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Singularity categories of derived categories of hereditary algebras are derived categories

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ABSTRACT

We show that for the path algebra A of an acyclic quiver, the singularity category of the derived category $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}A)$ is triangle equivalent to the derived category of the functor category of $\underline{\mathsf{mod}}\,A$, that is, $\mathsf{D}_{\mathrm{sg}}(\mathsf{D}^{\mathrm{b}}(\mathsf{mod}A)) \simeq \mathsf{D}^{\mathrm{b}}(\mathsf{mod}(\underline{\mathsf{mod}}\,A))$. This extends a result in [14] for the path algebra A of a Dynkin quiver. An important step is to establish a functor category analog of Happel's triangle equivalence for repetitive algebras.

1. Introduction

Let k be a field and A be a finite dimensional k-algebra. In [14], it was shown that if A is a representation-finite hereditary algebra, then there exists a triangle equivalence

$$\underline{\mathsf{mod}}\,\mathsf{D}^{\mathrm{b}}(\mathsf{mod}A) \simeq \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,B), \tag{1.1}$$

where B is the stable Auslander algebra of A, $\mathsf{mod}\,\mathsf{D}^{\mathrm{b}}(\mathsf{mod}A)$ is the Frobenius category of finitely presented functors from $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}A)$ to the category of abelian groups $\mathcal{A}b$, and $\underline{\mathsf{mod}}\,\mathsf{D}^{\mathrm{b}}(\mathsf{mod}A)$ is its stable category.

In this paper, we extend the triangle equivalence (1.1) to the case when A is a representation-infinite hereditary algebra. In this case, the role of the stable Auslander algebra is played by the stable category $\underline{\mathsf{mod}}\,A$ of finitely generated A-modules. We denote by $\mathsf{mod}(\underline{\mathsf{mod}}\,A)$ the category of finitely presented functors from the stable category $\underline{\mathsf{mod}}\,A$ to $\mathcal{A}b$. Our main result is the following.

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Theorem 1.1 *(Theorem 4.5). Let A be a finite dimensional hereditary k-algebra. We have a triangle equivalence*

$$\underline{\mathsf{mod}}\, \mathsf{D}^{\mathrm{b}}(\mathsf{mod} A) \simeq \mathsf{D}^{\mathrm{b}}(\mathsf{mod}(\underline{\mathsf{mod}}\, A)). \tag{1.2}$$

Note that for a triangulated category $\mathcal{T}$, the stable category $\underline{\mathsf{mod}}\, \mathcal{T}$ is triangle equivalent to the singularity category $\mathsf{D}_{\mathrm{sg}}(\mathcal{T}) = \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\mathcal{T})/\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, \mathcal{T})$ [9,25] (see Theorem 2.17). Thus (1.2) can be rewritten as $\mathsf{D}_{\mathrm{sg}}(\mathsf{D}^{\mathrm{b}}(\mathsf{mod} A)) \simeq \mathsf{D}^{\mathrm{b}}(\mathsf{mod}(\underline{\mathsf{mod}}\, A))$.

To prove Theorem 1.1, we need to give general preliminary results on functor categories and repetitive categories. The functor category $\mathsf{mod}(\underline{\mathsf{mod}}\, A)$ is an abelian category with enough projectives and enough injectives since the category $\underline{\mathsf{mod}}\, A$ forms a dualizing k-variety, which is a distinguished class of k-linear categories introduced by Auslander and Reiten [3]. A key role is played by the repetitive category $\mathsf{R}(\underline{\mathsf{mod}}\, A)$ of $\underline{\mathsf{mod}}\, A$. Our first result implies that $\mathsf{R}(\underline{\mathsf{mod}}\, A)$ is a dualizing k-variety.

Theorem 1.2 *(Theorem 3.7). Let $\mathcal{A}$ be a dualizing k-variety. Then the repetitive category $\mathsf{R}\mathcal{A}$ of $\mathcal{A}$ is a dualizing k-variety.*

In particular, $\mathsf{mod}\mathsf{R}\mathcal{A}$ is a Frobenius abelian category for any dualizing k-variety $\mathcal{A}$. We denote by $\underline{\mathsf{mod}}\, \mathsf{R}\mathcal{A}$ the stable category of $\mathsf{mod}\mathsf{R}\mathcal{A}$, which is triangulated.

In the case where A is a representation finite hereditary algebra, the following theorem by Happel [13] plays an important role in the proof of the triangle equivalence (1.1): for a finite dimensional k-algebra A of finite global dimension, the bounded derived category of A is triangle equivalent to the stable category of the repetitive algebra of A. In Section 3, we show a categorical analog of this triangle equivalence for dualizing k-varieties. In fact, we deal with the following more general class of categories including dualizing k-varieties. For a k-linear additive category $\mathcal{A}$, we denote by $\mathsf{proj}\, \mathcal{A}$ the category of finitely generated projective $\mathcal{A}$-modules and by $\mathsf{mod}\mathcal{A}$ the category of $\mathcal{A}$-modules having resolutions by $\mathsf{proj}\, \mathcal{A}$. We consider the following conditions:

(IFP) $\mathrm{D}\mathcal{A}(X, -)$ is in $\mathsf{mod}\mathcal{A}$ for each $X \in \mathcal{A}$, where $\mathrm{D} = \mathrm{Hom}_k(-, k)$.
(G) $\mathrm{D}\mathcal{A}(X, -)$ has finite projective dimension over $\mathcal{A}$ for each $X \in \mathcal{A}$.

For example, if $\mathcal{A}$ is a dualizing k-variety, then $\mathcal{A}$ satisfies the condition (IFP). On the other hand, the condition (G) is a categorical version of Gorensteinness. Gorenstein-projective modules (also known as Cohen-Macaulay modules, totally reflexive modules) are an important class of modules. We denote by $\mathsf{GP}(\mathsf{R}\mathcal{A}, \mathcal{A})$ the category of Gorenstein-projective $\mathsf{R}\mathcal{A}$-modules of finite projective dimension as $\mathcal{A}$-modules (see Subsection 3.2). We prove the following.

Theorem 1.3 *(Corollaries 3.17, 3.18). Let $\mathcal{A}$ be a k-linear, Hom-finite additive category.*

(a) *Assume that $\mathcal{A}$ and $\mathcal{A}^{\mathrm{op}}$ satisfy (IFP) and (G). Then we have a triangle equivalence*

$$\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, \mathcal{A}) \simeq \underline{\mathsf{GP}}(\mathsf{R}\mathcal{A}, \mathcal{A}).$$

(b) *Assume that $\mathcal{A}$ is a dualizing k-variety. If each object of $\mathsf{mod}\mathcal{A}$ and $\mathsf{mod}(\mathcal{A}^{\mathrm{op}})$ has finite projective dimension, then we have a triangle equivalence*

$$\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\mathcal{A}) \simeq \underline{\mathsf{mod}}\, \mathsf{R}\mathcal{A}.$$

We refer to [6,14,17,18,21,22,24,28] for recent results which realize stable categories as derived categories in different settings.

In Section 4, we show the following theorem, which together with Theorem 1.3 implies Theorem 1.1.

Theorem 1.4 *(Theorem 4.3). Let A be a finite dimensional representation-infinite hereditary k-algebra. Then we have an equivalence of additive categories*

$$\mathsf{R}(\underline{\mathsf{mod}}\, A) \simeq \mathsf{D}^{\mathrm{b}}(\mathsf{mod} A).$$

Notation. In this paper, we denote by k a field. All subcategories are full and closed under isomorphisms. Let $\mathcal{C}$ be an additive category and $\mathcal{S}$ be a subclass of objects of $\mathcal{C}$ or a subcategory of $\mathcal{C}$. We denote by $\mathsf{add}\,\mathcal{S}$ the subcategory of $\mathcal{C}$ whose objects are direct summands of finite direct sums of objects in $\mathcal{S}$. For subcategories $\mathcal{C}_i$ $(i \in I)$ of $\mathcal{C}$, we denote by $\bigvee_{i\in I} \mathcal{C}_i$ the smallest additive subcategory of $\mathcal{C}$ containing all $\mathcal{C}_i$ and closed under direct summands. For objects $X, Y \in \mathcal{C}$, we denote by $\mathcal{C}(X, Y)$ the set of morphisms from X to Y in $\mathcal{C}$. We call a category *skeletally small* if the class of isomorphism classes of objects is a set.

2. Preliminaries

Throughout this section, we assume that $\mathcal{A}$ is a skeletally small category.

2.1. Functor categories

In this subsection, we recall the definition of modules over categories. Let $\mathcal{A}$ be an additive category. An $\mathcal{A}$*-module* is a contravariant additive functor from $\mathcal{A}$ to $\mathcal{A}b$, where $\mathcal{A}b$ is the category of abelian groups. We denote by $\mathsf{Mod}\mathcal{A}$ the category of $\mathcal{A}$-modules, where morphisms of $\mathsf{Mod}\mathcal{A}$ are morphisms of functors. Since $\mathcal{A}$ is skeletally small, $\mathsf{Mod}\mathcal{A}$ is a category. It is well known that $\mathsf{Mod}\mathcal{A}$ is abelian.

For two morphisms $f : L \to M$ and $g : M \to N$ of $\mathsf{Mod}\mathcal{A}$, the sequence $L \to M \to N$ is exact in $\mathsf{Mod}\mathcal{A}$ if and only if the induced sequence $L(X) \to M(X) \to N(X)$ is exact in $\mathcal{A}b$ for any $X \in \mathcal{A}$.

Example 2.1. For each $X \in \mathcal{A}$, a representable functor $\mathcal{A}(-, X)$ is an $\mathcal{A}$-module. By Yoneda's lemma, $\mathcal{A}(-, X)$ is projective in $\mathsf{Mod}\mathcal{A}$.

The following notation is basic and used throughout this paper. We call an $\mathcal{A}$-module M *finitely generated* if there exists an epimorphism $\mathcal{A}(-, X) \to M$ in $\mathsf{Mod}\mathcal{A}$ for some $X \in \mathcal{A}$. We denote by $\mathsf{proj}\,\mathcal{A}$ the subcategory of $\mathsf{Mod}\mathcal{A}$ consisting of all finitely generated projective $\mathcal{A}$-modules. Note that finitely generated projective modules are precisely direct summands of representable functors. We need the following notation which is called FP_n in some sources (e.g. [7,8]).

Definition 2.2. Let $\mathcal{A}$ be an additive category and $n \geq 0$ be an integer.

(1) We denote by $\mathsf{mod}_n\mathcal{A}$ the subcategory of $\mathsf{Mod}\mathcal{A}$ consisting of all $\mathcal{A}$-modules M such that there exists an exact sequence

$$P_n \to \cdots \to P_1 \to P_0 \to M \to 0$$

in $\mathsf{Mod}\mathcal{A}$, where P_i is in $\mathsf{proj}\,\mathcal{A}$ for each $0 \leq i \leq n$.

(2) We denote by $\mathsf{mod}\mathcal{A}$ the subcategory of $\mathsf{Mod}\mathcal{A}$ consisting of all $\mathcal{A}$-modules M such that there exists an exact sequence

$$\cdots \to P_2 \to P_1 \to P_0 \to M \to 0$$

in $\mathsf{Mod}\mathcal{A}$, where P_i is in $\mathsf{proj}\,\mathcal{A}$ for each $i \geq 0$.

The following lemma is a basic observation on $\mathsf{mod}_n\mathcal{A}$, see [8, Chapter VIII, Proposition 4.3, Lemma 4.4] in the case where modules over a ring.

Lemma 2.3. *The following statements hold for an additive category $\mathcal{A}$.*

(a) *Let $M \in \mathsf{mod}_n\mathcal{A}$. Assume that there exists an exact sequence $P_l \to P_{l-1} \to \cdots \to P_0 \to M \to 0$ with $P_i \in \mathsf{proj}\,\mathcal{A}$ and $l \leq n$. Then there exist $P_{l+1}, \dots, P_n \in \mathsf{proj}\,\mathcal{A}$ and an exact sequence $P_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0$ extending the given exact sequence.*
(b) *Let $M \in \mathsf{Mod}\mathcal{A}$. Assume that there exist two exact sequences*

$$0 \to K \to P_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0,$$
$$0 \to L \to Q_n \to Q_{n-1} \to \cdots \to Q_0 \to M \to 0,$$

where $P_i, Q_i \in \mathsf{proj}\,\mathcal{A}$ for each $i \geq 0$. Then there exist $P, Q \in \mathsf{proj}\,\mathcal{A}$ such that $K \oplus P \simeq L \oplus Q$.

Proof. (a) This follows from (b).
(b) The case where $n = 0$ is well known as Schanuel's Lemma. The case where $n > 0$ is shown by an induction on n and by using the case where $n = 0$. □

The following lemma gives a sufficient condition when an $\mathcal{A}$-module is in $\mathsf{mod}_n\mathcal{A}$. For simplicity, we use the notation $\mathsf{mod}_{-1}\mathcal{A} := \mathsf{Mod}\mathcal{A}$, $\mathsf{mod}_\infty\mathcal{A} := \mathsf{mod}\mathcal{A}$ and $\infty - 1 := \infty$.

Lemma 2.4. *Let $\mathcal{A}$ be an additive category and M be an $\mathcal{A}$-module. Then we have the following properties.*

(a) *$\mathsf{mod}\mathcal{A} = \bigcap_{n\geq 0} \mathsf{mod}_n\mathcal{A}$ holds.*
(b) *Let $n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$. For an exact sequence $0 \to L \xrightarrow{f} M \to N \to 0$ in $\mathsf{Mod}\mathcal{A}$ with $L \in \mathsf{mod}_{n-1}\mathcal{A}$ and $M \in \mathsf{mod}_n\mathcal{A}$, we have $N \in \mathsf{mod}_n\mathcal{A}$.*
(c) *Let $n \geq 0$ be an integer. If there exists an exact sequence $X_n \xrightarrow{f_n} X_{n-1} \xrightarrow{f_{n-1}} \cdots \to X_0 \xrightarrow{f_0} M \to 0$ in $\mathsf{Mod}\mathcal{A}$ with $X_i \in \mathsf{mod}_{n-i}\mathcal{A}$ for any $0 \leq i \leq n$, then we have $M \in \mathsf{mod}_n\mathcal{A}$.*
(d) *If there exists an exact sequence $\cdots \to X_2 \to X_1 \to X_0 \to M \to 0$ in $\mathsf{Mod}\mathcal{A}$ with $X_i \in \mathsf{mod}\mathcal{A}$ for any $i \geq 0$, then we have $M \in \mathsf{mod}\mathcal{A}$.*

Proof. (a) In general $\mathsf{mod}\mathcal{A} \subset \mathsf{mod}_n\mathcal{A}$ holds for each $n \geq 0$. The converse follows from Lemma 2.3 (a).
(b) Assume that $n \neq \infty$. We have the following double complex

$$\begin{tikzcd}
 & Q_{n-1} \arrow[r, "g_{n-1}"] \arrow[d, "f_{n-1}"] & \cdots \arrow[r] & Q_1 \arrow[r, "g_1"] \arrow[d, "f_1"] & Q_0 \arrow[r, "\pi_L"] \arrow[d, "f_0"] & L \arrow[r] \arrow[d, "f"] & 0 \\
P_n \arrow[r, "h_n"'] & P_{n-1} \arrow[r, "h_{n-1}"'] & \cdots \arrow[r] & P_1 \arrow[r, "h_1"'] & P_0 \arrow[r, "\pi_M"'] & M \arrow[r] & 0
\end{tikzcd}$$

in $\mathsf{Mod}\mathcal{A}$, where all small squares are anticommutative, $N = \mathrm{Cok}(f)$, each horizontal sequence is exact except at P_n and Q_{n-1} respectively, and every P_i and Q_i are in $\mathsf{proj}\,\mathcal{A}$. We denote by $\mathbb{M}$ this double complex and denote by $\mathsf{Tot}(\mathbb{M})$ the total complex of $\mathbb{M}$, that is, $\mathsf{Tot}(\mathbb{M})_0 := M$, $\mathsf{Tot}(\mathbb{M})_{-1} := L \oplus P_0$, and $\mathsf{Tot}(\mathbb{M})_{-m} := Q_{m-2} \oplus P_{m-1}$ for $m = 2, \dots, n+1$, see [27, Subsection 1.2] more details. Since each horizontal sequence is exact except at P_n and Q_{n-1} respectively, we have $\mathrm{H}^{-i}(\mathsf{Tot}(\mathbb{M})) = 0$ for $i = 0, \dots, n$ by [27, Lemma 2.7.3]. Moreover, we denote by $\mathbb{P}$ a double complex which is obtained by removing L and M from $\mathbb{M}$. Namely, $\mathsf{Tot}(\mathbb{P})_{-m} = Q_{m-2} \oplus P_{m-1}$ for $m = 1, \dots, n+1$, where $Q_{-1} = 0$, and $\mathsf{Tot}(\mathbb{P})_{-m} = 0$ if otherwise. Let $\mathbb{X} := (L \xrightarrow{f} M)$. Then we have a morphism of chain complexes of $\mathsf{Mod}\mathcal{A}$, $\pi = \{\pi_L, \pi_M\} : \mathsf{Tot}(\mathbb{P})[-1] \to \mathbb{X}$ with the mapping cone $\mathsf{Tot}(\mathbb{M})$. By taking homologies, we have $\mathrm{H}^0(\mathsf{Tot}(\mathbb{P})[-1]) \simeq N$ and $\mathrm{H}^{-i}(\mathsf{Tot}(\mathbb{P})[-1]) = 0$ for $i = 1, \dots, n-1$. Since $(\mathsf{Tot}(\mathbb{P})[-1])_{-m} \in \mathsf{proj}\,\mathcal{A}$ for $m = 0, \dots, n$, N is an object of $\mathsf{mod}_n\mathcal{A}$. If $n = \infty$, then by a similar argument, we have the assertion.

(c) We show that $\mathrm{Im}(f_i) \in \mathsf{mod}_{n-i}\mathcal{A}$ by a descending induction on $i = n, n-1, \dots, 0$. Since $X_n \in \mathsf{mod}_0\mathcal{A}$, $\mathrm{Im}(f_n) \in \mathsf{mod}_0\mathcal{A}$ holds. Assume that $\mathrm{Im}(f_{i+1}) \in \mathsf{mod}_{n-i-1}\mathsf{mod}\mathcal{A}$ holds. We have a short exact sequence $0 \to \mathrm{Im}(f_{i+1}) \to X_i \to \mathrm{Im}(f_i) \to 0$. By (b), we have $\mathrm{Im}(f_i) \in \mathsf{mod}_{n-i}\mathcal{A}$. Therefore, $M = \mathrm{Im}(f_0) \in \mathsf{mod}_n\mathcal{A}$ holds.

(d) By (c), we have $M \in \mathsf{mod}_n\mathcal{A}$ for any $n \geq 0$. Thus by (a), $M \in \mathsf{mod}\mathcal{A}$ holds. $\square$

Let $\mathcal{A}$ be an abelian category and $\mathcal{B}$ be a subcategory of $\mathcal{A}$. We say that $\mathcal{B}$ is a *thick* subcategory of $\mathcal{A}$ if $\mathcal{B}$ is closed under direct summands and for any exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{A}$, if two of X, Y, Z are in $\mathcal{A}$, then so is the third. We have the following observation on the categories $\mathsf{mod}_n\mathcal{A}$.

Lemma 2.5. *Let $\mathcal{A}$ be an additive category. Then we have the following statements.*

(a) $\mathsf{mod}_n\mathcal{A}$ *is closed under extensions and direct summands in* $\mathsf{Mod}\mathcal{A}$ *for each* $n \geq 0$.
(b) (e.g. [12, Proposition 2.6]) $\mathsf{mod}\mathcal{A}$ *is a thick subcategory of* $\mathsf{Mod}\mathcal{A}$.

Proof. (a) By Horseshoe Lemma, $\mathsf{mod}_n\mathcal{A}$ is closed under extensions in $\mathsf{Mod}\mathcal{A}$. Let $X \oplus Y \in \mathsf{mod}_n\mathcal{A}$. We show that $X, Y \in \mathsf{mod}_n\mathcal{A}$ by an induction on n. If $n = 0$, then the claim is clear. Assume $n > 0$. Since $X \oplus Y \in \mathsf{mod}_n\mathcal{A} \subset \mathsf{mod}_{n-1}\mathcal{A}$ holds, by the inductive hypothesis, we have $X, Y \in \mathsf{mod}_{n-1}\mathcal{A}$. Then by Lemma 2.4 (b), we have $X, Y \in \mathsf{mod}_n\mathcal{A}$.

(b) By (a) and Lemma 2.4 (a), $\mathsf{mod}\mathcal{A}$ is closed under extensions and direct summands. Let $0 \to L \to M \to N \to 0$ be an exact sequence in $\mathsf{Mod}\mathcal{A}$. By Lemma 2.4 (b), if $L, M \in \mathsf{mod}\mathcal{A}$, then $N \in \mathsf{mod}\mathcal{A}$ holds. Assume that $M, N \in \mathsf{mod}\mathcal{A}$. There exists an exact sequence $0 \to \Omega N \to P \to N \to 0$ such that $P \in \mathsf{proj}\,\mathcal{A}$ and $\Omega N \in \mathsf{mod}\mathcal{A}$. By taking a pull-back diagram of $M \to N \leftarrow P$, we have an exact sequence $0 \to \Omega N \to P \oplus L \to M \to 0$. Since $\mathsf{mod}\mathcal{A}$ is closed under extensions and direct summands, we have $L \in \mathsf{mod}\mathcal{A}$. $\square$

2.2. Gorenstein-projective modules

We define Gorenstein-projective modules. Let $\mathcal{A}$ be an additive category. We first define a contravariant functor

$$(-)^* : \mathsf{Mod}\mathcal{A} \to \mathsf{Mod}(\mathcal{A}^{\mathrm{op}})$$

as follows: for $M \in \mathsf{Mod}\mathcal{A}$ and $X \in \mathcal{A}^{\mathrm{op}}$, let $(M)^*(X) := (\mathsf{Mod}\mathcal{A})(M, \mathcal{A}(-, X))$. By the same way, we define a contravariant functor $(-)^* : \mathsf{Mod}(\mathcal{A}^{\mathrm{op}}) \to \mathsf{Mod}\mathcal{A}$. If M is a representable functor from $\mathcal{A}$ to $\mathcal{A}b$, then $(M)^*$ is also a representable functor from $\mathcal{A}$ to $\mathcal{A}b$ by Yoneda's lemma. Let $P_\bullet := (P_i, d_i : P_i \to P_{i+1})_{i \in \mathbb{Z}}$ be a complex of finitely generated projective $\mathcal{A}$-modules. We say that $P_\bullet$ is *totally acyclic* if complexes $P_\bullet$ and $\cdots \to (P_{i+1})^* \to (P_i)^* \to (P_{i-1})^* \to \cdots$ are acyclic.

Definition 2.6. Let $\mathcal{A}$ be an additive category. An $\mathcal{A}$-module M is said to be *Gorenstein-projective* if there exists a totally acyclic complex $P_\bullet$ of finitely generated projective $\mathcal{A}$-modules such that $\mathrm{Im}\, d_0$ is isomorphic to M. We denote by $\mathsf{GP}\mathcal{A}$ the full subcategory of $\mathsf{Mod}\mathcal{A}$ consisting of all Gorenstein-projective $\mathcal{A}$-modules.

For instance, a finitely generated projective $\mathcal{A}$-module is Gorenstein-projective. In general, $\mathsf{GP}\mathcal{A} \subset \mathsf{mod}\mathcal{A}$ holds. We see a fundamental property of Gorenstein-projective modules.

Let $\mathcal{W}$ be a subcategory of $\mathsf{Mod}\mathcal{A}$. We denote by ${}^{\perp}\mathcal{W}$ the subcategory of $\mathsf{Mod}\mathcal{A}$ consisting of $\mathcal{A}$-modules M satisfying $\mathrm{Ext}^i_{\mathsf{Mod}\mathcal{A}}(M, W) = 0$ for any $W \in \mathcal{W}$ and any $i > 0$. We denote by $\mathcal{X}_{\mathcal{W}}$ the subcategory of ${}^{\perp}\mathcal{W}$ consisting of $\mathcal{A}$-modules M such that there exists an exact sequence $0 \to M \to W_0 \xrightarrow{f_0} W_1 \xrightarrow{f_1} \cdots$ with $W_i \in \mathcal{W}$ and $\mathrm{Im}\, f_i \in {}^{\perp}\mathcal{W}$ for any $i \geq 0$. By [4, Proposition 5.1], $\mathcal{X}_{\mathsf{proj}\,\mathcal{A}}$ is closed under extensions, direct summands and kernels of epimorphisms in $\mathsf{Mod}\mathcal{A}$.

Lemma 2.7. *Let $\mathcal{A}$ be an additive category. Then the following holds.*

(a) *The functor $(-)^* : \mathsf{Mod}\mathcal{A} \to \mathsf{Mod}(\mathcal{A}^{\mathrm{op}})$ induces a duality $(-)^* : \mathsf{GP}\mathcal{A} \to \mathsf{GP}(\mathcal{A}^{\mathrm{op}})$.*
(b) *$\mathcal{X}_{\mathsf{proj}\,\mathcal{A}} \cap \mathsf{mod}\mathcal{A} = \mathsf{GP}\mathcal{A}$ holds. In particular, $\mathsf{GP}\mathcal{A}$ is closed under extensions, direct summands and kernels of epimorphisms in $\mathsf{Mod}\mathcal{A}$.*

Proof. (a) This follows from the definition of $\mathsf{GP}\mathcal{A}$ and the fact that $(-)^*$ induces a duality between $\mathsf{proj}\,\mathcal{A}$ and $\mathsf{proj}(\mathcal{A}^{\mathrm{op}})$.

(b) By the definition of $\mathsf{GP}\mathcal{A}$, we have $\mathsf{mod}\mathcal{A} \supset \mathsf{GP}\mathcal{A}$. Let $M \in \mathsf{GP}\mathcal{A}$ and $P_\bullet := (P_i, d_i : P_i \to P_{i+1})_{i\in\mathbb{Z}}$ be a totally acyclic complex of finitely generated projective $\mathcal{A}$-modules such that $\mathrm{Im}\, d_0$ is isomorphic to M. Since $P_\bullet$ is a totally acyclic complex, $\mathrm{Im}\, d_i \in {}^{\perp}(\mathsf{proj}\,\mathcal{A})$ hold for every $i \in \mathbb{Z}$. This, together with the exact sequence $0 \to M \to P_1 \to P_2 \to \cdots$, implies that $M \in \mathcal{X}_{\mathsf{proj}\,\mathcal{A}}$.

Conversely, let $M \in \mathcal{X}_{\mathsf{proj}\,\mathcal{A}} \cap \mathsf{mod}\mathcal{A}$. Then there exists an exact sequence $P_\bullet = (P_i, d_i : P_i \to P_{i+1})_{i\in\mathbb{Z}}$, where $M \simeq \mathrm{Im}\, d_0$, $P_i \in \mathsf{proj}\,\mathcal{A}$ for any $i \in \mathbb{Z}$ and $\mathrm{Im}\, d_i \in {}^{\perp}(\mathsf{proj}\,\mathcal{A})$ for any $i \geq 1$. We have a projective resolution $\cdots \to P_{i-2} \to P_{i-1} \to P_i \to \mathrm{Im}\, d_i \to 0$ of $\mathrm{Im}\, d_i$ for each integer $i \geq 1$. Since $\mathrm{Im}\, d_i \in {}^{\perp}(\mathsf{proj}\,\mathcal{A})$, we have an exact sequence $0 \to (\mathrm{Im}\, d_i)^* \to (P_i)^* \to (P_{i-1})^* \to (P_{i-2})^* \to \cdots$. Therefore $P_\bullet$ is totally acyclic.

The last assertion follows from Lemma 2.5 and [4, Proposition 5.1]. $\square$

Let $\mathcal{B}$ be an extension closed subcategory of an abelian category $\mathcal{A}$. Then $\mathcal{B}$ has an induced structure of an exact category if we define a short exact sequence in $\mathcal{B}$ as a short exact sequence in $\mathcal{A}$ whose terms are in $\mathcal{B}$. We say that an object Z in $\mathcal{B}$ is *relative-projective* if any short exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{B}$ splits. Dually, we define *relative-injective* objects. We say that $\mathcal{B}$ has *enough projectives* if for any $X \in \mathcal{B}$, there exists a short exact sequence $0 \to Z \to P \to X \to 0$ in $\mathcal{B}$ such that P is relative-projective. We say that $\mathcal{B}$ has *enough injectives* if the dual condition is satisfied. An extension closed subcategory $\mathcal{B}$ of $\mathcal{A}$ is said to be *Frobenius* if $\mathcal{B}$ has enough projectives, enough injectives and the relative-projective objects coincide with the relative-injective objects.

The following observation is immediate (cf. [11]). Here, for the convenience of the reader, we give a proof.

Proposition 2.8. *Let $\mathcal{A}$ be an additive category. Then $\mathsf{GP}\mathcal{A}$ is a Frobenius category, where the relative-projective objects are precisely the finitely generated projective $\mathcal{A}$-modules.*

Proof. By Lemma 2.7 (b), $\mathsf{GP}\mathcal{A}$ is extension closed in $\mathsf{Mod}\mathcal{A}$. Clearly, the finitely generated projective $\mathcal{A}$-modules are contained in $\mathsf{GP}\mathcal{A}$ and are relative-projective in $\mathsf{GP}\mathcal{A}$. Conversely, let M be a relative-projective $\mathcal{A}$-module in $\mathsf{GP}\mathcal{A}$ and $P_\bullet := (P_i, d_i : P_i \to P_{i+1})_{i\in\mathbb{Z}}$ be a totally acyclic complex of finitely generated projective $\mathcal{A}$-modules such that $\mathrm{Im}\, d_0$ is isomorphic to M. Since M is relative-projective, the exact sequence $0 \to \mathrm{Im}(d_{-1}) \to P_0 \to M \to 0$ splits. Therefore the relative-projective objects in $\mathsf{GP}\mathcal{A}$ are precisely the finitely generated projective $\mathcal{A}$-modules. This implies that $\mathsf{GP}\mathcal{A}$ has enough projectives.

Since the duality $(-)^* : \mathsf{GP}\mathcal{A} \to \mathsf{GP}(\mathcal{A}^{\mathrm{op}})$ induces a duality between $\mathsf{proj}\,\mathcal{A}$ and $\mathsf{proj}(\mathcal{A}^{\mathrm{op}})$, the relative-injective objects in $\mathsf{GP}\mathcal{A}$ are precisely the finitely generated projective $\mathcal{A}$-modules and $\mathsf{GP}\mathcal{A}$ has enough injectives. Therefore $\mathsf{GP}\mathcal{A}$ is a Frobenius category such that the relative-projective objects are precisely the finitely generated projective $\mathcal{A}$-modules. $\square$

2.3. Dualizing k-varieties and Serre dualities

In this subsection, we recall the definition of dualizing k-varieties. Let $\mathcal{A}$ be an additive category. We call an object of $\mathsf{mod}_1\mathcal{A}$ a *finitely presented* $\mathcal{A}$-module.

A morphism $X \to Y$ in $\mathcal{A}$ is a *weak kernel* of a morphism $Y \to Z$ if the induced sequence $\mathcal{A}(-,X) \to \mathcal{A}(-,Y) \to \mathcal{A}(-,Z)$ is exact in $\mathsf{Mod}\mathcal{A}$. We say that $\mathcal{A}$ has weak kernels if each morphism in $\mathcal{A}$ has a weak kernel. The following lemma says when an additive category has weak kernels.

Lemma 2.9. *Let $\mathcal{A}$ be an additive category. The following statements are equivalent.*

(i) *$\mathcal{A}$ has weak kernels.*
(ii) *$\mathsf{mod}_1\mathcal{A}$ is an abelian subcategory of $\mathsf{Mod}\mathcal{A}$.*
(iii) *$\mathsf{mod}_1\mathcal{A} = \mathsf{mod}\mathcal{A}$ holds.*

Proof. It is well known that (i) implies (ii), see [2, Proposition 2.1] for instance. Clearly, (ii) implies (i). The statements (i) and (iii) are equivalent by [12, Proposition 2.7]. $\square$

Let $\mathcal{A}$ be an additive category and $X \in \mathcal{A}$. A morphism $e : X \to X$ in $\mathcal{A}$ is called an *idempotent* if $e^2 = e$. We call $\mathcal{A}$ *idempotent complete* if each idempotent of $\mathcal{A}$ has a kernel.

Let k be a field. A *k-linear category* $\mathcal{A}$ is a category such that $\mathcal{A}(X,Y)$ is equipped with a structure of k-module and the composition of morphisms of $\mathcal{A}$ is k-bilinear. A contravariant functor $F : \mathcal{A} \to \mathcal{B}$ between k-linear categories is called *k-functor* if $F_{X,Y} : \mathcal{A}(X,Y) \to \mathcal{B}(FY, FX)$ is k-linear for any $X, Y \in \mathcal{A}$. Let $\mathcal{A}$ be a k-linear additive category and M be an $\mathcal{A}$-module. For each $X \in \mathcal{A}$, since $\mathrm{End}_{\mathcal{A}}(X)$ acts on $M(X)$ and k acts on $\mathrm{End}_{\mathcal{A}}(X)$, $M(X)$ is a k-module. By this action, we regard any $\mathcal{A}$-module as a contravariant k-functor from $\mathcal{A}$ to $\mathsf{Mod}(k)$, where $\mathsf{Mod}(k)$ is the category of k-modules.

Let $\mathcal{A}$ be a k-linear additive category. We call $\mathcal{A}$ *Hom-finite* if $\mathcal{A}(X,Y)$ is finitely generated over k for any $X, Y \in \mathcal{A}$. We recall one proposition about the Krull-Schmidt property of k-linear additive categories (see [19] for details).

Proposition 2.10. *Let $\mathcal{A}$ be a k-linear, Hom-finite additive category. Then the following properties are equivalent.*

(i) *$\mathcal{A}$ is idempotent complete.*
(ii) *The endomorphism algebra of each indecomposable object in $\mathcal{A}$ is local.*
(iii) *$\mathcal{A}$ is Krull-Schmidt, that is, each object of $\mathcal{A}$ is a finite direct sum of objects whose endomorphism algebras are local.*

Moreover the decomposition of (iii) *is unique up to isomorphism.*

Proposition 2.11. *Let $\mathcal{A}$ be a k-linear, Hom-finite additive category. Then $\mathsf{mod}\mathcal{A}$ is Krull-Schmidt. In particular, each object of $\mathsf{mod}\mathcal{A}$ has a minimal projective resolution.*

Proof. By Lemma 2.5, $\mathsf{mod}\mathcal{A}$ is closed under direct summands in $\mathsf{Mod}\mathcal{A}$. Thus $\mathsf{mod}\mathcal{A}$ is idempotent complete. Since $\mathcal{A}$ is Hom-finite, $\mathsf{mod}\mathcal{A}$ is also Hom-finite. By Proposition 2.10, $\mathsf{mod}\mathcal{A}$ is Krull-Schmidt. $\square$

We recall the definition of dualizing k-varieties. We denote by $\mathrm{D} = \mathrm{Hom}_k(-, k)$ the standard k-dual of k-modules. Let $\mathcal{A}$ be a k-linear additive category. We have contravariant exact functors $\mathrm{D} : \mathsf{Mod}\mathcal{A} \to \mathsf{Mod}(\mathcal{A}^{\mathrm{op}})$ and $\mathrm{D} : \mathsf{Mod}(\mathcal{A}^{\mathrm{op}}) \to \mathsf{Mod}\mathcal{A}$ given by $(\mathrm{D}M)(X) := \mathrm{D}(M(X))$. We use the same letter D for the standard k-dual of k-modules and the contravariant functors between $\mathsf{Mod}\mathcal{A}$ and $\mathsf{Mod}(\mathcal{A}^{\mathrm{op}})$.

Definition 2.12. A *dualizing k-variety* is a k-linear, Hom-finite, idempotent complete additive category $\mathcal{A}$ such that the functor $\mathrm{D} : \mathsf{Mod}\mathcal{A} \to \mathsf{Mod}(\mathcal{A}^{\mathrm{op}})$ induces a duality between $\mathsf{mod}_1\mathcal{A}$ and $\mathsf{mod}_1(\mathcal{A}^{\mathrm{op}})$.

The following are typical examples of dualizing k-varieties.

Example 2.13.

(a) If $\mathcal{A}$ is a dualizing k-variety, then $\mathcal{A}^{\mathrm{op}}$ is a dualizing k-variety.
(b) Let A be a finite dimensional k-algebra and $\mathsf{mod}A$ be the category of finitely generated A-modules. Let $\mathsf{proj}\,A$ be the full subcategory of $\mathsf{mod}A$ consisting of all finitely generated projective A-modules. Then $\mathsf{mod}A$ and $\mathsf{proj}\,A$ are dualizing k-varieties by [3, Propositions 2.5, 2.6].

We state some properties of dualizing k-varieties.

Lemma 2.14. *Let $\mathcal{A}$ be a dualizing k-variety, then we have the following properties.*

(a) *$\mathcal{A}$ and $\mathcal{A}^{\mathrm{op}}$ have weak kernels.*
(b) *$\mathsf{mod}\mathcal{A}$ is a dualizing k-variety.*
(c) *Each object in $\mathsf{mod}\mathcal{A}$ has a projective cover and an injective hull.*

Proof. The statement (a) follows from [3, Theorem 2.4]. By Lemma 2.9, we have $\mathsf{mod}_1\mathcal{A} = \mathsf{mod}\mathcal{A}$. Then (b) follows from [3, Proposition 2.6]. By the definition of dualizing k-varieties, we have the duality D between $\mathsf{mod}\mathcal{A}$ and $\mathsf{mod}(\mathcal{A}^{\mathrm{op}})$. Thus by Proposition 2.11, (c) holds. $\square$

Let $\mathcal{A}$ be a k-linear, Hom-finite additive category. A *Serre functor* on $\mathcal{A}$ is a k-linear auto-equivalence $\mathbb{S} : \mathcal{A} \to \mathcal{A}$ together with k-linear isomorphisms

$$\eta_{X,Y} : \mathcal{A}(X, Y) \simeq \mathrm{D}\Big(\mathcal{A}\big(Y, \mathbb{S}(X)\big)\Big)$$

for any $X, Y \in \mathcal{A}$ which are functorial in X and Y. We denote by $\mathbb{S}^{-1}$ a quasi-inverse of $\mathbb{S}$. It is easy to see that if $\mathcal{A}$ has a Serre functor $\mathbb{S}$, then $\mathbb{S}^{-1}$ is a Serre functor on $\mathcal{A}^{\mathrm{op}}$.

If $\mathcal{A}$ has a Serre functor $\mathbb{S}$, then $(-)^*$ is described as in the following lemma. Since $\mathbb{S}$ is an auto-equivalence, we have an equivalence $\mathsf{Mod}\mathcal{A} \to \mathsf{Mod}\mathcal{A}$ given by $M \mapsto M \circ \mathbb{S}^{-1}$. By composing with the functor $\mathrm{D} : \mathsf{Mod}\mathcal{A} \to \mathsf{Mod}(\mathcal{A}^{\mathrm{op}})$, we have a contravariant functor $\mathsf{Mod}\mathcal{A} \to \mathsf{Mod}(\mathcal{A}^{\mathrm{op}})$ given by $M \mapsto \mathrm{D}(M \circ \mathbb{S}^{-1})$. We denote by $\mathsf{Mod}_{\mathrm{fg}}\mathcal{A}$ the subcategory of $\mathsf{Mod}\mathcal{A}$ consisting of $\mathcal{A}$-modules M such that $M(X)$ is finitely generated over k for any $X \in \mathcal{A}$. Note that D induces a duality $\mathsf{Mod}_{\mathrm{fg}}\mathcal{A} \to \mathsf{Mod}_{\mathrm{fg}}(\mathcal{A}^{\mathrm{op}})$ and that the categories $\mathsf{mod}_0\mathcal{A}$ and $\mathsf{GP}\mathcal{A}$ are contained in $\mathsf{Mod}_{\mathrm{fg}}\mathcal{A}$.

Lemma 2.15. *Let $\mathcal{A}$ be a k-linear, Hom-finite additive category with a Serre functor $\mathbb{S}$. Then the following statements hold.*

(a) *We have an isomorphism of functors $(-)^* \simeq \mathrm{D}(- \circ \mathbb{S}^{-1}) : \mathsf{Mod}_{\mathrm{fg}}\mathcal{A} \to \mathsf{Mod}_{\mathrm{fg}}(\mathcal{A}^{\mathrm{op}})$, and the functor $(-)^*$ is a duality.*
(b) *Let $M \in \mathsf{Mod}\mathcal{A}$. The following statements are equivalent.*

(i) $M \in \mathsf{GP}\mathcal{A}$.
(ii) $M \in \mathsf{mod}\mathcal{A}$ *and* $M^* \in \mathsf{mod}(\mathcal{A}^{\mathrm{op}})$.

Proof. (a) Let $M \in \mathsf{Mod}_{\mathrm{fg}}\mathcal{A}$ and $X \in \mathcal{A}$. We have the following isomorphisms

$$\begin{aligned}(M)^*(X) &= (\mathsf{Mod}\mathcal{A})(M, \mathcal{A}(-,X)) \\ &\simeq (\mathsf{Mod}\mathcal{A}^{\mathrm{op}})(\mathrm{D}\mathcal{A}(-,X), \mathrm{D}M) \\ &\simeq (\mathsf{Mod}\mathcal{A}^{\mathrm{op}})(\mathcal{A}(\mathbb{S}^{-1}(X), -), \mathrm{D}M) \\ &\simeq \mathrm{D}(M \circ \mathbb{S}^{-1})(X),\end{aligned}$$

which are functorial in X, where the last isomorphism is induced by Yoneda's lemma. Thus we have an isomorphism of functors $(-)^* \simeq \mathrm{D}(- \circ \mathbb{S}^{-1})$. This functor is a duality, since D is a duality and $\mathbb{S}$ is an equivalence.

(b) Assume that $M \in \mathsf{GP}\mathcal{A}$. By Lemma 2.7 (a), we have $M^* \in \mathsf{GP}(\mathcal{A}^{\mathrm{op}})$. In general $\mathsf{GP}\mathcal{A} \subset \mathsf{mod}\mathcal{A}$ holds, thus (i) implies (ii). Assume that (ii) holds. There exists an exact sequence $\cdots \to Q_2 \to Q_1 \to M^* \to 0$, where $Q_i \in \mathsf{proj}(\mathcal{A}^{\mathrm{op}})$. By (a), $(-)^*$ is an exact functor. Therefore we have an exact sequence

$$\cdots \to P_2 \to P_1 \to P_0 \xrightarrow{d} Q_1^* \to Q_2^* \to \cdots,$$

where $P_i, Q_i^* \in \mathsf{proj}\,\mathcal{A}$ and $\mathrm{Im}\, d \simeq M$. This exact sequence is totally acyclic, since $(-)^*$ is exact. We have $M \in \mathsf{GP}\mathcal{A}$. $\square$

Later we use the following characterization of dualizing k-varieties with Serre functors.

Proposition 2.16. *Let $\mathcal{A}$ be a k-linear, Hom-finite, idempotent complete additive category. Then the following statements are equivalent.*

(i) *$\mathcal{A}$ is a dualizing k-variety and has a Serre functor.*
(ii) *$\mathcal{A}$ and $\mathcal{A}^{\mathrm{op}}$ have weak kernels and $\mathcal{A}$ has a Serre functor.*
(iii) $\mathsf{GP}\mathcal{A} = \mathsf{mod}_1\mathcal{A}$, $\mathsf{GP}(\mathcal{A}^{\mathrm{op}}) = \mathsf{mod}_1(\mathcal{A}^{\mathrm{op}})$ *hold and* $\mathrm{D}\mathcal{A}(X,-) \in \mathsf{mod}_1\mathcal{A}$, $\mathrm{D}\mathcal{A}(-,X) \in \mathsf{mod}_1(\mathcal{A}^{\mathrm{op}})$ *hold for any* $X \in \mathcal{A}$.

Proof. By Lemma 2.14, (i) implies (ii). We show that (ii) implies (i). We show that, for any $M \in \mathsf{mod}_1\mathcal{A}$, $\mathrm{D}M$ is in $\mathsf{mod}_1(\mathcal{A}^{\mathrm{op}})$. There exists an exact sequence $P_1 \to P_0 \to M \to 0$ for some $P_1, P_0 \in \mathsf{proj}\,\mathcal{A}$. By the functor $\mathrm{D} : \mathsf{Mod}\mathcal{A} \to \mathsf{Mod}(\mathcal{A}^{\mathrm{op}})$, we have an exact sequence $0 \to \mathrm{D}M \to \mathrm{D}P_0 \to \mathrm{D}P_1$ in $\mathsf{Mod}\mathcal{A}$. Since $\mathcal{A}$ has a Serre functor, we have $\mathrm{D}P_1, \mathrm{D}P_0 \in \mathsf{proj}(\mathcal{A}^{\mathrm{op}})$. Since $\mathcal{A}^{\mathrm{op}}$ has weak kernels, by Lemma 2.9, $\mathrm{D}M$ is in $\mathsf{mod}_1(\mathcal{A}^{\mathrm{op}})$. By the dual argument, for any $N \in \mathsf{mod}_1(\mathcal{A}^{\mathrm{op}})$, we have $\mathrm{D}N \in \mathsf{mod}_1\mathcal{A}$. Thus $\mathrm{D} : \mathsf{mod}_1\mathcal{A} \to \mathsf{mod}_1(\mathcal{A}^{\mathrm{op}})$ is a duality.

We show that (i) implies (iii). Since $\mathcal{A}$ is a dualizing k-variety, $\mathrm{D}\mathcal{A}(X,-) \in \mathsf{mod}_1\mathcal{A}$, $\mathrm{D}\mathcal{A}(-,X) \in \mathsf{mod}_1(\mathcal{A}^{\mathrm{op}})$ hold for any $X \in \mathcal{A}$. By Lemma 2.9, we have $\mathsf{mod}\mathcal{A} = \mathsf{mod}_1\mathcal{A}$ and $\mathsf{mod}(\mathcal{A}^{\mathrm{op}}) = \mathsf{mod}_1(\mathcal{A}^{\mathrm{op}})$. In general $\mathsf{GP}\mathcal{A} \subset \mathsf{mod}\mathcal{A}$ holds. Let $M \in \mathsf{mod}\mathcal{A}$. We show that $M \in \mathsf{GP}\mathcal{A}$. Since $\mathcal{A}$ is a dualizing k-variety, $\mathrm{D}M \in \mathsf{mod}(\mathcal{A}^{\mathrm{op}})$ holds. By Lemma 2.15 (a), $M^* \in \mathsf{mod}(\mathcal{A}^{\mathrm{op}})$ holds. Thus by Lemma 2.15 (b), $M \in \mathsf{GP}\mathcal{A}$ holds.

We show that (iii) implies (ii). In general, $\mathsf{GP}\mathcal{A} \subset \mathsf{mod}\mathcal{A} \subset \mathsf{mod}_1\mathcal{A}$ holds. Therefore by Lemma 2.9, $\mathcal{A}$ and $\mathcal{A}^{\mathrm{op}}$ have weak kernels. Consider the functor $\mathrm{D} \circ (-)^* : \mathsf{Mod}\mathcal{A} \to \mathsf{Mod}\mathcal{A}$. This functor induces an equivalence $\mathsf{proj}\,\mathcal{A} \xrightarrow{\sim} \mathsf{proj}\,\mathcal{A}$. In fact, if $M \in \mathsf{proj}\,\mathcal{A}$, then $M^* \in \mathsf{proj}(\mathcal{A}^{\mathrm{op}})$. By the assumption, we have $\mathrm{D}(M^*) \in \mathsf{mod}_1\mathcal{A} = \mathsf{GP}\mathcal{A}$. Since $\mathrm{D} : \mathsf{Mod}_{\mathrm{fg}}(\mathcal{A}^{\mathrm{op}}) \to \mathsf{Mod}_{\mathrm{fg}}\mathcal{A}$ is a duality, $\mathrm{D}(M^*)$ is an injective object of $\mathsf{Mod}_{\mathrm{fg}}\mathcal{A}$. In particular, $\mathrm{D}(M^*)$ is a relative-injective object of $\mathsf{GP}\mathcal{A}$. Since $\mathsf{GP}\mathcal{A}$ is Frobenius, $\mathrm{D}(M^*)$ is an object of $\mathsf{proj}\,\mathcal{A}$. Thus we have a functor $\mathrm{D} \circ (-)^* : \mathsf{proj}\,\mathcal{A} \to \mathsf{proj}\,\mathcal{A}$. This is an equivalence, since its quasi-inverse is given by $(-)^* \circ \mathrm{D}$. Since $\mathcal{A}$ is idempotent complete, the Yoneda embedding $\mathcal{A} \to \mathsf{proj}\,\mathcal{A}$, $X \mapsto \mathcal{A}(-, X)$ is an equivalence. Thus there exists an equivalence $\mathbb{S} : \mathcal{A} \to \mathcal{A}$ such that the following diagram is commutative up to isomorphism of functors:

$$\begin{tikzcd} \mathsf{proj}\,\mathcal{A} \arrow[r, "\mathrm{D}\circ(-)^*"] & \mathsf{proj}\,\mathcal{A} \\ \mathcal{A} \arrow[u, "\simeq"] \arrow[r, "\mathbb{S}"] & \mathcal{A}. \arrow[u, "\simeq"'] \end{tikzcd}$$

For $X, Y \in \mathcal{A}$, we have the following isomorphisms which are functorial in X, Y:

$$\begin{aligned} \mathcal{A}(Y, \mathbb{S}X) &\simeq \mathrm{D}(\mathcal{A}(-, X)^*)(Y) \\ &\simeq \mathrm{D}(\mathsf{Mod}\mathcal{A}(\mathcal{A}(-, X), \mathcal{A}(-, Y))) \\ &\simeq \mathrm{D}\mathcal{A}(X, Y). \end{aligned}$$

This means that $\mathbb{S}$ is a Serre functor on $\mathcal{A}$. $\square$

2.4. Some observations on triangulated categories

In this subsection, we state some results on triangulated categories which we use later. Let $\mathcal{A}$ be an additive category and $\mathcal{B}$ be a subcategory of $\mathcal{A}$. For two objects $X, Y \in \mathcal{A}$, we denote by $\mathcal{A}_{\mathcal{B}}(X, Y)$ the subspace of $\mathcal{A}(X, Y)$ consisting of all morphisms which factor through an object of $\mathcal{B}$. We denote by $\mathcal{A}/[\mathcal{B}]$ the category defined as follows: the objects of $\mathcal{A}/[\mathcal{B}]$ are the same as $\mathcal{A}$ and the morphism space is defined by

$$(\mathcal{A}/[\mathcal{B}])(X, Y) := \mathcal{A}(X, Y)/\mathcal{A}_{\mathcal{B}}(X, Y),$$

for $X, Y \in \mathcal{A}$. We denote by $\mathsf{K}(\mathcal{A})$ the homotopy category of complexes in $\mathcal{A}$ and $\mathsf{K}^{\mathrm{b}}(\mathcal{A})$ (resp. $\mathsf{K}^{-}(\mathcal{A})$) the full subcategory of $\mathsf{K}(\mathcal{A})$ consisting of bounded complexes (resp. right bounded complexes).

Let $\mathcal{F}$ be a Frobenius category, $\mathcal{P}$ the full subcategory of $\mathcal{F}$ consisting of the projective objects in $\mathcal{F}$ and $\underline{\mathcal{F}} := \mathcal{F}/[\mathcal{P}]$. By Happel [13], it is known that $\underline{\mathcal{F}}$ is a triangulated category. Assume that $\mathcal{P}$ is idempotent complete. We denote by $\mathsf{K}^{-,\mathrm{b}}(\mathcal{P})$ the full subcategory of $\mathsf{K}(\mathcal{P})$ consisting of complexes $X = (X^i, d^i : X^i \to X^{i+1})$ satisfying the following conditions.

(1) There exists $n_X \in \mathbb{Z}$ such that $X^i = 0$ for any $i > n_X$.
(2) There exist $m_X \in \mathbb{Z}$ and exact sequences $0 \to Y^{i-1} \xrightarrow{a^{i-1}} X^i \xrightarrow{b^i} Y^i \to 0$ in $\mathcal{F}$ for any $i \le m_X$ such that $d^i = a^i b^i$ for any $i < m_X$.

We identify the category $\mathcal{F}$ with the full subcategory of $\mathsf{K}^{-,\mathrm{b}}(\mathcal{P})$ consisting of X such that there exist integers n_X and m_X satisfying (1), (2), respectively, and $n_X \le 0 \le m_X$. Then we have the following analogy of the well known equivalence due to [9,16,26].

Theorem 2.17. *[15, Corollary 2.2] Let $\mathcal{F}$ be a Frobenius category and $\mathcal{P}$ the full subcategory of $\mathcal{F}$ consisting of the projective objects. Assume that $\mathcal{P}$ is idempotent complete. Then the composite $\mathcal{F} \to \mathsf{K}^{-,\mathrm{b}}(\mathcal{P}) \to \mathsf{K}^{-,\mathrm{b}}(\mathcal{P})/\mathsf{K}^{\mathrm{b}}(\mathcal{P})$ induces a triangle equivalence $\underline{\mathcal{F}} \xrightarrow{\sim} \mathsf{K}^{-,\mathrm{b}}(\mathcal{P})/\mathsf{K}^{\mathrm{b}}(\mathcal{P})$.*

A complex $X = (X^i, d^i : X^i \to X^{i+1})$ of $\mathsf{K}(\mathcal{F})$ is said to be *acyclic* if there exist exact sequences $0 \to Y^{i-1} \xrightarrow{a^{i-1}} X^i \xrightarrow{b^i} Y^i \to 0$ in $\mathcal{F}$ for any $i \in \mathbb{Z}$ such that $d^i = a^i b^i$ for any $i \in \mathbb{Z}$. We denote by $\mathsf{K}^{\mathrm{a}}(\mathcal{F})$ the full subcategory of $\mathsf{K}(\mathcal{F})$ consisting of acyclic complexes. We use the following proposition in Section 3. We refer [23] for stable t-structures.

Proposition 2.18. *[10, Theorem 12.7] Let $\mathcal{F}$ be a Frobenius category and $\mathcal{P}$ the full subcategory of $\mathcal{F}$ consisting of the projective objects. Assume that $\mathcal{P}$ is idempotent complete. Then we have a stable t-structure $(\mathsf{K}^-(\mathcal{P}), \mathsf{K}^{\mathrm{a}}(\mathcal{F}) \cap \mathsf{K}^-(\mathcal{F}))$ on $\mathsf{K}^-(\mathcal{F})$.*

Let $\mathcal{U}$ be a triangulated category and $\mathcal{X}$ be a full subcategory of $\mathcal{U}$. We call $\mathcal{X}$ a *thick* subcategory of $\mathcal{U}$ if $\mathcal{X}$ is a triangulated subcategory of $\mathcal{U}$ and closed under direct summands. We denote by $\mathsf{thick}_{\mathcal{U}} \mathcal{X}$ the smallest thick subcategory of $\mathcal{U}$ which contains $\mathcal{X}$. Whenever if there is no danger of confusion, let $\mathsf{thick}_{\mathcal{U}} \mathcal{X} = \mathsf{thick}\, \mathcal{X}$.

Lemma 2.19. *[18, Appendix] Let $\mathcal{T}, \mathcal{U}$ be triangulated categories and $F : \mathcal{U} \to \mathcal{T}$ a triangle functor. Let $\mathcal{X}$ be a full subcategory of $\mathcal{U}$. Then the following holds.*

- *Assume that the map*

$$F_{M,N[n]} : \mathcal{U}(M, N) \to \mathcal{T}(FM, FN[n])$$

  is an isomorphism for any $M, N \in \mathcal{X}$ and any $n \in \mathbb{Z}$. Then $F : \mathsf{thick}\, \mathcal{X} \to \mathcal{T}$ is fully faithful.
- *If moreover $\mathcal{U}$ is idempotent complete, $\mathsf{thick}\, \mathcal{X} = \mathcal{U}$ and $\mathsf{thick}(\mathrm{Im}(F)) = \mathcal{T}$, then F is an equivalence.*

3. Repetitive categories

Throughout this section, we assume that $\mathcal{A}$ is a skeletally small category.

3.1. Repetitive categories

We recall the definition of repetitive categories of additive categories. The aim of this subsection is to show Theorem 3.7.

Definition 3.1. Let $\mathcal{A}$ be a k-linear additive category. The *repetitive category* $\mathsf{R}\mathcal{A}$ is the k-linear additive category generated by the following category: the class of objects is $\{(X, i) \mid X \in \mathcal{A}, i \in \mathbb{Z}\}$ and the morphism space is given by

$$\mathsf{R}\mathcal{A}\big((X,i),(Y,j)\big) = \begin{cases} \mathcal{A}(X,Y) & i = j, \\ \mathrm{D}\mathcal{A}(Y,X) & j = i+1, \\ 0 & \text{else.} \end{cases}$$

For $f \in \mathsf{R}\mathcal{A}\big((X,i),(Y,j)\big)$ and $g \in \mathsf{R}\mathcal{A}\big((Y,j),(Z,k)\big)$, the composition is given by

$$g \circ f = \begin{cases} g \circ f & i = j = k, \\ \big(\mathrm{D}\mathcal{A}(Z,f)\big)(g) & i = j = k-1, \\ \big(\mathrm{D}\mathcal{A}(g,X)\big)(f) & i+1 = j = k, \\ 0 & \text{else.} \end{cases}$$

We describe fundamental properties of repetitive categories of Hom-finite categories.

Lemma 3.2. *Let $\mathcal{A}$ be a k-linear, Hom-finite additive category. The following statements hold.*

(a) $\mathsf{R}\mathcal{A}$ *is* Hom-*finite.*
(b) $\mathsf{R}\mathcal{A}$ *has a Serre functor* $\mathbb{S}$ *which is defined by* $\mathbb{S}(X,i) := (X, i+1)$*.*
(c) *If* $\mathcal{A}$ *is idempotent complete, then so is* $\mathsf{R}\mathcal{A}$*.*

Proof. (a) (b) These are clear by the definition.

(c) By the definition, an object of $\mathsf{R}\mathcal{A}$ is indecomposable if and only if it is isomorphic to an object (X,i), where X is an indecomposable object of $\mathcal{A}$ and i is some integer. Let X be an indecomposable object of $\mathcal{A}$ and i be an integer. Since $\mathcal{A}$ is idempotent complete and Proposition 2.10, $\mathrm{End}_{\mathsf{R}\mathcal{A}}(X,i) = \mathrm{End}_{\mathcal{A}}(X)$ is local. Therefore again by Proposition 2.10, $\mathsf{R}\mathcal{A}$ is idempotent complete. □

We see a relation between the categories $\mathsf{mod}\mathcal{A}$ and $\mathsf{mod}\mathsf{R}\mathcal{A}$ and consequently, we show Theorem 3.7. Let $\mathcal{A}$ be a k-linear additive category and $i \in \mathbb{Z}$. Define the following full subcategory of $\mathsf{R}\mathcal{A}$:

$$\mathcal{A}_i := \mathsf{add}\{\, (X,i) \in \mathsf{R}\mathcal{A} \mid X \in \mathcal{A}\,\}.$$

The inclusion functor $\mathcal{A}_i \to \mathsf{R}\mathcal{A}$ induces an exact functor

$$\rho_i : \mathsf{Mod}\mathsf{R}\mathcal{A} \to \mathsf{Mod}\mathcal{A}_i.$$

Since the functor $\mathcal{A} \to \mathcal{A}_i$ defined by $X \mapsto (X,i)$ is an equivalence, whenever there is no danger of confusion, we denote an object (X,i) of $\mathcal{A}_i$ by X for simplicity.

Since we have a full dense functor $\mathsf{R}\mathcal{A} \to \mathcal{A}_i$ given by $(X,j) \mapsto X$ if $j = i$ and $(X,j) \mapsto 0$ if otherwise, we have a fully faithful functor from $\mathsf{Mod}\mathcal{A}_i$ to $\mathsf{Mod}\mathsf{R}\mathcal{A}$. Therefore we identify $\mathsf{Mod}\mathcal{A}_i$ with the full subcategory of $\mathsf{Mod}\mathsf{R}\mathcal{A}$ consisting of $\mathsf{R}\mathcal{A}$-modules M such that $M(X,j) = 0$ for any $j \neq i$ and any $X \in \mathcal{A}$.

For an object $(X,i) \in \mathcal{A}_i^{\mathrm{op}}$, we have $\mathcal{A}_i^{\mathrm{op}}(-,(X,i)) \in \mathsf{Mod}(\mathcal{A}_i^{\mathrm{op}})$ and $\mathrm{D}(\mathcal{A}_i^{\mathrm{op}}(-,(X,i))) = \mathrm{D}(\mathcal{A}_i((X,i),-)) \in \mathsf{Mod}\mathcal{A}_i$. For simplicity, let $\mathcal{A}_i(-,X) := \mathcal{A}_i(-,(X,i))$ and $\mathrm{D}\mathcal{A}_i(X,-) := \mathrm{D}(\mathcal{A}_i((X,i),-))$. We regard these functors as objects in $\mathsf{Mod}\mathsf{R}\mathcal{A}$ by the fully faithful functor from $\mathsf{Mod}\mathcal{A}_i$ to $\mathsf{Mod}\mathsf{R}\mathcal{A}$ as above.

Lemma 3.3. *Let $\mathcal{A}$ be an additive category and $i, j \in \mathbb{Z}$.*

(a) *We have* $\rho_j|_{\mathsf{Mod}\mathcal{A}_i} = \mathrm{id}_{\mathsf{Mod}\mathcal{A}_i}$ *if* $j = i$ *and* $\rho_j|_{\mathsf{Mod}\mathcal{A}_i} = 0$ *if otherwise.*
(b) *For any $X \in \mathcal{A}$, there exists a canonical exact sequence*

$$0 \to \mathrm{D}\mathcal{A}_{i-1}(X,-) \xrightarrow{\beta} \mathsf{R}\mathcal{A}(-,(X,i)) \xrightarrow{\alpha} \mathcal{A}_i(-,X) \to 0 \tag{3.1}$$

in $\mathsf{Mod}\mathsf{R}\mathcal{A}$*, where α and β are defined in the proof. In particular, we have* $\rho_j(P) \in \mathsf{add}\{\mathcal{A}_j(-,X), \mathrm{D}\mathcal{A}_j(X,-) \mid X \in \mathcal{A}\}$ *for any* $P \in \mathsf{proj}\,\mathsf{R}\mathcal{A}$ *and* $j \in \mathbb{Z}$*.*
(c) *Each finitely generated $\mathcal{A}_i$-module is a finitely generated* $\mathsf{R}\mathcal{A}$*-module.*

Proof. (a) The assertions follow from the definition of ρ_j.

(b) We construct morphisms α, β in $\mathsf{Mod}\mathsf{R}\mathcal{A}$. For a generating object (Y,j) of $\mathsf{R}\mathcal{A}$, define

$$\alpha_{(Y,j)} := \begin{cases} \mathrm{id}_{\mathcal{A}(Y,X)} & j = i, \\ 0 & \text{else,} \end{cases} \qquad \beta_{(Y,j)} := \begin{cases} \mathrm{id}_{\mathrm{D}\mathcal{A}(X,Y)} & j+1 = i, \\ 0 & \text{else,} \end{cases}$$

and extend α and β on $\mathsf{R}\mathcal{A}$ additively. It is easy to show that α and β are actually morphisms in $\mathsf{Mod}\mathsf{R}\mathcal{A}$. By definitions of α and β, for an object (Y,j) of $\mathsf{R}\mathcal{A}$, we have the following exact sequence

$$0 \to \mathrm{D}\mathcal{A}_{i-1}(X,(Y,j)) \xrightarrow{\beta_{(Y,j)}} \mathsf{R}\mathcal{A}((Y,j),(X,i)) \xrightarrow{\alpha_{(Y,j)}} \mathcal{A}_i((Y,j),X) \to 0$$

in $\mathsf{Mod}(k)$. Thus we have an exact sequence (3.1). Since ρ_j is exact, by applying ρ_j to the exact sequence (3.1) and by using (a), we have the assertion.

(c) This follows from (b) since α is an epimorphism. $\square$

By the following lemma, we construct a filtration on any module over a repetitive category. For $M \in \mathsf{Mod}\mathsf{R}\mathcal{A}$, let $\operatorname{Supp} M := \{\, i \in \mathbb{Z} \mid \rho_i(M) \neq 0 \,\}$.

Lemma 3.4. *Let $M \in \mathsf{Mod}\mathsf{R}\mathcal{A}$ and $i \in \mathbb{Z}$.*

(a) *If $\rho_{i-1}(M) = 0$, then there exists a short exact sequence*

$$0 \to \rho_i(M) \xrightarrow{\alpha} M \to N \to 0$$

in $\mathsf{Mod}\mathsf{R}\mathcal{A}$ such that $\rho_i(N) = 0$ and $\rho_j(N) = \rho_j(M)$ for any $j > i$.

(b) *Assume that $\operatorname{Supp} M$ is a non-empty finite set and put $m := \max \operatorname{Supp} M$ and $n := \min \operatorname{Supp} M$. Then there exists a sequence of subobjects of M:*

$$0 = M_{n-1} \subset M_n \subset \cdots \subset M_{m-1} \subset M_m = M$$

such that $M_i/M_{i-1} \simeq \rho_i(M)$ for any $i = n, n+1, \dots, m$.

Proof. (a) We construct a monomorphism $\alpha : \rho_i(M) \to M$ in $\mathsf{Mod}\mathsf{R}\mathcal{A}$. For a generating object (X,j) of $\mathsf{R}\mathcal{A}$, define

$$\alpha_{(X,j)} := \begin{cases} \mathrm{id}_{M(X,j)} & j = i, \\ 0 & \text{else}, \end{cases}$$

and extend this on $\mathsf{R}\mathcal{A}$ additively. Since $\rho_{i-1}(M) = 0$, α is a morphism of $\mathsf{Mod}\mathsf{R}\mathcal{A}$. By the definition, α is a monomorphism. Then we have an exact sequence $0 \to \rho_i(M) \to M \to N \to 0$ in $\mathsf{Mod}\mathsf{R}\mathcal{A}$, where $N := \operatorname{Cok}(\alpha)$. By Lemma 3.3, we have $\rho_j(\rho_i(M)) = \rho_i(M)$ if $j = i$ and $\rho_j(\rho_i(M)) = 0$ if else. Therefore by applying the functor ρ_j to this exact sequence, we have the assertion.

(b) This follows from (a). $\square$

By the following two lemmas, we see that the functors $\mathsf{Mod}\mathcal{A}_i \to \mathsf{Mod}\mathsf{R}\mathcal{A}$ and $\rho_i : \mathsf{Mod}\mathsf{R}\mathcal{A} \to \mathsf{Mod}\mathcal{A}_i$ restrict to functors between $\mathsf{mod}\mathcal{A}_i$ and $\mathsf{mod}\mathsf{R}\mathcal{A}$ under certain assumptions. For simplicity, we use the notation $\mathsf{mod}_{-1}\mathcal{A} := \mathsf{Mod}\mathcal{A}$, $\mathsf{mod}_{\infty}\mathcal{A} := \mathsf{mod}\mathcal{A}$ and $\infty - 1 := \infty$.

Lemma 3.5. *Let $\mathcal{A}$ be a k-linear, Hom-finite additive category and $n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$. Assume that $\mathrm{D}\mathcal{A}(X,-) \in \mathsf{mod}_{n-1}\mathcal{A}$ holds for any $X \in \mathcal{A}$. Then the inclusion functor $\mathsf{Mod}\mathcal{A}_i \to \mathsf{Mod}\mathsf{R}\mathcal{A}$ restricts to a functor $\mathsf{mod}_n\mathcal{A}_i \to \mathsf{mod}_n\mathsf{R}\mathcal{A}$ for any $i \in \mathbb{Z}$.*

Proof. Let $n \in \mathbb{Z}_{\geq 0}$. It is sufficient to show that $\mathsf{proj}\,\mathcal{A}_i \subset \mathsf{mod}_n\mathsf{R}\mathcal{A}$ for any $i \in \mathbb{Z}$. In fact, any $M \in \mathsf{mod}_n\mathcal{A}_i$ has an exact sequence $P_n \to \cdots \to P_0 \to M \to 0$ with $P_i \in \mathsf{proj}\,\mathcal{A}_i$ and hence M belongs to $\mathsf{mod}_n\mathsf{R}\mathcal{A}$ by Lemma 2.4 (c).

We show $\mathsf{proj}\,\mathcal{A}_i \subset \mathsf{mod}_n\mathsf{R}\mathcal{A}$ for any $i \in \mathbb{Z}$ by an induction on n. If $n = 0$, then by Lemma 3.3 (c), we have the assertion. Let $n > 0$, $X \in \mathcal{A}$ and $i \in \mathbb{Z}$. By Lemma 3.3 (b), there exists an exact sequence

$$0 \to \mathrm{D}\mathcal{A}_{i-1}(X, -) \to \mathsf{R}\mathcal{A}(-, (X, i)) \to \mathcal{A}_i(-, X) \to 0.$$

By the explanation in the first part of this proof, the assumption of this lemma and the inductive hypothesis, $\mathrm{D}\mathcal{A}_{i-1}(X, -) \in \mathsf{mod}_{n-1}\mathsf{R}\mathcal{A}$ holds. Therefore we have $\mathcal{A}_i(-, X) \in \mathsf{mod}_n\mathsf{R}\mathcal{A}$ by Lemma 2.4 (b).

By an argument similar to the above, the assertion holds when $n = \infty$. $\square$

Lemma 3.6. *Let $\mathcal{A}$ be a k-linear, Hom-finite additive category, $n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$. Assume that $\mathrm{D}\mathcal{A}(X, -) \in \mathsf{mod}_n\mathcal{A}$ holds for any $X \in \mathcal{A}$. Then the functor $\rho_i : \mathsf{ModR}\mathcal{A} \to \mathsf{Mod}\mathcal{A}_i$ restricts to a functor $\mathsf{mod}_n\mathsf{R}\mathcal{A} \to \mathsf{mod}_n\mathcal{A}_i$ for any $i \in \mathbb{Z}$.*

Proof. Let $n \in \mathbb{Z}_{\geq 0}$ and $M \in \mathsf{mod}_n\mathsf{R}\mathcal{A}$. We have an exact sequence $P_n \to \cdots \to P_1 \to P_0 \to M \to 0$ in $\mathsf{ModR}\mathcal{A}$, where $P_j \in \mathsf{proj}\,\mathsf{R}\mathcal{A}$ for each $j \geq 0$. Since ρ_i is exact, we have an exact sequence $\rho_i(P_n) \to \cdots \to \rho_i(P_1) \to \rho_i(P_0) \to \rho_i(M) \to 0$ in $\mathsf{Mod}\mathcal{A}_i$. By the assumption and Lemma 3.3 (b), $\rho_i(P_j) \in \mathsf{mod}_n\mathcal{A}_i$ holds for any $j \geq 0$. Therefore $\rho_i(M) \in \mathsf{mod}_n\mathcal{A}_i$ holds by Lemma 2.4 (c).

By an argument similar to the above, the assertion holds when $n = \infty$. $\square$

Note that in general $\mathsf{modR}\mathcal{A} = \mathsf{mod}_1\mathsf{R}\mathcal{A}$ does not hold for a k-linear additive category $\mathcal{A}$. This is the case where $\mathcal{A}$ is a dualizing k-variety by Theorem 3.7 below and Lemma 2.9. Note that there exists an isomorphism $(\mathsf{R}\mathcal{A})^{\mathrm{op}} \simeq \mathsf{R}(\mathcal{A}^{\mathrm{op}})$ given by $(X, i) \mapsto (X, -i)$.

Theorem 3.7. *Let $\mathcal{A}$ be a dualizing k-variety. Then the following statements hold.*

(a) $\mathsf{R}\mathcal{A}$ *and* $(\mathsf{R}\mathcal{A})^{\mathrm{op}}$ *have weak kernels.*
(b) $\mathsf{R}\mathcal{A}$ *is a dualizing k-variety.*

Proof. Note that since $\mathcal{A}$ is a dualizing k-variety, $\mathrm{D}\mathcal{A}(X, -) \in \mathsf{mod}_1\mathcal{A}$ holds for any $X \in \mathcal{A}$. By Lemmas 2.9 and 2.14, $\mathsf{mod}_1\mathcal{A} = \mathsf{mod}\mathcal{A}$ holds and this is an abelian subcategory of $\mathsf{Mod}\mathcal{A}$.

(a) Let $\mathsf{X}, \mathsf{Y} \in \mathsf{R}\mathcal{A}$ and $f : \mathsf{R}\mathcal{A}(-, \mathsf{X}) \to \mathsf{R}\mathcal{A}(-, \mathsf{Y})$ be a morphism of $\mathsf{modR}\mathcal{A}$. We show that $K := \mathrm{Ker}(f)$ is a finitely generated $\mathsf{R}\mathcal{A}$-module. For any $i \in \mathbb{Z}$, we have an exact sequence $0 \to \rho_i(K) \to \rho_i(\mathsf{R}\mathcal{A}(-, \mathsf{X})) \to \rho_i(\mathsf{R}\mathcal{A}(-, \mathsf{Y}))$ in $\mathsf{Mod}\mathcal{A}_i$. By Lemma 3.6, we have $\rho_i(\mathsf{R}\mathcal{A}(-, \mathsf{X})), \rho_i(\mathsf{R}\mathcal{A}(-, \mathsf{Y})) \in \mathsf{mod}\mathcal{A}_i$. Therefore $\rho_i(K) \in \mathsf{mod}\mathcal{A}_i$ for any $i \in \mathbb{Z}$, since $\mathsf{mod}_1\mathcal{A} = \mathsf{mod}\mathcal{A}$ is an abelian subcategory of $\mathsf{Mod}\mathcal{A}$. By Lemma 3.5, $\rho_i(K) \in \mathsf{modR}\mathcal{A}$ for any $i \in \mathbb{Z}$. Since K is a submodule of $\mathsf{R}\mathcal{A}(-, \mathsf{X})$, $\mathrm{Supp}\,K$ is a finite set. Thus by Lemma 3.4 (b), K has a finite filtration with subquotients $\rho_i(K) \in \mathsf{modR}\mathcal{A}$ and we have $K \in \mathsf{modR}\mathcal{A}$ by Lemma 2.5 (b). In particular, K is finitely generated and $\mathsf{R}\mathcal{A}$ has weak kernels. Since $(\mathsf{R}\mathcal{A})^{\mathrm{op}} \simeq \mathsf{R}(\mathcal{A}^{\mathrm{op}})$ holds and $\mathcal{A}^{\mathrm{op}}$ is a dualizing k-variety, $(\mathsf{R}\mathcal{A})^{\mathrm{op}}$ has weak kernels.

(b) By the definition of dualizing k-varieties, $\mathcal{A}$ is Hom-finite and idempotent complete. By Lemma 3.2, $\mathsf{R}\mathcal{A}$ is Hom-finite and idempotent complete with a Serre functor. Therefore by Proposition 2.16, $\mathsf{R}\mathcal{A}$ is a dualizing k-variety. $\square$

3.2. Tilting subcategories

The aim of this subsection is to show Theorem 3.10. Before stating the main theorem, we need the following definition.

Let $\mathcal{A}$ be a k-linear, Hom-finite additive category. We denote by

$$\rho : \mathsf{ModR}\mathcal{A} \to \mathsf{Mod}\mathcal{A}$$

a functor defined as $\rho(M) := \bigoplus_{i\in\mathbb{Z}} \rho_i(M)$ for any $M \in \mathsf{Mod}\mathsf{R}\mathcal{A}$, where we regard an $\mathcal{A}_i$-module $\rho_i(M)$ as an $\mathcal{A}$-module by the equivalence $\mathsf{Mod}\mathcal{A}_i \simeq \mathsf{Mod}\mathcal{A}$. Note that ρ is an exact functor. We denote by $\mathsf{GP}(\mathsf{R}\mathcal{A}, \mathcal{A})$ the full subcategory of $\mathsf{GP}(\mathsf{R}\mathcal{A})$ consisting of all objects M such that the projective dimension of $\rho(M)$ over $\mathcal{A}$ is finite. We consider the following Gorenstein condition on $\mathcal{A}$:

(G): the projective dimension of $\mathrm{D}\mathcal{A}(X, -)$ over $\mathcal{A}$ is finite for any $X \in \mathcal{A}$.

Proposition 3.8. *Let $\mathcal{A}$ be a k-linear, Hom-finite additive category. Then $\mathcal{A}$ satisfies (G) if and only if* $\mathsf{proj}\,\mathsf{R}\mathcal{A} \subset \mathsf{GP}(\mathsf{R}\mathcal{A}, \mathcal{A})$ *holds. In this case, the following statements hold.*

(a) $\mathsf{GP}(\mathsf{R}\mathcal{A}, \mathcal{A})$ *is a Frobenius category such that the projective objects are precisely the objects of* $\mathsf{proj}\,\mathsf{R}\mathcal{A}$.
(b) *The inclusion functor* $\mathsf{GP}(\mathsf{R}\mathcal{A}, \mathcal{A}) \to \mathsf{GP}(\mathsf{R}\mathcal{A})$ *induces a fully faithful triangle functor* $\underline{\mathsf{GP}}(\mathsf{R}\mathcal{A}, \mathcal{A}) \to \underline{\mathsf{GP}}(\mathsf{R}\mathcal{A})$.

Proof. The first assertion follows from Lemma 3.3 (b). Assume that $\mathcal{A}$ satisfies (G).

(a) By the definition and since ρ is exact, $\mathsf{GP}(\mathsf{R}\mathcal{A}, \mathcal{A})$ is an extension closed subcategory of ModRA. Clearly, any object of $\mathsf{proj}\,\mathsf{R}\mathcal{A}$ is relative projective of $\mathsf{GP}(\mathsf{R}\mathcal{A}, \mathcal{A})$. Let Q be a relative projective object of $\mathsf{GP}(\mathsf{R}\mathcal{A}, \mathcal{A})$. There exists an exact sequence $0 \to M \to P \to Q \to 0$ in $\mathsf{GP}(\mathsf{R}\mathcal{A})$ with $P \in \mathsf{proj}\,\mathsf{R}\mathcal{A}$ and $M \in \mathsf{GP}(\mathsf{R}\mathcal{A})$. We have $M \in \mathsf{GP}(\mathsf{R}\mathcal{A}, \mathcal{A})$ and therefore this sequence splits. Consequently, the relative projective objects of $\mathsf{GP}(\mathsf{R}\mathcal{A}, \mathcal{A})$ are the objects of $\mathsf{proj}\,\mathsf{R}\mathcal{A}$ and $\mathsf{GP}(\mathsf{R}\mathcal{A}, \mathcal{A})$ has enough projective objects. By Proposition 2.8, $\mathsf{proj}\,\mathsf{R}\mathcal{A}$ are relative injective objects of $\mathsf{GP}(\mathsf{R}\mathcal{A})$, and so of $\mathsf{GP}(\mathsf{R}\mathcal{A}, \mathcal{A})$. By Lemma 2.15 (a), the duality $(-)^* : \mathsf{GP}(\mathsf{R}\mathcal{A}) \to \mathsf{GP}((\mathsf{R}\mathcal{A})^{\mathrm{op}})$ sends $\mathsf{proj}\,\mathsf{R}\mathcal{A}$ to $\mathsf{proj}((\mathsf{R}\mathcal{A})^{\mathrm{op}})$. Therefore, by a similar argument as above, any relative injective object belongs to $\mathsf{proj}\,\mathsf{R}\mathcal{A}$. Thus, $\mathsf{GP}(\mathsf{R}\mathcal{A}, \mathcal{A})$ is a Frobenius category.

(b) This follows from (a). □

We regard $\underline{\mathsf{GP}}(\mathsf{R}\mathcal{A}, \mathcal{A})$ as a thick subcategory of $\underline{\mathsf{GP}}(\mathsf{R}\mathcal{A})$ by Proposition 3.8 (b) if $\mathcal{A}$ satisfies (G). Let $\mathcal{A}$ be a k-linear, Hom-finite additive category. We consider the following condition on $\mathcal{A}$:

(IFP): $\mathrm{D}\mathcal{A}(X, -) \in \mathsf{mod}\mathcal{A}$ holds for any $X \in \mathcal{A}$,

where (IFP) means that injective $\mathcal{A}$-modules have projective resolutions by finitely generated projective $\mathcal{A}$-modules. Note that if $\mathcal{A}$ is a dualizing k-variety, then $\mathcal{A}$ satisfies (IFP) by Lemmas 2.9 and 2.14. We denote by $\mathcal{M}$ the full subcategory of $\mathsf{Mod}\mathsf{R}\mathcal{A}$ given by

$$\mathcal{M} := \mathsf{add}\{\, \mathcal{A}_0(-, X) \mid X \in \mathcal{A}\,\}.$$

We recall the definition of tilting subcategories of a triangulated category.

Definition 3.9. Let $\mathcal{T}$ be a triangulated category. A full subcategory $\mathcal{N}$ of $\mathcal{T}$ is called a *tilting subcategory* of $\mathcal{T}$ if $\mathcal{T}(\mathcal{N}, \mathcal{N}[i]) = 0$ for any $i \neq 0$ and $\mathsf{thick}\,\mathcal{N} = \mathcal{T}$.

We establish the following result in the rest of this subsection.

Theorem 3.10. *Let $\mathcal{A}$ be a k-linear, Hom-finite additive category and assume that $\mathcal{A}$ and $\mathcal{A}^{\mathrm{op}}$ satisfy (IFP). Then the following hold.*

(a) *If $\mathcal{A}$ and $\mathcal{A}^{\mathrm{op}}$ satisfy (G), then* $\mathcal{M} \subset \mathsf{GP}(\mathsf{R}\mathcal{A}, \mathcal{A})$ *holds and $\mathcal{M}$ is a tilting subcategory of* $\underline{\mathsf{GP}}(\mathsf{R}\mathcal{A}, \mathcal{A})$.
(b) *If each object of* $\mathsf{mod}\mathcal{A}$ *and* $\mathsf{mod}(\mathcal{A}^{\mathrm{op}})$ *has finite projective dimension, then* $\mathcal{M} \subset \mathsf{GP}(\mathsf{R}\mathcal{A})$ *holds and $\mathcal{M}$ gives a tilting subcategory of* $\underline{\mathsf{GP}}(\mathsf{R}\mathcal{A})$.

In the case where $\mathcal{A}$ is a dualizing k-variety, we have the following corollary.

Corollary 3.11. *Let $\mathcal{A}$ be a dualizing k-variety. If each object of $\mathsf{mod}\mathcal{A}$ and $\mathsf{mod}(\mathcal{A}^{\mathrm{op}})$ has finite projective dimension, then $\mathcal{M}$ is a tilting subcategory of $\underline{\mathsf{mod}}\,\mathsf{R}\mathcal{A}$.*

Before starting the proof of Theorem 3.10, we prove two lemmas. Let $\mathcal{A}$ be a k-linear additive category and $i \in \mathbb{Z}$. Define the following full subcategories of $\mathsf{R}\mathcal{A}$:

$$\mathcal{A}_{<i} := \bigvee_{j<i} \mathcal{A}_j, \quad \mathcal{A}_{\geq i} := \bigvee_{j\geq i} \mathcal{A}_j.$$

For $M \in \mathsf{ModR}\mathcal{A}$ and $i \in \mathbb{Z}$, let $\rho_{<i}(M) := \bigoplus_{j<i} \rho_j(M)$ and $\rho_{\geq i}(M) := \bigoplus_{j\geq i} \rho_j(M)$.

Lemma 3.12. *Let $\mathcal{A}$ be a k-linear, Hom-finite additive category. Let M and N be finitely generated $\mathsf{R}\mathcal{A}$-modules and $i \in \mathbb{Z}$. Assume that $\rho_{\geq i}(M) = 0$ and $\rho_{<i}(N) = 0$.*

(a) *There exist epimorphisms*

$$\mathsf{R}\mathcal{A}(-, \mathsf{X}) \to M, \quad \mathsf{R}\mathcal{A}(-, \mathsf{Y}) \to N,$$

for some $\mathsf{X} \in \mathcal{A}_{<i}$ and $\mathsf{Y} \in \mathcal{A}_{\geq i}$.

(b) *We have $(\mathsf{ModR}\mathcal{A})(M, N) = 0$ and $(\mathsf{ModR}\mathcal{A})(N, M) = 0$.*

(c) *Assume $M \in \mathsf{modR}\mathcal{A}$. Let*

$$\cdots \to P_2 \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \to 0 \tag{3.2}$$

be a minimal projective resolution of M in $\mathsf{modR}\mathcal{A}$. Then we have $\rho_{\geq i}(\operatorname{Ker} f_l) = 0$ for $l \geq 0$. Moreover by applying the functor ρ_{i-1}, we have a minimal projective resolution of $\rho_{i-1}(M)$ in $\mathsf{mod}\mathcal{A}_{i-1}$.

Proof. (a) Since M and N are finitely generated, there exist epimorphisms $\mathsf{R}\mathcal{A}(-, \mathsf{X}) \to M$ and $\mathsf{R}\mathcal{A}(-, \mathsf{Y}) \to N$, where X and Y are in $\mathsf{R}\mathcal{A}$. Let W be an object of $\mathcal{A}_{\geq i}$. By Yoneda's lemma and the assumption, we have $(\mathsf{ModR}\mathcal{A})(\mathsf{R}\mathcal{A}(-, \mathsf{W}), M) \simeq M(\mathsf{W}) = 0$. Therefore we can replace X with an object of $\mathcal{A}_{<i}$. Similarly, we can replace Y with an object of $\mathcal{A}_{\geq i}$.

(b) By (a), there exists an epimorphism $\mathsf{R}\mathcal{A}(-, \mathsf{X}) \to M$, where $\mathsf{X} \in \mathcal{A}_{<i}$. We have a monomorphism $(\mathsf{ModR}\mathcal{A})(M, N) \to (\mathsf{ModR}\mathcal{A})(\mathsf{R}\mathcal{A}(-, \mathsf{X}), N)$. Since $(\mathsf{ModR}\mathcal{A})(\mathsf{R}\mathcal{A}(-, \mathsf{X}), N) \simeq N(\mathsf{X}) = 0$, $(\mathsf{ModR}\mathcal{A})(M, N) = 0$ holds. Similarly, by applying $(\mathsf{ModR}\mathcal{A})(-, M)$ to an epimorphism $\mathsf{R}\mathcal{A}(-, \mathsf{Y}) \to N$, we have $(\mathsf{ModR}\mathcal{A})(N, M) = 0$.

(c) By (a), there exists $\mathsf{X}_0 \in \mathcal{A}_{<i}$ such that P_0 is a direct summand of $\mathsf{R}\mathcal{A}(-, \mathsf{X}_0)$. We have $\rho_{\geq i}(\mathsf{R}\mathcal{A}(-, \mathsf{X}_0)) = 0$. Therefore the submodule $\operatorname{Ker} f_0$ of $\mathsf{R}\mathcal{A}(-, \mathsf{X}_0)$ satisfies $\rho_{\geq i}(\operatorname{Ker} f_0) = 0$. By using this argument inductively, we have that there exist $\mathsf{X}_l \in \mathcal{A}_{<i}$ such that P_l is a direct summand of $\mathsf{R}\mathcal{A}(-, \mathsf{X}_l)$ for any $l \geq 0$. Therefore we have $\rho_{\geq i}(\operatorname{Ker} f_l) = 0$ for $l \geq 0$.

For any $l \geq 0$, by Lemma 3.3, $\rho_{i-1}(P_l)$ is a direct sum of $\mathcal{A}_{i-1}(-, X)$ for some $X \in \mathcal{A}$ and zero objects. Therefore each $\rho_{i-1}(P_l)$ is a finitely generated projective $\mathcal{A}_{i-1}$-module. Thus, by applying the functor ρ_{i-1} to the resolution (3.2), we have a projective resolution $\cdots \to \rho_{i-1}(P_1) \to \rho_{i-1}(P_0) \to \rho_{i-1}(M) \to 0$ of $\rho_{i-1}(M)$ in $\mathsf{mod}\mathcal{A}_{i-1}$, and this is minimal, since the resolution (3.2) is minimal. □

We explain when $\mathsf{GP}(\mathsf{R}\mathcal{A})$ contains all representable modules $\mathcal{A}_i(-, X)$. Note that there exists an isomorphism $s : \mathsf{R}(\mathcal{A}^{\mathrm{op}}) \xrightarrow{\sim} (\mathsf{R}\mathcal{A})^{\mathrm{op}}$ given by $(X, i) \mapsto (X, -i)$. Thus we have a duality

$$s^*\mathrm{D} : \mathsf{Mod}_{\mathrm{fg}}\mathsf{R}\mathcal{A} \xrightarrow{\mathrm{D}} \mathsf{Mod}_{\mathrm{fg}}((\mathsf{R}\mathcal{A})^{\mathrm{op}}) \xrightarrow{s^*:=(-)\circ s} \mathsf{Mod}_{\mathrm{fg}}\mathsf{R}(\mathcal{A}^{\mathrm{op}}).$$

Under this duality $s^*\mathrm{D}$, the full subcategory $\mathsf{mod}\mathcal{A}_i$ of $\mathsf{modR}\mathcal{A}$ goes to the full subcategory $\mathsf{mod}(\mathcal{A}^{\mathrm{op}}_{-i})$ of $\mathsf{modR}(\mathcal{A}^{\mathrm{op}})$. Since s is an isomorphism, s^* induces equivalences between $\mathsf{proj}((\mathsf{R}\mathcal{A})^{\mathrm{op}})$ and $\mathsf{proj}(\mathsf{R}(\mathcal{A}^{\mathrm{op}}))$, $\mathsf{mod}((\mathsf{R}\mathcal{A})^{\mathrm{op}})$ and $\mathsf{mod}(\mathsf{R}(\mathcal{A}^{\mathrm{op}}))$, and $\mathsf{GP}((\mathsf{R}\mathcal{A})^{\mathrm{op}})$ and $\mathsf{GP}(\mathsf{R}(\mathcal{A}^{\mathrm{op}}))$, respectively.

Lemma 3.13. *Let $\mathcal{A}$ be a k-linear, Hom-finite additive category.*

(a) *The following statements are equivalent.*
  (i) *$\mathcal{A}$ and $\mathcal{A}^{\mathrm{op}}$ satisfy (IFP).*
  (ii) *$\mathcal{A}_i(-,X) \in \mathsf{GP}(\mathsf{R}\mathcal{A})$ and $\mathcal{A}_i(X,-) \in \mathsf{GP}(\mathsf{R}(\mathcal{A}^{\mathrm{op}}))$ hold for any $X \in \mathcal{A}$ and $i \in \mathbb{Z}$.*
  (iii) *$\mathrm{D}\mathcal{A}_i(X,-) \in \mathsf{GP}(\mathsf{R}\mathcal{A})$ and $\mathrm{D}\mathcal{A}_i(-,X) \in \mathsf{GP}(\mathsf{R}(\mathcal{A}^{\mathrm{op}}))$ hold for any $X \in \mathcal{A}$ and $i \in \mathbb{Z}$.*

(b) *If $\mathcal{A}$ and $\mathcal{A}^{\mathrm{op}}$ satisfy (IFP), then $\rho_i(M) \in \mathsf{GP}(\mathsf{R}\mathcal{A})$ holds for any $M \in \mathsf{GP}(\mathsf{R}\mathcal{A})$ and $i \in \mathbb{Z}$.*

Proof. (a) By Lemma 3.2, $\mathsf{R}\mathcal{A}$ has a Serre functor $\mathbb{S}$. Thus by Lemma 2.15, we have an isomorphism of dualities $s^*(-)^* \simeq s^*\mathrm{D}(- \circ \mathbb{S}^{-1}) : \mathsf{Mod}_{\mathrm{fg}}\mathsf{R}\mathcal{A} \to \mathsf{Mod}_{\mathrm{fg}}\mathsf{R}(\mathcal{A}^{\mathrm{op}})$. We have

$$s^*(\mathcal{A}_i(-,X))^* \simeq \mathrm{D}(\mathcal{A}^{\mathrm{op}})_{-i-1}(X,-) = \mathrm{D}\mathcal{A}_{-i-1}(-,X) \tag{3.3}$$

for any $X \in \mathcal{A}$ and $i \in \mathbb{Z}$. Therefore, by Lemma 2.7 (a) and since s^* induces an equivalence between $\mathsf{GP}((\mathsf{R}\mathcal{A})^{\mathrm{op}})$ and $\mathsf{GP}(\mathsf{R}(\mathcal{A}^{\mathrm{op}}))$, $\mathcal{A}_i(-,X) \in \mathsf{GP}(\mathsf{G}\mathcal{A})$ if and only if $\mathrm{D}\mathcal{A}_{-i-1}(-,X) \in \mathsf{GP}(\mathsf{R}(\mathcal{A}^{\mathrm{op}}))$. Similarly, $\mathrm{D}\mathcal{A}_i(X,-) \in \mathsf{GP}(\mathsf{R}\mathcal{A})$ if and only if $\mathcal{A}_{-i+1}(X,-) \in \mathsf{GP}(\mathsf{R}(\mathcal{A}^{\mathrm{op}}))$. Namely, (ii) and (iii) are equivalent.

We show that (i) implies (ii). Let $X \in \mathcal{A}$. By Lemma 3.5, $\mathcal{A}_i(-,X) \in \mathsf{modR}\mathcal{A}$ holds. We have $(\mathcal{A}_i(-,X))^* \in \mathsf{mod}(\mathsf{R}\mathcal{A})^{\mathrm{op}}$, since $\mathcal{A}^{\mathrm{op}}$ satisfies (IFP), together with the equality (3.3) and Lemma 3.5. Therefore by Lemma 2.15 (b), we have $\mathcal{A}_i(-,X) \in \mathsf{GP}(\mathsf{R}\mathcal{A})$. Dually, we have $\mathcal{A}_i(X,-) \in \mathsf{GP}(\mathsf{R}(\mathcal{A}^{\mathrm{op}}))$.

We show that (ii) implies (i). Let $X \in \mathcal{A}$. Take a minimal projective resolution of $\mathcal{A}_i(-,X)$ in $\mathsf{modR}\mathcal{A}$:

$$\cdots \to Q_2 \to Q_1 \xrightarrow{d_1} \mathsf{R}\mathcal{A}(-,(X,i)) \to \mathcal{A}_i(-,X) \to 0.$$

By Lemma 3.3 (b), we have $\operatorname{Im} d_1 = \mathrm{D}\mathcal{A}_{i-1}(X,-)$. By Lemma 3.12 (c), applying ρ_{i-1}, we have $\mathrm{D}\mathcal{A}_{i-1}(X,-) \in \mathsf{mod}\mathcal{A}_{i-1}$. This means $\mathrm{D}\mathcal{A}(X,-) \in \mathsf{mod}\mathcal{A}$. Dually, we have $\mathrm{D}\mathcal{A}(-,X) \in \mathsf{mod}(\mathcal{A}^{\mathrm{op}})$.

(b) By Lemma 3.3 (b), we have $\rho_i(P) \in \mathsf{add}\{\mathcal{A}_i(-,X), \mathrm{D}\mathcal{A}_i(X,-) \mid X \in \mathcal{A}\}$ for any $P \in \mathsf{proj}\,\mathsf{R}\mathcal{A}$. In particular, we have $\rho_i(P) \in \mathsf{mod}\mathcal{A}_i$ for any $i \in \mathbb{Z}$ and any $P \in \mathsf{proj}\,\mathsf{R}\mathcal{A}$. Therefore $s^*(\rho_i(P))^* \in \mathsf{mod}(\mathcal{A}^{\mathrm{op}}_{-i-1})$ holds by the equality (3.3) and the assumption. Let $M \in \mathsf{GP}(\mathsf{R}\mathcal{A})$ and $P_\bullet = (P_j, d_j : P_j \to P_{j+1})$ be a totally acyclic complex such that $\operatorname{Im} d_0 = M$, where $P_j \in \mathsf{proj}\,\mathsf{R}\mathcal{A}$. By applying ρ_i, we have an exact sequence $\rho_i(P_\bullet) = (\rho_i(P_j), \rho_i(d_j) : \rho_i(P_j) \to \rho_i(P_{j+1}))$ such that $\operatorname{Im} \rho_i(d_0) = \rho_i(M)$. We have an exact sequence $\cdots \to \rho_i(P_{-1}) \to \rho_i(P_0) \to \rho_i(M) \to 0$. By Lemmas 2.4 (d) and 3.5, $\rho_i(M) \in \mathsf{modR}\mathcal{A}$ holds. By applying the functor $(-)^*$ to $0 \to \rho_i(M) \to \rho_i(P_1) \to \rho_i(P_2) \to \cdots$, and using Lemmas 2.4 (d) and 3.5 to the resulting exact sequence, we have $(\rho_i(M))^* \in \mathsf{mod}(\mathsf{R}\mathcal{A})^{\mathrm{op}}$. Therefore we have $\rho_i(M) \in \mathsf{GP}(\mathsf{R}\mathcal{A})$ by Lemma 2.15 (b). □

By Lemma 3.13, if $\mathcal{A}$ and $\mathcal{A}^{\mathrm{op}}$ satisfy (IFP), then $\mathcal{M} \subset \mathsf{GP}(\mathsf{R}\mathcal{A})$ holds. We also denote by $\mathcal{M}$ the subcategory of $\underline{\mathsf{GP}}(\mathsf{R}\mathcal{A})$ consisting of direct summands of a finite direct sum of objects $\mathcal{A}_0(-,X_i)$ for $X_i \in \mathcal{A}$. Then we show Theorem 3.10. We divide the proof into two propositions.

Proposition 3.14. *Let $\mathcal{A}$ be a k-linear, Hom-finite additive category and assume that $\mathcal{A}$ and $\mathcal{A}^{\mathrm{op}}$ satisfy (IFP). Let $\mathcal{T} := \underline{\mathsf{GP}}(\mathsf{R}\mathcal{A})$. Then we have $\mathcal{T}(\mathcal{M}, \mathcal{M}[i]) = 0$ for any $i \neq 0$.*

Proof. By Lemma 3.13, since $\mathcal{A}$ and $\mathcal{A}^{\mathrm{op}}$ satisfy (IFP), we have $\mathcal{A}_0(-, X) \in \mathsf{GP}(\mathsf{R}\mathcal{A})$ for any $X \in \mathcal{A}$. Let

$$\cdots \to P_2 \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} \mathcal{A}_0(-, X) \to 0$$

be a minimal projective resolution in $\mathsf{modR}\mathcal{A}$. Put $K^i := \mathrm{Ker}(f_{i-1})$ for $i \geq 1$. By Lemmas 3.3 (b) and 3.12 (c), we have $\rho_{\geq 0}(K^i) = 0$ for $i \geq 1$. Let $Y \in \mathcal{A}$. Since $\rho_{<0}(\mathcal{A}_0(-, Y)) = 0$ and Lemma 3.12 (b), we have

$$(\mathsf{ModR}\mathcal{A})(K^i, \mathcal{A}_0(-, Y)) = 0, \quad (\mathsf{ModR}\mathcal{A})(\mathcal{A}_0(-, Y), K^i) = 0,$$

for any $i \geq 1$. Therefore we have

$$\mathcal{T}(\mathcal{A}_0(-, Y), \mathcal{A}_0(-, X)[-i]) = \mathcal{T}(\mathcal{A}_0(-, Y), K^i) = 0,$$
$$\mathcal{T}(\mathcal{A}_0(-, X), \mathcal{A}_0(-, Y)[i]) = \mathcal{T}(K^i, \mathcal{A}_0(-, Y)) = 0,$$

for any $i \geq 1$. □

Proposition 3.15. *Let $\mathcal{A}$ be a k-linear, Hom-finite additive category and assume that $\mathcal{A}$ and $\mathcal{A}^{\mathrm{op}}$ satisfy (IFP). Let $\mathcal{T} := \underline{\mathsf{GP}}(\mathsf{R}\mathcal{A})$. If $\mathcal{A}$ and $\mathcal{A}^{\mathrm{op}}$ satisfy (G), then we have $\mathsf{thick}_{\mathcal{T}}\, \mathcal{M} = \underline{\mathsf{GP}}(\mathsf{R}\mathcal{A}, \mathcal{A})$.*

Proof. Since $\mathcal{A}$ and $\mathcal{A}^{\mathrm{op}}$ satisfy (IFP), we have $\mathcal{M} \subset \underline{\mathsf{GP}}(\mathsf{R}\mathcal{A}, \mathcal{A})$ by Lemma 3.13. Therefore we have $\mathsf{thick}\, \mathcal{M} := \mathsf{thick}_{\mathcal{T}}\, \mathcal{M} \subset \underline{\mathsf{GP}}(\mathsf{R}\mathcal{A}, \mathcal{A})$.

Let $i \in \mathbb{Z}$ and $N \in \mathsf{mod}\mathcal{A}_i$. Assume that N has finite projective dimension over $\mathcal{A}_i$. By Lemma 3.5, we have an inclusion $\mathsf{mod}\mathcal{A}_i \to \mathsf{modR}\mathcal{A}$ which is exact. Thus we obtain a finite resolution of N by objects that are direct summands of objects of the form $\mathcal{A}_i(-, X)$, $(X \in \mathcal{A})$ in $\mathsf{modR}\mathcal{A}$. Therefore if N is an object of $\mathsf{GP}(\mathsf{R}\mathcal{A}, \mathcal{A})$, then N is in $\mathsf{thick}\, \mathcal{M}$ if $\mathcal{A}_i(-, X)$ is in $\mathsf{thick}\, \mathcal{M}$ for any $X \in \mathcal{A}$.

Let $M \in \mathsf{GP}(\mathsf{R}\mathcal{A}, \mathcal{A})$. Since M is a factor module of a finitely generated projective $\mathsf{R}\mathcal{A}$-module, $\mathrm{Supp}\, M$ is a finite set. Thus by Lemma 3.4 (b), M has a finite filtration by $\rho_i(M)$ for $i = n, n+1, \ldots, m$, where $n = \min \mathrm{Supp}\, M$ and $m = \max \mathrm{Supp}\, M$. By Lemma 3.13 (b) and since $\rho(M)$ has finite projective dimension over $\mathcal{A}$, $\rho_i(M) \in \mathsf{GP}(\mathsf{R}\mathcal{A}, \mathcal{A})$ for any $i \in \mathbb{Z}$. Therefore M is in $\mathsf{thick}\, \mathcal{M}$ if $\mathcal{A}_i(-, X)$ is in $\mathsf{thick}\, \mathcal{M}$ for any $X \in \mathcal{A}$ and $i = n, n+1, \ldots, m$.

We show that $\mathcal{A}_i(-, X)$ is in $\mathsf{thick}\, \mathcal{M}$ for any $X \in \mathcal{A}$ and $i \in \mathbb{Z}$ by an induction on i. We first show $\mathcal{A}_i(-, X) \in \mathsf{thick}\, \mathcal{M}$ for $i \geq 0$. Since $\mathcal{A}_0(-, X) \in \mathcal{M}$, we have $\mathcal{A}_0(-, X) \in \mathsf{thick}\, \mathcal{M}$. Assume that $\mathcal{A}_j(-, X) \in \mathsf{thick}\, \mathcal{M}$ for $0 \leq j \leq i-1$. By Lemma 3.3, we have an exact sequence in $\mathsf{GP}(\mathsf{R}\mathcal{A})$

$$0 \to \mathrm{D}\mathcal{A}_{i-1}(X, -) \to \mathsf{R}\mathcal{A}(-, (X, i)) \to \mathcal{A}_i(-, X) \to 0.$$

Since $\mathrm{D}\mathcal{A}_{i-1}(X, -)$ has finite projective dimension over $\mathcal{A}$ by the property (G), and by the inductive hypothesis, we have $\mathrm{D}\mathcal{A}_{i-1}(X, -) \in \mathsf{thick}\, \mathcal{M}$. Therefore $\mathcal{A}_i(-, X)$ is in $\mathsf{thick}\, \mathcal{M}$.

Next we show that $\mathcal{A}_{-i}(-, X) \in \mathsf{thick}\, \mathcal{M}$ for $i > 0$. Assume that $\mathcal{A}_{-j}(-, X) \in \mathsf{thick}\, \mathcal{M}$ for $0 \leq j \leq i-1$. By Lemma 3.5, $\mathcal{A}_{-i}(-, X)$ belongs to $\mathsf{modR}\mathcal{A}$. By applying the duality $s^*\mathrm{D}$ to this module, we have $s^*\mathrm{D}(\mathcal{A}_{-i}(-, X)) \simeq \mathrm{D}\mathcal{A}^{\mathrm{op}}_i(X, -) \in \mathsf{mod}(\mathcal{A}^{\mathrm{op}}_i) \subset \mathsf{modR}(\mathcal{A}^{\mathrm{op}})$. Let n be the projective dimension of $\mathrm{D}\mathcal{A}^{\mathrm{op}}_i(X, -)$ in $\mathsf{mod}(\mathcal{A}^{\mathrm{op}}_i)$ and

$$Q_n \xrightarrow{f} \cdots \to Q_1 \to Q_0 \to \mathrm{D}\mathcal{A}^{\mathrm{op}}_i(X, -) \to 0$$

be the first n arrows of a minimal projective resolution of $\mathrm{D}\mathcal{A}^{\mathrm{op}}_i(X, -)$ in $\mathsf{modR}(\mathcal{A}^{\mathrm{op}})$. Put $K := \mathrm{Ker}\, f$. We have $K \in \mathsf{GP}(\mathsf{R}(\mathcal{A}^{\mathrm{op}}))$ by Lemmas 2.7 (b) and 3.13 (a). By applying ρ to this resolution, we have $K \in \mathsf{GP}(\mathsf{R}(\mathcal{A}^{\mathrm{op}}), \mathcal{A}^{\mathrm{op}})$. Since the projective dimension of $\mathrm{D}\mathcal{A}^{\mathrm{op}}_i(X, -)$ in $\mathsf{mod}(\mathcal{A}^{\mathrm{op}}_i)$ is n and by Lemma 3.12 (c), we have $\rho_i(K) = 0$. Moreover by Lemma 3.12 (c), we have $\rho_{\geq i+1}(K) = 0$. Therefore an $\mathsf{R}\mathcal{A}$-module $K' := (\mathrm{D}(s^*)^{-1})(K) \in \mathsf{GP}(\mathsf{R}\mathcal{A}, \mathcal{A})$ satisfies $\rho_{<-i+1}(K') = 0$. Since K' is a finitely generated $\mathsf{R}\mathcal{A}$-module, $\operatorname{Supp} K'$ is finite. Thus by Lemma 3.4 (b), K' has a finite filtration with subquotients $\rho_j(K')$ for $-i+1 \leq j \leq m$, where $m = \max \operatorname{Supp} K'$. By the inductive hypothesis, $K' \in \mathsf{thick}\,\mathcal{M}$ holds. We have an exact sequence in $\mathsf{GP}(\mathsf{R}\mathcal{A})$

$$0 \to \mathcal{A}_{-i}(-, X) \to Q'_0 \to Q'_1 \to \cdots \to Q'_n \to K' \to 0,$$

where each $Q'_l := (\mathrm{D}(s^*)^{-1})(Q_l)$ is a projective $\mathsf{R}\mathcal{A}$-module for $0 \leq l \leq n$ by Lemma 2.15 (a). This means $\mathcal{A}_{-i}(-, X) \simeq K'[-n-1]$ in $\underline{\mathsf{GP}}(\mathsf{R}\mathcal{A}, \mathcal{A})$. Therefore we have $\mathcal{A}_{-i}(-, X) \in \mathsf{thick}\,\mathcal{M}$. □

Proof of Theorem 3.10. (a) This follows from Propositions 3.14 and 3.15.

(b) Let $M \in \mathsf{GP}(\mathsf{R}\mathcal{A})$. By Lemma 3.6, we have $\rho(M) \in \mathsf{mod}\mathcal{A}$. Therefore, since each object of $\mathsf{mod}\mathcal{A}$ has finite projective dimension, we have $\mathsf{GP}(\mathsf{R}\mathcal{A}, \mathcal{A}) = \mathsf{GP}(\mathsf{R}\mathcal{A})$ and $\underline{\mathsf{GP}}(\mathsf{R}\mathcal{A}, \mathcal{A}) = \underline{\mathsf{GP}}(\mathsf{R}\mathcal{A})$. Since $\mathcal{A}$ and $\mathcal{A}^{\mathrm{op}}$ satisfy (IFP) and each object of $\mathsf{mod}\mathcal{A}$ and $\mathsf{mod}(\mathcal{A}^{\mathrm{op}})$ has finite projective dimension, $\mathcal{A}$ and $\mathcal{A}^{\mathrm{op}}$ satisfy (G). Thus the assertion follows from (a). □

Proof of Corollary 3.11. Since $\mathcal{A}$ and $\mathcal{A}^{\mathrm{op}}$ are dualizing k-varieties, both of them satisfy (IFP). By Lemmas 2.9, 3.2, Proposition 2.16 and Theorem 3.7, $\mathsf{GP}(\mathsf{R}\mathcal{A}) = \mathsf{modR}\mathcal{A}$ holds. The assertion directly follows from Theorem 3.10. □

3.3. Happel's theorem for functor categories

As an application of Theorem 3.10, we show Happel's theorem for functor categories. We need the following lemma.

Lemma 3.16. *Let $\mathcal{A}$ be a k-linear, Hom-finite additive category and assume that $\mathcal{A}$ and $\mathcal{A}^{\mathrm{op}}$ satisfy (IFP). Let $X, Y \in \mathcal{A}$, $\mathcal{T} := \underline{\mathsf{GP}}(\mathsf{R}\mathcal{A})$. We have the following equality:*

$$\mathcal{T}(\mathcal{A}_0(-, X), \mathcal{A}_0(-, Y)[n]) \simeq \begin{cases} \mathcal{A}(X, Y) & n = 0, \\ 0 & \text{else.} \end{cases}$$

Proof. By Proposition 3.14, $\mathcal{T}(\mathcal{A}_0(-, X), \mathcal{A}_0(-, Y)[n]) = 0$ holds for any $n \neq 0$. Moreover we have

$$\begin{aligned} (\mathsf{ModR}\mathcal{A})(\mathcal{A}_0(-, X), \mathsf{R}\mathcal{A}(-, (Y, 0))) &\simeq (\mathsf{Mod}(\mathsf{R}\mathcal{A})^{\mathrm{op}})(\mathrm{D}\mathsf{R}\mathcal{A}(-, (Y, 0)), \mathrm{D}\mathcal{A}_0(-, X)) \\ &\simeq (\mathsf{Mod}(\mathsf{R}\mathcal{A})^{\mathrm{op}})(\mathsf{R}\mathcal{A}((Y, -1), -), \mathrm{D}\mathcal{A}_0(-, X)) \\ &\simeq \mathrm{D}\mathcal{A}_0((Y, -1), X) = 0, \end{aligned} \tag{3.4}$$

where we use Lemma 3.2 (b) and Yoneda's lemma. By Lemma 3.3 (b), if a morphism $f : \mathcal{A}_0(-, X) \to \mathcal{A}_0(-, Y)$ in $\mathsf{ModR}\mathcal{A}$ factors through an object of $\mathsf{proj}\,\mathsf{R}\mathcal{A}$, then f factors through $\mathsf{R}\mathcal{A}(-, (Y, 0))$. Thus by the equality (3.4), we have

$$\mathcal{T}(\mathcal{A}_0(-, X), \mathcal{A}_0(-, Y)) = (\mathsf{ModR}\mathcal{A})(\mathcal{A}_0(-, X), \mathcal{A}_0(-, Y)).$$

By applying the functor $(\mathsf{ModR}\mathcal{A})(-, \mathcal{A}_0(-, Y))$ to the exact sequence of Lemma 3.3 (b), since $(\mathsf{ModR}\mathcal{A})(\mathrm{D}\mathcal{A}_{-1}(X, -), \mathcal{A}_0(-, Y)) = 0$ holds, we have

$$\begin{aligned}(\mathsf{ModR}\mathcal{A})(\mathcal{A}_0(-,X),\mathcal{A}_0(-,Y)) &\simeq (\mathsf{ModR}\mathcal{A})(\mathsf{R}\mathcal{A}(-,(X,0)),\mathcal{A}_0(-,Y))\\ &\simeq \mathcal{A}_0((X,0),Y)\\ &\simeq \mathcal{A}(X,Y). \quad \square\end{aligned}$$

We have the following result, which is a functor category version of Happel's theorem.

Corollary 3.17. *Let $\mathcal{A}$ be a k-linear, Hom-finite additive category and assume that $\mathcal{A}$ and $\mathcal{A}^{\mathrm{op}}$ satisfy (IFP).*

(a) *If $\mathcal{A}$ and $\mathcal{A}^{\mathrm{op}}$ satisfy (G), then we have a triangle equivalence*

$$\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\mathcal{A}) \simeq \underline{\mathsf{GP}}(\mathsf{R}\mathcal{A},\mathcal{A}).$$

(b) *If each object of $\mathsf{mod}\mathcal{A}$ and $\mathsf{mod}(\mathcal{A}^{\mathrm{op}})$ has finite projective dimension, then we have a triangle equivalence*

$$\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\mathcal{A}) \simeq \underline{\mathsf{GP}}(\mathsf{R}\mathcal{A}).$$

Proof. (a) Let $\mathcal{F} := \mathsf{GP}(\mathsf{R}\mathcal{A},\mathcal{A})$ and $\mathcal{P} := \mathsf{proj}\,\mathsf{R}\mathcal{A}$. By Proposition 3.8, $\mathcal{F}$ is a Frobenius category such that the projective objects are precisely the objects of $\mathcal{P}$. Since the inclusion functor $\mathsf{proj}\,\mathcal{A} \simeq \mathsf{proj}\,\mathcal{A}_0 \to \mathcal{F}$ is exact, it induces a triangle functor $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\mathcal{A}) \to \mathsf{K}^{\mathrm{b}}(\mathcal{F})$. By restricting the stable t-structure obtained from Proposition 2.18 to $\mathsf{K}^{-,\mathrm{b}}(\mathcal{F})$, we have a stable t-structure $(\mathsf{K}^{-,\mathrm{b}}(\mathcal{P}), \mathsf{K}^{\mathrm{a}}(\mathsf{F}) \cap \mathsf{K}^{-,\mathrm{b}}(\mathcal{F}))$ on $\mathsf{K}^{-,\mathrm{b}}(\mathcal{F})$. Therefore, there exists a triangle functor $\mathsf{K}^{-,\mathrm{b}}(\mathcal{F}) \to \mathsf{K}^{-,\mathrm{b}}(\mathcal{P})$ such that the inclusion functor $\mathsf{K}^{-,\mathrm{b}}(\mathcal{P}) \to \mathsf{K}^{-,\mathrm{b}}(\mathcal{F})$ is a left adjoint. Then we have the following triangle functors

$$F : \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\mathcal{A}) \to \mathsf{K}^{-,\mathrm{b}}(\mathcal{P}) \to \mathsf{K}^{-,\mathrm{b}}(\mathcal{P})/\mathsf{K}^{\mathrm{b}}(\mathcal{P}) \to \underline{\mathcal{F}},$$

where the third functor is a quasi-inverse of the equivalence obtained from Theorem 2.17. We denote by F the composite of these functors. We show that F is an equivalence by using Lemma 2.19.

Put $\mathcal{U} := \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\mathcal{A})$ and $\mathcal{T} := \underline{\mathsf{GP}}(\mathsf{R}\mathcal{A},\mathcal{A}) = \underline{\mathcal{F}}$. Note that $\mathsf{proj}\,\mathcal{A}$ is a subcategory of $\mathcal{U}$. We show that the map

$$F_{M,N[n]} : \mathcal{U}(M, N[n]) \to \mathcal{T}(FM, FN[n])$$

is an isomorphism for any $M, N \in \mathsf{proj}\,\mathcal{A}$ and $n \in \mathbb{Z}$. By Theorem 2.17, a quasi-inverse of $\mathsf{K}^{-,\mathrm{b}}(\mathcal{P})/\mathsf{K}^{\mathrm{b}}(\mathcal{P}) \to \underline{\mathcal{F}}$ is induced from the composite of the canonical functors $\mathcal{F} \to \mathsf{K}^{-,\mathrm{b}}(\mathcal{P}) \to \mathsf{K}^{-,\mathrm{b}}(\mathcal{P})/\mathsf{K}^{\mathrm{b}}(\mathcal{P})$. Therefore we have $F(\mathcal{A}(-,X)) = \mathcal{A}_0(-,X)$ for any $X \in \mathcal{A}$. For any $X, Y \in \mathcal{A}$, we have

$$\mathcal{U}(\mathcal{A}(-,X),\mathcal{A}(-,Y)) = \mathcal{A}(X,Y), \quad \mathcal{U}(\mathcal{A}(-,X),\mathcal{A}(-,Y)[n]) = 0,$$

for $n \neq 0$. Consequently, by Lemma 3.16, $F_{M,N[n]}$ is an isomorphism for any $M, N \in \mathsf{proj}\,\mathcal{A}$ and $n \in \mathbb{Z}$.

We show that $\mathcal{U} = \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\mathcal{A})$ is idempotent complete. For an additive category $\mathcal{B}$, we denote by $\mathsf{C}^{\mathrm{b}}(\mathcal{B})$ the category of bounded complexes of $\mathcal{B}$. Since $\mathsf{proj}\,\mathcal{A}$ is Hom-finite, so is $\mathsf{C}^{\mathrm{b}}(\mathsf{proj}\,\mathcal{A})$. Because $\mathsf{C}^{\mathrm{b}}(\mathsf{proj}\,\mathcal{A})$ is closed under taking direct summands in $\mathsf{C}^{\mathrm{b}}(\mathsf{Mod}\mathcal{A})$, $\mathsf{C}^{\mathrm{b}}(\mathsf{proj}\,\mathcal{A})$ is idempotent complete. Thus by Proposition 2.10, $\mathsf{C}^{\mathrm{b}}(\mathsf{proj}\,\mathcal{A})$ is a Krull-Schmidt category. By Proposition 2.10 (iii) and since there exists a full dense functor from $\mathsf{C}^{\mathrm{b}}(\mathsf{proj}\,\mathcal{A})$ to $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\mathcal{A})$, the latter category is also Krull-Schmidt. Therefore by Proposition 2.10, $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\mathcal{A})$ is idempotent complete. Clearly we have $\mathsf{thick}_{\mathcal{U}}(\mathsf{proj}\,\mathcal{A}) = \mathcal{U}$. Since $\mathrm{Im}(F|_{\mathsf{proj}\,\mathcal{A}}) = \mathcal{M}$ holds, we have $\mathsf{thick}(\mathrm{Im}(F)) = \mathcal{T}$ by Theorem 3.10 (a). Therefore F is an equivalence by Lemma 2.19.

(b) Since each object of $\mathsf{mod}\mathcal{A}$ has finite projective dimension, we have $\mathsf{GP}(\mathsf{R}\mathcal{A},\mathcal{A}) = \mathsf{GP}(\mathsf{R}\mathcal{A})$ and thus $\underline{\mathsf{GP}}(\mathsf{R}\mathcal{A},\mathcal{A}) = \underline{\mathsf{GP}}(\mathsf{R}\mathcal{A})$. Moreover, by the same assumption, $\mathcal{A}$ and $\mathcal{A}^{\mathrm{op}}$ satisfy (G). Therefore we have the assertion by (a). $\square$

Corollary 3.18. *Let $\mathcal{A}$ be a dualizing k-variety. If each object of* $\mathsf{mod}\mathcal{A}$ *and* $\mathsf{mod}(\mathcal{A}^{\mathrm{op}})$ *has finite projective dimension, then we have a triangle equivalence*

$$\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\mathcal{A}) \simeq \underline{\mathsf{mod}}\,\mathsf{R}\mathcal{A}.$$

Proof. If $\mathcal{A}$ is a dualizing k-variety, then $\mathsf{GP}(\mathsf{R}\mathcal{A}) = \mathsf{mod}\mathsf{R}\mathcal{A}$ holds. Since the natural inclusion $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\mathcal{A}) \to \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\mathcal{A})$ is a triangle equivalence, the assertion directly follows from Corollary 3.17. □

4. Proof of Theorem 1.1

Throughout this section, let A be a finite dimensional hereditary k-algebra, that is, $\mathrm{gldim}(A) \leq 1$. In this section, we apply Corollary 3.18 to $\underline{\mathsf{mod}}\,A$ and show Theorem 4.5.

We denote by $\mathsf{mod}A$ the category of the finitely generated A-modules and denote by τ and τ^{-1} the Auslander-Reiten translations of A. We call an indecomposable A-module M *preprojective* (resp. *preinjective*) if there exists an indecomposable projective A-module P (resp. injective A-module I) and an integer $i \leq 0$ (resp. $i \geq 0$) such that $M \simeq \tau^i(P)$ (resp. $M \simeq \tau^i(I)$). We call an indecomposable A-module M *regular* if M is neither preprojective nor preinjective. Then it is easy to see that an indecomposable A-module M is regular if and only if $\tau^i(M) \neq 0$ for any $i \in \mathbb{Z}$ (see [5, VIII. 4] for instance). Define the following subcategories of $\mathsf{mod}A$:

$$\mathcal{P} := \mathsf{add}\{M \in \mathsf{mod}A \mid M \text{ is a preprojective module}\},$$
$$\mathcal{I} := \mathsf{add}\{M \in \mathsf{mod}A \mid M \text{ is a preinjective module}\},$$
$$\mathcal{R} := \mathsf{add}\{M \in \mathsf{mod}A \mid M \text{ is a regular module}\}.$$

We denote by $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}A)$ the bounded derived category of $\mathsf{mod}A$ and denote by $\mathbb{S} = \mathrm{D}A \otimes^{\mathbb{L}}_A (-)$ a Serre functor on $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}A)$. We regard $\mathsf{mod}A$ as a full subcategory of $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}A)$ by the canonical inclusion. Thus for any $X \in \mathsf{D}^{\mathrm{b}}(\mathsf{mod}A)$, $X \in \mathsf{mod}A$ if and only if $\mathrm{H}^i(X) = 0$ for any $i \neq 0$.

The following proposition is well known (see [1, Chapter VIII. 2.1. Proposition] [13, Chapter I, 5.2, Lemma]).

Proposition 4.1. *Let A be a finite dimensional representation-infinite hereditary k-algebra. Then we have the following equalities.*

$$\mathsf{D}^{\mathrm{b}}(\mathsf{mod}A) = \bigvee_{i \in \mathbb{Z}} (\mathsf{mod}A)[i],$$
$$\mathsf{mod}A = \mathcal{P} \vee \mathcal{R} \vee \mathcal{I}.$$

We denote by $\mathsf{mod}_{\mathrm{p}}A$ the full subcategory of $\mathsf{mod}A$ consisting of modules without non-zero projective direct summands. The following lemma is used to define a functor from $\mathsf{R}(\mathsf{mod}_{\mathrm{p}}A)$ to $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}A)$.

Lemma 4.2. *Let A be a finite dimensional representation-infinite hereditary k-algebra. For any $i < 0$ and $j > 1$, we have*

$$\mathbb{S}^i(\mathsf{mod}_{\mathrm{p}}A) \subset \mathsf{add}(A) \vee \bigvee_{l<0} \mathsf{mod}A\,[l], \quad \mathbb{S}^j(\mathsf{mod}_{\mathrm{p}}A) \subset \mathsf{add}(\mathrm{D}A) \vee \bigvee_{l>1} \mathsf{mod}A\,[l].$$

Proof. Let $M \in \mathsf{mod}_{\mathrm{p}}A$ be an indecomposable A-module. By the description of the Serre functor $\mathbb{S}$ and the first equation of Proposition 4.1, we have $\mathbb{S}^{-1}(M) \in \mathsf{proj}\,A$ if M is injective, or $\mathbb{S}^{-1}(M) \in \mathsf{mod}A[-1]$

if otherwise. This, together with $\mathbb{S}^{-1}(\mathsf{proj}\,A) \subset \mathsf{mod}A[-1]$, implies the first inclusion. Similarly, the second inclusion follows. □

We define an additive functor

$$\Phi : \mathsf{R}(\mathsf{mod}_{\mathrm{p}}A) \to \mathsf{D}^{\mathrm{b}}(\mathsf{mod}A)$$

as follows. For $X \in \mathsf{mod}_{\mathrm{p}}A$ and $i \in \mathbb{Z}$, let $\Phi(X,i) := \mathbb{S}^i(X)$. For $X, Y \in \mathsf{mod}_{\mathrm{p}}A$ and $i,j \in \mathbb{Z}$, since $\mathbb{S}$ is a Serre functor of $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}A)$, we have

$$\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathsf{mod}A)}\big(\mathbb{S}^i(X), \mathbb{S}^j(Y)\big) \simeq \begin{cases} \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathsf{mod}A)}(X,Y) & i=j, \\ \mathrm{D}\,\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathsf{mod}A)}(Y,X) & j=i+1, \\ 0 & \text{else}, \end{cases}$$

where the last isomorphism follows from Lemma 4.2. By using these isomorphisms, we define a map

$$\Phi_{(X,i),(Y,j)} : \mathrm{Hom}_{\mathsf{R}(\mathsf{mod}_{\mathrm{p}}A)}\big((X,i),(Y,j)\big) \to \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathsf{mod}A)}\big(\mathbb{S}^i(X), \mathbb{S}^j(Y)\big),$$

and we extend Φ on $\mathsf{R}(\mathsf{mod}_{\mathrm{p}}A)$ additively. Then Φ is actually a functor, since a Serre duality is bifunctorial.

The first theorem of this section is the following. Put $\mathbb{S}_1 := \mathbb{S} \circ [-1]$. By the description of the Serre functor $\mathbb{S}$ and the definition of the Auslander-Reiten translation τ, $H^0(\mathbb{S}_1(M)) \simeq \tau(M)$ and $H^0(\mathbb{S}_1^{-1}(M)) \simeq \tau^{-1}(M)$ hold for any $M \in \mathsf{mod}A$, see also [1, Chapter IV. 2.4. Proposition]. Since A is hereditary, the canonical functor $\mathsf{mod}_{\mathrm{p}}A \to \underline{\mathsf{mod}}\,A$ induces an equivalence $\mathsf{mod}_{\mathrm{p}}A \simeq \underline{\mathsf{mod}}\,A$.

Theorem 4.3. *Let A be a finite dimensional representation-infinite hereditary k-algebra. Then the functor $\Phi : \mathsf{R}(\mathsf{mod}_{\mathrm{p}}A) \to \mathsf{D}^{\mathrm{b}}(\mathsf{mod}A)$ is an equivalence of additive categories. In particular, we have an equivalence $\mathsf{R}(\underline{\mathsf{mod}}\,A) \to \mathsf{D}^{\mathrm{b}}(\mathsf{mod}A)$ of additive categories.*

Proof. By the definition, Φ is fully faithful. We show that Φ is dense. Let X be an indecomposable object of $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}A)$. By Proposition 4.1, there exist an indecomposable A-module M and an integer l such that $X \simeq M[l]$.

Assume that M is a preprojective module. If P is an indecomposable projective module, then $\mathbb{S}_1^{-i}(P)$ has cohomology concentrated in degree zero for any $i \geq 0$. Therefore, there exist an indecomposable projective A-module P and $i \geq 0$ such that $M \simeq \mathbb{S}_1^{-i}(P)$. If $i + l > 0$, then we have $\mathbb{S}_1^{-(i+l)}(P) \in \mathsf{mod}_{\mathrm{p}}A$ and

$$\begin{aligned} \Phi(\mathbb{S}_1^{-(i+l)}(P), l) &= \mathbb{S}^l(\mathbb{S}_1^{-(i+l)}(P)) \\ &= \mathbb{S}_1^{-i}(P)[l] \\ &\simeq X. \end{aligned}$$

If $i + l \leq 0$, then for an injective A-module $\mathbb{S}(P)$, we have $\mathbb{S}_1^{-(i+l)}(\mathbb{S}(P)) \in \mathsf{mod}_{\mathrm{p}}A$ and

$$\begin{aligned} \Phi(\mathbb{S}_1^{-(i+l)}(\mathbb{S}(P)), l-1) &= \mathbb{S}^{l-1}(\mathbb{S}_1^{-(i+l)}(\mathbb{S}(P))) \\ &= \mathbb{S}_1^{-i}(P)[l] \\ &\simeq X. \end{aligned}$$

Next assume that M is a preinjective module. There exist an indecomposable injective A-module I and $i \geq 0$ such that $M \simeq \mathbb{S}_1^i(I)$. If $i - l \geq 0$, then we have $\mathbb{S}_1^{i-l}(I) \in \mathsf{mod}_{\mathrm{p}}A$ and

$$\begin{aligned}\Phi(\mathbb{S}_1^{i-l}(I), l) &= \mathbb{S}^l(\mathbb{S}_1^{i-l}(I)) \\ &= \mathbb{S}_1^i(I)[l] \\ &\simeq X.\end{aligned}$$

If $i - l < 0$, then we have $\mathbb{S}_1^{i-l}(\mathbb{S}^{-1}(I)) \in \mathsf{mod}_\mathrm{p} A$ and

$$\begin{aligned}\Phi(\mathbb{S}_1^{i-l}(\mathbb{S}^{-1}(I)), l+1) &= \mathbb{S}^{l+1}(\mathbb{S}_1^{i-l}(\mathbb{S}^{-1}(I))) \\ &= \mathbb{S}_1^i(I)[l] \\ &\simeq X.\end{aligned}$$

Assume that M is a regular module. Then we have $\mathbb{S}_1^{-l}(M) \in \mathcal{R} \subset \mathsf{mod}_\mathrm{p} A$ and $\Phi(\mathbb{S}_1^{-l}(M), l) = \mathbb{S}^l(\mathbb{S}_1^{-l}(M)) = M[l]$ holds. Therefore the functor $\Phi : \mathsf{R}(\mathsf{mod}_\mathrm{p} A) \to \mathsf{D}^\mathrm{b}(\mathsf{mod} A)$ is dense. The last assertion follows from the equivalence $\mathsf{mod}_\mathrm{p} A \simeq \underline{\mathsf{mod}}\, A$. $\square$

There is the well known equivalence $\mathsf{D}^\mathrm{b}(\mathcal{H}) \simeq \mathsf{Rep}\,\mathcal{H}$ for a hereditary abelian category $\mathcal{H}$ [20, Theorem 3.1]. Theorem 4.3 is an analog of this equivalence, in the sense that both equivalences give "repetitive shape" of the derived category of a hereditary algebra. But these equivalences are quite different, since the equivalence of Theorem 4.3 is induced from the Serre functor, on the other hand, the equivalence $\mathsf{D}^\mathrm{b}(\mathcal{H}) \simeq \mathsf{Rep}\,\mathcal{H}$ is induced from an inclusion functor.

We recall the following proposition.

Proposition 4.4. *Let $\mathcal{C}$ be a skeletally small dualizing k-variety and $\mathcal{D} := \mathsf{mod}\mathcal{C}$. Let $\mathcal{P}$ be the full subcategory of $\mathcal{D}$ consisting of all projective $\mathcal{C}$-modules. Then the following statements hold.*

(a) *$\mathcal{D}/[\mathcal{P}]$ is a dualizing k-variety.*
(b) *Assume that the global dimension of $\mathsf{mod}\mathcal{C}$ is at most n, then the global dimension of $\mathsf{mod}(\mathcal{D}/[\mathcal{P}])$ is at most $3n - 1$.*

Proof. (a) This follows from [3, Proposition 6.2].

(b) Throughout only this proof, we use the following notation, which is compatible with that of [3]. We denote by $\underline{\mathsf{mod}}\,\mathcal{D}$ the subcategory of $\mathsf{mod}\mathcal{D}$ consisting of objects M such that $M(Q) = 0$ for any $Q \in \mathcal{P}$. Then this is an abelian subcategory of $\mathsf{mod}\mathcal{D}$. We have an equivalence $\mathsf{mod}(\mathcal{D}/[\mathcal{P}]) \simeq \underline{\mathsf{mod}}\,\mathcal{D}$, see [3, Section 6]. Thus the assertion follows from [3, Proposition 10.2]. $\square$

Then we apply Corollary 3.18 to $\underline{\mathsf{mod}}\, A$.

Theorem 4.5. *Let A be a finite dimensional hereditary k-algebra. Then we have the following triangle equivalences*

$$\underline{\mathsf{mod}}\,\mathsf{D}^\mathrm{b}(\mathsf{mod} A) \simeq \underline{\mathsf{mod}}\,\mathsf{R}(\underline{\mathsf{mod}}\, A) \simeq \mathsf{D}^\mathrm{b}(\mathsf{mod}(\underline{\mathsf{mod}}\, A)).$$

Proof. If A is a representation-finite algebra, then equivalences are shown by Iyama-Oppermann [14, Corollary 4.11]. Assume that A is a representation-infinite algebra. The first equivalence comes from Theorem 4.3. By Proposition 4.4, $\underline{\mathsf{mod}}\, A$ is a dualizing k-variety such that the global dimension of $\mathsf{mod}(\underline{\mathsf{mod}}\, A)$ is at most two. Therefore we can apply Corollary 3.18 to the dualizing k-variety $\underline{\mathsf{mod}}\, A$. We have the second equivalence. $\square$

We say that two dualizing k-varieties $\mathcal{A}$ and $\mathcal{A}'$ are derived equivalent if the derived categories of $\mathsf{mod}\mathcal{A}$ and $\mathsf{mod}\mathcal{A}'$ are triangle equivalent.

Corollary 4.6. *Let A, A' be representation-infinite hereditary algebras. If A and A' are derived equivalent, then $\underline{\mathsf{mod}}\, A$ and $\underline{\mathsf{mod}}\, A'$ are derived equivalent.*

Remark 4.7. If A is a representation-finite hereditary algebra, then Theorems 4.3, 4.5 and Corollary 4.6 were shown by Iyama-Oppermann, see [14, Theorem 4.7, Corollary 4.11].

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