}-k} e_{jl} e_i^k &= 0, \\ \sum_{k=0}^{1-la_{ij}} (-1)^k \begin{bmatrix} 1-la_{ij} \\ k \end{bmatrix}_i f_i^{1-la_{ij}-k} f_{jl} f_i^k &= 0 \quad \text{for } i \in I^{\text{re}}, i \neq (j, l) \in I^\infty, \\ e_{ik} e_{jl} - e_{jl} e_{ik} &= 0, \quad f_{ik} f_{jl} - f_{jl} f_{ik} = 0 \quad \text{for } a_{ij} = 0. \end{aligned} \quad (1.4)$$

In [1], Bozec showed that one can define an algebra homomorphism called the (*co-multiplication*)  $\Delta : \widehat{U} \rightarrow \widehat{U} \otimes \widehat{U}$  given by

$$\begin{aligned} \Delta(q^h) &= q^h \otimes q^h, \\ \Delta(e_{il}) &= \sum_{m+n=l} q_{(i)}^{mn} e_{im} \otimes K_i^{-m} e_{in}, \\ \Delta(f_{il}) &= \sum_{m+n=l} q_{(i)}^{-mn} f_{im} K_i^n \otimes f_{in}, \end{aligned} \quad (1.5)$$

where  $K_i = q^{s_i h_i}$  ( $i \in I$ ). Furthermore, Bozec also showed that one can extend  $(\ , \ )_L$  to a symmetric bilinear form  $(\ , \ )_L$  on  $\widehat{U}$  satisfying

$$\begin{aligned} (q^h, K_j)_L &= q^{-\langle h, \alpha_j \rangle}, \\ (q^h, e_{il})_L &= (q^h, f_{il})_L = 0, \\ (e_{ik}, e_{jl})_L &= (f_{ik}, f_{jl})_L = \delta_{ij} \delta_{kl} \tau_{ik}. \end{aligned} \quad (1.6)$$

Define an involution  $\omega : \widehat{U} \rightarrow \widehat{U}$  by

$$\omega(q^h) = q^{-h}, \quad \omega(e_{il}) = f_{il}, \quad \omega(f_{il}) = e_{il} \quad \text{for } h \in P^\vee, (i, l) \in I^\infty. \quad (1.7)$$

For  $x \in \widehat{U}$ , following the Sweedler's notation [15], write

$$\Delta(x) = \sum x_{(1)} \otimes x_{(2)}. \quad (1.8)$$

**Definition 1.2.** Given a Borchers-Cartan datum, the *quantum Borchers-Bozec algebra*  $U_q(\mathfrak{g})$  is defined to be the quotient algebra of  $\widehat{U}$  by the defining relations

$$\sum (a_{(1)}, b_{(2)})_L \omega(b_{(1)}) a_{(2)} = \sum (a_{(2)}, b_{(1)})_L a_{(1)} \omega(b_{(2)}) \quad \text{for all } a, b \in \widehat{U}. \quad (1.9)$$

## 2. Quantum string algebras

Recall that for all  $(i, k), (j, l) \in I^\infty$ , we have the co-multiplication formulas on  $\widehat{U}$ :

$$\Delta(f_{ik}) = \sum_{m+n=k} q_{(i)}^{-mn} f_{im} K_i^n \otimes f_{in}, \quad \Delta(f_{jl}) = \sum_{r+s=l} q_{(j)}^{-rs} f_{jr} K_j^s \otimes f_{js}.$$

Then the defining relation (1.9) yields

$$\begin{aligned} & \sum_{\substack{m+n=k \\ r+s=l}} q_{(i)}^{-mn} q_{(j)}^{-rs} (f_{im} K_i^n, f_{js})_L e_{jr} K_j^{-s} f_{in} \\ &= \sum_{\substack{m+n=k \\ r+s=l}} q_{(i)}^{-mn} q_{(j)}^{-rs} (f_{in}, f_{jr} K_j^s)_L f_{im} K_i^n e_{js}. \end{aligned} \quad (2.1)$$

Suppose  $i \neq j$ . Then we have  $(f_{im} K_i^n, f_{js})_L = 0$  unless  $m = 0, s = 0$ , in which case,  $n = k, r = l$ . Hence the left-hand side of (2.1) is equal to  $e_{jl} f_{ik}$ . Similarly,  $(f_{in}, f_{jr} K_j^s)_L = 0$  implies  $n = r = 0, m = k, s = l$ , and the right-hand side of (2.1) is the same as  $f_{ik} e_{jl}$ . Hence we obtain

$$e_{jl} f_{ik} = f_{ik} e_{jl} \quad \text{for all } i \neq j, k, l > 0. \quad (2.2)$$

Now we will deal with the case when  $i = j$ . For each  $i \in I$ , we define the *quantum  $i$ -string algebra*  $U_{(i)}$  to be the subalgebra of  $U_q(\mathfrak{g})$  generated by  $e_{il}, f_{ik}, K_i^{\pm 1}$  ( $k, l > 0$ ). We denote by  $U_{(i)}^+$  (resp.  $U_{(i)}^-$ ) the subalgebra generated by  $e_{il}$  (resp.  $f_{il}$ ) for  $(i, l) \in I^\infty$ .

We return to the relation (2.1). Since  $i = j$ , we have

$$\begin{aligned} & \sum_{\substack{m+n=k \\ r+s=l}} q_{(i)}^{-mn-rs} (f_{im} K_i^n, f_{is})_L e_{ir} K_i^{-s} f_{in} \\ &= \sum_{\substack{m+n=k \\ r+s=l}} q_{(i)}^{-mn-rs} (f_{in}, f_{ir} K_i^s)_L f_{im} K_i^n e_{is}. \end{aligned} \quad (2.3)$$

Let us denote by  $L$  and  $R$  the left-hand side and right-hand side of (2.3), respectively. Since  $(f_{im} K_i^n, f_{is})_L = 0$  unless  $m = s$ , we have

$$L = \sum_{\substack{m+n=k \\ r+m=l}} q_{(i)}^{-m(n+r)} (f_{im} K_i^n, f_{im})_L e_{ir} K_i^{-m} f_{in} = \sum_{\substack{m+n=k \\ r+m=l}} q_{(i)}^{-m(n+r)} \tau_{im} e_{ir} K_i^{-m} f_{in}.$$

Note that

$$K_i^{-m} f_{in} = q_i^{mn a_{ii}} f_{in} K_i^{-m} = q_{(i)}^{2mn} f_{in} K_i^{-m}.$$

Hence we get

$$L = \sum_{\substack{m+n=k \\ r+m=l}} q_{(i)}^{m(n-r)} \tau_{im} e_{ir} f_{in} K_i^{-m}.$$

On the other hand, since  $(f_{in}, f_{ir} K_i^n)_L = 0$  unless  $n = r$  and  $K_i^n e_{is} = q_{(i)}^{2ns} e_{is} K_i^n$ , we have

$$R = \sum_{\substack{m+n=k \\ n+s=l}} q_{(i)}^{n(s-m)} \tau_{in} f_{im} e_{is} K_i^n.$$

By rearranging the indices in  $L$ , we obtain

**Lemma 2.1.** For all  $k, l > 0$ , we have the following relations in  $U_{(i)}$ :

$$\sum_{\substack{m+n=k \\ n+s=l}} q_{(i)}^{n(m-s)} \tau_{in} e_{is} f_{im} K_i^{-n} = \sum_{\substack{m+n=k \\ n+s=l}} q_{(i)}^{-n(m-s)} \tau_{in} f_{im} e_{is} K_i^n. \quad (2.4)$$

Let us analyze the implication of Lemma 2.1 in more detail.

**Example 2.2.** Suppose  $a_{ii} = 2$ . In this case,  $k = l = 1$  and we have the following two cases:

- (i)  $m = 0, n = 1, s = 0$
- (ii)  $m = 1, n = 0, s = 1$ .

Therefore (2.4) implies

$$\tau_{i,1} K_i^{-1} + \tau_{i,0} e_i f_i = \tau_{i,1} K_i + \tau_{i,0} f_i e_i.$$

Take  $\tau_{i,0} = 1$ ,  $\tau_{i,1} = \frac{1}{1 - q_i^2}$  and replace  $f_i$  by  $F_i = -q_i f_i$ . Then we obtain

$$e_i F_i - F_i e_i = \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}.$$

Hence  $U_{(i)} \cong U_q(sl_2)$  and the set  $\mathbf{B} = \{f_i^n \mid n \geq 0\}$  is a basis of  $U_{(i)}^-$ .

**Example 2.3.** Suppose  $a_{ii} = 0$ . In this case, by the defining relations of quantum Borchers-Bozec algebras, we have

$$e_{ik} e_{il} = e_{il} e_{ik}, \quad f_{ik} f_{il} = f_{il} f_{ik}, \quad K_i^{\pm 1} e_{il} = e_{il} K_i^{\pm 1}, \quad K_i^{\pm 1} f_{il} = f_{il} K_i^{\pm 1}.$$

By Lemma 2.1, the algebra  $U_{(i)}$  has the additional relation

$$\sum_{\substack{m+n=k \\ n+s=l}} \tau_{in} e_{is} f_{im} K_i^{-n} = \sum_{\substack{m+n=k \\ n+s=l}} \tau_{in} f_{im} e_{is} K_i^n.$$

We will call  $U_{(i)}$  the *quantum twisted Heisenberg algebra*.

For each  $l > 0$ , let  $\mathbf{c} = (c_1, c_2, \dots, c_l)$  be a *partition* of  $l$  and define  $f_{i,\mathbf{c}} := f_{i,c_1} f_{i,c_2} \cdots f_{i,c_l}$ . Set  $\mathbf{B}_l = \{f_{i,\mathbf{c}} \mid \mathbf{c} \text{ is a partition of } l\}$ . Then  $\mathbf{B} := \bigcup_{l \geq 0} \mathbf{B}_l$  is a basis of  $U_{(i)}^-$ , where  $\mathbf{B}_0 = \{1\}$ .

**Example 2.4.** Suppose  $a_{ii} < 0$ . In this case, there are no relations other than (2.4). In particular,  $U_{(i)}^+$  (resp.  $U_{(i)}^-$ ) is the free associative algebra generated by  $e_{ik}$  (resp.  $f_{ik}$ ) for  $k > 0$ .

For each  $l > 0$ , let  $\mathbf{c} = (c_1, c_2, \dots, c_l)$  be a *composition* of  $l$  and define  $f_{i,\mathbf{c}} := f_{i,c_1} f_{i,c_2} \cdots f_{i,c_l}$ . Set  $\mathbf{B}_l = \{f_{i,\mathbf{c}} \mid \mathbf{c} \text{ is a composition of } l\}$ . Then  $\mathbf{B} := \bigcup_{l \geq 0} \mathbf{B}_l$  is a basis of  $U_{(i)}^-$ , where  $\mathbf{B}_0 = \{1\}$ .

For simplicity, we will often write  $U$  for the quantum Borchers-Bozec algebra  $U_q(\mathfrak{g})$ . We now prove that  $\Delta$  passes down to the co-multiplication on  $U$ .

**Proposition 2.5.** *The co-multiplication  $\Delta$  on  $\hat{U}$  defines an algebra homomorphism*

$$\Delta : U \longrightarrow U \otimes U. \quad (2.5)$$

**Proof.** We need to prove  $\Delta$  preserves the defining relations of  $U$ .

When  $i \neq j$ , we have already seen that

$$f_{ik} e_{jl} = e_{jl} f_{ik} \quad \text{for all } k, l > 0.$$

Recall that

$$\Delta(f_{ik}) = \sum_{m+n=k} q_{(i)}^{-mn} f_{im} K_i^n \otimes f_{in}, \quad \Delta(e_{jl}) = \sum_{r+s=l} q_{(j)}^{rs} e_{jr} \otimes K_j^{-s} e_{js}.$$

Hence

$$\begin{aligned} \Delta(f_{ik}) \Delta(e_{jl}) &= \sum_{\substack{m+n=k \\ n+s=l}} (q_{(i)}^{-mn} f_{im} K_i^n \otimes f_{in}) (q_{(j)}^{rs} e_{jr} \otimes K_j^{-r} e_{js}) \\ &= \sum_{\substack{m+n=k \\ n+s=l}} q_{(i)}^{-mn} q_{(j)}^{rs} (f_{im} K_i^n e_{jr} \otimes f_{in} K_j^{-r} e_{js}), \end{aligned}$$

which coincides with the expression in the triangular decomposition of  $U$ .

On the other hand,

$$\begin{aligned} \Delta(e_{jl}) \Delta(f_{ik}) &= \sum_{\substack{m+n=k \\ n+s=l}} (q_{(j)}^{rs} e_{jr} \otimes K_j^{-r} e_{js}) (q_{(i)}^{-mn} f_{im} K_i^n \otimes f_{in}) \\ &= \sum_{\substack{m+n=k \\ n+s=l}} q_{(i)}^{-mn} q_{(j)}^{rs} (e_{jr} f_{im} K_i^n \otimes K_j^{-r} e_{js} f_{in}) \\ &= \sum_{\substack{m+n=k \\ n+s=l}} q_{(i)}^{-mn} q_{(j)}^{rs} (f_{im} e_{jr} K_i^n \otimes K_j^{-r} f_{in} e_{js}). \end{aligned}$$

We need to change the order of the expression  $e_{jr} K_i^n$  and  $K_j^{-r} f_{in}$ . Note that

$$e_{jr} K_i^n = q_i^{-r n a_{ij}} K_i^n e_{jr}, \quad K_i^{-r} f_{in} = q^{r n a_{ji}} f_{in} K_j^{-r}.$$

Since  $A$  is symmetrizable, we have  $q_i^{a_{ij}} = q^{s_i a_{ij}} = q^{s_j a_{ji}} = q_j^{a_{ji}}$ , which implies

$$\begin{aligned}\Delta(e_{jl})\Delta(f_{ik}) &= \sum_{\substack{m+n=k \\ n+s=l}} q_{(i)}^{-mn} q_{(j)}^{rs} q_i^{-rna_{ij}} q_j^{rna_{ji}} (f_{im} K_i^n e_{jr} \otimes f_{in} K_j^{-r} e_{js}) \\ &= \sum_{\substack{m+n=k \\ n+s=l}} q_{(i)}^{-mn} q_{(j)}^{rs} (f_{im} K_i^n e_{jr} \otimes f_{in} K_j^{-r} e_{js}).\end{aligned}$$

Hence we have

$$\Delta(f_{ik})\Delta(e_{jl}) = \Delta(e_{jl})\Delta(f_{ik}),$$

as desired.

We will move to the case when  $i = j$ .

**Case 1:** Suppose  $k = l$ .

In this case, by Lemma 2.1, for all  $k > 0$ , we have

$$\sum_{n=0}^k \tau_{in}(e_{i,k-n} f_{i,k-n} K_i^{-n} - f_{i,k-n} e_{i,k-n} K_i^n) = 0. \quad (2.6)$$

We would like to prove

$$\Delta\left(\sum_{n=0}^k \tau_{in}(e_{i,k-n} f_{i,k-n} K_i^{-n} - f_{i,k-n} e_{i,k-n} K_i^n)\right) = 0. \quad (2.7)$$

We will use induction on  $k$ .

If  $k = l = 1$ , we have

$$\tau_{i0}(e_{i1} f_{i1} - f_{i1} e_{i1}) + \tau_{i1}(K_i^{-1} - K_i) = 0.$$

Hence (1.5) gives

$$\begin{aligned}\Delta(e_{i1})\Delta(f_{i1}) &= (e_{i1} \otimes K_i^{-1} + 1 \otimes e_{i1})(f_{i1} \otimes 1 + K_i \otimes f_{i1}) \\ &= e_{i1} f_{i1} \otimes K_i^{-1} + q_{(i)}^2 e_{i1} K_i \otimes f_{i1} K_i^{-1} + f_{i1} \otimes e_{i1} + K_i \otimes e_{i1} f_{i1}, \\ \Delta(f_{i1})\Delta(e_{i1}) &= (f_{i1} \otimes 1 + K_i \otimes f_{i1})(e_{i1} \otimes K_i^{-1} + 1 \otimes e_{i1}) \\ &= f_{i1} e_{i1} \otimes K_i^{-1} + f_{i1} \otimes e_{i1} + q_{(i)}^2 e_{i1} K_i \otimes f_{i1} K_i^{-1} + K_i \otimes f_{i1} e_{i1}.\end{aligned}$$

Therefore we obtain

$$\begin{aligned}\Delta(\tau_{i0}(e_{i1} f_{i1} - f_{i1} e_{i1}) + \tau_{i1}(K_i^{-1} - K_i)) \\ &= \tau_{i0}(e_{i1} f_{i1} - f_{i1} e_{i1}) \otimes K_i^{-1} + \tau_{i0}(K_i \otimes (e_{i1} f_{i1} - f_{i1} e_{i1})) \\ &\quad + \tau_{i1}(K_i^{-1} \otimes K_i^{-1} - K_i \otimes K_i) \\ &= \tau_{i1}(K_i - K_i^{-1}) \otimes K_i^{-1} + K_i \otimes \tau_{i1}(K_i - K_i^{-1}) \\ &\quad + \tau_{i1}(K_i^{-1} \otimes K_i^{-1} - K_i \otimes K_i) = 0.\end{aligned}$$



Suppose  $k > 1$  and set

$$\begin{aligned} A_0 &= \tau_{i1}(e_{i,k-1}f_{i,k-1} - f_{i,k-1}e_{i,k-1}) \\ &\quad + \tau_{i2}(e_{i,k-2}f_{i,k-2}K_i^{-1} - f_{i,k-2}e_{i,k-2}K_i) \\ &\quad + \cdots + \tau_{ik}(K_i^{-k+1} - K_i^{k-1}). \end{aligned} \quad (2.8)$$

By Lemma 2.1,  $A_0 = 0$ . Multiply  $A_0$  by  $K_i^{-1} + K_i$ . Then we obtain

$$A := A_0(K_i^{-1} + K_i) = B + C = 0, \quad (2.9)$$

where

$$\begin{aligned} B &= \tau_{i1}(e_{i,k-1}f_{i,k-1}K_i^{-1} - f_{i,k-1}e_{i,k-1}K_i) \\ &\quad + \tau_{i2}(e_{i,k-2}f_{i,k-2}K_i^{-2} - f_{i,k-2}e_{i,k-2}K_i^2) \\ &\quad + \cdots + \tau_{ik}(K_i^{-k} - K_i^k), \end{aligned} \quad (2.10)$$

$$\begin{aligned} C &= \tau_{i1}(e_{i,k-1}f_{i,k-1}K_i - f_{i,k-1}e_{i,k-1}K_i^{-1}) \\ &\quad + \tau_{i2}(e_{i,k-2}f_{i,k-2} - f_{i,k-2}e_{i,k-2}K_i^{-1}) \\ &\quad + \cdots + \tau_{ik}(K_i^{-k+2} - K_i^{k-2}). \end{aligned} \quad (2.11)$$

The induction hypothesis gives  $\Delta(A_0) = 0$ . Since  $\Delta$  is an algebra homomorphism on  $\widehat{U}$ , we get

$$\begin{aligned} 0 &= \Delta(A_0)\Delta(K_i^{-1} + K_i) = \Delta(A_0(K_i^{-1} + K_i)) \\ &= \Delta(A) = \Delta(B + C) = \Delta(B) + \Delta(C). \end{aligned} \quad (2.12)$$

The relation (2.6) gives

$$\tau_{i0}(e_{ik}f_{ik} - f_{ik}e_{ik}) + B = 0$$

and we would like to prove

$$\Delta(\tau_{i0}(e_{ik}f_{ik} - f_{ik}e_{ik}) + B) = \Delta(\tau_{i0}(e_{ik}f_{ik} - f_{ik}e_{ik})) + \Delta(B) = 0.$$

Since  $B + C = 0$ , we have

$$\tau_{i0}(e_{ik}f_{ik} - f_{ik}e_{ik}) = -B = C,$$

which implies

$$\Delta(\tau_{i0}(e_{ik}f_{ik} - f_{ik}e_{ik})) = \Delta(C).$$

Therefore, by (2.12), we obtain

$$\begin{aligned} \Delta(\tau_{i0}(e_{ik}f_{ik} - f_{ik}e_{ik}) + B) &= \Delta(\tau_{i0}(e_{ik}f_{ik} - f_{ik}e_{ik})) + \Delta(B) \\ &= \Delta(C) + \Delta(B) = 0. \end{aligned}$$

**Case 2:** Suppose  $k < l$ .

In this case, by Lemma 2.1, for all  $k > 0$ , we have

$$\sum_{n=0}^k \tau_{in} \left( q_{(i)}^{-n(l-k)} e_{i,l-n} f_{i,k-n} K_i^{-n} - q_{(i)}^{n(l-k)} f_{i,k-n} e_{i,l-n} K_i^n \right) = 0. \quad (2.13)$$

We need to prove

$$\Delta \left( \sum_{n=0}^k \tau_{in} (q_{(i)}^{-n(l-k)} e_{i,k-n} f_{i,k-n} K_i^{-n} - q_{(i)}^{n(l-k)} f_{i,k-n} e_{i,k-n} K_i^n) \right) = 0. \quad (2.14)$$

We will use induction on  $k$ .

If  $k = 1, l > 1$ , we have

$$\tau_{i0}(e_{il}f_{i1} - f_{i1}e_{il}) + \tau_{i1}e_{i,l-1}(q_{(i)}^{-l+1}K_i^{-1} - q_{(i)}^{l-1}K_i) = 0. \quad (2.15)$$

We will verify

$$\Delta \left( \tau_{i0}(e_{il}f_{i1} - f_{i1}e_{il}) + \tau_{i1}e_{i,l-1}(q_{(i)}^{-l+1}K_i^{-1} - q_{(i)}^{l-1}K_i) \right) = 0$$

by a direct calculation.

Recall that

$$\begin{aligned} \Delta(e_{il}) &= e_{il} \otimes K_i^{-l} + q_{(i)}^{l-1} e_{i,l-1} \otimes K_i^{-l+1} e_{i1} + q_{(i)}^{2(l-2)} e_{i,l-2} \otimes K_i^{-l+2} e_{i,2} \\ &\quad + \cdots + q_{(i)}^{2(l-2)} e_{i2} \otimes K_i^{-2} e_{i,l-2} + q_{(i)}^{l-1} e_{i1} \otimes K_i^{-1} e_{i,l-1} + 1 \otimes e_{il}, \\ \Delta(f_{i1}) &= f_{i1} \otimes 1 + K_i \otimes f_{i1}. \end{aligned}$$

Thus we have

$$\begin{aligned} &\tau_{i0}(\Delta(e_{il})\Delta(f_{i1}) - \Delta(f_{i1})\Delta(e_{il})) \\ &= \tau_{i0}(e_{il}f_{i1} - f_{i1}e_{il}) \otimes K_i^{-l} \\ &\quad + q_{(i)}^{l-1} \tau_{i0}(e_{i,l-1}f_{i1} - f_{i1}e_{i,l-1}) \otimes K_i^{-l+1} e_{i,1} \\ &\quad + \cdots + q_{(i)}^{l-1} \tau_{i0}(e_{i1}f_{i1} - f_{i1}e_{i1}) \otimes K_i^{-l} e_{i,l-1} \\ &\quad + q_{(i)}^{l-1} e_{i,l-1} K_i \otimes \tau_{i0}(e_{i1}f_{i1} - f_{i1}e_{i1}) K_i^{-l+1} \\ &\quad + e_{i,l-2} K_i \otimes \tau_{i0}(e_{i2}f_{i1} - f_{i1}e_{i2}) K_{-l+2} \\ &\quad + \cdots + q_{(i)}^{-(l-3)} e_i K_i \otimes \tau_{i0}(e_{i,l-1}f_{i1} - f_{i1}e_{i,l-1}) K_i^{-1} \\ &\quad + K_i \otimes \tau_{i0}(e_{il}f_{i1} - f_{i1}e_{il}). \end{aligned}$$

Using the relation (2.15), we get

$$\begin{aligned} &\tau_{i0}(\Delta(e_{il})\Delta(f_{i1}) - \Delta(f_{i1})\Delta(e_{il})) \\ &= \tau_{i1}(q_{(i)}^{l-1} e_{i,l-1} K_i \otimes K_i^{-l+2} + q_{(i)} e_{i,l-2} K_i \otimes e_{i1} K_i^{-l+3} \\ &\quad + \cdots + q_{(i)} e_{i,1} K_i \otimes e_{i,l-2} + q_{(i)}^{l-1} K_i \otimes e_{i,l-1} K_i) \\ &\quad - \tau_{i1}(q_{(i)}^{-l+1} e_{i,l-1} K_i^{-1} \otimes K_i^{-l} + q_{(i)}^{-2l+3} e_{i,l-2} K_i^{-1} \otimes e_{i1} K_i^{-l+1} \\ &\quad + \cdots + q_{(i)}^{-2l+3} e_{i1} K_i^{-1} \otimes e_{i,l-2} K_i^{-2} + q_{(i)}^{-l+1} K_i^{-1} \otimes e_{i,l-1} K_i^{-1}). \end{aligned} \quad (2.16)$$

Now we consider the co-multiplication of  $e_{i,l-1}(q_{(i)}^{-l+1}K_i^{-1} - q_{(i)}^{l-1}K_i)$ . Note that

$$\Delta(q_{(i)}^{-l+1}K_i^{-1} - q_{(i)}^{l-1}K_i) = q_{(i)}^{-l+1}(K_i^{-1} \otimes K_i^{-1}) - q_{(i)}^{l-1}(K_i \otimes K_i).$$

Hence we have

$$\begin{aligned} & \Delta(\tau_1(e_{i,l-1}(q_{(i)}^{-l+1}K_i^{-1} - q_{(i)}^{l-1}K_i))) \\ &= \tau_{i1}(q_{(i)}^{-l+1}e_{i,l-1}K_i^{-1} \otimes K_i^{-l} + q_{(i)}^{-2l+3}e_{i,l-2}K_i^{-1} \otimes e_{i,1}K_i^{-l+1} \\ & \quad + \cdots + q_{(i)}^{-2l+3}e_{i1}K_i^{-1} \otimes e_{i,l-2}K_i^{-2} + q_{(i)}^{-l+1}K_i^{-1} \otimes e_{i,l-1}K_i^{-1}) \\ & \quad - \tau_{i1}(q_{(i)}^{l-1}e_{i,l-1}K_i \otimes K_i^{-l+2} + q_{(i)}e_{i,l-2}K_i \otimes e_{i1}K_i^{-l+3} \\ & \quad + \cdots + q_{(i)}e_{i1}K_i \otimes e_{i,l-2} + q_{(i)}^{l-1}K_i \otimes e_{i,l-1}K_i). \end{aligned} \quad (2.17)$$

Therefore, combining (2.16) and (2.17), we obtain

$$\Delta\left(\tau_{i0}(e_{il}f_{i1} - f_{i1}e_{il}) + \tau_{i1}e_{i,l-1}(q_{(i)}^{-l+1}K_i^{-1} - q_{(i)}^{l-1}K_i)\right) = 0.$$

Assume that  $k > 1$ . Our argument is similar to the case when  $k = l$ . We already know

$$\begin{aligned} & \tau_{i0}(e_{il}f_{ik} - f_{ik}e_{il}) + \tau_{i1}\left(q_{(i)}^{-(l-k)}e_{i,l-1}f_{i,k-1}K_i^{-1} - q_{(i)}^{l-k}f_{i,k-1}e_{i,l-1}K_i\right) \\ & + \tau_{i2}\left(q_{(i)}^{-2(l-k)}e_{i,l-2}f_{i,k-2}K_i^{-2} - q_{(i)}^{2(l-k)}f_{i,k-2}e_{i,l-2}K_i^2\right) \\ & + \cdots + \tau_{ik}\left(q_{(i)}^{-k(l-k)}e_{i,l-k}K_i^{-1} - q_{(i)}^{k(l-k)}e_{i,l-k}K_i^k\right) = 0, \end{aligned}$$

and we need to show  $\Delta$  preserves the above relation.

Set

$$\begin{aligned} A_0 &= \tau_{i1}(e_{i,l-1}f_{i,k-1} - f_{i,k-1}e_{i,l-1}) \\ & + \tau_{i2}\left(q_{(i)}^{-(l-k)}e_{i,l-2}f_{i,k-2}K_i^{-1} - q_{(i)}^{l-k}f_{i,k-2}e_{i,l-2}K_i\right) \\ & + \cdots + \tau_{ik}\left(q_{(i)}^{-(k-1)(l-k)}e_{i,l-k}K_i^{-1} - q_{(i)}^{(k-1)(l-k)}e_{i,l-k}K_i^k\right), \end{aligned}$$

which is equal to 0.

Multiply  $A_0$  by  $q_{(i)}^{-(l-k)}K_i^{-1} + q_{(i)}^{l-k}K_i$  to obtain

$$A := A_0\left(q_{(i)}^{-(l-k)}K_i^{-1} + q_{(i)}^{l-k}K_i\right) = B + C = 0,$$

where

$$\begin{aligned} B &= \tau_{i1}\left(q_{(i)}^{-(l-k)}e_{i,l-1}f_{i,k-1}K_i^{-1} - q_{(i)}^{l-k}f_{i,k-1}e_{i,l-1}K_i\right) \\ & + \tau_{i2}\left(q_{(i)}^{-2(l-k)}e_{i,l-2}f_{i,k-2}K_i^{-2} - q_{(i)}^{2(l-k)}f_{i,k-2}e_{i,l-2}K_i^2\right) \\ & + \cdots + \tau_{ik}\left(q_{(i)}^{-k(l-k)}e_{i,l-k}K_i^{-1} - q_{(i)}^{k(l-k)}e_{i,l-k}K_i^k\right), \\ C &= \tau_{i1}\left(q_{(i)}^{l-k}e_{i,l-1}f_{i,k-1}K_i - q_{(i)}^{-(l-k)}f_{i,k-1}e_{i,l-1}K_i^{-1}\right) \\ & + \tau_{i2}(e_{i,l-2}f_{i,k-2} - f_{i,k-2}e_{i,l-2}) \\ & + \cdots + \tau_{ik}\left(q_{(i)}^{-(k-2)(l-k)}e_{i,l-k}K_i^{-1} - q_{(i)}^{(k-2)(l-k)}e_{i,l-k}K_i^k\right). \end{aligned}$$

As in the case of  $k = l$ , since  $B + C = 0$ , we have

$$\tau_{i0}(e_{il}f_{ik} - f_{ik}e_{il}) = C$$

and

$$\Delta(\tau_{i0}(e_{il}f_{ik} - f_{ik}e_{il})) = \Delta(C).$$

By the induction hypothesis, we have

$$\begin{aligned}\Delta(A) &= \Delta(B) + \Delta(C) = \Delta(B + C) = \Delta\left(A_0(q_{(i)}^{-(l-k)}K_{-1} + q_{(i)}^{l-k}K_i)\right) \\ &= \Delta(A_0)\Delta\left(q_{(i)}^{-(l-k)}K_{-1} + q_{(i)}^{l-k}K_i\right) = 0.\end{aligned}$$

Therefore,

$$\Delta(\tau_{i0}(e_{il}f_{ik} - f_{ik}e_{il}) + B) = \Delta(\tau_{i0}(e_{il}f_{ik} - f_{ik}e_{il})) + \Delta(B) = \Delta(C) + \Delta(B) = 0,$$

which proves our claim.

**Case 3:** If  $k > l$ , we can prove our claim almost in the same way as we did above.

Therefore, we obtain a co-multiplication on  $U$

$$\Delta : U \longrightarrow U \otimes U$$

as desired.  $\square$

### 3. Triangular decomposition

Let  $U^+$  (resp.  $U^-$ ) be the subalgebra of  $U_q(\mathfrak{g})$  generated by  $e_{il}$  (resp.  $f_{il}$ ) for  $(i, l) \in I^\infty$ . We will denote by  $U^0$  the subalgebra generated by  $q^h$  ( $h \in P^\vee$ ). It is easy to see that  $U^0 = \bigoplus_{h \in P^\vee} \mathbf{Q}(q)q^h$ . We will prove that the quantum Borchers-Bozec algebra has a *triangular decomposition*.

We first prove the following lemma.

**Lemma 3.1.** *Let  $U^{\geq 0}$  (resp.  $U^{\leq 0}$ ) be the subalgebra of  $U_q(\mathfrak{g})$  generated by  $U^0$  and  $U^+$  (resp.  $U^-$  and  $U^0$ ). Then we have the isomorphisms:*

$$U^{\leq 0} \cong U^- \otimes U^0, \quad U^{\geq 0} \cong U^0 \otimes U^+. \quad (3.1)$$

**Proof.** We will prove the first isomorphism only because the second one would follow from a similar argument.

Since  $U^-$  is spanned by the monomials in  $f_{il}$  ( $(i, l) \in I^\infty$ ), we can extract a monomial basis  $\mathbf{B}^- = \{f_\tau \mid \tau \in \Omega\}$  of  $U^-$  indexed by an ordered set  $\Omega$ . By the defining relations (1.4), we have a surjective homomorphism

$$U^- \otimes U^0 \longrightarrow U^{\leq 0}$$

given by

$$f_\tau \otimes q^h \longmapsto f_\tau q^h \quad (\tau \in \Omega, h \in P^\vee).$$

Hence we need to show  $f_\tau q^h$  ( $\tau \in \Omega, h \in P^\vee$ ) are linearly independent.

Note that  $\mathbf{B}^-$  can be decomposed into a disjoint union  $\mathbf{B}^- = \bigsqcup_{\beta \in Q_+} \mathbf{B}_{-\beta}$ , where  $\mathbf{B}_{-\beta}$  consists of the monomials  $f_\tau$  with  $\deg f_\tau = -\beta$ .

Now consider the linear dependence relation

$$\sum_{\tau, h} c_{\tau, h} f_\tau q^h = 0 \quad \text{with } \tau \in \Omega, h \in P^\vee, c_{\tau, h} \in \mathbf{Q}(q). \quad (3.2)$$

By the above observation, the relation (3.2) can be written as

$$\sum_{\beta \in Q_+} \left( \sum_{\substack{\deg f_\tau = -\beta \\ h \in P^\vee}} c_{\tau, h} f_\tau q^h \right) = 0,$$

which yields

$$\sum_{\substack{\deg f_\tau = -\beta \\ h \in P^\vee}} c_{\tau, h} f_\tau q^h = 0 \quad \text{for all } \beta \in Q_+. \quad (3.3)$$

Let us write  $f_\tau = f_{i_1, l_1} \cdots f_{i_r, l_r}$  with  $l_1 \alpha_{i_1} + \cdots + l_r \alpha_{i_r} = \beta$  and recall that

$$\begin{aligned} \Delta(f_{il}) &= \sum_{m+n=l} q_{(i)}^{-mn} f_{im} K_i^n \otimes f_{in} \\ &= f_{il} \otimes 1 + q_{(i)}^{-(l-1)} f_{i, l-1} K_i \otimes f_{i, 1} + q_{(i)}^{-2(l-2)} f_{i, l-2} K_i^2 \otimes f_{i, 2} \\ &\quad + \cdots + q_{(i)}^{-(l-1)} f_{i, 1} K_i^{l-1} \otimes f_{i, l-1} + K_i^l \otimes f_{i, l}. \end{aligned}$$

Hence we may write

$$\begin{aligned} \Delta(f_\tau) &= f_{i_1, l_1} \cdots f_{i_r, l_r} \otimes 1 + \cdots + (\text{intermediate terms}) \\ &\quad + \cdots + K_{i_1}^{l_1} \cdots K_{i_r}^{l_r} \otimes f_{i_1, l_1} \cdots f_{i_r, l_r}, \\ &= f_\tau \otimes 1 + \cdots + (\text{intermediate terms}) + \cdots + q^{h_\tau} \otimes f_\tau, \end{aligned}$$

where  $q^{h_\tau} = K_{i_1}^{l_1} \cdots K_{i_r}^{l_r}$ . Applying the co-multiplication  $\Delta$  in (3.3), we obtain

$$\begin{aligned} 0 &= \sum_{\tau, h} c_{\tau, h} \Delta(f_\tau) (q^h \otimes q^h) \\ &= \sum_{\tau, h} c_{\tau, h} (f_\tau q^h \otimes q^h + (\text{intermediate terms}) + q^{h_\tau+h} \otimes f_\tau q^h). \end{aligned}$$

Let us focus on the terms of bi-degree  $(0, -\beta)$ :

$$\begin{aligned} 0 &= \sum_{\tau, h} c_{\tau, h} q^{h_\tau+h} \otimes f_\tau q^h = \sum_h \left( \sum_\tau c_{\tau, h} q^{h_\tau+h} \otimes f_\tau q^h \right) \\ &= \sum_h \left( q^{h_\tau+h} \otimes \left( \sum_\tau c_{\tau, h} f_\tau q^h \right) \right). \end{aligned}$$

Since  $q^{h_\tau+h}$  ( $h \in P^\vee$ ) are linearly independent, by an elementary property of tensor product, we conclude

$$\sum_{\tau} c_{\tau,h} f_{\tau} q^h = 0 \quad \text{for all } h \in P^\vee,$$

which implies  $\sum_{\tau} c_{\tau,h} f_{\tau} = 0$ . But  $f_{\tau}$  ( $\tau \in \Omega$ ) are linearly independent. Hence  $c_{\tau,h} = 0$  for all  $\tau \in \Omega$ ,  $h \in P^\vee$  as desired.  $\square$

We prove our main theorem in this section.

**Theorem 3.2.** *The quantum Borchers-Bozec algebra  $U_q(\mathfrak{g})$  has the following triangular decomposition:*

$$U_q(\mathfrak{g}) \cong U^- \otimes U^0 \otimes U^+. \quad (3.4)$$

**Proof.** We first show that there exists a surjective homomorphism

$$U^- \otimes U^0 \otimes U^+ \longrightarrow U_q(\mathfrak{g}). \quad (3.5)$$

That is, every element  $u \in U_q(\mathfrak{g})$  can be written as

$$u = \sum u^- u^0 u^+, \quad (3.6)$$

where  $u^0 \in U^0$ ,  $u^\pm \in U^\pm$ .

By the defining relations of  $U_q(\mathfrak{g})$ , we have only to verify that  $e_{jl} f_{ik}$  for  $(i, k), (j, l) \in I^\infty$  can be written in the form of (3.6). If  $i \neq j$ , we have seen that  $e_{jl} f_{ik} = f_{ik} e_{jl}$  and (3.6) is verified. So we will focus on the case when  $i = j$ .

As in Section 2, we denote by  $L$  and  $R$  the left-hand side and the right-hand side of the relation (2.4):

$$\begin{aligned} L &= \sum_{\substack{m+n=k \\ n+s=l}} q_{(i)}^{n(m-s)} \tau_{in} e_{is} f_{im} K_i^{-n}, \\ R &= \sum_{\substack{m+n=k \\ n+s=l}} q_{(i)}^{-n(m-s)} \tau_{in} f_{im} e_{is} K_i^n. \end{aligned}$$

Note that

$$e_{is} K_i^n = q_i^{-nsa_{ii}} e_{is} = q_{(i)}^{-2ns} K_i^n e_{is}.$$

Hence

$$R = \sum_{\substack{m+n=k \\ n+s=l}} q_{(i)}^{-n(m+s)} \tau_{in} f_{im} K_i^n e_{is},$$

which is in the form of (3.6).

For all  $k, l > 0$ , we will show that

$$e_{il} f_{ik} = R - S,$$

where  $S$  is in the form of (3.6).

Since

$$e_{is} K_i^{-n} = q_i^{nsa_{ii}} K_i^n e_{is} = q_{(i)}^{2ns} K_i^{-n} e_{is},$$

we have

$$L = \sum_{\substack{m+n=k \\ n+s=l}} q_{(i)}^{-n(m-s)} \tau_{in} K_i^{-n} e_{is} f_{im}.$$

**Case 1:** Suppose  $k = l$ .

In this case,

$$\begin{aligned} L &= \sum_{n=0}^k \tau_{in} K_i^{-n} e_{i,k-n} f_{i,k-n} \\ &= \tau_{i0} e_{ik} f_{ik} + \tau_{i1} K_i^{-1} e_{i,k-1} f_{i,k-1} + \tau_{i2} K_i^{-2} e_{i,k-2} f_{i,k-2} \\ &\quad + \cdots + \tau_{i,k-1} K_i^{-k+1} e_{i1} f_{i1} + \tau_{ik} K_i^{-k}. \end{aligned}$$

We will prove our assertion by induction on  $k$ .

If  $k = 1$ , we have

$$L = \tau_{i0} e_{i1} f_{i1} + \tau_{i1} K_i^{-1} = R,$$

which implies

$$e_{i1} f_{i1} = \frac{1}{\tau_{i0}} (R - \tau_{i1} K_i^{-1}) \in U^- U^0 U^+,$$

as desired.

If  $k > 1$ , set

$$S = \tau_{i1} K_i^{-1} e_{i,k-1} f_{i,k-1} + \cdots + \tau_{i,k-1} K_i^{-k+1} e_{i1} f_{i1} + \tau_{ik} K_i^{-k},$$

so that

$$L = \tau_{i0} e_{ik} f_{ik} + S = R.$$

By induction hypothesis, all  $e_{i1} f_{i1}, e_{i2} f_{i2}, \dots, e_{i,k-1} f_{i,k-1}$  can be written in the form of (3.6). Thus we can verify that  $S \in U^- U^0 U^+$ , which yields

$$e_{ik} f_{ik} = \frac{1}{\tau_{i0}} (R - S) \in U^- U^0 U^+.$$

**Case 2:** Suppose  $k < l$ .

In this case,

$$\begin{aligned} L &= \sum_{n=0}^k q_{(i)}^{n(l-k)} \tau_{in} K_i^{-n} e_{i,l-n} f_{i,k-n} \\ &= \tau_{i0} e_{il} f_{ik} + q_{(i)}^{l-k} \tau_{i1} K_i^{-1} e_{i,l-1} f_{i,k-1} + q_{(i)}^{2(l-k)} \tau_{i2} K_i^{-2} e_{i,l-2} f_{i,k-2} \\ &\quad + \cdots + q_{(i)}^{(k-1)(l-k)} \tau_{i,k-1} K_i^{-k+1} e_{i,l-k+1} f_{i1} + q_{(i)}^{k(l-k)} \tau_{ik} K_i^{-k} e_{i,l-k}. \end{aligned}$$

If  $k = 1$ , then

$$L = \tau_{i0} e_{il} f_{i1} + q_{(i)}^{l-1} \tau_{i1} K_i^{-1} e_{i,l-1} = R,$$

which implies

$$e_{i,l-1} f_{i1} = \frac{1}{\tau_{i0}} \left( R - q_{(i)}^{l-1} \tau_{i1} K_i^{-1} e_{i,l-1} \right) \in U^- U^0 U^+.$$

Assume that  $k > 1$  and set

$$S = q_{(i)}^{l-k} \tau_{i1} K_i^{-1} e_{i,l-1} f_{i,k-1} + \cdots + q_{(i)}^{k(l-k)} \tau_{ik} K_i^{-k} e_{i,l-k}.$$

Then using the induction hypothesis, we conclude

$$e_{il} f_{ik} = \frac{1}{\tau_{i0}} (R - S) \in U^- U^0 U^+.$$

**Case 3:** When  $k > l$ , we can prove our assertion using the same argument as above.

We now prove the injectivity of the homomorphism (3.5). Let  $\mathbf{B}^+ = \{e_\tau \mid \tau \in \Omega\}$  denote a monomial basis of  $U^+$ . We need to show that the set  $\mathbf{B} = \{f_\tau q^h e_\mu \mid \tau, \mu \in \Omega, h \in P^\vee\}$  is linearly independent. As in Lemma 3.1, we have only to consider the linear dependence relation

$$\sum_{\substack{h \in P^\vee \\ \deg f_\tau + \deg e_\mu = \gamma}} c_{\tau,h,\mu} f_\tau q^h e_\mu = 0 \quad (3.7)$$

for all  $\gamma \in Q$ .

Write

$$\begin{aligned} \Delta(e_\mu) &= e_\mu \otimes q^{-h_\mu} + (\text{intermediate terms}) + 1 \otimes e_\mu, \\ \Delta(f_\tau) &= f_\tau \otimes 1 + (\text{intermediate terms}) + q^{h_\tau} \otimes f_\tau \end{aligned}$$

so that we have

$$\begin{aligned} 0 &= \Delta \left( \sum_{\substack{h \in P^\vee \\ \deg f_\tau + \deg e_\mu = \gamma}} c_{\tau,h,\mu} f_\tau q^h e_\mu \right) \\ &= \sum_{\substack{h \in P^\vee \\ \deg f_\tau + \deg e_\mu = \gamma}} c_{\tau,h,\mu} (f_\tau \otimes 1 + (\text{intermediate terms}) + q^{h_\tau} \otimes f_\tau) \\ &\quad \times (q^h \otimes q^h) (e_\mu \otimes q^{-h_\mu} + (\text{intermediate terms}) + 1 \otimes e_\mu). \end{aligned}$$

Take a total ordering  $\leq$  on  $Q$  given by the height and lexicographic ordering. Let  $\Omega_0$  (resp.  $\Omega_1$ ) be the set of all  $\tau \in \Omega$  (resp.  $\mu \in \Omega$ ) such that  $\deg f_\tau$  (resp.  $\deg e_\mu$ ) is minimal (resp. maximal) among the terms appearing in (3.7) with respect to  $\leq$ . Since  $\deg f_\tau \in Q_-$ ,  $\deg e_\mu \in Q_+$  and  $\deg f_\tau + \deg e_\mu = \gamma$ , it is clear that  $\tau \in \Omega_0$  if and only if  $\mu \in \Omega_1$ .

We now focus on the terms of bi-degree  $(\max, \min)$  in (3.7), which gives

$$0 = \sum_{\substack{h \in P^\vee \\ \tau \in \Omega_0 \\ \mu \in \Omega_1}} c_{\tau,h,\mu} q^{h_\tau+h} e_\mu \otimes f_\tau q^{h-h_\tau} = \sum_{\substack{\tau \in \Omega_0 \\ h \in P^\vee}} \left( \sum_{\mu \in \Omega_1} c_{\tau,h,\mu} q^{h_\tau+h} e_\mu \right) \otimes f_\tau q^{h-h_\tau}.$$



Since  $f_\tau q^{h-h_\mu}$  ( $\tau \in \Omega_0, h \in P^\vee$ ) are linearly independent, we have

$$\sum_{\mu \in \Omega_1} c_{\tau, h, \mu} q^{h_\tau + h} e_\mu = 0.$$

Therefore  $c_{\tau, h, \mu} q^{h_\tau + h} = 0$  and hence  $c_{\tau, h, \mu} = 0$  for all  $\tau \in \Omega_0, \mu \in \Omega_1$ .

Repeat this process along with the total ordering  $\leq$  on  $Q$ , we conclude  $c_{\tau, h, \mu} = 0$  for all  $h \in P^\vee, \tau, \mu \in \Omega$ , which proves our theorem.  $\square$

**Remark 3.3.** The surjectivity of the homomorphism (3.5) can be proved using [2, Proposition 3.10 (ii)]. In [2, Remark 3.23], it was mentioned that the quantum Borcherds-Bozec algebras have a triangular decomposition.

#### 4. Highest weight representation theory

Let  $U_q(\mathfrak{g})$  be a quantum Borcherds-Bozec algebra and let  $M$  be a  $U_q(\mathfrak{g})$ -module. We say that  $M$  has a *weight space decomposition* if

$$M = \bigoplus_{\mu \in P} M_\mu, \quad \text{where } M_\mu = \{m \in M \mid q^h m = q^{\langle h, \mu \rangle} m \text{ for all } h \in P^\vee\}.$$

We denote  $\text{wt}(M) := \{\mu \in \mathfrak{h}^* \mid M_\mu \neq 0\}$ . When  $\dim M_\mu < \infty$  for all  $\mu \in P$ , we define the *character* of  $M$  to be

$$\text{ch}(M) = \sum_{\mu \in P} (\dim M_\mu) e^\mu,$$

where  $e^\mu$  ( $\mu \in P$ ) are multiplicative basis vectors of the group algebra of  $P$ . A non-zero vector  $m \in M_\mu$  is said to be of *weight*  $\mu$ . If  $m$  is annihilated by all  $e_{i,l}$  ( $(i, l) \in I^\infty$ ),  $m$  is called a *maximal vector* of weight  $\mu$ .

A  $U_q(\mathfrak{g})$ -module  $V$  is called a *highest weight module with highest weight*  $\lambda$  if there is a non-zero vector  $v_\lambda$  in  $V$  such that

- (i)  $q^h v_\lambda = q^{\langle h, \lambda \rangle} v_\lambda$  for all  $h \in P^\vee$ ,
- (ii)  $e_{i,l} v_\lambda = 0$  for all  $(i, l) \in I^\infty$ ,
- (iii)  $V = U_q(\mathfrak{g})v_\lambda$ .

Such a vector  $v_\lambda$  is called a *highest weight vector* with highest weight  $\lambda$ . Note that  $V_\lambda = \mathbf{Q}(q)v_\lambda$  and  $V$  has a weight space decomposition  $V = \bigoplus_{\mu \leq \lambda} V_\mu$ . If a  $U_q(\mathfrak{g})$ -module  $M$  has a weight space decomposition, a maximal vector of weight  $\lambda$  would generate a highest weight submodule with highest weight  $\lambda$ .

For  $\lambda \in P$ , let  $J(\lambda)$  be the left ideal of  $U_q(\mathfrak{g})$  generated by the elements  $q^h - q^{\langle h, \lambda \rangle} \mathbf{1}$  ( $h \in P^\vee$ ) and  $e_{i,l}$  ( $(i, l) \in I^\infty$ ). Set  $M(\lambda) := U_q(\mathfrak{g})/J(\lambda)$ . Then  $M(\lambda)$  becomes a  $U_q(\mathfrak{g})$ -module, called the *Verma module*, via left multiplication. The following properties of  $M(\lambda)$  are straightforward consequences of the definition.

**Proposition 4.1.** *The Verma module  $M(\lambda)$  satisfies the following properties.*

- (a)  $M(\lambda)$  is a highest weight module with highest weight  $\lambda$ .
- (b)  $M(\lambda)$  has a unique maximal submodule.
- (c)  $M(\lambda)$  is a free  $U^-$ -module of rank 1.
- (d) Every highest weight module with highest weight  $\lambda$  is a quotient module of  $M(\lambda)$ .

Let  $R(\lambda)$  be the unique maximal submodule of  $M(\lambda)$  and let  $V(\lambda) := M(\lambda)/R(\lambda)$  the irreducible quotient of  $M(\lambda)$ , which is also a highest weight module with highest weight  $\lambda$ . The following proposition is one of the most important ingredients of the integrable representation theory of quantum Borchers-Bozec algebras.

**Proposition 4.2.** *Let  $\lambda \in P^+$  be a dominant integral weight and let  $V(\lambda) = U_q(\mathfrak{g}) v_\lambda$  be the irreducible highest weight module with highest weight  $\lambda$  and highest weight vector  $v_\lambda$ .*

*Then the following statements hold.*

- (a) *If  $i \in I^{re}$ , then  $f_i^{\langle h_i, \lambda \rangle + 1} v_\lambda = 0$ .*
- (b) *If  $i \in I^{im}$  and  $\langle h_i, \lambda \rangle = 0$ , then  $f_{ik} v_\lambda = 0$  for all  $k > 0$ .*

**Proof.** (a) If  $i \in I^{re}$ , by the  $U_q(sl_2)$ -representation theory, it is known that  $e_i f_i^{\langle h_i, \lambda \rangle + 1} v_\lambda = 0$ . If  $(j, l) \neq i$ , then  $j \neq i$  and by (2.2), we have

$$e_{jl} f_i^{\langle h_i, \lambda \rangle + 1} v_\lambda = f_i^{\langle h_i, \lambda \rangle + 1} e_{jl} v_\lambda = 0.$$

Hence if  $f_i^{\langle h_i, \lambda \rangle + 1} v_\lambda \neq 0$ , it is a maximal vector and would generate a highest weight submodule of  $V(\lambda)$  with highest weight  $\lambda - (\langle h_i, \lambda \rangle + 1)\alpha_i < \lambda$ . Since  $V(\lambda)$  is irreducible, this is a contradiction.

(b) For each  $(i, k) \in I^\infty$ , we will show that  $e_{jl} f_{ik} v_\lambda = 0$  for all  $(j, l) \in I^\infty$ .

If  $j \neq i$ , as in (2.2), we have

$$e_{jl} f_{ik} v_\lambda = f_{ik} e_{jl} v_\lambda = 0 \quad \text{for all } l > 0. \quad (4.1)$$

So we will concentrate on the case when  $j = i$ . In this case, Lemma 2.1 yields

$$\sum_{\substack{m+n=k \\ n+s=l}} q_{(i)}^{n(m-s)} \tau_{in} e_{is} f_{im} K_i^{-n} = \sum_{\substack{m+n=k \\ n+s=l}} q_{(i)}^{-n(m-s)} \tau_{in} f_{im} e_{is} K_i^n. \quad (4.2)$$

As in Lemma 2.1, let  $L$  (resp.  $R$ ) denote the left-hand side (resp. right-hand side) of the above equation. We will prove our claim in 3 steps. Note that, since  $\langle h_i, \lambda \rangle = 0$ , we have  $K_i^{\pm 1} v_\lambda = v_\lambda$ .

**Step 1:** Suppose  $k = l$ . In this case, we have  $m = s$  and

$$R(v_\lambda) = \sum_{n=0}^k \tau_{in} f_{i,k-n} e_{i,k-n} v_\lambda = \tau_{ik} v_\lambda.$$

On the other hand,

$$L(v_\lambda) = \sum_{n=0}^k \tau_{in} e_{i,k-n} f_{i,k-n} v_\lambda = \sum_{n=0}^{k-1} \tau_{in} e_{i,k-n} f_{i,k-n} v_\lambda + \tau_{ik} v_\lambda.$$

It follows that

$$\sum_{n=0}^{k-1} \tau_{in} e_{i,k-n} f_{i,k-n} v_\lambda = 0.$$

Hence by induction on  $k$ , we obtain

$$e_{ik} f_{ik} v_\lambda = 0 \quad \text{for all } k > 0. \quad (4.3)$$

**Step 2:** Suppose  $k < l$ . In this case, we have

$$R(v_\lambda) = \sum_{n=0}^k q_{(i)}^{-n(k-l)} \tau_{in} f_{i,k-n} e_{i,l-n} v_\lambda = 0$$

because  $l - n > 0$  for all  $n = 0, 1, \dots, k$ , which implies

$$L(v_\lambda) = \sum_{n=0}^k q_{(i)}^{n(k-l)} \tau_{in} e_{i,l-n} f_{i,k-n} v_\lambda = 0.$$

If  $k = 1$ , we have  $l > 1$  and

$$0 = \tau_{i,0} e_{i,l} f_{i,1} v_\lambda + q_{(i)}^{1-l} \tau_{i,1} e_{i,l-1} f_{i,0} v_\lambda = \tau_{i,0} e_{i,l} f_{i,1} v_\lambda.$$

Hence  $e_{i,l} f_{i,1} v_\lambda = 0$  for  $l > 1$ . Note that (4.3) gives  $e_{i,1} f_{i,1} v_\lambda = 0$ . Therefore, we get  $e_{i,l} f_{i,1} v_\lambda = 0$  for all  $l \geq 1$ , which implies  $f_{i,1} v_\lambda = 0$ , for otherwise, it would generate a highest weight submodule with highest weight  $\lambda - \alpha_i < \lambda$ .

Assume that

$$f_{i,1} v_\lambda = 0, \dots, f_{i,k-1} v_\lambda = 0. \quad (4.4)$$

Then we have

$$\begin{aligned} 0 = L(v_\lambda) &= \sum_{n=0}^k q_{(i)}^{n(k-l)} \tau_{in} e_{i,l-n} f_{i,k-n} v_\lambda \\ &= \tau_{i,0} e_{i,l} f_{i,k} v_\lambda + q_{(i)}^{k-l} \tau_{i,1} e_{i,l-1} f_{i,k-1} v_\lambda + q_{(i)}^{2(k-l)} \tau_{i,2} e_{i,l-2} f_{i,k-2} v_\lambda \\ &\quad + \dots + q_{(i)}^{(k-1)(k-l)} \tau_{i,k-1} e_{i,l-k+1} f_{i,1} v_\lambda + q_{(i)}^{k(k-l)} \tau_{i,k} e_{i,l} f_{i,0} v_\lambda \\ &= \tau_{i,0} e_{i,l} f_{i,k} v_\lambda. \end{aligned}$$

Therefore, combined with (4.3), we obtain

$$e_{i,l} f_{i,k} v_\lambda = 0 \text{ for all } l \geq k.$$

**Step 3:** Suppose  $k > l$ . Then, since  $k - l < k$ , by the induction hypothesis (4.4), the relation (4.2) implies

$$R(v_\lambda) = \sum_{n=0}^l q_{(i)}^{-n(k-l)} \tau_{i,n} f_{i,k-n} e_{i,l-n} v_\lambda = q_{(i)}^{-l(k-l)} f_{i,k-l} v_\lambda = 0.$$

On the other hand, by the induction hypothesis (4.4) again, we have

$$\begin{aligned} L(v_\lambda) &= \sum_{n=0}^l q_{(i)}^{n(k-l)} \tau_{i,n} e_{i,l-n} f_{i,k-n} v_\lambda \\ &= \tau_{i,0} e_{i,l} f_{i,k} v_\lambda + q_{(i)}^{k-l} \tau_{i,1} e_{i,l-1} f_{i,k-1} v_\lambda + \dots + q_{(i)}^{l(k-l)} \tau_{i,l} e_{i,0} f_{i,k-l} v_\lambda \\ &= \tau_{i,0} e_{i,l} f_{i,k} v_\lambda = 0, \end{aligned}$$

which yields  $e_{i,l} f_{i,k} v_\lambda = 0$  for all  $l < k$ .

Therefore, combining (4.1), (Step 1) and (Step 2), we obtain

$$e_{j,l} f_{i,k} v_\lambda = 0 \quad \text{for all } (j, l) \in I^\infty.$$

Hence by induction on  $k$ , we conclude  $f_{i,k} v_\lambda = 0$  for all  $k > 0$ , which proves our claim.  $\square$

**Example 4.3.** In this example, we briefly describe the structure of the irreducible highest weight module  $V(\lambda)$  over the quantum string algebra  $U_{(i)}$  for  $i \in I^{\text{im}}$ . If  $\langle h_i, \lambda \rangle = 0$ ,  $V(\lambda)$  is the 1-dimensional trivial representation.

If  $\langle h_i, \lambda \rangle > 0$ , then  $V(\lambda)$  is isomorphic to the Verma module  $M(\lambda)$  and  $\mathbf{B}(\lambda) := \{b v_\lambda \mid b \in \mathbf{B}\}$  is a basis of  $V(\lambda)$ , where  $\mathbf{B}$  is the basis of  $U_{(i)}^-$  given in Example 2.3 and Example 2.4.

**Proposition 4.4.** Let  $\lambda \in P^+$  be a dominant integral weight and let  $V(\lambda) = U_q(\mathfrak{g})v_\lambda$  be the irreducible highest weight  $U_q(\mathfrak{g})$ -module with highest weight  $\lambda$  and highest weight vector  $v_\lambda$ . For  $i \in I^{\text{im}}$  and  $\mu \in \text{wt}(V(\lambda))$ , the following statements hold.

- (a)  $\langle h_i, \mu \rangle \geq 0$ .
- (b) If  $\langle h_i, \mu \rangle = 0$ , then  $V(\lambda)_{\mu - l\alpha_i} = 0$  for all  $l > 0$ .
- (c) If  $\langle h_i, \mu \rangle = 0$ , then  $f_{il}(V(\lambda)_\mu) = 0$ .
- (d) If  $\langle h_i, \mu \rangle \leq -la_{ii}$ , then  $e_{il}(V(\lambda)_\mu) = 0$ .

**Proof.** (a) Since  $\mu \in \text{wt}(V(\lambda))$ ,  $\mu = \lambda - \beta$  for some  $\beta \in Q_+$ . Write  $\beta = l_1\alpha_{i_1} + \cdots + l_r\alpha_{i_r}$ . Then since  $a_{ij} \leq 0$  for all  $j \in I$ , we have

$$\langle h_i, \mu \rangle = \langle h_i, \lambda \rangle - \langle h_i, \beta \rangle = \langle h_i, \lambda \rangle - (l_1 a_{i, i_1} + \cdots + l_r a_{i, i_r}) \geq 0.$$

(b) If  $\langle h_i, \mu \rangle = 0$ , then  $\langle h_i, \lambda \rangle = 0$  and  $a_{i, i_k} = 0$  for all  $k = 1, \dots, r$ . Let  $u = f_{i_1, l_1} \cdots f_{i_r, l_r}$  be a monomial such that  $u v_\lambda \in V(\lambda)_\mu$ . Since  $a_{i, i_k} = 0$  for all  $k = 1, \dots, r$ , by the defining relation of quantum Borcherds-Bozec algebras,  $f_{il} f_{i_k, l_k} = f_{i_k, l_k} f_{il}$ . Hence  $f_{il} u v_\lambda = u f_{il} v_\lambda = 0$  by Proposition 4.2(b).

(c) Our statement follows immediately from (b).

(d) Suppose  $e_{il}(V(\lambda)_\mu) \neq 0$ . Then  $\mu + l\alpha_i \in \text{wt}(V(\lambda))$  and by (a)

$$0 \leq \langle h_i, \mu + l\alpha_i \rangle = \langle h_i, \mu \rangle + la_{ii} \leq 0,$$

which implies  $\langle h_i, \mu + l\alpha_i \rangle = 0$ . Then by (b),  $\mu = (\mu + l\alpha_i) - l\alpha_i$  would not be a weight of  $V(\lambda)$ , which is a contradiction. Hence  $e_{il}(V(\lambda)_\mu) = 0$ .  $\square$

**Remark 4.5.** There is a sign error in [7, Proposition 2.2 (b)].

## 5. The category $\mathcal{O}_{\text{int}}$

We introduce the notion of integrable representations.

**Definition 5.1.** The category  $\mathcal{O}_{\text{int}}$  consists of  $U_q(\mathfrak{g})$ -modules  $M$  such that

- (i)  $M$  has a weight space decomposition  $M = \bigoplus_{\mu \in P} M_\mu$  with  $\dim_{\mathbf{Q}(q)} M_\mu < \infty$  for all  $\mu \in P$ .
- (ii) there exist finitely many weights  $\lambda_1, \dots, \lambda_s \in P$  such that

$$\text{wt}(M) \subset \bigcup_{j=1}^s (\lambda_j - Q_+),$$

- (iii) if  $i \in I^{\text{re}}$ ,  $f_i$  is locally nilpotent on  $M$ ,
- (iv) if  $i \in I^{\text{im}}$ , we have  $\langle h_i, \mu \rangle \geq 0$  for all  $\mu \in \text{wt}(M)$ ,
- (v) if  $i \in I^{\text{im}}$  and  $\langle h_i, \mu \rangle = 0$ , then  $f_{il}(M_\mu) = 0$  for all  $l \in \mathbf{Z}_{>0}$ ,
- (vi) if  $i \in I^{\text{im}}$  and  $\langle h_i, \mu \rangle \leq -la_{ii}$ , then  $e_{il}(M_\mu) = 0$  for all  $l \in \mathbf{Z}_{>0}$ .

**Remark 5.2.**

- (a) By (ii),  $e_{il} ((i, l) \in I^\infty)$  are locally nilpotent.
- (b) If  $i \in I^{\text{im}}$ ,  $f_{il}$  are not necessarily locally nilpotent.
- (c) The irreducible highest weight  $U_q(\mathfrak{g})$ -module  $V(\lambda)$  with  $\lambda \in P^+$  belongs to the category  $\mathcal{O}_{\text{int}}$ .
- (d) A submodule or a quotient module of a  $U_q(\mathfrak{g})$ -module in the category  $\mathcal{O}_{\text{int}}$  is again an object of  $\mathcal{O}_{\text{int}}$ .
- (e) A finite number of direct sums or a finite number of tensor products of  $U_q(\mathfrak{g})$ -modules in the category  $\mathcal{O}_{\text{int}}$  is again an object of  $\mathcal{O}_{\text{int}}$ .

Now we would like to give a character formula for  $V(\lambda)$  with  $\lambda \in P^+$ . For this purpose, we need some preparation [3]

For a dominant integral weight  $\lambda \in P^+$ , let  $F_\lambda$  be the set of elements of the form  $s = \sum_{k=1}^r l_k \alpha_{i_k}$  ( $r \geq 0$ ) such that

- (i)  $i_k \in I^{\text{im}}$ ,  $l_k \in \mathbf{Z}_{>0}$  for all  $1 \leq k \leq r$ ,
- (ii)  $(\alpha_{i_p}, \alpha_{i_q}) = 0$  for all  $1 \leq p, q \leq r$ ,
- (iii)  $(\alpha_{i_k}, \lambda) = 0$  for all  $1 \leq k \leq r$ .

When  $r = 0$ , we understand  $s = 0$ .

For  $s = \sum s_k \alpha_{i_k} \in F_\lambda$ , we define

$$\begin{aligned}
 d_i(s) &= \begin{cases} \#\{k \mid i_k = i\} & \text{if } i \notin I^{\text{iso}}, \\ \sum_{i_k=i} s_k & \text{if } i \in I^{\text{iso}}, \end{cases} \\
 \epsilon(s) &= \prod_{i \notin I^{\text{iso}}} (-1)^{d_i(s)} \prod_{i \in I^{\text{iso}}} \phi(d_i(s)) \\
 &= (-1)^{\#(\text{supp}(s) \cap I \setminus I^{\text{iso}})} \prod_{i \in I^{\text{iso}}} \phi(d_i(s)),
 \end{aligned} \tag{5.1}$$

where  $\phi(n)$  are given by  $\prod_{k=1}^\infty (1 - q^k) = \sum_{n \geq 0} \phi(n) q^n$ .

Define

$$S_\lambda = \sum_{s \in F_\lambda} \epsilon(s) e^{-s}. \tag{5.2}$$

If  $F_\lambda = \{0\}$ , we understand  $S_\lambda = 1$ .

**Example 5.3.** Let  $i \in I^{\text{im}}$  and consider the quantum string algebra  $U_{(i)}$ .

- (a) If  $\langle h_i, \lambda \rangle > 0$ , then  $F_\lambda = \{0\}$  and  $S_\lambda = 1$ .
- (b) if  $i \in I^{\text{iso}}$  and  $\langle h_i, \lambda \rangle = 0$ , then  $\lambda = 0$ ,  $F_0 = \{l\alpha_i \mid l \geq 0\}$  and  $d_i(l\alpha_i) = l$ . Hence

$$S_0 = \sum_{l \geq 0} \epsilon(l\alpha_i) e^{-l\alpha_i} = \sum_{l \geq 0} \phi(l) e^{-l\alpha_i} = \prod_{k=1}^\infty (1 - e^{-k\alpha_i}).$$

(c) If  $i \in I^{\text{im}} \setminus I^{\text{iso}}$  and  $\langle h_i, \lambda \rangle = 0$ , then  $\lambda = 0$ ,  $F_0 = \{l\alpha_i \mid l \geq 0\}$  and

$$d_i(l\alpha_i) = \begin{cases} 0 & \text{if } l = 0, \\ 1 & \text{if } l \geq 1, \end{cases} \quad \epsilon(l\alpha_i) = \begin{cases} 1 & \text{if } l = 0, \\ -1 & \text{if } l \geq 1, \end{cases}$$

which implies

$$S_0 = 1 - (e^{-\alpha_i} + e^{-2\alpha_i} + \dots) = 1 - e^{-\alpha_i} \frac{1}{1 - e^{-\alpha_i}} = \frac{1 - 2e^{-\alpha_i}}{1 - e^{-\alpha_i}}.$$

Choose a linear functional  $\rho$  on  $\mathfrak{h}$  such that  $\langle h_i, \rho \rangle = 1$  for all  $i \in I$ . For each  $w \in W$ , we denote by  $l(w)$  the length of  $w$  and set  $\epsilon(w) = (-1)^{l(w)}$ . The following proposition gives a character formula for  $V(\lambda)$ .

**Proposition 5.4.** [3] *Let  $V = U_q(\mathfrak{g})v_\lambda$  be a highest weight  $U_q(\mathfrak{g})$ -module with highest weight  $\lambda \in P^+$ . If  $V$  satisfies the conditions in Proposition 4.2, then the character of  $V$  is given by the following formula:*

$$\begin{aligned} \text{ch}V &= \frac{\sum_{w \in W} \epsilon(w) e^{w(\lambda + \rho) - \rho w} (S_\lambda)}{\prod_{\alpha \in \Delta_+} (1 - e^{-\alpha})^{\dim \mathfrak{g}_\alpha}} \\ &= \frac{\sum_{w \in W} \sum_{s \in F_\lambda} \epsilon(w) \epsilon(s) e^{w(\lambda + \rho - s) - \rho}}{\prod_{\alpha \in \Delta_+} (1 - e^{-\alpha})^{\dim \mathfrak{g}_\alpha}}. \end{aligned} \quad (5.3)$$

In particular, the character of  $V(\lambda)$  is given by this formula.

**Proof.** The proof given in [3, Theorem 6.1] depends only on the conditions in Proposition 4.2, not on the irreducibility of  $V$ . Hence their argument works for any highest weight  $U_q(\mathfrak{g})$ -module with highest weight  $\lambda \in P^+$  satisfying the conditions in Proposition 4.2.  $\square$

**Remark 5.5.** Here,  $\mathfrak{g}_\alpha$  ( $\alpha \in \Delta_+$ ) denotes the root space of the Borcherds-Bozec algebra  $\mathfrak{g}$  associated with the same Borcherds-Cartan datum as  $U_q(\mathfrak{g})$ .

**Corollary 5.6.** *Let  $V = U_q(\mathfrak{g})v_\lambda$  be a highest weight module with highest weight  $\lambda \in P^+$ . If  $V$  satisfies the conditions in Proposition 4.2, then  $V \cong V(\lambda)$ .*

**Proof.** This is an immediate consequence of Proposition 5.4.  $\square$

**Corollary 5.7.**

- (a) *Let  $V = U_q(\mathfrak{g})v_\lambda$  be a highest weight module with highest weight  $\lambda \in P$ . If  $V$  is an object in the category  $\mathcal{O}_{\text{int}}$ , then  $\lambda \in P^+$  and  $V \cong V(\lambda)$ .*
- (b) *Every simple object in the category  $\mathcal{O}_{\text{int}}$  is isomorphic to some  $V(\lambda)$  with  $\lambda \in P^+$ .*

**Proof.** (a) Suppose that  $V$  is an object in  $\mathcal{O}_{\text{int}}$ . If  $i \in I^{\text{re}}$ ,  $f_i$  is locally nilpotent on  $V$  and by the standard  $U_q(\mathfrak{sl}_2)$ -theory, we have  $\langle h_i, \lambda \rangle \geq 0$ ,  $f_i^{(h_i, \lambda) + 1} v_\lambda = 0$ .

If  $i \in I^{\text{im}}$ , by the condition (iv) in Definition 5.1, we have  $\langle h_i, \lambda \rangle \geq 0$  and  $\langle h_i, \lambda \rangle = 0$  implies  $f_{il} v_\lambda = 0$  for all  $(i, l) \in I^\infty$ .

Hence  $\lambda \in P^+$  and  $V$  satisfies the conditions in Proposition 4.2, which proves our claim.

(b) Let  $V$  be an irreducible  $U_q(\mathfrak{g})$ -module in the category  $\mathcal{O}_{\text{int}}$ . Then  $V$  must be a highest weight module because by Definition 5.1, there is at least one maximal vector in  $V$  and any maximal vector would generate a highest weight submodule. By (a),  $V \cong V(\lambda)$  for some  $\lambda \in P^+$ .  $\square$

We will now prove that every  $U_q(\mathfrak{g})$ -module in the category  $\mathcal{O}_{\text{int}}$  is completely reducible following the outline given in [5] and [6].

Define an *anti*-involution  $\varphi : U_q(\mathfrak{g}) \rightarrow U_q(\mathfrak{g})$  by

$$e_{il} \mapsto f_{il}, \quad f_{il} \mapsto e_{il}, \quad q^h \mapsto q^{-h} \quad \text{for all } (i, l) \in I^\infty, \quad h \in P^\vee.$$

Let  $M = \bigoplus_{\mu \in P} M_\mu$  be a  $U_q(\mathfrak{g})$ -module in the category  $\mathcal{O}_{\text{int}}$  and set

$$M^* := \bigoplus_{\mu \in P} M_\mu^*, \quad \text{where } M_\mu^* = \text{Hom}_{\mathbf{Q}(q)}(M_\mu, \mathbf{Q}(q)).$$

We define a  $U_q(\mathfrak{g})$ -module structure on  $M^*$  by

$$\langle x \psi, m \rangle := \langle \psi, \varphi(x) m \rangle \quad \text{for } x \in U_q(\mathfrak{g}), \quad \psi \in M^*, \quad m \in M.$$

**Lemma 5.8.** *Let  $M = \bigoplus_{\mu \in P} M_\mu$  be a  $U_q(\mathfrak{g})$ -module in the category  $\mathcal{O}_{\text{int}}$ .*

- (a) *There is a canonical isomorphism  $(M^*)^* \cong M$  as  $U_q(\mathfrak{g})$ -modules.*
- (b)  *$(M^*)_\mu = M_\mu^*$  for all  $\mu \in P$ .*
- (c)  *$\text{wt}(M^*) = \text{wt}(M)$ .*
- (d)  *$M^*$  is an object of  $\mathcal{O}_{\text{int}}$ .*

**Proof.** (a) They are canonically isomorphic as vector spaces. Hence it suffices to verify that the canonical linear map commutes with the  $U_q(\mathfrak{g})$ -module action, which is straightforward.

(b) is clear and (c) follows from (b).

(d) Clearly,  $M^*$  satisfies the conditions (i) - (iv) in Definition 5.1. Thus we will check the conditions (v) and (vi).

Let  $i \in I^{\text{im}}$ ,  $\langle h_i, \mu \rangle = 0$  and  $\psi \in M_\mu^*$ . Since  $\text{wt}(f_{il}(M_\mu^*)) = \mu - l\alpha_i$ , we have only to check the condition (v) for the vectors  $m \in M_{\mu-l\alpha_i}$ . Since  $\langle h_i, \mu - l\alpha_i \rangle = -la_{ii}$ , by the definition of  $U_q(\mathfrak{g})$ -module action on  $M^*$  and the condition (vi) in Definition 5.1 for  $M$ , we have  $\langle f_{il} \psi, m \rangle = \langle \psi, e_{il} m \rangle = 0$ , which verifies the condition (v) for  $M^*$ .

Similarly, suppose  $i \in I^{\text{im}}$ ,  $\langle h_i, \mu \rangle \leq -la_{ii}$  and let  $\psi \in M_\mu^*$ . Since  $\text{wt}(e_{il} \psi) = \mu + l\alpha_i$ , we take a non-zero vector  $m \in M_{\mu+l\alpha_i}$ . Then we must have  $\langle h_i, \mu + l\alpha_i \rangle \geq 0$ . On the other hand,  $\langle h_i, \mu + l\alpha_i \rangle = \langle h_i, \mu \rangle + la_{ii} \leq 0$ , which implies  $\langle h_i, \mu + l\alpha_i \rangle = 0$ . Hence by the condition (v) in Definition 5.1 for  $M$ , we have

$$\langle e_{il} \psi, m \rangle = \langle \psi, f_{il} m \rangle = 0 \quad \text{for all } l > 0,$$

as desired.  $\square$

Let us proceed to prove the complete reducibility of  $U_q(\mathfrak{g})$ -modules in the category  $\mathcal{O}_{\text{int}}$ . Let  $M$  be a  $U_q(\mathfrak{g})$ -module in the category  $\mathcal{O}_{\text{int}}$  and let  $v_\lambda$  be a maximal vector of weight  $\lambda$  in  $M$ . Then the submodule  $V$  generated by  $v_\lambda$  is isomorphic to  $V(\lambda)$  and  $\lambda \in P^+$ . Take a linear functional  $\psi_\lambda \in M_\lambda^*$  such that  $\langle \psi_\lambda, v_\lambda \rangle = 1$ ,  $\langle \psi_\lambda, M_\mu \rangle = 0$  for all  $\mu \neq \lambda$ . Then it is easy to verify that  $\psi_\lambda$  is a maximal vector of weight  $\lambda$  in  $M^*$ . Hence the submodule  $W := U_q(\mathfrak{g}) \psi_\lambda$  of  $M^*$  is isomorphic to  $V(\lambda)$ .

The following Lemma is a critical ingredient in proving our main result.

**Lemma 5.9.** *Let  $M$  be a  $U_q(\mathfrak{g})$ -module in the category  $\mathcal{O}_{\text{int}}$  and let  $V$  be the submodule of  $M$  generated by a maximal vector  $v_\lambda$  of weight  $\lambda$ . Then we have*

$$M \cong V \oplus M/V.$$

**Proof.** Consider the short exact sequence

$$0 \longrightarrow V \xhookrightarrow{\iota} M \longrightarrow M/V \longrightarrow 0. \quad (5.4)$$

We need to prove the sequence (5.4) splits.

Let  $W = U_q(\mathfrak{g})\psi_\lambda$  be the submodule of  $M^*$  described above. Take the dual of the inclusion  $W \hookrightarrow M^*$  to get a homomorphism  $M^{**} \rightarrow W^*$ . Let  $\eta : M \xrightarrow{\sim} M^{**} \rightarrow W^*$  be the composition of homomorphisms to obtain

$$\eta \circ \iota : V \hookrightarrow M \xrightarrow{\sim} M^{**} \rightarrow W^*.$$

Note that the image of the maximal vector of  $V$  is non-zero under the homomorphism  $\eta \circ \iota$ . Since  $W^* \cong W \cong V(\lambda) \cong V$ , by Schur's Lemma,  $\eta \circ \iota$  is an isomorphism. Hence we get a homomorphism

$$(\eta \circ \iota)^{-1} \circ \eta : M \xrightarrow{\sim} M^{**} \rightarrow W^* \rightarrow V.$$

Clearly, the composition of the homomorphisms

$$V \xhookrightarrow{\iota} M \xrightarrow{\eta} W^* \xrightarrow{(\eta \circ \iota)^{-1}} V$$

is the identity and hence the exact sequence (5.4) splits.  $\square$

Now we prove that the category  $\mathcal{O}_{\text{int}}$  is semi-simple.

**Theorem 5.10.** *Every  $U_q(\mathfrak{g})$ -module in the category  $\mathcal{O}_{\text{int}}$  is completely reducible.*

**Proof.** Let  $M$  be a  $U_q(\mathfrak{g})$ -module in the category  $\mathcal{O}_{\text{int}}$ . We will prove our claim in two steps.

**Step 1:** If  $M = U_q(\mathfrak{g})V$  for some finite-dimensional  $U^{\geq 0}$ -submodule  $V$ , then  $M$  is completely reducible.

We will use induction on the dimension of  $V$ . If  $V = 0$ , our claim is trivial. If  $V \neq 0$ , there a maximal vector  $v$  in  $V$  with a dominant integral weight  $\lambda \in P^+$ . Then the submodule  $W$  generated by  $v$  is isomorphic to  $V(\lambda)$  and by Lemma 5.9, we have  $M \cong W \oplus M/W$ . Since  $M/W \cong U_q(\mathfrak{g})(V/V \cap W)$  and  $\dim_{\mathbf{Q}(q)}(V/V \cap W) < \dim_{\mathbf{Q}(q)} V$ ,  $M/W$  is completely reducible.

**Step 2:** For every  $v \in M$ , set  $V(v) := U^{\geq 0}v$ , which is a finite-dimensional  $U^{\geq 0}$ -module due to the condition (ii) in Definition 5.1. By Step 1,  $U_q(\mathfrak{g})v = U_q(\mathfrak{g})V(v)$  is completely reducible. Hence  $M = \sum_{v \in M} U_q(\mathfrak{g})v$  is a sum of irreducible  $U_q(\mathfrak{g})$ -submodules. Now by a general argument on semi-simplicity [4, Proposition 3.12], a sum of irreducible submodules is a direct sum. Hence we conclude  $M$  is completely reducible.  $\square$

As an immediate consequence, we obtain the following corollary.

**Corollary 5.11.** *Let  $M$  be a  $U_q(\mathfrak{g})$ -module in the category  $\mathcal{O}_{\text{int}}$  and let  $U_{(i)}$  be the quantum string subalgebra corresponding to  $i \in I$ .*

- (a) *If  $i \in I^{\text{re}}$ ,  $M$  is isomorphic to a direct sum of finite-dimensional irreducible  $U_q(\mathfrak{sl}_2)$ -modules.*
- (b) *If  $i \in I^{\text{im}}$ ,  $M$  is isomorphic to a direct sum of 1-dimensional trivial modules and infinite-dimensional irreducible highest weight modules over  $U_{(i)}$ .*

**Proof.** For each  $i \in I$ , using the same argument in this section, one can verify that  $M$  is completely reducible as a  $U_{(i)}$ -module. Our assertions follow from the observation in Example 4.3.  $\square$



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