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Permutation modules for cellularly stratified algebras

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ABSTRACT

Permutation modules play an important role in the representation theory of the symmetric group. Hartmann and Paget defined permutation modules for Brauer algebras. We generalise their construction to a wider class of algebras, namely cellularly stratified algebras, satisfying certain conditions. We give a decomposition into indecomposable summands, the Young modules, and show that permutation modules and Young modules admit cell filtrations (with well-defined filtration multiplicities). Partition algebras are shown to satisfy the given conditions, provided the characteristic of the underlying field is large enough. Thus we obtain a definition of permutation modules for partition algebras as an application.

1. Introduction

The Specht modules S^λ are cornerstones of the representation theory of symmetric groups Σ_r. In characteristic zero, they form a complete set of simple modules ([10, Theorem 3]). In arbitrary characteristic p, the simple modules occur as top quotients $S^\lambda/S^\lambda \cap S^{\lambda^\perp}$ of Specht modules, in case λ is a p-regular partition of r ([10, Theorem 2]); for p-singular partitions λ, $S^\lambda/S^\lambda \cap S^{\lambda^\perp}$ is zero. In the more general case of cellular algebras, introduced by Graham and Lehrer [4] in 1996, the cell modules $\Theta(\lambda)$ adopt the role of Specht modules S^λ or their duals S_λ.

Another cornerstone in the representation theory of symmetric groups are the permutation modules $M^\lambda = k\Sigma_r \underset{k\Sigma_\lambda}{\otimes} k$. By James' Submodule Theorem ([10, Theorem 1]), M^λ has a unique direct summand Y^λ, called Young module, containing S^λ as a submodule. Since Young modules are self-dual (cf. [3, 2.2.1 (b)]), Y^λ can also be characterised as the only direct summand of M^λ with quotient S_λ. Young modules

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for different partitions are non-isomorphic ([12, Theorem 3.1 (iii)]). All direct summands of M^λ are Young modules Y^μ, with $\mu \leq \lambda$ and Y^λ appears exactly once ([12, Theorem 3.1 (i)]).

Cellularly stratified algebras, introduced by Hartmann, Henke, König and Paget ([5]) in 2010, are cellular algebras with additional structure. The aim of this article is to generalise the well-known results about permutation modules for symmetric groups to cellularly stratified algebras containing group algebras of symmetric groups, or their Hecke algebras, as subalgebras. We will need to make some further structural assumptions (see Section 20) for the results to hold. Young modules for cellularly stratified algebras have already been used in [5]. They were defined abstractly via iterated universal extensions. While this definition is useful for theoretical considerations, the construction of iterated universal extensions might be hard in examples. Extending the construction of Young modules for Brauer algebras of Hartmann and Paget ([8]), we present an explicit construction of Young modules (Theorem 1), which coincides, under additional assumptions stated in Section 20, with the abstract definition in [5] (Corollary 23). These assumptions are satisfied by Brauer algebras and partition algebras, thus we completely recover the results from [8] in a more general setting. This provides new proofs for some results of [5], e.g. a method of finding all indecomposable (relative) projective modules ([5, Proposition 12.3]) and Schur-Weyl duality ([5, Theorem 13.1]). The fact that two Young modules with different indices are non-isomorphic follows from the construction (Corollary 19).

The structural main result of this article is the decomposition of permutation modules $M(l, \lambda)$ into Young modules $Y(m, \mu)$ (Theorem 4). In order to decompose permutation modules for symmetric groups, James used Schur algebras via Schur-Weyl duality and PIMs. There is a Schur-Weyl duality between cellularly stratified algebras and certain quasi-hereditary algebras, which can be regarded as Schur algebras associated to the cellularly stratified algebras, by [5, Theorem 13.1].

Our homological main result is to show that the Young modules $Y(l, \lambda)$ admit filtrations by cell modules (Theorem 2) and are relative projective in the category $\mathcal{F}(\Theta)$ of modules admitting cell filtrations (Theorem 3). These statements hold provided the cellularly stratified algebra satisfies the additional assumptions stated in Section 20. This generalises a result from Hemmer and Nakano [7, Proposition 4.1.1] for Hecke algebras and enables us to prove the analogue of James' theorem on the decomposition of permutation modules.

This article was inspired by the results of Hartmann and Paget [8] for Brauer algebras. We apply the theory developed here to Brauer algebras (Section 5.1) and recover their results (Theorem 5), thus providing new proofs.

Further applications to partition algebras (Section 5.2) show that, provided the characteristic of the field is large enough, we can construct permutation modules for partition algebras with the desired properties (Theorem 6). In order to have the homological Hemmer-Nakano-type results, we need filtrations of restrictions of cell modules to symmetric groups ([20, Theorem 1]) and filtrations of restrictions of permutation modules to symmetric groups. In Proposition 29 we show that the restriction of a permutation module to a group algebra of a symmetric group is isomorphic to a direct sum of permutation modules over this symmetric group. In the appendix, there is an example (B) and a `GAP` algorithm (C) to compute the occurring permutation modules.

The approach fails for BMW algebras, the third main example for cellularly stratified algebras in [5], since the appearing Hecke algebras are not subalgebras of BMW algebras. However, this is satisfied for q-Brauer algebras, another deformation of Brauer algebras, and there is hope that the theory applies in this case.

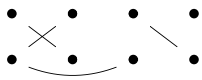
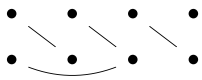
2. Preliminaries

In this article, we study a large class of abstract algebras. These are cellularly stratified algebras with certain extra conditions. The definition and some preliminary structural properties of cellularly stratified algebras can be found in Subsection 2.2, the additional assumptions are, in full detail, stated in Section 4. Throughout this article, our main examples will be partition algebras $P_k(r,\delta)$ with $\delta \neq 0$, which we discuss in Subsections 2.1 and 2.3. Important steps in our arguments will be illustrated by small partition algebras.

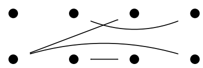
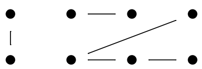
2.1. Partition algebras

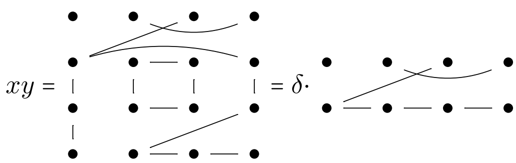
Partition algebras were independently defined by Jones [13] and Martin [16] to describe the Potts model in statistical mechanics. They are diagram algebras containing, as subalgebras, group algebras of symmetric groups and Brauer algebras. For further details on partition algebras, we advise the reader to see, for example, [23] or [9].

Definition 1. Let k be an algebraically closed field of arbitrary characteristic. Let $r \in \mathbb{N}$ and $\delta \in k$. The partition algebra $P_k(r,\delta)$ is the algebra with basis given by all set partitions of $\{1,\ldots,r,1',\ldots,r'\}$. To each set partition, we associate an equivalence class of diagrams consisting of two rows of r dots each. Two dots a and b are connected via a path $a - \ldots - b$ if and only if they belong to the same part of the set partition. Two diagrams are equivalent, if they correspond to the same set partition. Multiplication is given by concatenation of diagrams. Parts which are not connected to either top or bottom row (called *inner circles*) are replaced by a factor $\delta \in k$.

Example *(Equivalence of diagrams).* The set partition $\{\{1,2'\},\{2,1',3'\},\{3,4'\},\{4\}\}$ corresponds to the diagram  with path $2-1'-3'$ as well as to the diagram  with path $2-3'-1'$, and the diagrams are equivalent to each other.

We choose to write all diagrams such that the paths are ordered decreasingly with respect to the order $r > r-1 > \ldots > 1 > 1' > 2' > \ldots > r'$, like in the first diagram of the above example.

Example *(Concatenation of diagrams).* Consider the diagrams $x =$  and $y =$  in $P_k(4,\delta)$. Their product is



Note that multiplication of diagrams can decrease the number of *propagating parts*, i.e. parts connecting top and bottom row, but never increase the number of propagating parts.

2.2. Cellularly stratified algebras

Let k be an algebraically closed field, r a natural number and A an associative k-algebra. We denote the symmetric group on r letters by Σ_r; its Iwahori-Hecke algebra is denoted by $\mathcal{H}_{k,q}(\Sigma_r)$, for some unit $q \in k$. Let h be the smallest integer such that $\sum_{i=0}^{h-1} q^i = 0$. If $q = 1$, then $h = \mathrm{char} k$. If q is a primitive $n$throot of unity, then $h = n$.

Definition 2 *([5], Definition 2.1).* An algebra A is called *cellularly stratified* if the following holds.

(1) For each $l = 0, ..., r$ there is a cellular algebra B_l and a vector space V_l such that $A = \bigoplus_{l=0}^{r} B_l \otimes_k V_l \otimes_k V_l$ as a vector space, respecting within each layer the multiplication of A, i.e. A is an iterated inflation of the cellular algebras B_l along the vector spaces V_l as defined in [14].
(2) For all $l = 0, ..., r$ there are elements $u_l, v_l \in V_l \setminus \{0\}$ such that $e_l := 1_{B_l} \otimes u_l \otimes v_l$ is an idempotent and $e_l e_{l'} = e_l = e_{l'} e_l$ for all $l' \geq l$.

The tuple $(B_0, V_0, ..., B_r, V_r)$ is called *stratification data* of A.

It follows from the first part of the definition that A is cellular with a chain of two-sided ideals

$$0 = J_{-1} \subseteq J_0 \subseteq J_1 \subseteq ... \subseteq J_r = A$$

such that $J_l/J_{l-1} = B_l \otimes_k V_l \otimes_k V_l$ as a non-unital algebra ([14, Proposition 3.1 and § 3.2]) which we call the *lth layer* of A, and $J_l = Ae_lA$ ([5, Lemma 2.2]). The product of $x \in J_l \setminus J_{l-1}$ and $y \in J_{l'} \setminus J_{l'-1}$ lies in J_t, where $t = \min\{l, l'\}$ by [14, § 3.2]. The iterated inflation structure tells us that multiplication within a layer $B_l \otimes_k V_l \otimes_k V_l$ is given by

$$(b \otimes x \otimes y)(b' \otimes x' \otimes y') = (b\varphi(y, x')b' \otimes x \otimes y') + \textit{lower terms}$$

where *lower terms* refers to elements from layers with smaller index and φ is a bilinear form with $\varphi(u_l, v_l) = 1 = \varphi(v_l, u_l)$ coming from the inflation data, cf [14, § 3.2] and [5, § 2.1].

There is an involution j on A, which is compatible with the involutions i_n of the input algebras B_n, via $j(b \otimes x \otimes y) = i_n(b) \otimes y \otimes x$ for $b \in B_n$ and $x, y \in V_n$, cf. [5, § 2.1].

A reader who is not familiar with cellularly stratified algebras might find it helpful to check the example of cellular stratification of partition algebras in Subsection 2.3 before reading the rest of this subsection.

We will make excessive use of the following lemma, whose proof will be sketched here for completeness.

Lemma 3 *([5], Lemma 2.3). If A is cellularly stratified, then*

$$B_l \simeq e_l A e_l / e_l J_{l-1} e_l$$

with 1_{B_l} mapped to e_l.

Proof. As an algebra, $e_l A e_l / e_l J_{l-1} e_l$ is isomorphic to $(e_l + J_{l-1})(J_l/J_{l-1})(e_l + J_{l-1})$. Using the fact that $J_l/J_{l-1} \simeq B_l \otimes_k V_l \otimes_k V_l$ and the multiplication in A, we get that the elements of $(e_l + J_{l-1})(J_l/J_{l-1})(e_l + J_{l-1})$ are of the form

$$\varphi(v_l, x) b \varphi(y, u_l) \otimes u_l \otimes v_l + J_{l-1}$$

with $b \in B_l$ and $x, y \in V_l$. Choosing $x = u_l$ and $y = v_l$, we obtain any element in $B_l \otimes \langle u_l \rangle \otimes \langle v_l \rangle + J_{l-1}$. Thus, the map $B_l \to (e_l + J_{l-1})(J_l/J_{l-1})(e_l + J_{l-1})$ given by $b \mapsto b \otimes u_l \otimes v_l + J_{l-1}$ is bijective. Multiplication in A shows that this map is a homomorphism. □

Throughout this article, we assume that the input algebras B_l are isomorphic to subalgebras of $e_l A e_l$.

Corollary 4. *The input algebra B_l is isomorphic to a subalgebra of $e_l A e_l$ if and only if the multiplication map*

$$\begin{aligned} m_l : (B_l \otimes_k V_l \otimes_k V_l) \times (B_l \otimes_k V_l \otimes_k V_l) &\to e_l A e_l \\ (b \otimes x \otimes y, b' \otimes x' \otimes y') &\mapsto (b\varphi(y, x')b' \otimes x \otimes y') + \text{lower terms} \end{aligned}$$

restricted to $(B_l \otimes \langle u_l\rangle \otimes \langle v_l\rangle) \times (B_l \otimes \langle u_l\rangle \otimes \langle v_l\rangle)$ *has no lower terms in its image. In this case,*

$$be_l = b = e_l b$$

for all $b \in B_l$*.*

Proof. By Lemma 3, we have an isomorphism $B_l \simeq e_l A e_l / e_l J_{l-1} e_l$ sending 1 to $e_l + e_l J_{l-1} e_l$ which factors through $e_l A e_l$:

$$\begin{array}{ccccc} B_l & \to & e_l A e_l & \to & e_l A e_l / e_l J_{l-1} e_l \\ 1 & \mapsto & e_l & \mapsto & e_l + e_l J_{l-1} e_l \end{array}$$

In particular, the map $f : B_l \to e_l A e_l$ is injective. It is an embedding of algebras if and only if $f(b)f(b') = f(bb')$ for all $b, b' \in B_l$, i.e. if and only if

$$(b \otimes u_l \otimes v_l)(b' \otimes u_l \otimes v_l) = bb' \otimes u_l \otimes v_l$$

without any lower terms.

If B_l is isomorphic to a subalgebra of $e_l A e_l$, then $b \in B_l$ can be regarded as an element $b \otimes u_l \otimes v_l \in A$, hence

$$be_l = (b \otimes u_l \otimes v_l)(1 \otimes u_l \otimes v_l) = b\varphi(v_l, u_l) \otimes u_l \otimes v_l = b \otimes u_l \otimes v_l = \varphi(v_l, u_l)b \otimes u_l \otimes v_l = e_l b. \quad \square$$

Remark. If B_l is isomorphic to a subalgebra of $e_l A e_l$, then 1_{B_l} is mapped to e_l, the identity in $e_l A e_l$, so the composition

$$B_l \hookrightarrow e_l A e_l \twoheadrightarrow e_l A e_l / e_l J_{l-1} e_l$$

is an isomorphism.

Proposition 5. *Let A be cellularly stratified such that B_l is isomorphic to a subalgebra of $e_l A e_l$ for some $l < r$. If $B_n \subseteq B_{n+1}$ for all $n < l$, and e_l is fixed by the involution j of A, then the algebra $e_l A e_l$ is cellularly stratified. The stratification data is $(B_0, V_0^l, \dots, B_l, V_l^l)$, where $V_n^l \subseteq V_n$ is a subspace such that $e_n \in B_n \otimes_k V_n^l \otimes_k V_n^l$, i.e. $u_n, v_n \in V_n^l$.*

Proof. Let $A = \bigoplus_{n=0}^{r} B_n \otimes_k V_n \otimes_k V_n$ as a vector space. Then

$$\begin{aligned} e_l A e_l &= e_l \Big(\bigoplus_{n=0}^{r} B_n \otimes_k V_n \otimes_k V_n\Big) e_l \\ &= e_l \Big(\bigoplus_{n=0}^{l} B_n \otimes_k V_n \otimes_k V_n\Big) e_l \\ &\subseteq B_l \oplus \Big(\bigoplus_{n=0}^{l-1} B_n \otimes_k V_n \otimes_k V_n\Big) \end{aligned}$$

where the inclusion holds up to the isomorphism $B_l \simeq B_l \otimes_k \langle u_l\rangle_k \otimes_k \langle v_l\rangle_k$. Assumption 3.4 in [14] shows that

$$(b \otimes x \otimes y)(1_{B_l} \otimes u_l \otimes v_l) \in B_n \otimes x \otimes V_n \oplus \textit{lower layers}$$

and

$$(1_{B_l} \otimes u_l \otimes v_l)(b \otimes x \otimes y) \in B_n \otimes V_n \otimes y \oplus \textit{lower layers}$$

for all $b \in B_n$, $x, y \in V_n$. In particular, right multiplication with e_l fixes the first V_n and left multiplication with e_l fixes the second V_n. Let $V_n^{l,1}$ be the smallest subspace of V_n such that

(I) $(1_{B_l} \otimes u_l \otimes v_l)(b \otimes x \otimes y) \in B_n \otimes V_n^{l,1} \otimes y \oplus \textit{lower layers}$

and let $V_n^{l,2}$ be the smallest subspace of V_n such that

(II) $(b \otimes x \otimes y)(1_{B_l} \otimes u_l \otimes v_l) \in B_n \otimes x \otimes V_n^{l,2} \oplus \textit{lower layers}$

for all $b \in B_n$, $x, y \in V_n$. Then

$$\begin{aligned}(1 \otimes u_l \otimes v_l)(b \otimes x \otimes y)(1 \otimes u_l \otimes v_l) &\overset{(II)}{\equiv} (b' \otimes w_1 \otimes y)(1 \otimes u_l \otimes v_l) \\ &\qquad\qquad \text{with } w_1 \in V_n^{l,1}, b' \in B_n \\ &\overset{(I)}{\equiv} b'' \otimes w_1 \otimes w_2 \\ &\qquad\qquad \text{with } w_2 \in V_n^{l,2}, b'' \in B_n\end{aligned}$$

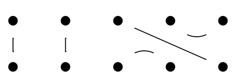
modulo lower layers. Then $j((1 \otimes u_l \otimes v_l)(b \otimes x \otimes y)(1 \otimes u_l \otimes v_l)) \equiv j(b'' \otimes w_1 \otimes w_2) = i_n(b'') \otimes w_2 \otimes w_1$ modulo lower layers. On the other hand,

$$\begin{aligned}j((1 \otimes u_l \otimes v_l)(b \otimes x \otimes y)(1 \otimes u_l \otimes v_l)) &= j(1 \otimes u_l \otimes v_l) j(b \otimes x \otimes y) j(1 \otimes u_l \otimes v_l) \\ &= (1 \otimes u_l \otimes v_l)(i_n(b) \otimes y \otimes x)(1 \otimes u_l \otimes v_l) \\ &= \bar{b} \otimes z_1 \otimes z_2 + \textit{lower terms}\end{aligned}$$

with $z_1 \in V_n^{l,1}$ and $z_2 \in V_n^{l,2}$. It follows that $V_n^{l,1} = V_n^{l,2} =: V_n^l$.

It follows that $e_l A e_l = \bigoplus_{n=0}^{l} B_n^l \otimes_k V_n^l \otimes_k V_n^l$ for some $B_n^l \subseteq B_n$ and $V_n^l \subseteq V_n$. Using $B_n^l \subseteq B_n \subseteq B_l$, we get $be_l = e_l b$ for $b \in B_n$ by Corollary 4. For any $b \in B_n$ we have $b \otimes u_n \otimes v_n = be_n = be_l e_n e_l = e_l b e_n e_l = e_l(b \otimes u_n \otimes v_n)e_l \in e_l(B_n \otimes V_n \otimes V_n)e_l = B_n^l \otimes_k V_n^l \otimes_k V_n^l$. Hence $B_n^l = B_n$. □

Remark. The assumption that $j(e_l) = e_l$ is necessary to show $V_n^{l,1} = V_n^{l,2}$. However, if $j(e_l) \neq e_l$, there might be an isomorphism between $V_n^{l,1}$ and $V_n^{l,2}$ making $e_l A e_l$ a cellularly stratified algebra, e.g. for the Brauer algebra $A = B_k(5,0)$, we have

$e_3 =$  $= 1 \otimes \mathrm{top}(e_3) \otimes \mathrm{bottom}(e_3)$

$= 1 \otimes$  $\otimes$  .

For $n = 1$, the vector spaces $V_n^{l,1}$ and $V_n^{l,2}$ have basis partial diagrams with two arcs, such that each $v \in V_n^{l,1}$ contains the arc in $\mathrm{top}(e_3)$ and each $v \in V_n^{l,2}$ contains the arc in $\mathrm{bottom}(e_3)$. Thus the two vector spaces have trivial intersection. However, there is an isomorphism $V_n^{l,2} \xrightarrow{\sim} V_n^{l,1}$, $v \mapsto v(3\,5)$.

Proposition 6. *Let A be cellularly stratified such that B_l is isomorphic to a subalgebra of e_lAe_l for some $0 \leq l \leq r$. Furthermore, assume that $B_n \subseteq B_{n+1}$ for all $0 \leq n < l$ and e_l is fixed by the involution j of A. Then e_lAe_l has the following (B_l, B_l)-bimodule decomposition:*

$$e_lAe_l \simeq B_l \oplus e_lJ_{l-1}e_l.$$

Proof. By Proposition 5, e_lAe_l is cellularly stratified and $e_lAe_l = \bigoplus_{n=0}^{l} B_n \otimes_k V_n^l \otimes_k V_n^l$ as vector space. On the other hand, the top layer of e_lAe_l is isomorphic to B_l by Lemma 3. Hence we have a vector space decomposition $e_lAe_l \simeq B_l \oplus e_lJ_{l-1}e_l$. The composition of algebra homomorphisms

$$\begin{array}{cccc} e_lAe_l \twoheadrightarrow & e_lAe_l/e_lJ_{l-1}e_l \simeq & B_l \hookrightarrow & e_lAe_l \\ e_l \mapsto & e_l + e_lJ_{l_1}e_l \mapsto & 1 \mapsto & e_l \end{array}$$

respects the (B_l, B_l)-bimodule structure in every step,

$$\begin{array}{cccc} e_lAe_l \twoheadrightarrow & B_l \hookrightarrow & & e_lAe_l \\ be_lb' = bb' \otimes u_l \otimes v_l \mapsto & bb' \mapsto & bb' \otimes u_l \otimes v_l = be_lb' \end{array}$$

so B_l is a direct summand of e_lAe_l as (B_l, B_l)-bimodule and the claim follows. □

2.3. Cellular stratification of partition algebras

Proposition 7 *([5], Proposition 2.6). If $\delta \neq 0$, the partition algebra is cellularly stratified with stratification data*

$$(k, V_0, k, V_1, k\Sigma_2, V_2, \ldots, k\Sigma_r, V_r).$$

The idempotents are given by

$$e_0 := \frac{1}{\delta} \cdot \begin{array}{c} \bullet^1 - \bullet - \ldots - \bullet - \bullet^r \\ \\ \bullet_{1'} - \bullet - \ldots - \bullet - \bullet_{r'} \end{array}, \quad e_n := \begin{array}{ccccc} \bullet^1 & \ldots & \bullet & \bullet^n - \ldots - \bullet^r \\ | & & | & | \\ \bullet_{1'} & \ldots & \bullet & \bullet_{n'} - \ldots - \bullet_{r'} \end{array} \quad \text{for } n \geq 1.$$

In order to define the vector spaces V_l from the stratification data, we need further notation and definitions.

A diagram consisting of only one row with r dots and arbitrary connections is called *partial diagram*. We have to distinguish certain parts from others; we say they are *labelled* and write the dots as empty circles ∘ instead of dots •. When we complete a partial diagram to a full diagram with two rows of dots, the labelled parts become propagating, i.e. they are connected to the other row. We count the parts from left to right, according to the leftmost dot of each part. We define V_n to be the vector space with basis all partial diagrams with exactly n labelled parts (and possibly further unlabelled parts). For example, • ∘ • — • ∘ — ∘ • is a basis element of V_2, with $r = 7$; the labelled singleton ∘ is the first labelled part, the part ∘ – ∘ is the second. Writing the idempotents e_l in the form $1_{B_l} \otimes u_l \otimes v_l$, we get

$$u_l = v_l = \circ \quad \ldots \quad \circ \quad \circ - \circ - \ldots - \circ\ ,$$

the partial diagram with $l-1$ labelled singletons followed by one labelled part of size $r-l+1$. The two-sided ideal $J_l = Ae_lA$ is generated, as a vector space, by the diagrams with at most l propagating parts.

We write $\mathrm{top}(d)$ to denote the top row of a diagram $d \in P_k(r,\delta)$ and $\mathrm{bottom}(d)$ for its bottom row. The permutation induced by the propagating parts is denoted by $\Pi(d)$. It is well-defined by the convention to connect labelled top and bottom row parts via their respective leftmost dots. Thus a diagram d (with exactly l propagating parts) is uniquely determined by the tensor product

$$\Pi(d) \otimes \mathrm{top}(d) \otimes \mathrm{bottom}(d) \in k\Sigma_l \otimes_k V_l \otimes_k V_l$$

as predicted by the vector space decomposition $A = \bigoplus_{l=0}^{r} B_l \otimes V_l \otimes V_l$.

Lemma 8. *For each $0 \leq l \leq r$, there is an algebra isomorphism $P_k(l,\delta) \to e_l P_k(r,\delta) e_l$.*

Proof. This isomorphism is given by attaching $r-l$ dots to the right of both top and bottom row of a diagram in $P_l(l,\delta)$ and connecting the new dots to the rightmost dots of top and bottom row respectively of the original diagram. This map sends each diagram in $P_k(l,\delta)$ (i.e. each basis element) to a diagram (i.e. a basis element) in $e_l P_k(r,\delta) e_l$ and each diagram in $e_l P_k(r,\delta) e_l$ is of this form. Both algebras use the same multiplication. □

The partition algebra $P_k(r,\delta)$ contains the Brauer algebra $B_k(r,\delta)$ and the group algebra $k\Sigma_r$ of the symmetric group Σ_r as subalgebras. The Brauer algebra is the subalgebra with basis given by all diagrams where each dot is connected to exactly one other dot. We call such a connection *(horizontal) arc* if it connects two dots within the same row. A permutation $\sigma \in \Sigma_r$ corresponds to the diagram connecting the ith dot of the top row to the $\sigma(i)$th dot of the bottom row.

Corollary 9. *Each symmetric group algebra $k\Sigma_l$ with $0 \leq l \leq r$ is isomorphic to a subalgebra of $P_k(l,\delta) \simeq e_l P_k(r,\delta) e_l$.*

2.4. Functors

Let A be cellularly stratified with stratification data $(B_0, V_0, \ldots, B_r, V_r)$ where the B_l are isomorphic to group algebras of symmetric groups or their Iwahori-Hecke algebras, such that for each $l \in \{0,\ldots,r\}$ we have an embedding $B_l \hookrightarrow e_l A e_l$ of algebras. This is satisfied for Brauer algebras and partition algebras, but not for BMW-algebras, the third main example of cellularly stratified algebras in [5]. However, it is satisfied for another deformation of Brauer algebras: the q-Brauer algebras defined by Wenzl in [22].

Furthermore, we assume that for each layer l, the idempotent e_l is fixed by the involution j of A. We choose as cell modules for the cellular algebras B_l the dual Specht modules S_λ.

Lemma 10. *Let A be cellularly stratified.*

(1) Any B_l-module has also an $e_l A e_l$-module structure.
(2) Assume additionally, that B_l is isomorphic to a subalgebra of $e_l A e_l$. Then any $e_l A e_l$-module has a B_l-module structure.

Proof. (1) By Lemma 3, B_l is isomorphic to a quotient algebra $e_l A e_l / e_l J_{l-1} e_l$ of $e_l A e_l$. The action of $e_l A e_l$ on M is defined via this quotient map.
(2) The action of $e_l A e_l$ restricts to an action of B_l in this case. □

We need two types of induction and two types of restriction functors, which we define as follows. We attach small examples of these functors in the appendix.

Definition 11. Let A be cellularly stratified such that for each $0 \le l \le r$ the input algebra B_l is isomorphic to a subalgebra of $e_l A e_l$. We define the functors

$$\begin{aligned} ind_l : B_l - \text{mod} &\to A - \text{mod} & Ind_l : B_l - \text{mod} &\to A - \text{mod} \\ M &\mapsto Ae_l \underset{e_l A e_l}{\otimes} M & M &\mapsto Ae_l \underset{B_l}{\otimes} M \\ res_l : A - \text{mod} &\to B_l - \text{mod} & Res_l : A - \text{mod} &\to B_l - \text{mod} \\ N &\mapsto e_l(A/J_{l-1}) \underset{A}{\otimes} N & N &\mapsto e_l A \underset{A}{\otimes} N \simeq e_l N \end{aligned}$$

where J_l denotes the two-sided ideal Ae_lA and $e_l(A/J_{l-1})$ is a short notation for e_lA/e_lJ_{l-1}.

For each B_l-module X, we have $X \simeq B_l \underset{B_l}{\otimes} X \simeq B_l \underset{e_lAe_l}{\otimes} X$, where e_lAe_l acts on both X and B_l via the quotient map $e_lAe_l \twoheadrightarrow e_l(A/J_{l-1})e_l \simeq B_l$. Thus, the layer induction ind_l corresponds to the functor $G_l := Ae_l \underset{e_lAe_l}{\otimes} B_l \underset{e_lAe_l}{\otimes} -$, defined in [5]. Hence, we can apply [5, Lemma 3.4] to get an isomorphism

$$ind_l X \simeq (A/J_{l-1})e_l \underset{e_lAe_l}{\otimes} X$$

of A-modules. We will make extensive use of the isomorphisms

$$ind_l X \simeq G_l X \simeq (A/J_{l-1})e_l \underset{e_lAe_l}{\otimes} X \simeq (A/J_{l-1})e_l \underset{B_l}{\otimes} X$$

without special mention.

Let $N \in A - \text{mod}$. We call the subquotient $(J_n/J_{n-1}) \underset{A}{\otimes} N$ of N the *nth layer* of N. Since $ind_l M \simeq (A/J_{l-1})e_l \underset{e_lAe_l}{\otimes} M \simeq (J_l/J_{l-1}) \underset{A}{\otimes} Ae_l \underset{e_lAe_l}{\otimes} M = (J/J_{l-1}) \underset{A}{\otimes} ind_l M$, $ind_l M$ lives in the lth layer of A. We call ind_l the *layer induction functor*.

The induction functor Ind_l sends a B_l-module M to an A-module with a usually non-zero action of J_{l-1}, i.e. $Ind_l M$ lives in all layers n with $n \le l$.

While res_l removes the lower layers (with $n < l$) of the A-module N, Res_l keeps all layers of the module.

Proposition 12 *([5], Propositions 4.1–4.3; Corollary 7.4; Propositions 8.1 and 8.2). The functor ind_l has the following properties.*

(1) It is exact.
(2) The set $\{ind_l S_\lambda | l = 0, \dots, r; S_\lambda \text{ cell module of } B_l\}$ is a complete set of cell modules for A.
(3) $\mathrm{Hom}_{B_l}(X,Y) \simeq \mathrm{Hom}_A(ind_l X, ind_l Y)$ for all $X, Y \in B_l - \text{mod}$.
(4) $\mathrm{Ext}^i_A(M,N) \simeq \mathrm{Ext}^i_{A/J_l}(M,N)$ for all $i > 0$ and $M, N \in A/J_l - \text{mod}$.
(5) $\mathrm{Ext}^j_{B_l}(X,Y) \simeq \mathrm{Ext}^j_A(ind_l X, ind_l Y)$ for all $j \ge 0$ and $X, Y \in B_l - \text{mod}$.

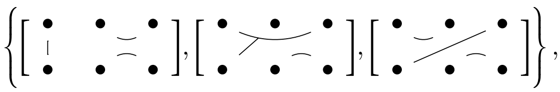
If $l < m$ then

(6) $\mathrm{Hom}_A(ind_l X, ind_m Y) = 0$ for all $X \in B_l - \text{mod}, Y \in B_m - \text{mod}$.
(7) $\mathrm{Ext}^i_A(ind_l X, ind_m Y) = 0$ for all $i \ge 1$ and $X \in B_l - \text{mod}, Y \in B_m - \text{mod}$.

The induction Ind_l is not exact in general and does not send cell modules to cell modules. However, we will give sufficient conditions for Ind_l to send cell filtered modules to cell filtered modules in Section 3. Theorem 3 will tell us that, under additional conditions, Ind_l sends relative projective modules to relative projective modules, cf. Definition 15.

The following properties of the functors are straightforward calculations. The layer restriction res_l is right-exact, but in general not exact. It is left adjoint to $\mathrm{Hom}_{B_l}(e_l(A/J_{l-1}), -)$ and left inverse to both ind_l and Ind_l. The restriction Res_l is exact, since e_lA is projective as right A-module. It is left adjoint to $\mathrm{Hom}_{B_l}(e_lA, -)$ and right adjoint to Ind_l, i.e. we have a triple $(Ind_l, Res_l, \mathrm{Hom}_{B_l}(e_lA, -))$ of adjoint functors. Furthermore, Res_l is left inverse to ind_l, but in general not to Ind_l; the layers added by Ind_l are not removed by Res_l.

For example, if A is the Brauer algebra $B_{\mathbb{C}}(3,\delta)$ with $\delta \neq 0$ and $l = 3$, and X is the trivial $\mathbb{C}\Sigma_3$-module $\mathbb{C}$, then $e_3J_1e_3 = J_1 = Ae_1A$, which consists of all linear combinations of Brauer diagrams with exactly one horizontal arc per row. The left $\mathbb{C}\Sigma_3$-module $Res_3Ind_3\mathbb{C}$ contains $Ae_1A \underset{\mathbb{C}\Sigma_3}{\otimes} \mathbb{C}$ which has a basis



where the brackets denote residue classes containing all three bottom row configurations. In particular, $Ae_1A \underset{\mathbb{C}\Sigma_3}{\otimes} \mathbb{C}$ is non-zero and not isomorphic to X.

Proposition 13. *If X is a cell module of A, then res_lX is a cell module of B_l or zero.*

Proof. Let X be a cell module of A. By Proposition 12, part *(2)*, we have $X \simeq ind_nS_\nu$ for some $1 \leq n \leq r$, where S_ν is a dual Specht module in $B_n-\mathrm{mod}$. So $res_lX \simeq res_lind_nS_\nu \simeq e_l(A/J_{l-1}) \underset{A}{\otimes} (A/J_{n-1})e_n \underset{e_nAe_n}{\otimes} S_\nu \simeq e_l(A/J_m)e_n \underset{e_nAe_n}{\otimes} S_\nu$, where $m = \max\{l-1, n-1\}$. If $n < l$, then $e_n \in J_m = J_{l-1}$ and if $n > l$, then $e_l \in J_m = J_{n-1}$. So, in both cases we have $res_lX = 0$. For $n = l$, we have $res_lind_lS_\nu \simeq e_l(A/J_{l-1}) \underset{A}{\otimes} (A/J_{l-1})e_l \underset{e_lAe_l}{\otimes} S_\nu \simeq e_l(A/J_{l-1})e_l \underset{B_l}{\otimes} S_\nu \simeq S_\nu$. Thus, the layer restriction of a cell module from the same layer is a cell module, while cell modules from other layers vanish under restriction. □

2.5. Further definitions and notation

Let $\Lambda_r := \{(l,\lambda) | 0 \leq l \leq r, \lambda \vdash l'\}$, where l' is the index of the symmetric group related to B_l and $\lambda \vdash l'$ means that λ is a partition of l'. We define an order $\prec$ on Λ_r by setting

$$(n,\nu) \prec (l,\lambda) \Leftrightarrow n \geq l \text{ and if } n = l \text{ then } \nu \leq \lambda \text{ in the dominance order.}$$

Let $(l,\lambda) \in \Lambda_r$ and let M^λ be the corresponding permutation module in $B_l - \mathrm{mod}$.

Definition 14. We call the A-module $M(l,\lambda) := Ind_lM^\lambda$ *permutation module* for A.

Let $\Theta := \{\Theta(l,\lambda) := ind_lS_\lambda \,|\, (l,\lambda) \in \Lambda_r\}$ denote the set of cell modules. The category of A-modules with a cell filtration, i.e. modules M admitting a chain of submodules $M = M_n \supset M_{n-1} \supset \ldots \supset M_1 \supset M_0 = 0$ such that the subquotients M_m/M_{m-1} are isomorphic to cell modules, is denoted by $\mathcal{F}(\Theta)$. The category of B_l-modules admitting a filtration by dual Specht modules is denoted by $\mathcal{F}_l(S)$.

Definition 15 *([5], Definition 11.2)*. Let $M, M' \in \mathcal{F}(\Theta)$. We say that M is *relative projective in* $\mathcal{F}(\Theta)$, if

$$\mathrm{Ext}^1_A(M,N) = 0 \text{ for all } N \in \mathcal{F}(\Theta).$$

A relative projective module $M \in \mathcal{F}(\Theta)$ is the *relative projective cover* of M', if M is minimal with respect to the property that there is an epimorphism $f : M \twoheadrightarrow M'$ with $\ker f \in \mathcal{F}(\Theta)$.

3. Young modules

In this section, we define Young modules as direct summands of permutation modules, following the definitions given for Brauer algebras by Hartmann and Paget, [8]. This allows us to extend the results of James for group algebras of symmetric groups to cellularly stratified algebras whose input algebras are isomorphic to group algebras of symmetric groups or their Hecke algebras.

Theorem 1. *Fix $l \leq r$ and let A be a cellularly stratified algebra with input algebras B_n isomorphic to group algebras of symmetric groups or their Hecke algebras such that $B_n \subseteq B_{n+1}$ for all $0 \leq n < l$ and each B_n embeds into $e_n A e_n$ as subalgebra. Assume further that e_l is fixed by the involution j of A which is compatible with the involutions of the B_n. Then $Ind_l M^\lambda$ has a unique direct summand with quotient isomorphic to $ind_l Y^\lambda$.*

Proof. It is well-known that the B_l-permutation module M^λ decomposes into a direct sum of indecomposable Young modules Y^μ with multiplicities a_μ, where $a_\lambda = 1$ and $a_\mu \neq 0$ implies $\mu \geq \lambda$ ([12, Theorem 3.1]). Therefore, we have $Ind_l M^\lambda = \bigoplus_{(l,\mu)\in\Lambda_r} (Ind_l Y^\mu)^{a_\mu}$. Decompose $Ind_l Y^\lambda$ further into a direct sum of indecomposables $(Ind_l Y^\lambda)\epsilon_i$, given by primitive idempotents $\epsilon_i \in \mathrm{End}_A(Ind_l Y^\lambda)$ such that $\sum_{i=1}^{s} \epsilon_i = 1_{End_A(Ind_l Y^\lambda)}$.

Claim 1. $Ind_l Y^\lambda$ has a direct summand with quotient isomorphic to $ind_l Y^\lambda$.

Let $\pi_i : Ind_l Y^\lambda \twoheadrightarrow (Ind_l Y^\lambda)\epsilon_i$ be the canonical projection onto $(Ind_l Y^\lambda)\epsilon_i$ and $\iota_i : (Ind_l Y^\lambda)\epsilon_i \hookrightarrow Ind_l Y^\lambda$ the canonical inclusion of $(Ind_l Y^\lambda)\epsilon_i$. The functor Res_l is exact, so applying it to the composition $\iota_i \circ \pi_i$ gives maps

$$Res_l(\iota_i \circ \pi_i): \ e_l A \underset{A}{\otimes} A e_l \underset{B_l}{\otimes} Y^\lambda \xrightarrow{e_l A \otimes \pi_i} e_l A \underset{A}{\otimes} Y_i \xrightarrow{e_l A \otimes \iota_i} e_l A \underset{A}{\otimes} A e_l \underset{B_l}{\otimes} Y^\lambda \ .$$

By Proposition 6, we have a decomposition $e_l A e_l \simeq B_l \oplus e_l J_{l-1} e_l$ of (B_l, B_l)-bimodules. Thus the homomorphism $Res_l(\iota_i \circ \pi_i)$ is given by a matrix, where the top left entry is an endomorphism $f_i \in \mathrm{End}_{B_l}(Y^\lambda)$. This gives a commutative diagram

$$\begin{tikzcd}
A e_l \underset{B_l}{\otimes} Y^\lambda \arrow[r, two heads, "\pi_i"] \arrow[d, "Res_l"] & (A e_l \underset{B_l}{\otimes} Y^\lambda)\epsilon_i \arrow[r, hook, "\iota_i"] & A e_l \underset{B_l}{\otimes} Y^\lambda \arrow[d, "Res_l"] & \text{in } A-\mathrm{mod} \\
e_l A e_l \underset{B_l}{\otimes} Y^\lambda \arrow[rr, "Res_l(\iota_i \circ \pi_i)"] \arrow[d, "\wr"] & & e_l A e_l \underset{B_l}{\otimes} Y^\lambda \arrow[d, "\wr"] & \text{in } B_l-\mathrm{mod} \\
Y^\lambda \oplus (e_l J_{l-1} e_l \underset{B_l}{\otimes} Y^\lambda) \arrow[rr, "\begin{pmatrix} f_i & h_i' \\ g_i & h_i \end{pmatrix}"] & & Y^\lambda \oplus (e_l J_{l-1} e_l \underset{B_l}{\otimes} Y^\lambda) & \text{in } B_l-\mathrm{mod}
\end{tikzcd}$$

where the isomorphisms from second to third row come from the decomposition $e_l A e_l \simeq B_l \oplus e_l J_{l-1} e_l$, see Proposition 6.

Let $y \in Y^\lambda$. Then $\pi_i(e_l \otimes y) = \epsilon_i(e_l \otimes y) \in A e_l \underset{B_l}{\otimes} Y^\lambda$. Since ϵ_i is an A-homomorphism, we have

$$\epsilon_i(e_l \otimes y) = \epsilon_i(e_l^2 \otimes y) = e_l \epsilon_i(e_l \otimes y) = e_l\left(\sum_{j=1}^{t} a_j e_l \otimes x_j\right) = \sum_{j=1}^{t} e_l a_j e_l \otimes x_j$$

for some $t \in \mathbb{N}$, $a_j \in A$ and $x_j \in Y^\lambda$. If $e_l a_j e_l \in J_l \setminus J_{l-1}$ then $e_l a_j e_l$ corresponds to an element $b_j \in B_l$ via the isomorphism

$$\begin{aligned} e_l A e_l \setminus e_l J_{l-1} e_l &\longrightarrow e_l A e_l / e_l J_{l-1} e_l \longrightarrow B_l \\ e_l a e_l &\mapsto e_l a e_l + e_l J_{l-1} e_l \qquad \mapsto b \end{aligned}$$

where the last map is the isomorphism from Lemma 3. In particular, $e_l a_j e_l \otimes x = e_l b_j \otimes x = e_l \otimes b_j x$ in this case. It follows that

$$\pi_i(e_l \otimes y) = \epsilon_i(e_l \otimes y) = e_l \otimes x + \textit{lower terms}$$

for some $x \in Y^\lambda$. By *lower terms* we mean terms in $e_l J_{l-1} e_l \underset{B_l}{\otimes} Y^\lambda$. It follows that

$$Res_l(\epsilon_i(e_l \otimes y)) = e_l \otimes x + \textit{lower terms}.$$

On the other hand, $e_l \otimes y \in Res_l(Ind_l Y^\lambda)$ is sent to $(y, 0)$ under the isomorphism in Proposition 6, so $Res_l(\epsilon_i(e_l \otimes y))$ is the preimage of $\begin{pmatrix} f_i & h_i' \\ g_i & h_i \end{pmatrix}\begin{pmatrix} y \\ 0 \end{pmatrix} = f_i(y) + g_i(y) = f_i(y) + \textit{lower terms}$ under the isomorphism in Proposition 6, i.e.

$$Res_l(\epsilon_i(e_l \otimes y)) = e_l \otimes f_i(y) + \textit{lower terms}.$$

This shows that

$$\pi_i(e_l \otimes y) = e_l \otimes f_i(y) + \textit{lower terms}.$$

The identity on $Ind_l Y^\lambda$ is $\sum_{i=1}^{s} (\iota_i \circ \pi_i)$, so

$$e_l \otimes y = \sum_{i=1}^{s} \iota_i(\pi_i(e_l \otimes y)) = \sum_{i=1}^{s} (e_l \otimes f_i(y)) + \textit{lower terms}$$

for any $y \in Y^\lambda$. Since there are no lower terms on the left hand side, they vanish on the right hand side and we have

$$e_l \otimes y = \sum_{i=1}^{s} (e_l \otimes f_i(y)) = e_l \otimes \left(\sum_{i=1}^{s} f_i(y) \right).$$

Hence there is a $\xi \in \Sigma_\lambda$ such that $\xi \sum_{i=1}^{s} f_i(y) = y$, so $\sum_{i=1}^{s} (\xi f_i)$ is the identity on Y^λ.

Y^λ is finite dimensional and indecomposable, so $\mathrm{End}_{B_l}(Y^\lambda)$ is local. Therefore, at least one of the summands ξf_i must be invertible. We now assume without loss of generality that ξf_1 is invertible, thus f_1 is invertible.

Let

$$\begin{aligned} \varphi :\ Ind_l Y^\lambda &\longrightarrow \quad ind_l Y^\lambda \\ e_l \otimes y \quad &\longmapsto (e_l + J_{l-1} e_l) \otimes y \end{aligned}$$

and $\varphi' := \varphi \circ \iota_1 \circ \pi_1$ its restriction to $(Ind_l Y^\lambda)\epsilon_1$. Then

$$\varphi'(e_l \otimes y) = \varphi(e_l \otimes f_1(y) + \textit{lower terms}) = (e_l + J_{l-1} e_l) \otimes f_1(y).$$

Surjectivity of f_1 implies that the A-homomorphism φ' is surjective, so $ind_l Y^\lambda$ is a quotient of $(Ind_l Y^\lambda)\epsilon_1$.

Claim 2. $(Ind_l Y^\lambda)\epsilon_1$ is the only summand of $Ind_l Y^\lambda$ with quotient isomorphic to $ind_l Y^\lambda$.

Suppose there is another summand $(Ind_l Y^\lambda)\epsilon_2$ of $Ind_l Y^\lambda$ such that there is an epimorphism $\psi : Ind_l Y^\lambda \twoheadrightarrow ind_l Y^\lambda$ with

$$\begin{aligned} \psi((Ind_l Y^\lambda)\epsilon_2) &= ind_l Y^\lambda \\ \text{and } \psi((Ind_l Y^\lambda)\epsilon_j) &= 0 \text{ for all } j \neq 2. \end{aligned}$$

By tensor-hom adjunction, ψ is an element in

$$\begin{aligned} \mathrm{Hom}_A(Ind_l Y^\lambda, ind_l Y^\lambda) &\simeq \mathrm{Hom}_{B_l}(Y^\lambda, \mathrm{Hom}_A(Ae_l, Ae_l \underset{e_l A e_l}{\otimes} Y^\lambda)) \\ &\simeq \mathrm{Hom}_{B_l}(Y^\lambda, e_l A e_l \underset{e_l A e_l}{\otimes} Y^\lambda) \simeq \mathrm{End}_{B_l}(Y^\lambda), \end{aligned}$$

so ψ is given by

$$\psi(e_l \otimes y) = (e_l + J_{l-1} e_l) \otimes g(y)$$

for some $g \in \mathrm{End}_{B_l}(Y^\lambda)$. Let $y \in Y^\lambda$, $y \neq 0$. The surjectivity of ψ provides the existence of a preimage $v = \sum_{i=1}^{t} (a_i e_l \otimes y_i) \in (Ind_l Y^\lambda)\epsilon_2$ of $(e_l + J_{l-1}e_l) \otimes y \in ind_l Y^\lambda$ with $a_i \in A$ and $y_i \in Y^\lambda$ for all $i \in \{1, ..., t\}$ and some $t \in \mathbb{N}$. Since $e_l A e_l$ decomposes into $B_l \oplus e_l J_{l-1} e_l$ as (B_l, B_l)-bimodule by Proposition 6, we can write any element $e_l a e_l \in e_l A e_l$ as $b + e_l j e_l$ with $b \in B_l$ and $j \in J_{l-1}$. Thus

$$e_l v = e_l(\sum_{i=1}^{t} a_i e_l \otimes y_i) = \sum_{i=1}^{t} e_l a_i e_l \otimes y_i = e_l \otimes w + \textit{lower terms}$$

for some $w \in Y^\lambda$. Application of ψ yields

$$\psi(e_l v) = \psi(e_l \otimes w + \textit{lower terms}) = (e_l + J_{l-1}e_l) \otimes g(w).$$

On the other hand,

$$\psi(e_l v) = e_l \psi(v) = e_l((e_l + J_{l-1}e_l) \otimes y) = (e_l + J_{l-1}e_l) \otimes y,$$

so $g(w) = 0$ would imply $y = 0$, which we excluded. Hence $w \neq 0$. But $e_l v \in (Ind_l Y^\lambda)\epsilon_2$ and

$$\varphi'(e_l v) = \varphi'(e_l \otimes w + \textit{lower terms}) = (e_l + J_{l-1}e_l) \otimes f_1(w) \neq 0$$

since $w \neq 0$ and f_1 is a unit, in particular injective. So $\varphi'((Ind_l Y^\lambda)\epsilon_2) \neq 0$, which contradicts the definition of φ'.

Claim 3. There is no summand of $Ind_l Y^\mu$ with quotient $ind_l Y^\lambda$ for $\mu \neq \lambda$.

Assume there is a direct summand Y^μ of M^λ with $\mu > \lambda$ such that $ind_l Y^\lambda$ is a quotient of $Ind_l Y^\mu$. An arbitrary homomorphism $\Phi : Ind_l Y^\mu \to ind_l Y^\lambda$ is given by $\Phi(e_l \otimes y) = (e_l + J_{l-1}e_l) \otimes \varphi(y)$ for some

$\varphi \in \mathrm{Hom}_{B_l}(Y^\mu, Y^\lambda)$ by the adjunction $\mathrm{Hom}_A(Ind_l Y^\mu, ind_l Y^\lambda) \simeq \mathrm{Hom}_{B_l}(Y^\mu, Y^\lambda)$. Φ is surjective only if φ is surjective.[1]

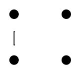
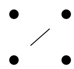
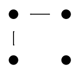
The rest of the proof can be copied from [8] in case $B_l = k\Sigma_{l'}$. We give here a similar proof for Iwahori-Hecke algebras $\mathcal{H} := \mathcal{H}_{k,q}(\Sigma_l)$, inspired by the one for group algebras of symmetric groups, using notation from [1].

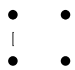
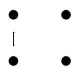
Suppose there is an epimorphism $\varphi : Y^\mu \twoheadrightarrow Y^\lambda$, which we extend to an epimorphism $\hat{\varphi} : M^\mu \to Y^\lambda$ such that $\hat{\varphi}$ is zero on all summands other than Y^μ, i.e. $\hat{\varphi}$ is the projection from M^μ onto the direct summand Y^μ, followed by the map φ. Recall (e.g. from [1]) that $\mathcal{H}$ is generated by elements T_π, $\pi \in \Sigma_l$, and $M^\mu = \mathcal{H}x_\mu$, where $x_\mu = \sum_{\omega \in \Sigma_\mu} T_\omega$. For $y_{\lambda'} = \sum_{\omega \in \Sigma_{\lambda'}} (-q)^{l(\omega)} T_\omega$, where l is the length function on symmetric group elements and λ' is the conjugate of the partition λ, we have that $y_{\lambda'} T_\pi x_\mu \neq 0$ implies $\lambda = \lambda'' \geq \mu$ by [1, Lemma 4.1]. So for $\mu > \lambda$, we have $y_{\lambda'} M^\mu = 0$. Then $0 = \hat{\varphi}(0) = \hat{\varphi}(y_{\lambda'} M^\mu) = y_{\lambda'} \hat{\varphi}(M^\mu) = y_{\lambda'} Y^\lambda$. But $y_{\lambda'} Y^\lambda$ contains the generator $y_{\lambda'} T_{w_\lambda} x_\lambda = z_\lambda$ of S^λ, in particular $y_{\lambda'} Y^\lambda \neq 0$.

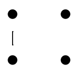
This concludes the proof of Theorem 1. $\square$

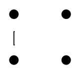
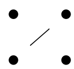
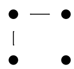
Definition 16. We denote the unique summand of $Ind_l Y^\lambda$ with quotient $ind_l Y^\lambda$ constructed above by $Y(l, \lambda)$, in analogy to [8], and call it *Young module* for A with respect to $(l, \lambda) \in \Lambda_r$.

Example. Let $A = P_k(2, \delta)$ with $\delta \neq 0$ and $\mathrm{char} k \neq 2, 3$. Let $\lambda = (1,1) = (1^2)$. Then $Y^{(1^2)} = k_{sgn}$ is the sign-module for $k\Sigma_2$ and

$$Ind_2 Y^{(1^2)} = A \underset{k\Sigma_2}{\otimes} k_{sgn} =_k \Big\langle$$  $\otimes 1,$  $\otimes 1,$  $\otimes 1,\ e_2 \otimes 1 \Big\rangle.$

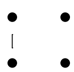
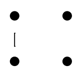
The idempotents $\epsilon_1 : e_2 \otimes 1 \mapsto \Big(e_2 - \frac{1}{\delta}$  $\Big) \otimes 1$ and $\epsilon_2 : e_2 \otimes 1 \mapsto \frac{1}{\delta}$  $\otimes 1$ give the following decomposition into indecomposables:

$$(Ind_2 Y^{(1^2)})\epsilon_1 =_k \Big\langle \Big(e_2 - \frac{1}{\delta}$$  $\Big) \otimes 1 \Big\rangle$

$$(Ind_2 Y^{(1^2)})\epsilon_2 =_k \Big\langle$$  $\otimes 1,$  $\otimes 1,$  $\otimes 1 \Big\rangle = J_1 \underset{k\Sigma_2}{\otimes} k_{sgn}$

The summand $(Ind_2 Y^{(1^2)})\epsilon_2$ lies in the kernel of any map

$$Ind_2 Y^{(1^2)} \to ind_2 Y^{(1^2)} = A/J_1 \underset{k\Sigma_2}{\otimes} k_{sgn}$$

so the only candidate for $Y(2, (1^2))$ is $(Ind_2 Y^{(1^2)})\epsilon_1$. Since $\ker(\pi_1) = J_1 \underset{k\Sigma_2}{\otimes} k_{sgn}$, the second column of the matrix in the commutative diagram in Claim 1 is zero. For the generator $e_2 \otimes 1$ we have $\pi_1(e_2 \otimes 1) = \Big(e_2 - \frac{1}{\delta}$  $\Big) \otimes 1$ and $Res_2(e_2 \otimes 1)$ corresponds to $(1, 0)$ under the vertical isomorphism in the commutative diagram. This shows that the matrix is $\begin{pmatrix} 1 & 0 \\ g & 0 \end{pmatrix}$ with $g(1) = -\frac{1}{\delta}$  $\otimes 1$. The epimorphism $\varphi' : Ind_2 Y^{(1^2)} \to Y(2, (1^2)) \to ind_2 Y^{(1^2)}$ is given by

[1] Assume there is $w \in Y^\lambda$ such that $\varphi(y) \neq w$ for all $y \in Y^\mu$. Let $\sum (a_i e_l \otimes y_i)$ be an arbitrary element of $Ind_l Y^\mu$ and suppose that $\Phi(\sum(a_i e_l \otimes y_i)) = \sum(a_i e_l \otimes \varphi(y_i)) = e_l \otimes w$. Then for each i we have $a_i e_l = e_l b_i$ for some $b_i \in B_l$ and thus $\sum(a_i e_l \otimes \varphi(y_i)) e_l = \sum(e_l \otimes b_i \varphi(y_i)) = e_l \otimes \varphi(\sum b_i y_i)) = e_l \otimes w \Rightarrow \varphi(\sum b_i y_i) = w$ ↯.

$$\begin{aligned}\varphi'(e_2 \otimes 1) &= \varphi\left(\left(e_2 - \frac{1}{\delta}\begin{array}{cc}\bullet & \bullet \\ \vdots & \\ \bullet & \bullet\end{array}\right)\otimes 1\right) \\ &= \left(\left(e_2 - \frac{1}{\delta}\begin{array}{cc}\bullet & \bullet \\ \vdots & \\ \bullet & \bullet\end{array}\right) + J_1\right)\otimes 1 \\ &= (e_2 + J_1) \otimes 1.\end{aligned}$$

There cannot be any surjective A-homomorphism $Ind_2 Y^{(2)} \twoheadrightarrow ind_2 Y^{(1^2)}$ since any such ψ would send the generator $e_2 \otimes 1$ to a scalar $s \in k$ and

$$\psi((1\,2)e_2 \otimes 1) = (1\,2)\psi(e_2 \otimes 1) = (1\,2)s = -s,$$

but on the other hand,

$$(1\,2)e_2 \otimes 1 = e_2(1\,2) \otimes 1 = e_2 \otimes 1$$

holds in $Ind_2 Y^{(2)}$ so $\psi((1\,2)e_2 \otimes 1) = s$. Thus $\mathrm{Hom}_A(Ind_2 Y^{(2)}, ind_2 Y^{(1^2)}) = 0$.

We now collect conditions for a Young module $Y(m, \mu)$ to appear as a summand of $M(l, \lambda)$. They generalise the conditions from [8, Lemmas 17 and 18] for $A = B_k(r, \delta)$. The fact that these are the only direct summands of permutation modules is our main result (Theorem 4) and will be proven using results from the next Section.

Lemma 17. *If $(l, \lambda), (m, \mu) \in \Lambda_r$ with $l < m$, then $Y(m, \mu)$ does not appear as a summand of $M(l, \lambda)$.*

Proof. Ind_l is left adjoint to Res_l, so

$$\begin{aligned}\mathrm{Hom}_A(Ind_l M^\lambda, ind_m Y^\mu) &\simeq \mathrm{Hom}_{B_l}(M^\lambda, Res_l ind_m Y^\mu) \\ &\simeq \mathrm{Hom}_{B_l}(M^\lambda, e_l(A/J_{m-1})e_m \underset{e_m A e_m}{\otimes} Y^\mu).\end{aligned}$$

For $l < m$, $e_l \in J_{m-1}$, so $Res_l ind_m Y^\mu = 0$. Thus, there cannot be a non-zero map

$$Ind_l M^\lambda \to Y(m, \mu)$$

since it would extend to a non-zero map $Ind_l M^\lambda \to ind_m Y^\mu$. □

Lemma 18. *If $(l, \lambda), (l, \kappa) \in \Lambda_r$, then $Y(l, \lambda)$ occurs as a direct summand of $M(l, \kappa)$ if and only if Y^λ is a direct summand of M^κ. This can only occur if $\lambda \trianglerighteq \kappa$.*

Proof. If Y^λ is a direct summand of M^κ, then $Y(l, \lambda)$, as a direct summand of $Ind_l Y^\lambda$, is a direct summand of $Ind_l M^\kappa = M(l, \kappa)$.

If $Y(l, \lambda)$ is a direct summand of $M(l, \kappa)$ and $M^\kappa = \bigoplus (Y^\mu)^{a_\mu}$, then $Y(l, \lambda)$ is a summand of $Ind_l Y^\mu$ for some μ.

It follows from Theorem 1, Claim 3, that $\mu = \lambda$, so Y^λ is a direct summand of M^κ. □

Corollary 19. *If $(l, \lambda) \neq (l, \kappa)$, then $Y(l, \lambda) \not\simeq Y(l, \kappa)$.*

Proof. Let $(l, \lambda) \neq (l, \kappa)$. Then $Y^\lambda \not\simeq Y^\kappa$, see for example [15, Section 7.6], so $Ind_l Y^\lambda \not\simeq Ind_l Y^\kappa$ since otherwise $res_l Ind_l Y^\lambda \simeq Y^\lambda$ would be isomorphic to $res_l Ind_l Y^\kappa \simeq Y^\kappa$. Assume that $Y(l, \lambda) \simeq Y(l, \kappa)$. Then

$Y(l,\kappa)$ is a direct summand of $M(l,\lambda)$ and by Lemma 18, Y^κ is a direct summand of M^λ. So $Ind_l Y^\kappa$ is a summand of $M(l,\lambda)$ and has a summand $Y(l,\kappa)$ with quotient $ind_l Y^\kappa$. But $Y(l,\kappa)$ is isomorphic to $Y(l,\lambda)$ with quotient $ind_l Y^\lambda$, so $Ind_l Y^\kappa$ has a direct summand with quotient isomorphic to $ind_l Y^\lambda$ and $\kappa \neq \lambda$. This contradicts Claim 3 from Theorem 1. □

4. Properties

Each Young module $Y(l,\lambda)$ is a direct summand of the permutation module $M(l,\lambda) = Ind_l M^\lambda$ by definition. In this section, we show that the indecomposable direct summands of permutation modules are exactly the Young modules, as in the symmetric group case, provided A satisfies the additional assumptions defined below. The results extend the results on Brauer algebras stated in [8] to our setup.

We give conditions under which the permutation modules for our cellularly stratified algebra A admit a cell filtration in Subsection 4.1. In Subsection 4.2, we show that permutation modules are relative projective in the subcategory $\mathcal{F}(\Theta)$ of cell filtered A-modules, provided a further condition is satisfied. Then the Young module $Y(l,\lambda)$ is the relative projective cover of the cell module $\Theta(l,\lambda) := ind_l S_\lambda$ (Theorem 3). As a corollary of this, we recover a result about Schur-Weyl duality from [5] in Subsection 4.3. Finally, we can prove Theorem 4, the decomposition of the permutation module $M(l,\lambda)$ into a direct sum of Young modules $Y(l,\lambda)$, in Subsection 4.4.

A crucial point in the study of a category $\mathcal{F}(\Delta)$ of Δ-filtered A-modules is that it is closed under direct summands if the set Δ with ordered index set $(I,\le)$ forms a *standard system*,[2] i.e. for all $l,m \in I$

- A crucial point in the study of] $\mathrm{End}_A(\Delta(l))$ is a division ring.
- $\mathrm{Hom}_A(\Delta(l),\Delta(m)) \neq 0$ implies $l \ge m$.
- $\mathrm{Ext}^1_A(\Delta(l),\Delta(m)) \neq 0$ implies $l > m$.

The statement follows from [21, Theorem 2].

Lemma 20. *Let A be cellularly stratified with stratification data $(B_0,V_0,\dots,B_r,V_r)$ where the B_l are isomorphic to group algebras of symmetric groups or their Iwahori-Hecke algebras. Let* $\mathrm{char}k = p \in \mathbb{Z}_{\ge 0} \setminus \{2,3\}$ *if the input algebras are group algebras of symmetric groups and let $h \ge 4$ if the input algebras B_l are isomorphic to Hecke algebras $\mathcal{H}_{k,q}(\Sigma_l)$. Then the cell modules Θ of A form a standard system with respect to the order $\prec$ defined in Subsection 2.5.*

Proof. Dual Specht modules for symmetric groups form a standard system by [7, Proposition 4.2.1] and [11, Corollary 13.17]. Dual Specht modules for Iwahori-Hecke algebras of symmetric groups form a standard system by [7, Proposition 4.2.1] and [17, Exercise 4.11]. The statement follows from [5, Theorem 10.2 (a)]. □

Assumptions. We give names to the following assumptions that we make on A in order to prove the desired properties for permutation modules and Young modules. Furthermore, we often assume that $\mathrm{char}k \in \mathbb{Z}_{\ge 0} \setminus \{2,3\}$ (or $h \ge 4$, in case the B_l are Iwahori-Hecke algebras) to be able to use Lemma 20.

Let A be cellularly stratified with stratification data $(B_0,V_0,\dots,B_r,V_r)$ where the B_l are isomorphic to group algebras of symmetric groups or their Iwahori-Hecke algebras, such that for each $l \in \{0,\dots,r\}$ we have an embedding $B_l \hookrightarrow e_l A e_l$ of algebras. Let n,l with $1 \le n \le l \le r$ be arbitrary.

(I) $J_n e_l \simeq J_{n-1} e_l \oplus (J_n/J_{n-1}) e_l$ as right B_l-modules.
(II) $(J_n/J_{n-1}) e_l \simeq (A/J_{n-1}) e_n \underset{e_n A e_n}{\otimes} e_n (A/J_{n-1}) e_l$ as right B_l-modules.

[2] cf. [2, Section 3] or [5, Definition 10.1].

(III) Layer-removing restriction to B_n-mod of a permutation module from layer l is dual Specht filtered:

$$res_n Ind_l M^\lambda \in \mathcal{F}_n(S)$$

(IV) Classical restriction to B_l-mod of a cell module from layer n is dual Specht filtered:

$$Res_l ind_n S_\nu \in \mathcal{F}_l(S)$$

Remark. Assumption (IV) implies that for any $X \in \mathcal{F}_n(S)$, $Res_l ind_n X \in \mathcal{F}_l(S)$: The functor ind_n is exact and sends dual Specht modules to cell modules, so $ind_n X$ has a cell filtration. Res_l is exact, so $Res_l ind_n X$ has a filtration by modules of the form $Res_l ind_n S_\nu \in \mathcal{F}_l(S)$. The statement follows since $\mathcal{F}_l(S)$ is extension-closed.

Lemma 21. *Instead of* (II), *we can assume*

(II') $(J_n/J_{n-1})e_l \simeq B_n \otimes_k V_n \otimes_k V_n^l$ *as vector spaces.*

Proof. By Proposition 5, the algebra $e_l A e_l$ is cellularly stratified with idempotents $e_n = 1_{B_n} \otimes u_n \otimes v_n \in B_n \otimes_k V_n^l \otimes_k V_n^l \subseteq B_n \otimes_k V_n \otimes_k V_n$. Then $e_n(A/J_{n-1})e_l = e_n(e_l A e_l / e_l J_{n-1} e_l)$ is free of rank $\dim V_n^l$ over B_n by [5, Proposition 3.5] and $ind_n(e_n(A/J_{n-1})e_l) \simeq (A/J_{n-1})e_n \underset{B_n}{\otimes} e_n(A/J_{n-1})e_l \simeq \bigoplus_{i=1}^{\dim V_n^l} (A/J_{n-1})e_n$ as left A-modules. Hence, $\dim(ind_n(e_n(A/J_{n-1})e_l)) = \dim((A/J_{n-1})e_n) \cdot \dim V_n^l = \dim B_n \cdot \dim V_n \cdot \dim V_n^l$, since $(A/J_{n-1})e_n$ is free of rank $\dim V_n$ over B_n.

The multiplication map

$$(A/J_{n-1})e_n \underset{B_n}{\otimes} e_n(A/J_{n-1})e_l \longrightarrow (J_n/J_{n-1})e_l$$

$$(a + J_{n-1})e_n \otimes e_n(b + J_{n-1})e_l \longmapsto (ae_n b + J_{n-1})e_l$$

is an epimorphism of (A, B_l)-bimodules and $\dim(ind_n(e_n(A/J_{n-1})e_l)) = \dim V_n^l \cdot \dim V_n \cdot \dim B_n = \dim((J_n/J_{n-1})e_l)$ by (II'), so (II) is satisfied. □

4.1. Cell filtrations

Theorem 2. *Let A be cellularly stratified, such that the input algebras B_l are isomorphic to group algebras of symmetric groups or their Hecke algebras and $B_n \subseteq B_{n+1}$ for all $0 \le n < l$. Assume that the idempotent e_l is fixed by the involution j of A which is compatible with the involutions of the B_n and that A satisfies* (I),(II) *and* (III). *Then the permutation module $M(l, \lambda)$ has a filtration by cell modules.*
If, in addition, $\operatorname{char} k \in \mathbb{Z}_{\ge 0} \setminus \{2,3\}$ *or $h \ge 4$, then the direct summands of $Ind_l M^\lambda$ have cell filtrations.*

Proof. $A = J_r \supset J_{r-1} \supset \ldots \supset J_1 \supset J_0 = 0$ is a filtration of A (with quotients isomorphic to $B_n \otimes_k V_n \otimes_k V_n$), so we have short exact sequences

$$0 \to J_{n-1} \to J_n \to J_n/J_{n-1} \to 0$$

of (A, A)-bimodules for $1 \le n \le r$. Application of the exact restriction functor $- \underset{A}{\otimes} Ae_l$ gives exact sequences

$$0 \to J_{n-1}e_l \to J_n e_l \to (J_n/J_{n-1})e_l \to 0$$

of $(A, e_l A e_l)$-bimodules for $n \le l$, which are split exact as sequences of right B_l-modules by assumption (I). Hence, we get exact sequences

$$0 \to J_{n-1}e_l \underset{B_l}{\otimes} M^\lambda \to J_n e_l \underset{B_l}{\otimes} M^\lambda \to (J_n/J_{n-1})e_l \underset{B_l}{\otimes} M^\lambda \to 0$$

of left A-modules, which give rise to a filtration

$$Ae_l \underset{B_l}{\otimes} M^\lambda \supset J_{l-1}e_l \underset{B_l}{\otimes} M^\lambda \supset \dots \supset J_1 e_l \underset{B_l}{\otimes} M^\lambda \supset 0$$

of $M(l,\lambda) = Ind_l M^\lambda$ with quotients $M^n(l,\lambda) := (J_n/J_{n-1})e_l \underset{B_l}{\otimes} M^\lambda$, the nth layer of $M(l,\lambda)$. Assumption (II) gives

$$\begin{aligned} M^n(l,\lambda) &\simeq ind_n(e_n(A/J_{n-1})e_l) \underset{B_l}{\otimes} M^\lambda \\ &\simeq ind_n(e_n(A/J_{n-1})e_l \underset{B_l}{\otimes} M^\lambda) \\ &\simeq ind_n(res_n Ind_l M^\lambda). \end{aligned}$$

By assumption (III), $res_n Ind_l M^\lambda \in \mathcal{F}_n(S)$. The functor ind_n is exact and sends dual Specht modules to cell modules by Proposition 12, so $M^n(l,\lambda) \in \mathcal{F}(\Theta)$ for all $1 \le n \le l$, in particular $M(l,\lambda) = M^l(l,\lambda) \in \mathcal{F}(\Theta)$.

If char$k$ is different from 2 and 3, then the cell modules of A form a standard system by Lemma 20. In this case, $\mathcal{F}(\Theta)$ is closed under direct summands by [21, Theorem 2], so all direct summands of $Ind_l M^\lambda$, in particular the Young modules $Y(l,\lambda)$, admit cell filtrations. □

4.2. Relative projectivity

An important property of the permutation modules $M^\lambda \in B_l - \mathrm{mod}$ is their relative projectivity in the category $\mathcal{F}_l(S)$, as shown by Hemmer and Nakano in [7, Proposition 4.1.1], in case $h \ge 4$. This property is translated to the permutation modules $M(l,\lambda)$ of A, in case the conditions (I) to (IV) are satisfied. Furthermore, the Young modules are relative projective covers of the cell modules.

Theorem 3. *Let A be cellularly stratified, such that the input algebras B_l are isomorphic to group algebras of symmetric groups or their Hecke algebras and $B_n \subseteq B_{n+1}$ for all $0 \le n < l$. Assume that the idempotent e_l is fixed by the involution j of A which is compatible with the involutions of the B_n and that A satisfies (I) to (IV). Then the permutation module $Ind_l M^\lambda$ is relative projective in $\mathcal{F}(\Theta)$. If, in addition,* char$k \in \mathbb{Z}_{\ge 0} \setminus \{2,3\}$ *(or $h \ge 4$), then all direct summands of $Ind_l M^\lambda$ are relative projective in $\mathcal{F}(\Theta)$. Furthermore, $Y(l,\lambda)$ is the relative projective cover of $\Theta(l,\lambda)$ in the category $\mathcal{F}(\Theta)$ of cell filtered modules.*

Proof. By Theorem 2, $M(l,\lambda)$ and all its direct summands (provided char$k \ne 2,3$ or $h \ge 4$) are in $\mathcal{F}(\Theta)$ if A satisfies conditions (I) to (III). We have to show that $\mathrm{Ext}^1_A(M(l,\lambda), X) = 0$ for all $X \in \mathcal{F}(X)$. Let $X \in \mathcal{F}(\Theta)$ and let

$$(*) : 0 \to X \to Y \to Ind_l M^\lambda \to 0$$

be a short exact sequence in $\mathrm{Ext}^1_A(M(l,\lambda), X)$.

Apply the exact functor Res_l on $(*)$ to get a short exact sequence

$$(**) : 0 \to e_l X \to e_l Y \to e_l A e_l \underset{B_l}{\otimes} M^\lambda \to 0$$

in B_l – mod. Now we apply the left exact functor $\mathrm{Hom}_{B_l}(M^\lambda, -)$ to get a long exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Hom}_{B_l}(M^\lambda, e_lX) \arrow[r] & \mathrm{Hom}_{B_l}(M^\lambda, e_lY) \arrow[r] & \mathrm{Hom}_{B_l}(M^\lambda, e_lAe_l \underset{B_l}{\otimes} M^\lambda) \arrow[dll, rounded corners] \\ & \mathrm{Ext}^1_{B_l}(M^\lambda, e_lX) \arrow[r] & \dots & \end{tikzcd}$$

It follows from assumption (IV) and the exactness of Res_l that $e_lX \in \mathcal{F}_l(S)$ for $X \in \mathcal{F}(\Theta)$. Since M^λ is relative projective in $\mathcal{F}_l(S)$, we get $\mathrm{Ext}^1_{B_l}(M^\lambda, e_lX) = 0$, in particular we get a short exact sequence

$$0 \to \mathrm{Hom}_{B_l}(M^\lambda, e_lX) \to \mathrm{Hom}_{B_l}(M^\lambda, e_lY) \to \mathrm{Hom}_{B_l}(M^\lambda, e_lAe_l \underset{B_l}{\otimes} M^\lambda) \to 0$$

which is isomorphic to the short exact sequence

$$(\diamond): 0 \to \mathrm{Hom}_A(Ind_lM^\lambda, X) \to \mathrm{Hom}_A(Ind_lM^\lambda, Y) \xrightarrow{f} \mathrm{End}_A(Ind_lM^\lambda) \to 0$$

since Res_l is right adjoint to Ind_l.

Consider

$$\begin{tikzcd} (*): & 0 \arrow[r] & X \arrow[r] & Y \arrow[r, "\alpha"] & Ind_lM^\lambda \arrow[r] & 0 \\ & & & & Ind_lM^\lambda \arrow[u, equal] \arrow[ul, dashed, "\exists\beta"] & \end{tikzcd}$$

then β exists (such that the diagram commutes) by surjectivity of the map f in $(\diamond)$. This shows that $(*)$ splits and so $\mathrm{Ext}^1_A(M(l,\lambda), X) = 0$. In particular, $M(l,\lambda)$ is relative projective in $\mathcal{F}(\Theta)$.

Now let Z be a direct summand of $M(l,\lambda)$ with $\pi : Ind_lM^\lambda \to Z$ the projection onto Z and $\iota : Z \to Ind_lM^\lambda$ the inclusion of Z into $M(l,\lambda)$. With the same strategy as above, applied to the short exact sequence

$$(\star): 0 \to X \to Y \to Z \to 0,$$

we see that the map $\mathrm{Hom}_A(Ind_lM^\lambda, Y) \to \mathrm{Hom}_A(Ind_lM^\lambda, Z)$ is surjective, which provides the existence of a map $f : Ind_lM^\lambda \to Y$ such that $\pi = gf$:

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r] & Y \arrow[r, "g"] & Z \arrow[r] \arrow[d, hook, shift left, "\iota"] & 0 \\ & & & Ind_lM^\lambda \arrow[u, two heads, shift left, "\pi"] \arrow[ul, dashed, "\exists f"] & \end{tikzcd}$$

But $\pi\iota = id_Z$, so $gf\iota = id_Z$ and $f\iota$ is right inverse to g. Therefore, the sequence $(\star)$ splits and $\mathrm{Ext}^1_A(Z, X) = 0$, so all direct summands of Ind_lM^λ are relative projective in $\mathcal{F}(\Theta)$.

In order to prove that $Y(l,\lambda)$ is the relative projective cover of $\Theta(l,\lambda)$, we have to show that there is an epimorphism

$$\Psi : Y(l,\lambda) \twoheadrightarrow \Theta(l,\lambda)$$

with $\ker(\Psi) \in \mathcal{F}(\Theta)$ and that $Y(l,\lambda)$ is minimal with respect to this property. Once we have established an epimorphism whose kernel is in $\mathcal{F}(\Theta)$, the minimality condition is immediately satisfied since $Y(l,\lambda)$ is indecomposable, and then $Y(l,\lambda)$ is a relative projective cover of $\Theta(l,\lambda)$.

The B_l-module Y^λ has a dual Specht filtration with top quotient S_λ, so the kernel of the map $Y^\lambda \twoheadrightarrow S_\lambda$ lies in $\mathcal{F}_l(S)$. The functor ind_l is exact and sends dual Specht modules to cell modules, so the kernel of the epimorphism

$$\psi : ind_l Y^\lambda \twoheadrightarrow ind_l S_\lambda = \Theta(l,\lambda)$$

has a cell filtration.

Recall from the proof of Theorem 1 that there is an epimorphism

$$\phi : Y(l,\lambda) \overset{\iota}{\hookrightarrow} Ind_l Y^\lambda \overset{\varphi}{\to} ind_l Y^\lambda.$$

Consider the commutative diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & \ker\phi \arrow[d] & & \\
0 \arrow[r] & \ker\Psi \arrow[r] & Y(l,\lambda) \arrow[r, "\Psi=\phi\psi"] \arrow[d, "\phi"] & \Theta(l,\lambda) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \ker\psi \arrow[r] & ind_l Y^\lambda \arrow[r, "\psi"] \arrow[d] & \Theta(l,\lambda) \arrow[r] & 0 \\
 & & 0 & &
\end{tikzcd}$$

with $\ker\psi$, $Y(l,\lambda)$, $ind_l Y^\lambda$ and $\Theta(l,\lambda)$ in $\mathcal{F}(\Theta)$. The composition

$$\ker\Psi \to Y(l,\lambda) \overset{\phi}{\to} ind_l Y^\lambda \overset{\psi}{\to} \Theta(l,\lambda)$$

is zero, so the universal property of the kernel of ψ provides a unique morphism $\ker\Psi \to \ker\psi$, with kernel K, making the diagram

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & K \arrow[r, "\sim"] \arrow[d] & \ker\phi \arrow[d] & & \\
0 \arrow[r] & \ker\Psi \arrow[r] \arrow[d] & Y(l,\lambda) \arrow[r, "\Psi"] \arrow[d, "\phi"] & \Theta(l,\lambda) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \ker\psi \arrow[r] \arrow[d] & ind_l Y^\lambda \arrow[r, "\psi"] \arrow[d] & \Theta(l,\lambda) \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}
$$

commutative. The map $K \to \ker\phi$ is given by the universal property of the kernel of ϕ and is an isomorphism by the snake lemma. The snake lemma also asserts surjectivity of the map $\ker\Psi \to \ker\psi$.

Thus, we have a short exact sequence

$$0 \to \ker\phi \to \ker\Psi \to \ker\psi \to 0$$

with $\ker\psi \in \mathcal{F}(\Theta)$. If we can show that $\ker\phi = \ker\varphi\iota \in \mathcal{F}(\Theta)$, then $\ker\Psi \in \mathcal{F}(\Theta)$ since $\mathcal{F}(\Theta)$ is extension-closed.

Consider the commutative diagram

$$
\begin{tikzcd}
0 \arrow[r] & \ker\varphi\iota \arrow[r] & Y(l,\lambda) \arrow[r, "\varphi\iota"] \arrow[d, hook, "\iota"] & ind_l Y^\lambda \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \ker\varphi \arrow[r] & Ind_l Y^\lambda \arrow[r, "\varphi"] & ind_l Y^\lambda \arrow[r] & 0
\end{tikzcd}
$$

We have $\iota(\ker\varphi\iota) \subseteq \ker\varphi$, so ι restricts to $\ker\varphi\iota \to \ker\varphi$.

Now, we consider the commutative diagram

$$
\begin{tikzcd}
0 \arrow[r] & \ker\varphi\iota \arrow[r] & Y(l,\lambda) \arrow[r, "\varphi\iota"] & ind_l Y^\lambda \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \ker\varphi \arrow[r] & Ind_l Y^\lambda \arrow[r, "\varphi"] \arrow[u, two heads, "\pi"] & ind_l Y^\lambda \arrow[r] & 0
\end{tikzcd}
$$

where π is the projection from $Ind_l Y^\lambda$ onto its summand $Y(l,\lambda)$. We see that $\pi(\ker\varphi) \subseteq \ker\varphi\iota$, so π restricts to $\ker\varphi \to \ker\varphi\iota$.

$$
\begin{tikzcd}
0 \arrow[r] & \ker\varphi\iota \arrow[r] \arrow[d, hook, shift left, "\iota"] & Y(l,\lambda) \arrow[r, "\varphi\iota"] \arrow[d, hook, shift left, "\iota"] & ind_l Y^\lambda \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \ker\varphi \arrow[r] \arrow[u, two heads, shift left, "\pi"] & Ind_l Y^\lambda \arrow[r, "\varphi"] \arrow[u, two heads, shift left, "\pi"] & ind_l Y^\lambda \arrow[r] & 0
\end{tikzcd}
$$

In particular, $\ker\varphi\iota$ is a direct summand of $\ker\varphi = J_{l-1}e_l \underset{B_l}{\otimes} Y^\lambda$. By the proof of Theorem 2, the module $J_{l-1}e_l \underset{B_l}{\otimes} M^\lambda$ has a cell filtration. By the assumption on the characteristic of the field, cell filtrations restrict to direct summands, so $\ker\varphi$ and $\ker\varphi\iota$ lie in $\mathcal{F}(\Theta)$. Since $\mathcal{F}(\Theta)$ is extension-closed, we get $\ker\Psi \in \mathcal{F}(\Theta)$ and so $Y(l,\lambda)$ is a relative projective cover of $\Theta(l,\lambda)$. □

Corollary 22 *([5], Corollary 12.4). If B_l is a group algebra of a symmetric group $\Sigma_{l'}$ for some $l' \in \mathbb{N}$,* char$k = p \in \mathbb{Z}_{\geq 0} \smallsetminus \{2,3\}$*, and A satisfies (I) to (IV), then $Y(l,\lambda)$ is projective if and only if λ is p-restricted.*

4.3. Schur-Weyl duality

In [5], the Young modules $Y_{pr}(l,\lambda)$ of a cellularly stratified algebra A are defined as the relative projective covers of the cell modules $\Theta(l,\lambda)$, in the case where the cell modules of the input algebras B_l form standard systems. Since we assumed B_l to be isomorphic to $k\Sigma_{l'}$ or $\mathcal{H}_{k,q}(\Sigma_{l'})$ for some $l' \in \mathbb{N}$ and char$k \in \mathbb{Z}_{\geq 0} \smallsetminus \{2,3\}$, respectively $h \geq 4$, we are in this situation (Lemma 20). Therefore, we have the following corollary of Theorem 3.

Corollary 23. *The Young modules $Y_{pr}(l,\lambda)$, defined abstractly in [5], coincide with the explicitly defined Young modules $Y(l,\lambda)$ of this article.*

In particular, we are in the situation of Theorem 13.1 from [5]:

Corollary 24. *Let A be cellularly stratified, such that the input algebras B_l are isomorphic to group algebras of symmetric groups or their Hecke algebras and $B_n \subseteq B_{n+1}$ for all $0 \leq n < r$. Assume that the idempotents e_l are fixed by the involution j of A and that the assumptions(I) to (IV) are satisfied. Let* char$k \in \mathbb{Z}_{\geq 0} \smallsetminus \{2,3\}$ *(or $h \geq 4$). Then the following holds.*

(1) Each $M \in \mathcal{F}(\Theta)$ has well-defined filtration multiplicities.
(2) The category $\mathcal{F}_A(\Theta)$ of cell filtered A-modules is equivalent, as exact category, to the category $\mathcal{F}_{\mathrm{End}_A(Y)}(\Delta)$ of standard filtered modules over the quasi-hereditary algebra $\mathrm{End}_A(Y)$, where

$$Y = \bigoplus_{(l,\lambda)\in\Lambda_r} Y(l,\lambda)^{n_{l,\lambda}}$$

and $n_{l,\lambda} = \begin{cases} \dim L(l,\lambda) & \textit{if there is a simple module } L(l,\lambda) \\ 1 & \textit{otherwise.} \end{cases}$

(3) There is a Schur-Weyl duality between A and $\mathrm{End}_A(Y)$. In particular, we have $A = \mathrm{End}_{\mathrm{End}_A(Y)}(Y)$.

Remark. The multiplicities $n_{l,\lambda}$ of the Young modules $Y(l,\lambda)$ in Y are chosen to be minimal such that all Young modules appear at least once and such that the projective Young modules appear as often as they appear in A, i.e. such that there is a $D \in A-\mathrm{mod}$ with $Y = A \oplus D$.

4.4. Decomposition of permutation modules

Using the results of the previous subsections, we are finally able to prove that permutation modules for A decompose into a direct sum of Young modules, just like permutation modules for B_l decompose into direct sums of Young modules.

Theorem 4. *Let A be cellularly stratified, such that the input algebras B_l are isomorphic to group algebras of symmetric groups or their Hecke algebras and $B_n \subseteq B_{n+1}$ for all $0 \leq n < l$. Assume that the idempotent e_l is fixed by the involution j of A which is compatible with the involutions of the B_n and that A satisfies the assumptions (I) to (IV). Let* char$k \in \mathbb{Z}_{\geq 0} \smallsetminus \{2,3\}$ *(or $h \geq 4$). Let $(l,\lambda) \in \Lambda_r$. Then there is a decomposition*

$$Ind_l M^\lambda = \bigoplus_{(m,\mu)\geq(l,\lambda)} Y(m,\mu)^{a_{m,\mu}}$$

with non-negative integers $a_{m,\mu}$. *Moreover,* $a_{l,\lambda} = 1$.

Proof. By Lemma 20, the set Θ forms a standard system. Corollary 24 says that there is a quasi-hereditary algebra $C = \mathrm{End}_A(Y)$ such that the categories $\mathcal{F}_A(\Theta)$ of cell filtered A-modules and $\mathcal{F}_C(\Delta)$ of standard filtered C-modules are equivalent, which was first established in [2]. To prove this equivalence, Dlab and Ringel show that there is a one-to-one correspondence between the modules in the standard system $\{\Theta\}$ and the indecomposable relative projective modules in $\mathcal{F}(\Theta)$. By Theorem 3, the Young modules $Y(l,\lambda)$ are indecomposable relative projective. The one-to-one correspondence shows that these are all indecomposable relative projective A-modules, since for each $(l,\lambda)\in\Lambda_r$ there is exactly one Young module and exactly one cell module, and these are all cell modules, cf. Proposition 12 part *(2)*, Theorem 1 and Corollary 19. The algebra C is quasi-hereditary, so the relative projective C-modules are exactly the projective C-modules, cf. [21, Corollary 2], and they correspond under the equivalence to the relative projective A-modules. Hence, the projective C-modules are indexed by Λ_r.

The permutation module $M(l,\lambda)$ is relative projective in $\mathcal{F}(\Theta)$, so its image under the equivalence $\mathcal{F}(\Theta)\xrightarrow{\sim}\mathcal{F}(\Delta)$ is a projective C-module P. Let $P = \bigoplus_{(n,\nu)\in\Lambda_r} P(n,\nu)^{a_{n,\nu}}$ be a decomposition of P into indecomposable modules. Sending $P(n,\nu)$ back to $\mathcal{F}(\Theta)$ through the equivalence, its image must be an indecomposable relative projective module $Y(m,\mu)$. Thus, $M(l,\lambda) = \bigoplus_{(m,\mu)\in\Lambda_r} Y(m,\mu)^{a_{m,\mu}}$ for some non-negative integers $a_{m,\mu}$. $a_{l,\lambda}=1$ by definition of $Y(l,\lambda)$. Lemmas 17 and 18 show that we only have to sum over those Young modules $Y(m,\mu)$ with $(m,\mu)\geq(l,\lambda)$. □

5. Applications

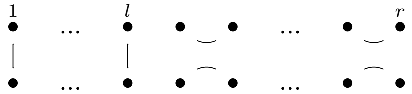
There are three main examples of cellularly stratified algebras in [5]: Brauer algebras, partition algebras and Birman-Murakami-Wenzl algebras (BMW algebras), a deformation of Brauer algebras. The results for Brauer algebras first appeared in [8]. With the theory from this article, we can recover their results, using less combinatorics specific to Brauer algebras but the more structural properties of cellularly stratified algebras, which have been introduced after the work of Hartmann and Paget on Brauer algebras appeared. We recover the results for Brauer algebras in Subsection 5.1, thus providing new proofs. In Subsection 5.2, we show that the results hold for partition algebras under certain additional assumptions. The theory fails for BMW algebras, since we need the cellular algebras $B_l = \mathcal{H}_{k,q}(\Sigma_{l'})$ to be subalgebras. However, the q-Brauer algebras, defined by Wenzl in [22], are another deformation of Brauer algebras which fit into this setting. They are cellularly stratified as shown by Nguyen in his PhD thesis [18] and contain Hecke algebras as subalgebras. We do not prove that the q-Brauer algebras satisfy the assumptions in this article.

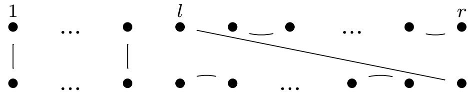
5.1. Recovering results for Brauer algebras

Let $A = B_k(r,\delta)\subseteq P_k(r,\delta)$ be the Brauer algebra on r dots with $\delta\in k$. If r is even, let $\delta\neq 0$. Then by [5, Proposition 2.4], A is cellularly stratified with stratification data

$$(k\Sigma_t, V_t, k\Sigma_{t+2}, V_{t+2}, \ldots, k\Sigma_{r-2}, V_{r-2}, k\Sigma_r, V_r),$$

where $t=0$ if r is even and $t=1$ if r is odd, and V_l is the vector space with basis consisting of partial diagrams with exactly $\frac{r-l}{2}$ horizontal arcs. The idempotents e_l are defined as $e_l = \frac{1}{\delta^{\frac{r-l}{2}}}\cdot$

 for $\delta \neq 0$. For $\delta = 0$ (and r odd), we use

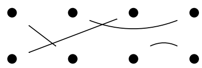
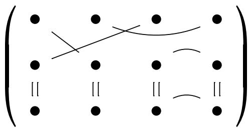
$e_l =$  .

In case $\delta = 0$ we do not have $j(e_l) = e_l$, where j is the involution flipping a diagram over a horizontal axis running between the rows of dots, but we explained in the remark below Proposition 5 how we can still apply the proposition in this case.

We want to recover the results from [8], so we have to show that the Young modules defined here coincide with those defined in [8] as indecomposable submodules of $Ind_l Y^\lambda$ with quotient $V_l \underset{k}{\otimes} Y^\lambda$. The module structure on $V_l \underset{k}{\otimes} X$ is defined as follows. Let $b \in B_k(r, \delta)$ be a basis element and let $v \otimes x \in V_l \underset{k}{\otimes} X$. Then

$$b(v \otimes x) = (bv) \otimes \pi(b, v)x$$

where bv is the partial diagram obtained by writing b on top of v, identifying bottom(b) with v and following the new connections in top(b), multiplying by δ for each closed loop. If the result is not in V_l, set $bv = 0$. The permutation $\pi(b, v)$ is given by the permutation of the free dots of v in bv.

Example. Let b =  $\in B_k(4, \delta)$ and v =  $\in V_2$. Then $bv = \delta^{\#\text{closed loops}}\text{top}$  $= \delta$  and $\pi(b, v) = (1, 2)$.

Proposition 25. *For any $X \in k\Sigma_l -$ mod, there is an isomorphism $ind_l X \simeq V_l \underset{k}{\otimes} X$ of $B_k(r, \delta)$-modules.*

Proof. Let $X \in k\Sigma_l -$ mod and consider the map

$$\begin{aligned} \varphi : V_l \otimes_k X &\longrightarrow (A/J_{l-2})e_l \underset{k\Sigma_l}{\otimes} X \\ v \otimes x \quad &\longmapsto (d^v + J_{l-2}) \otimes x, \end{aligned}$$

where d^v is the diagram in $J_l e_l \setminus J_{l-2} e_l$ with top$(d^v) = v$ and non-crossing propagating lines.[3] Let $(ae_l + J_{l-2}) \otimes x \in (A/J_{l-2})e_l \underset{k\Sigma_l}{\otimes} X$, with $ae_l + J_{l-2}$ corresponding to $b \otimes w \otimes v_l$ under the isomorphism $J_l/J_{l-2} \simeq k\Sigma_l \otimes_k V_l \otimes_k V_l$, i.e. $ae_l + J_{l-2} = d^w b + J_{l-2}$. Then $\varphi(w \otimes bx) = (d^w + J_{l-2}) \otimes bx = (d^w b + J_{l-2}) \otimes x = (ae_l + J_{l-2}) \otimes x$, so φ is surjective. By [5, Proposition 3.5], $\dim((A/J_{l-2})e_l \underset{k\Sigma_l}{\otimes} X) = \dim(k\Sigma_l^{\dim V_l} \underset{k\Sigma_l}{\otimes} X) = \dim V_l \cdot \dim X = \dim(V_l \otimes_k X)$. Hence, φ is bijective. To see that φ is an isomorphism, we have to check that it is A-linear. Let $a \in A$ and $v \otimes x \in V_l \otimes_k X$. Then

$$\begin{aligned} \varphi(a(v \otimes x)) &= \varphi(av \otimes \pi(a, v)x) \\ &= (d^{av} + J_{l-2}) \otimes \pi(a, v)x \\ &= (d^{av}\pi(a, v) + J_{l-2}) \otimes x \end{aligned}$$

and

[3] Since d^v is in $J_l e_l$, its bottom row is fixed: l free dots followed by $\frac{r-l}{2}$ horizontal arcs sitting side by side.

$$\begin{aligned} a\varphi(v \otimes x) &= a((d^v + J_{l-2}) \otimes x) \\ &= (ad^v + J_{l-2}) \otimes x. \end{aligned}$$

If $ad^v \in J_{l-2}$ then $a\varphi(v \otimes x) = (ad^v + J_{l-2}) \otimes x = 0$. On the other hand, $ad^v \in J_{l-2}$ implies that av has more than $\frac{r-l}{2}$ horizontal arcs, so $\varphi(a(v \otimes x)) = \varphi(av \otimes \pi(a,v)x) = \varphi(0) = 0$. If ad^v has l propagating lines, then $a \in J_m \setminus J_{l-2}$ for some $m \geq l$ and $m - l$ of the free dots[4] of $\mathrm{top}(a)$ are bound by horizontal arcs in ad^v since the product lies in J_l. The remaining l free dots of $\mathrm{top}(a)$ are end points of propagating lines in ad^v. Therefore, the permutation of the propagating lines of ad^v is $\pi(a,v)$. This shows $a\varphi(v \otimes x) = (ad^v + J_{l-2}) \otimes x = (d^{av}\pi(a,v) + J_{l-2}) \otimes x = \varphi(a(v \otimes x))$ and φ is A-linear. □

Corollary 26. *The cell, Young and permutation modules defined here coincide with those defined in* [8].

It remains to verify that $A = B_k(r,\delta)$, with $\delta \neq 0$ if r is even, satisfies the assumptions (I) to (IV). Let $0 \leq n \leq l \leq r$.

The right action of $k\Sigma_l - \mathrm{mod}$ on $J_n e_l$ permutes the dots of the bottom row, but it never changes the amount of horizontal arcs, so assumption (I) is satisfied. Assumption (II) holds by [6, Lemma 4.3]. By [6, Lemma 4.2], $e_n(J_n/J_{n-2})e_l \simeq k \underset{H \times k\Sigma_n}{\otimes} k\Sigma_l$, where $H := k(C_2 \wr \Sigma_{\frac{l-n}{2}})$. We get the following isomorphisms of $k\Sigma_n$-modules

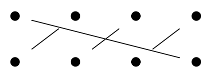
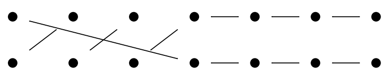
$$res_n Ind_l M^\lambda \simeq e_n(J_n/J_{n-2})e_l \underset{k\Sigma_l}{\otimes} M^\lambda \simeq k \underset{H \times k\Sigma_n}{\otimes} k\Sigma_l \underset{k\Sigma_l}{\otimes} M^\lambda \simeq k \underset{H \times k\Sigma_n}{\otimes} k\Sigma_l \underset{k\Sigma_\lambda}{\otimes} k.$$

The last module is equal to a direct sum of $k\Sigma_n$-permutation modules M^ν by [6, Lemma 4.5]. Therefore, $res_n Ind_l M^\lambda \in \mathcal{F}_n(S)$ and assumption (III) is satisfied. The restriction of a cell module $ind_n S_\nu$ to $k\Sigma_l - \mathrm{mod}$, with $l \geq n$, is dual Specht filtered by [19, Proposition 8], thus A satisfies assumption (IV). This gives a new proof for the following theorem.

Theorem 5 *([8]). Let* $\mathrm{char} k \neq 2, 3$. *The Brauer algebra* $B_k(r,\delta)$, *with* $\delta \neq 0$ *if* r *is even, has permutation modules* $M(l,\lambda)$, *which are a direct sum of indecomposable Young modules. The Young modules are the relative projective covers of the cell modules* $ind_l S_\lambda$. *Every module admitting a cell filtration has well-defined filtration multiplicities.*

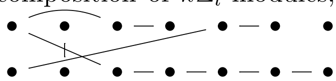
5.2. New results for partition algebras

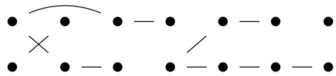
Now, let $A = P_k(r,\delta)$ be the partition algebra on r dots with $\delta \in k \setminus \{0\}$. Then A is cellularly stratified by [5, Proposition 2.6]. The cellularly stratified structure was described in Section 2.3. We use the following embedding of $k\Sigma_l$ into $P_k(r,\delta)$. Let $d(\pi) \in P_k(l,\delta)$ be the diagram describing the permutation $\pi \in \Sigma_l$, i.e. the dot i in the top row is connected to the dot $\pi(i)$ in the bottom row, and these are all the connections. Then $d(\pi)$ becomes an element of $P_k(r,\delta)$ by attaching dots $l+1, ..., r$ to the right of the top row and connecting all these new dots to the lth dot of the top row. Do the same for the bottom row. This embedding agrees with the isomorphism $P_k(l,\delta) \simeq e_l P_k(r,\delta) e_l$ from Lemma 8. In particular, for each $0 \leq l \leq r$, the input algebra $k\Sigma_l$ of the cellularly stratified structure is a subalgebra of $e_l A e_l$ (see Corollary 9).

Example. Let $\pi = (1\,4\,3\,2) \in \Sigma_4$ and let $r = 7$. Then $d(\pi) =$  is clearly an element of $P_k(4,\delta)$. The corresponding element in $P_k(7,\delta)$ is  .

[4] In this case, a free dot is a dot which does not belong to a horizontal arc.

Furthermore, $j(e_l) = e_l$ for each l, where j is the involution flipping a diagram upside-down, so we can apply Proposition 5. It remains to show that A satisfies conditions (I) to (IV). Fix some l between 0 and r and remember that J_l denotes the two-sided ideal Ae_lA, generated, as a vector space, by diagrams with at most l propagating parts. Set $J_{-1} := 0$. There is a bijective map $J_n \setminus J_{n-1} \to J_n/J_{n-1}$ sending a diagram with exactly n propagating parts to its residue class. Hence, we can ignore the residue classes in many cases.

The right action of $k\Sigma_l$ on a partition diagram $d \in Ae_l$ permutes the bottom row of d, but it never changes the size of a part of d. In particular, the number of propagating lines remains invariant under the $k\Sigma_l$-action,[5] so $J_ne_l \simeq (J_n/J_{n-1})e_l \oplus J_{n-1}e_l$ is a decomposition of $k\Sigma_l$-modules, so assumption (I) is satisfied. For example, if $\pi = (1\,4\,3\,2) \in k\Sigma_4$ and $d =$  $\in (J_3 \setminus J_2)e_4$, then

$d\pi$  $\in (J_3 \setminus J_2)e_4$. Note that $\pi d \in J_2e_4$.

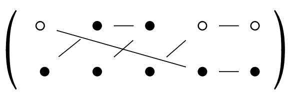
For $0 \le n \le l$, the basis diagrams of $(J_n/J_{n-1})e_l$ have exactly n propagating parts and the last $r-l+1$ dots of the bottom row belong to the same part. Hence, we have an isomorphism of vector spaces $(J_n/J_{n-1})e_l \simeq k\Sigma_n \otimes_k V_n \otimes_k V_n^l$, where V_n is the vector space of partial diagrams with exactly n labelled parts and V_n^l is the subspace of V_n where the last $r-l+1$ dots belong to the same part. For example, the diagram d above corresponds to the tensor product $(1\,3) \otimes \mathrm{top}(d) \otimes \mathrm{bottom}(d)$. This shows assumption (II′) is satisfied and thus, by Lemma 21, assumption (II) is satisfied as well.

Assumption (IV) holds by [20, Theorem 1] in case $\mathrm{char} k > \lfloor \frac{r}{3} \rfloor$. The condition on the characteristic is sufficient, but potentially too strong, as explained in [20].

We now prove that assumption (III) is satisfied, i.e. we show that the left $k\Sigma_n$-module $res_n Ind_l M^\lambda \simeq e_n(A/J_{n-1})e_l \underset{k\Sigma_l}{\otimes} M^\lambda$ admits a filtration by dual Specht modules for $n \le l$.

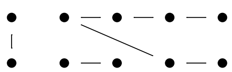
Fix $0 \le n \le l \le r$. When dealing with the size of a part in a partial diagram, we will from now on count the last $r-l+1$ dots as one. Let $v, w \in V_n^l$. We say that v is equivalent to w, $v \sim w$, if and only if there is a $\pi \in \Sigma_l$ such that $v\pi = w$, where $v\pi$ is defined as follows. Write the diagram π below v and identify $\mathrm{top}(\pi)$ with v. Then $v\pi$ is the bottom row of this diagram, where a part is labelled if and only if it contains at least one labelled dot. In diagrams, this means that v and w are equivalent if and only if for each size, the number of labelled parts and the number of unlabelled parts of v and w coincide. Remember that the last $r-l+1$ dots count as one.

Example. Let $r = 5$. The partial diagram $v =$ $\in V_2^4$ is equivalent to

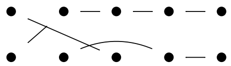
$v\pi = \mathrm{bottom}$  $=$

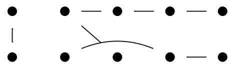
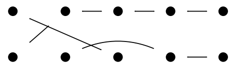
where $\pi = (1\,4\,3\,2) \in \Sigma_4$. Both v and $v\pi$ have two labelled singletons and one unlabelled part of size two.

For $v \in V_n^l$, we define d_v to be the diagram in $P_k(r,\delta)$ with $\mathrm{top}(d_v) = \mathrm{top}(e_n)$, $\mathrm{bottom}(d_v) = v$ and $\Pi(d_v) = 1_{k\Sigma_n}$. Let $b \in e_n(A/J_{n-1})e_l$ be a diagram with $\mathrm{bottom}(b) \sim v$. By definition, there is a $\pi \in \Sigma_l$ such that $\mathrm{bottom}(b) = v\pi$. Then $b = \Pi(b)\Pi(d_v\pi)^{-1}d_v\pi$.

Example *(continued)*. $d_v =$  .

[5] Note that this statement is usually wrong for $d \in A \setminus Ae_l$, since two different labelled parts of $\mathrm{bottom}(d)$ containing dots $\ge l$ will be identified by right multiplication with $\pi = e_l\pi e_l$.

For $b =$  we have $\mathrm{bottom}(b) = v(3\,4)$ and

$$\Pi(b)\Pi(d_v\pi)^{-1}d_v\pi = (1\,2)$$  $=$  $= b.$

Let U_v be the $(k\Sigma_n, k\Sigma_l)$-bimodule generated by d_v.

Lemma 27 *([20, Lemma 1]). The $(k\Sigma_n, k\Sigma_l)$-bimodule $e_n(A/J_{n-1})e_l$ decomposes into $\bigoplus_{v\in V_n^l/\sim} U_v$.*

Example *(continued).* As symmetric group bimodule, $e_2(A/J_1)e_4$ is spanned by the diagrams with bottom rows equivalent to one of

$$v_1 = \circ \quad \circ \quad \bullet - \bullet - \bullet\,, \quad v_2 = \circ \quad \circ - \circ \quad \bullet - \bullet\,,$$
$$v_3 = \circ \quad \circ - \circ - \circ - \circ\,, \quad v_4 = \circ - \circ \quad \circ - \circ - \circ$$

Left multiplication of d_v with Σ_2 affects only the propagating lines, while right multiplication of d_v with Σ_4 affects both propagating lines and the bottom row. However, the sizes of the parts remain invariant, so the sum is direct and we have $e_2(A/J_1)e_4 = \bigoplus_{i=1}^{4} U_{v_i}$. The partial diagram v we studied before is equivalent to v_1, so $d_v \in U_{v_1}$.

Fix a partial diagram $v \in V_n^l$ and set $d := d_v$. Let α_i be the number of labelled parts of size i and β_i the number of unlabelled parts of size i of v, where again the last $r - l + 1$ dots count as one dot. Set $\alpha := (\alpha_1, \alpha_2, \ldots)$ and $\beta := (\beta_1, \beta_2, \ldots)$ be the corresponding compositions. Then $\sum_i(\alpha_i \cdot i) + \sum_i(\beta_i \cdot i) = l$ and $\sum_i \alpha_i = n$. Without loss of generality, assume that the parts of v are ordered as follows. The labelled parts are on the left hand side, the unlabelled parts on the right hand side. The parts are then ordered increasingly from left to right.

Example *(continued).* The partial diagrams v_i are all ordered in this way. We have $\alpha(v_1) = (2)$, $\alpha(v_2) = (1,1)$, $\alpha(v_3) = (1,0,1)$, $\alpha(v_4) = (0,2)$ and $\beta(v_1) = (0,1)$, $\beta(v_2) = (1)$, $\beta(v_3) = \beta(v_4) = ()$.

Let $\mathcal{S}_i^j \subseteq \{1,\ldots,l\}$ be the set of dots of v belonging to the jth labelled part of size i and let $\mathcal{T}_i^j \subseteq \{1,\ldots,l\}$ be the set of dots of v belonging to the jth unlabelled part of size i. Then $\prod_\alpha := \prod_{i\geq 1,\alpha_i\neq 0}((\Sigma_{\mathcal{S}_i^1} \times \ldots \times \Sigma_{\mathcal{S}_i^{\alpha_i}}) \rtimes \Sigma_{\alpha_i})$ is the stabilizer subgroup of Σ_l which stabilizes exactly the labelled parts of v. Similarly, the stabilizer subgroup of Σ_l which stabilizes the unlabelled parts of v is $\prod_\beta := \prod_{i\geq 1,\beta_i\neq 0}((\Sigma_{\mathcal{T}_i^1} \times \ldots \times \Sigma_{\mathcal{T}_i^{\beta_i}}) \rtimes \Sigma_{\beta_i})$. In particular, $\prod_\beta$ stabilizes d, while $\prod_\alpha$ permutes the propagating lines of d. Note that $\prod_\alpha \simeq \prod_{i\geq 1,\alpha_i\neq 0}(\Sigma_i \wr \Sigma_{\alpha_i})$ and $\prod_\beta \simeq \prod_{i\geq 1,\beta_i\neq 0}(\Sigma_i \wr \Sigma_{\beta_i})$, where $\wr$ denotes the wreath product. Define a right-action of $\prod_\alpha \times \prod_\beta$ on $k\Sigma_n$ via $\eta \cdot \zeta := \eta\Pi(d\zeta)$ for $\eta \in \Sigma_n$ and $\zeta \in \prod_\alpha \times \prod_\beta$, i.e. $\prod_\alpha \times \prod_\beta$ acts on $k\Sigma_n$ via the canonical epimorphism

$$\rho : \prod_{i\geq 1,\alpha_i\neq 0}(\Sigma_i \wr \Sigma_{\alpha_i}) \times \prod_{i\geq 1,\beta_i\neq 0}(\Sigma_i \wr \Sigma_{\beta_i}) \twoheadrightarrow \Sigma_\alpha.$$

Then we can define the tensor product $k\Sigma_n \underset{k\prod_\alpha \times k\prod_\beta}{\otimes} k\Sigma_l$.

Example *(continued).* For v_1 we have $\prod_\alpha = (\Sigma_{\{1\}} \times \Sigma_{\{2\}}) \rtimes \Sigma_2 \simeq \Sigma_1 \wr \Sigma_2 \simeq \Sigma_2$ and $\prod_\beta = \Sigma_{\{3,4\}} \rtimes \Sigma_1 \simeq \Sigma_2$ and the canonical projection $\rho : \Sigma_2 \times \Sigma_2 \twoheadrightarrow \Sigma_2$ (projection onto the first factor).

For v_4 we have $\prod_\alpha = (\Sigma_{\{1,2\}} \times \Sigma_{\{3,4\}}) \rtimes \Sigma_2 \simeq \Sigma_2 \wr \Sigma_2$ and $\prod_\beta = \langle 1 \rangle$ and the canonical projection $\rho : \Sigma_2 \wr \Sigma_2 \twoheadrightarrow \Sigma_2$ (factoring out the base group of the wreath product).

Lemma 28 *([20, Lemma 2]). There is an isomorphism of $(k\Sigma_n, k\Sigma_l)$-bimodules $k\Sigma_n \underset{k\prod_\alpha \times k\prod_\beta}{\otimes} k\Sigma_l \longrightarrow U_v$ given by $\eta \otimes \tau \longmapsto \eta d\tau$.*

We want to understand the summands $k\Sigma_n \underset{k\prod_\alpha \times k\prod_\beta}{\otimes} k\Sigma_l \underset{k\Sigma_\lambda}{\otimes} k$ of $res_n Ind_l M^\lambda = e_n(A/J_{n-1})e_l \underset{k\Sigma_l}{\otimes} k\Sigma_l \underset{k\Sigma_\lambda}{\otimes} k$ for a partition λ of l. Fix double coset representatives $\pi_1, ..., \pi_q$ of $(\prod_\alpha \times \prod_\beta)\backslash\Sigma_l/\Sigma_\lambda$. To each π_i, we attach a composition ν^i as follows. Set $\prod_{\nu^i} := (\prod_\alpha \times \prod_\beta) \cap \pi_i \Sigma_\lambda \pi_i^{-1}$. Then $\zeta \in \prod_\alpha \times \prod_\beta$ is in $\prod_{\nu^i}$ if and only if there is a $\vartheta \in \Sigma_\lambda$ such that $\zeta\pi_i = \pi_i\vartheta$. Since $\pi_i\Sigma_\lambda\pi_i^{-1}$ is isomorphic to Σ_λ, it is a Young subgroup of Σ_l, and $\prod_\alpha \times \prod_\beta$ is a direct product of wreath products of symmetric groups. Thus the intersection $(\prod_\alpha \times \prod_\beta) \cap \pi\Sigma_\lambda\pi^{-1}$ is a subgroup of a product of wreath products. The image of $\prod_{\nu^i}$ under the canonical epimorphism ρ is a Young subgroup of Σ_n, which we denote by Σ_{ν^i}.

Example *(continued).* For $v = v_4$ and $\lambda = (2^2) = (2, 2)$, we get the coset representatives $\pi_1 = 1$ and $\pi_2 = (2, 3)$. Then $\prod_{\nu^1} = \Sigma_\lambda$, $\Sigma_{\nu^1} = \langle 1 \rangle$ and $\prod_{\nu^2} = \langle (1\,3)(2\,4) \rangle$, $\Sigma_{\nu^2} = \Sigma_2$.

A bigger example for Π_{ν^i}, Σ_{ν^i} and a GAP-algorithm to compute them can be found in Appendix B and C.

Proposition 29. *The left $k\Sigma_n$-module $k\Sigma_n \underset{k(\Pi_\alpha \times \Pi_\beta)}{\otimes} k\Sigma_l \underset{k\Sigma_\lambda}{\otimes} k$ is isomorphic to the direct sum $\bigoplus_{i=1}^{q} (k\Sigma_n \underset{k\Sigma_{\nu^i}}{\otimes} k)$ of various permutation modules. In particular, it admits a filtration by dual $k\Sigma_n$-Specht modules.*

Proof. We define a map

$$\varphi : k\Sigma_n \underset{k(\Pi_\alpha \times \Pi_\beta)}{\otimes} k\Sigma_l \underset{k\Sigma_\lambda}{\otimes} k \longrightarrow \bigoplus_{i=1}^{q} (k\Sigma_n \underset{k\Sigma_{\nu^i}}{\otimes} k)$$

as follows. Let $\eta \in \Sigma_n$ and $\tau \in \Sigma_l$ with $\tau = \zeta\pi_i\vartheta$ for some $\zeta \in \prod_\alpha \times \prod_\beta$ and $\vartheta \in \Sigma_\lambda$. Set $\varphi(\eta \otimes \tau \otimes 1) = (0, ..., 0, \eta\Pi(d\zeta) \otimes 1, 0, ..., 0) =: (\eta\Pi(d\zeta) \otimes 1)^{(i)}$ with non-zero entry only in the ith summand. Extend this $k\Sigma_n$-linearly to get a $k\Sigma_n$-homomorphism.

We have to show that this map is well-defined, that is we have to show that whenever two elements $\eta \otimes \tau \otimes 1$ and $\eta' \otimes \tau' \otimes 1$ are equivalent in $k\Sigma_n \underset{k(\Pi_\alpha \times \Pi_\beta)}{\otimes} k\Sigma_l \underset{k\Sigma_\lambda}{\otimes} k$, then their images are equivalent in $\bigoplus_{i=1}^{q} (k\Sigma_n \underset{k\Sigma_{\nu^i}}{\otimes} k)$.

Let $\eta \otimes \tau \otimes 1 = \eta' \otimes \tau' \otimes 1$ with $\eta, \eta' \in \Sigma_n$ and $\tau, \tau' \in \Sigma_l$ and let $\tau = \zeta\pi_i\vartheta$ and $\tau' = \zeta'\pi_j\vartheta'$. Since $\eta \otimes \tau \otimes 1 = \eta' \otimes \tau' \otimes 1$, we have $i = j$ and $\eta\Pi(d\zeta) = \eta'\Pi(d\zeta')$. It follows that $\varphi(\eta \otimes \tau \otimes 1) = \varphi(\eta \otimes \zeta\pi_i\vartheta \otimes 1) = (\eta\Pi(d\zeta) \otimes 1)^{(i)} = (\eta'\Pi(d\zeta') \otimes 1)^{(i)} = \varphi(\eta' \otimes \tau' \otimes 1)$, so φ is well-defined.

The inverse is given by

$$\psi : \bigoplus_{i=1}^{q} (k\Sigma_n \underset{k\Sigma_{\nu^i}}{\otimes} k) \longrightarrow k\Sigma_n \underset{k(\Pi_\alpha \times \Pi_\beta)}{\otimes} k\Sigma_l \underset{k\Sigma_\lambda}{\otimes} k$$

with $\psi(\sum_{i=1}^{q} (\eta_i \otimes 1)^{(i)}) = \sum_{i=1}^{q} \eta_i \otimes \pi_i \otimes 1$ for $\eta_i \in \Sigma_n$:

$$(\psi \circ \varphi)(\eta \otimes \zeta\pi_i\vartheta \otimes 1) = \psi((\eta\Pi(d\zeta) \otimes 1)^{(i)}) = \eta\Pi(d\zeta) \otimes \pi_i \otimes 1 = \eta \otimes \zeta\pi_i\vartheta \otimes 1$$

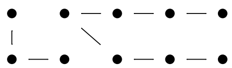
and

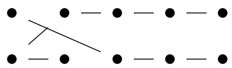
$$(\varphi \circ \psi)((\eta \otimes 1)^{(i)}) = \varphi(\eta \otimes \pi_i \otimes 1) = (\eta \otimes 1)^{(i)}$$

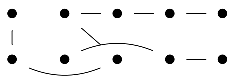
for $\eta \in \Sigma_n$, $\zeta \in \prod_\alpha \times \prod_\beta$ and $\vartheta \in \Sigma_\lambda$.

It remains to show that ψ is well-defined. Let $\eta, \eta' \in \Sigma_n$ such that $\eta \otimes 1$ and $\eta' \otimes 1$ are equivalent in $k\Sigma_n \underset{k\Sigma_{\nu^i}}{\otimes} k$ for some i. Then there is a $\xi \in \Sigma_{\nu^i}$ such that $\eta' = \eta\xi$. It follows that $\psi((\eta' \otimes 1)^{(i)}) = \eta' \otimes \pi_i \otimes 1 = \eta\xi \otimes \pi_i \otimes 1 = \eta \otimes \hat{\xi}\pi_i \otimes 1$ for some $\hat{\xi} \in \prod_\alpha$ with $\Pi(d\hat{\xi}) = \xi$. By definition of Σ_{ν^i} as the image of the canonical projection $\prod_{\nu^i} \to \Sigma_n$, we have $\hat{\xi} \in \pi_i \Sigma_\lambda \pi_i^{-1}$. So there is a $\vartheta \in \Sigma_\lambda$ such that $\hat{\xi}\pi_i = \pi_i\vartheta$. Therefore we have $\psi((\eta' \otimes 1)^{(i)}) = \eta \otimes \pi_i\vartheta \otimes 1 = \eta \otimes \pi_i \otimes 1 = \psi((\eta \otimes 1)^{(i)})$ and $\psi = \varphi^{-1}$ is well-defined. $\square$

Example *(continued).* $U_{v_4} \underset{k\Sigma_{(2^2)}}{\otimes} k \simeq k\Sigma_2 \underset{k(\Sigma_2 \wr \Sigma_2)}{\otimes} k\Sigma_4 \underset{k\Sigma_{(2^2)}}{\otimes} k \simeq k\Sigma_2 \underset{k\Sigma_1}{\otimes} k \oplus k\Sigma_2 \underset{k\Sigma_2}{\otimes} k \simeq k\Sigma_2 \oplus k$ as left $k\Sigma_2$-modules. In detail, these isomorphisms, evaluated at basis diagrams, look as follows.

 $\otimes 1 \mapsto 1 \otimes 1 \otimes 1 = 1 \otimes \pi_1 \otimes 1 \mapsto (1 \otimes 1, 0) \mapsto (1, 0)$

 $\otimes 1 \mapsto (1\,2) \otimes 1 \otimes 1 = (1\,2) \otimes \pi_1 \otimes 1 \mapsto ((1\,2) \otimes 1, 0) \mapsto ((1\,2), 0)$

 $\otimes 1 \mapsto 1 \otimes (2\,3) \otimes 1 = 1 \otimes \pi_2 \otimes 1 \mapsto (0, 1 \otimes 1) \mapsto (0, 1)$

All diagrams in U_{v_4} which are not displayed here are equivalent to the diagram in the right row in $U_{v_4} \underset{k\Sigma_\lambda}{\otimes} k$. It is left for the reader to check that they are all sent to $(0, 1)$ via the isomorphism.

Corollary 30. *Layer restriction of a permutation module is isomorphic to a direct sum of permutation modules. In particular, layer restriction of a permutation module has a dual Specht filtration.*

Proof. For $n > l$, we have $res_n Ind_l M^\lambda = 0$, so the statement is true. For $n \leq l$, we can apply Lemmas 27, 28 and Proposition 29 to get a decomposition

$$res_n Ind_l M^\lambda \simeq \bigoplus_{v \in V_n^l/\sim} \bigoplus_{i=1}^{q(v)} (k\Sigma_n \underset{k\Sigma_{\nu^i(v)}}{\otimes} k) \in \mathcal{F}_n(S). \quad \square$$

This shows that assumption (III) is satisfied and we can conclude the following theorem.

Theorem 6. *Let $r \in \mathbb{N}$ and let k be an algebraically closed field with* $\mathrm{char} k = 0$ *or at least* $\max\{5, \lfloor \frac{r}{3} \rfloor\}$. *Then the partition algebra $P_k(r, \delta)$, with $\delta \neq 0$, has permutation modules $M(l, \lambda)$, which are a direct sum of indecomposable Young modules. The Young modules are relative projective covers of the cell modules $ind_l S_\lambda$. Every module admitting a cell filtration has well-defined filtration multiplicities.*

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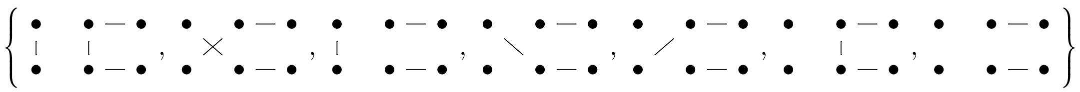
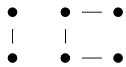
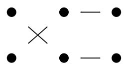
Appendix A. Example for the functors defined in §2.4

We want to illustrate the functors ind_l, res_l, Ind_l and Res_l on small examples. Let A be the partition algebra $P_k(3, \delta)$ for some algebraically closed field k of characteristic $\neq 2$ and let $B_l = k\Sigma_2$.

First, let us understand the structures Ae_2, $J_1 = Ae_1A$, A/J_1, $(A/J_1)e_2$ and e_2Ae_2. The set e_2Ae_2 is an idempotent subalgebra of A and a $(k\Sigma_2, k\Sigma_2)$-bimodule. As a vector space, e_2Ae_2 has basis



The set Ae_2 is an (A, e_2Ae_2)-bimodule, generated by diagrams with arbitrary top row and bottom row (with arbitrary labelling), and $\dim_k Ae_2 = 32$. The set $J_1 = Ae_1A$ is an (A, A)-bimodule generated by all diagrams with at most one propagating part, and $\dim_k J_1 = 125$. The quotient A/J_1 is the (A, A)-bimodule generated by residue classes of diagrams with at least two propagating lines, modulo J_1, and $\dim_k A/J_1 = 78$. Multiplication with e_2 (from the right) restricts the number of propagating lines to be at most two (and thus exactly two) and fixes the bottom row to be . Thus the diagrams in $(A/J_1)e_2$ are uniquely determined by their top rows (with exactly two labelled parts) and whether or not the propagating lines cross. The action of $k\Sigma_2$ on a diagram is given by the embedding

$1 \mapsto$ , $(1\,2) \mapsto$ .

Thus the right action of $(1\,2)$ on $(A/J_1)e_2$ changes whether or not the propagating lines cross.

Let $S_{(1,1)}$ be the Specht module in $k\Sigma_2 - \text{mod}$ generated by the polytabloid $s := \begin{array}{c}\overline{1}\\ \underline{2}\end{array} - \begin{array}{c}\overline{2}\\ \underline{1}\end{array}$, i.e. $S_{(1,1)}$ is the sign-module for $k\Sigma_2$. Then

$$ind_2 S_{(1,1)} \simeq (A/J_1)e_2 \underset{k\Sigma_2}{\otimes} S_{(1,1)}$$

is generated by tensor products $d \otimes s$ where s is the generator of $S_{(1,1)}$ and d is a diagram in $(A/J_1)e_2$ with non-crossing propagating lines.[6] In particular,

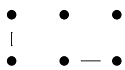
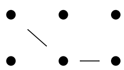
$$ind_2 S_{(1,1)} \simeq V_2^3$$

as a vector space, and $\dim_k ind_2 S_{(1,1)} = 2$. On the other hand, we have

$$Ind_2 S_{(1,1)} = Ae_2 \underset{k\Sigma_2}{\otimes} S_{(1,1)}$$

generated by tensor products $d \otimes s$ with $d \in Ae_2$ with non-crossing propagating lines. In particular, d might have less than two propagating lines. As a vector space, we have

$$Ind_2 S_{(1,1)} \simeq ind_2 S_{(1,1)} \oplus (J_1 e_2 \otimes S_{(1,1)}).$$

If $d \in J_0 e_2 \subset J_1 e_2$ then $d = d(1\,2)$, so $d \otimes s = d(1\,2) \otimes s = d \otimes -s = -(d \otimes s)$. Therefore, $J_0 e_2 \underset{k\Sigma_2}{\otimes} S_{(1,1)} = 0$. The set $(J_1 \setminus J_0)e_2$ is 20-dimensional, and $J_1 e_2 \underset{k\Sigma_2}{\otimes} S_{(1,1)}$ identifies each basis diagram with the negative of another one, e.g.  $\otimes s = -$  $\otimes s$, so $\dim_k J_1 e_2 \underset{k\Sigma_2}{\otimes} S_{(1,1)} = 10$ which implies $\dim_k Ind_2 S_{(1,1,)} = 12$.

[6] If d and d' differ only by the crossing of propagating lines, then $d' \otimes s = d(1\,2) \otimes s = d \otimes -s = -(d \otimes s)$.

Now, let N be the projective left A-module Ae_1. Then

$$res_2 N = e_2(A/J_1) \underset{A}{\otimes} Ae_1 = e_2(A/J_{l-1}) \underset{A}{\otimes} J_1 e_1 = e_2(A/J_1)J_1 \underset{A}{\otimes} Ae_1 = 0,$$

while

$$Res_2 N = e_2 A \underset{A}{\otimes} Ae_1 = e_2 Ae_1$$

is generated by diagrams with at most one propagating part, top row ∘ •—• or • ∘—∘ and bottom row •—•—• or ∘—∘—∘ . In particular, $Res_2 N \neq 0$.

Appendix B. Example for Proposition 29, calculated by hand

Example. Let $v =$ ∘ ∘ ∘ ∘—∘ ∘—∘ •—• $\in V_5^9$. The summand U_v of $e_5(A/J_4)e_9 \underset{k\Sigma_{(7,2)}}{\otimes} k$ is isomorphic to

$$(k\Sigma_5 \underset{k\Sigma_{(3,2)}}{\otimes} k)^2 \oplus (k\Sigma_5 \underset{k\Sigma_{(3,1^2)}}{\otimes} k)^2 \oplus (k\Sigma_5 \underset{k\Sigma_{(2^2,1)}}{\otimes} k)^2 \oplus (k\Sigma_5 \underset{k\Sigma_{(2,1^3)}}{\otimes} k).$$

This can be verified as follows. We have $\prod_\alpha \times \prod_\beta = \Sigma_3 \times (\Sigma_2 \times \Sigma_2) \times \Sigma_2$ and the set of double coset representatives is

$$\{id, (7\,8), (6\,8)(7\,9), (5\,8\,6)(7\,9), (3\,8\,7\,6\,5\,4), (3\,8\,6\,5\,4)(7\,9), (2\,8\,6\,4)(3\,9\,7\,5)\}.$$

The only transpositions in $\Sigma_\lambda = \Sigma_{(7,2)}$ leaving $v\pi$ invariant are those with both end points belonging to the same part λ_i. In the partial diagram $v\pi$, mark these dots as $*$ if they were labelled. The only products of two disjoint transpositions $(a\,b)(c\,d)$ leaving $v\pi$ invariant are those where a and c (or a and d) belong to the same part λ_i and b and d (or b and c, respectively) belong to the same part λ_j. Note that here, $\lambda_i = \lambda_j$ is possible.[7] Mark these dots as $\diamond$ if they were labelled. Put vertical lines at the end of each part λ_i. Translate this back to $v = v\pi\pi^{-1}$. We can read off $\Pi_\nu = \prod_\alpha \cap \pi\Sigma_\lambda\pi^{-1}$ from the given information by the labelling of dots: $*$s of the same size become symmetric groups, $\diamond$s become wreath products, if both end points a, b lie in the same part λ_i in $v\pi$ and the group generated by $(a\,b)(c\,d)$ otherwise.[8] We do this for each double coset representative in Table 1.

Appendix C. GAP code to compute summands of restriction of permutation modules for partition algebras

For a given summand U_v of $e_n(A/J_{n-1})e_l$, the following GAP code calculates which Young subgroups Σ_{ν^i} appear in the decomposition of $k\Sigma_n \underset{k(\prod_\alpha \times \prod_\beta)}{\otimes} k\Sigma_l \underset{k\Sigma_\lambda}{\otimes} k \simeq U_v \underset{k\Sigma_\lambda}{\otimes} k$, given in Proposition 29.

As input, we need `G=` Σ_l, `H=` $\prod_\alpha \times \prod_\beta$ and `K=` Σ_λ, as well as the list `imgs` of images of the generators of H under the canonical epimorphism $\prod_\alpha \times \prod_\beta \twoheadrightarrow \Sigma_n$, sending ζ to $\Pi(d\zeta)$. We state the code for the example in Appendix B.

[7] In this case, the transpositions $(a\,b)$ and $(c\,d)$ belong to the first group of dots ($*$ or $\star$) as well.
[8] It does not make a difference for Σ_ν which of the two cases we have, since the projection onto Σ_n is the same.

Table 1
Diagrammatic deduction of Young subgroups Σ_ν.

π_i	$v\pi_i$ v	Π_ν	Σ_ν
1	$\ast\ \ast\ \ast\ \diamond - \diamond\ \ \diamond - \diamond \mid \bullet - \bullet$ $\ast\ \ast\ \ast\ \diamond - \diamond\ \ \diamond - \diamond \mid \bullet - \bullet$	$\Sigma_3 \times (\Sigma_2 \wr \Sigma_2) = \prod_\alpha$	$\Sigma_{(3,2)}$
$(7\,8)$	$\ast\ \ast\ \ast\ \ast - \ast\ \ \circ\ \ \bullet \mid \circ\ \ \bullet$ $\ast\ \ast\ \ast\ \ast - \ast\ \ \circ - \circ \mid \bullet - \bullet$	$\Sigma_{(3,2,1^2)}$	$\Sigma_{(3,1^2)}$
$(6\,8)(7\,9)$	$\ast\ \ast\ \ast\ \ast - \ast\ \ \bullet - \bullet \mid \ast - \ast$ $\ast\ \ast\ \ast\ \ast - \ast\ \ \ast - \ast \mid \bullet - \bullet$	$\Sigma_{(3,2^2)}$	$\Sigma_{(3,1^2)}$
$(5\,8\,6)(7\,9)$	$\ast\ \ast\ \ast\ \diamond\ \ \diamond\ \ \bullet - \bullet \mid \diamond\ \ \diamond$ $\ast\ \ast\ \ast\ \diamond - \diamond\ \ \diamond - \diamond \mid \bullet - \bullet$	$\Sigma_3 \times \langle(4\,6)(5\,7)\rangle$	$\Sigma_{(3,2)}$
$(3\,8\,7\,6\,5\,4)$	$\ast\ \ast\ \diamond - \diamond\ \ \diamond - \diamond\ \ \bullet \mid \circ\ \ \bullet$ $\ast\ \ast\ \circ\ \ \diamond - \diamond\ \ \diamond - \diamond \mid \bullet - \bullet$	$\Sigma_{(2,1)} \times (\Sigma_2 \wr \Sigma_2)$	$\Sigma_{(2,1,2)}$
$(3\,8\,6\,5\,4)(7\,9)$	$\ast\ \ast\ \ast - \ast\ \ \circ\ \ \bullet - \bullet \mid \circ - \circ$ $\ast\ \ast\ \circ\ \ \ast - \ast\ \ \circ - \circ \mid \bullet - \bullet$	$\Sigma_{(2,1,2,1^2)}$	$\Sigma_{(2,1^3)}$
$(2\,8\,6\,4)(3\,9\,7\,5)$ +	$\circ\ \ \diamond - \diamond\ \ \diamond - \diamond\ \ \bullet - \bullet \mid \ast\ \ \ast$ $\circ\ \ \ast\ \ast\ \diamond - \diamond\ \ \diamond - \diamond \mid \bullet - \bullet$	$\Sigma_{(1,2)} \times (\Sigma_2 \wr \Sigma_2)$	$\Sigma_{(1,2^2)}$

```
INPUT: S2:=SymmetricGroup(2); S3:=SymmetricGroup(3);
S5:=SymmetricGroup(5); S7:=SymmetricGroup(7);            # abbreviations
G:= SymmetricGroup(14); H:=DirectProduct(S3,WreathProduct(S2,S2),S2);
K:=DirectProduct(S7,S2);                      # G = Σ_l, H = ∏_α × ∏_β, K = Σ_λ.

gens:=GeneratorsOfGroup(H);
                                              # to each generator gens[i], set
imgs:=[(1,2,3),(1,2),(),(),(4,5),()];         # imgs[i]:=the image of gens[i]
                                              # under the canonical epimorphism
                                              # ∏_α × ∏_β ↠ Σ_5.
hom:=GroupHomomorphismByImages(H,S5,gens,imgs);

iso=function(G,H,K)
local L, r, R, Pinu, Snu;
L:=[]; R:=List(DoubleCosets(G,H,K),Representative);
for r in R do
Pinu:=Intersection(H,ConjugateSubgroup(K,r^-1));
Snu:=Image(hom,Pinu);
Add(L,Snu);
od;
return L;
end;
```

OUTPUT: list L of all appearing Young subgroups Σ_{ν^i} =Snu of Σ_5 =S5.

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