Projective normality of torus quotients of flag varieties

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A R T I C L E   I N F O

A B S T R A C T

Let $G = SL_n(\mathbb{C})$ and T be a maximal torus in G. We show that the quotient $T\backslash\backslash G/P_{\alpha_1} \cap P_{\alpha_2}$ is projectively normal with respect to the descent of a suitable line bundle, where P_{α_i} is the maximal parabolic subgroup in G associated to the simple root α_i, $i = 1, 2$. We give a degree bound of the generators of the homogeneous coordinate ring of $T\backslash\backslash(G_{3,6})_T^{ss}(\mathcal{L}_{2\varpi_3})$. If $G = Spin_7$, we give a degree bound of the generators of the homogeneous coordinate ring of $T\backslash\backslash(G/P_{\alpha_2})_T^{ss}(\mathcal{L}_{2\varpi_2})$ whereas we prove that the quotient $T\backslash\backslash(G/P_{\alpha_3})_T^{ss}(\mathcal{L}_{4\varpi_3})$ is projectively normal with respect to the descent of the line bundles $\mathcal{L}_{4\varpi_3}$.

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1. Introduction

For the action of a maximal torus T on the Grassmannian $G_{r,n}$ the quotients $T\backslash\backslash G_{r,n}$ have been studied extensively. Allen Knutson called them weight varieties in his thesis [17]. In [8] Hausmann and Knutson identified the GIT quotient of the Grassmannian $G_{2,n}$ by the natural action of the maximal torus with the moduli space of polygons in $\mathbb{R}^3$ and this GIT quotient can also be realized as the GIT quotient of an n-fold product of projective lines by the diagonal action of $PSL(2, \mathbb{C})$. In the symplectic geometry literature these spaces are known as polygon spaces as they parameterize the n-sides polygons in $\mathbb{R}^3$ with fixed edge length up to rotation. More generally, $T\backslash\backslash G_{r,n}$ can be identified with the GIT quotient of $(\mathbb{P}^{r-1})^n$ by the diagonal action of $PSL(r, \mathbb{C})$ via the Gelfand-MacPherson correspondence. In [15] and [16] Kapranov studied the Chow quotient of the Grassmannians and he showed that the Grothendieck-Knudsen moduli space $\overline{M}_{0,n}$ of stable n-pointed curves of genus zero arises as the Chow quotient of the maximal torus action on the Grassmannian $G_{2,n}$.

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In [6] Dabrowski has proved that for any parabolic subgroup P of G, the Zariski closure of a generic T-orbit in G/P is normal. For a precise statement, see [6, Theorem 3.2, pg. 327]. In [4] Carrell and Kurth proved that if G is of type A_n, D_4 or B_2 and P is any maximal parabolic subgroup of G, then every T orbit closure in G/P is normal. In the context of a problem on projective normality for torus actions, Howard proved that for any parabolic subgroup P of $SL_n(\mathbb{C})$, the Zariski closure $\overline{T.x}$ of the T-orbit of any point x in $SL_n(\mathbb{C})/P$ is projectively normal for the choice of any ample line bundle $\mathcal{L}$ on $SL_n(\mathbb{C})/P$. For a precise statement, see [9, Theorem 5.4, pg. 540].

In [14] the authors consider the quotients of a projective space X for the linear action of finite solvable groups and for finite groups acting by pseudo reflections. They prove that $X//G$ is projectively normal with respect to the descent of $\mathcal{O}_X(1)^{|G|}$. In [5] these results were obtained for every finite group but with a larger power of the descent of $\mathcal{O}_X(1)^{|G|}$. In [13] there was an attempt to study the projective normality of $T\backslash\backslash(G_{2,n})$ (n odd) with respect to the descent of the line bundle corresponding to the fundamental weight ω_2. There it was proved that the homogeneous coordinate ring of $T\backslash\backslash(G_{2,n})$ is a finite module over the subring generated by the degree one elements. In [2] and [10] the authors show that the quotient $T\backslash\backslash G_{2,n}$ is projectively normal with respect to the descent of the line bundle corresponding to $n\varpi_2$ using the Kempe embedding.

In this paper we give a short proof of the projective normality of the quotient $T\backslash\backslash G_{2,n}$ with respect to the descent of the line bundle $\mathcal{L}_{n\varpi_2}$ using Standard Monomial Theory and some graph theoretic techniques. We also prove that the quotient $T\backslash\backslash G/P_{\alpha_1} \cap P_{\alpha_2}$ is projectively normal with respect to the descent of a suitable line bundle, where P_{α_i} is the maximal parabolic subgroup in G associated to the simple root α_i, $i = 1, 2$, which is the main ingredient of this paper. We give a degree bound of the generators of the homogeneous coordinate ring of $T\backslash\backslash(G_{3,6})_T^{ss}(\mathcal{L}_{2\varpi_3})$. If $G = Spin_7$, we give a degree bound of the generators of the homogeneous coordinate ring of $T\backslash\backslash(G/P_{\alpha_2})_T^{ss}(\mathcal{L}_{2\varpi_2})$ whereas we prove that $T\backslash\backslash(G/P_{\alpha_3})_T^{ss}(\mathcal{L}_{4\varpi_3})$ is projectively normal with respect to the descent of the line bundles $\mathcal{L}_{4\varpi_3}$.

The layout of the paper is as follows. Section 2 consists of preliminary definitions and notation. In Section 3 we recall some preliminaries of Standard Monomial Theory and in Section 4 we recall some preliminaries of graph theory. In Section 5 we show that the GIT quotients $T\backslash\backslash(G_{r,n})$ and $T\backslash\backslash(G_{n-r,n})$ are isomorphic. In Section 6 we give a proof of the projective normality of the quotient $T\backslash\backslash(G_{2,n})_T^{ss}(\mathcal{L}_{n\varpi_2})$ with respect to the descent of the line bundle $\mathcal{L}_{n\varpi_2}$ and we give a degree bound of the generators of the homogeneous coordinate ring of $T\backslash\backslash(G_{3,6})_T^{ss}(\mathcal{L}_{2\varpi_3})$. In Section 7 for $G = SL_n$ we prove projective normality of the quotient $T\backslash\backslash G/P_{\alpha_1} \cap P_{\alpha_2}$ with respect to the descent of a suitable line bundle and in Section 8 for $G = Spin_7$, we give a degree bound of the generators of the homogeneous coordinate ring of $T\backslash\backslash(G/P_{\alpha_2})_T^{ss}(\mathcal{L}_{2\varpi_2})$ and we prove that $T\backslash\backslash(G/P_{\alpha_3})_T^{ss}(\mathcal{L}_{4\varpi_3})$ is projectively normal with respect to the descent of the line bundle $\mathcal{L}_{4\varpi_3}$.

2. Preliminaries

In this section we set up some preliminaries and notation. We refer to [11], [12], [25] for preliminaries in Lie algebras and algebraic groups. Let G be a semi-simple algebraic group over $\mathbb{C}$. We fix a maximal torus T of G and a Borel subgroup B of G containing T. Let $N_G(T)$ be the normaliser of T in G. Let $W = N_G(T)/T$ be the Weyl group of G with respect to T. Let R denote the set of roots with respect to T. Let $S = \{\alpha_1, \alpha_2, \ldots, \alpha_n\} \subset R$ be the set of simple roots and for a subset $I \subseteq S$ we denote by P_I the parabolic subgroup of G generated by B and $\{n_\alpha : \alpha \in S \setminus I\}$, where n_α is a representative of s_α in $N_G(T)$. Let $X(T)$ (resp. $Y(T)$) denote the set of characters of T (resp. one parameter subgroups of T). Let $E_1 := X(T) \otimes \mathbb{R}$, $E_2 := Y(T) \otimes \mathbb{R}$. Let $\langle .,. \rangle : E_1 \times E_2 \to \mathbb{R}$ be the canonical non-degenerate bilinear form. For all homomorphism $\phi_\alpha : SL_2 \to G$, $(\alpha \in R)$, we have $\check{\alpha} : \mathbb{G}_m \to G$ defined by

$$\check{\alpha}(t) = \phi_\alpha(\begin{bmatrix} t & 0 \\ 0 & t^{-1} \end{bmatrix}).$$

We also have $s_\alpha(\chi) = \chi - \langle \chi, \check{\alpha} \rangle \alpha$ for all $\alpha \in R$ and $\chi \in E_1$. Set $s_i = s_{\alpha_i}$ for all $i = 1, 2, \ldots, n$. Let $\{\varpi_i : i = 1, 2, \ldots, n\} \in E_1$ be the fundamental weights; i.e. $\langle \varpi_i, \check{\alpha}_j \rangle = \delta_{ij}$ for all $i, j = 1, 2, \ldots n$.

For a simply connected semi-simple algebraic group G and for a parabolic subgroup P, the quotient space G/P is a homogeneous space for the left action of G. The quotient G/P is called a generalized flag variety. When $G = SL_n(\mathbb{C})$ and P_r is the maximal parabolic subgroup corresponding to the simple root α_r, the quotient can be identified with $G_{r,n}$, the Grassmannian of r dimensional subspaces of $\mathbb{C}^n$.

Now we recall the definition of projective normality of a projective variety. A projective variety $X \subset \mathbb{P}^n$ is said to be projectively normal if the affine cone $\hat{X}$ over X is normal at its vertex. For a reference, see exercise 3.18, page 23 of [7]. For the practical purpose we need the following fact about projective normality of a polarized variety.

A polarized variety $(X, \mathcal{L})$ where $\mathcal{L}$ is a very ample line bundle is said to be projectively normal if its homogeneous coordinate ring $\oplus_{n \in \mathbb{Z}_{\geq 0}} H^0(X, \mathcal{L}^{\otimes n})$ is integrally closed and it is generated as a $\mathbb{C}$-algebra by $H^0(X, \mathcal{L})$ (see Exercise 5.14, Chapter II of [7]). Projective normality depends on the particular projective embedding of the variety.

Example. The projective line $\mathbb{P}^1$ is obviously projectively normal since its cone is the affine plane $\mathbb{C}^2$ (which is non-singular). However it can also be embedded in $\mathbb{P}^3$ as the quartic curve, namely,

$$V_+ = \{(a^4, a^3 b, a b^3, b^4) \in \mathbb{P}^3 \mid (a, b) \in \mathbb{P}^1\},$$

then it is normal but not projectively normal (see [7], Chapter 1, Ex. 3.18).

For preliminaries in Geometric Invariant Theory we refer to [23] and [24]. Let X be a projective variety which is acted upon by a reductive group G. Let $\mathcal{L}$ be a G-linearized very ample line bundle on X. The GIT quotient $X//G$ is by definition the uniform categorical quotient of the (open) set of semistable points $X_G^{ss}(\mathcal{L})$ by G. We denote the GIT quotient of X by G with respect to $\mathcal{L}$ by $X_G^{ss}(\mathcal{L})//G$. Assume that the line bundle $\mathcal{L}$ descends to the quotient $X_G^{ss}(\mathcal{L})//G$ and denote the descent by $\mathcal{L}'$. Then the polarized variety $(X_G^{ss}(\mathcal{L})//G, \mathcal{L}')$ is $Proj(\oplus_{n \in \mathbb{Z}_{\geq 0}} (H^0(X, \mathcal{L}^{\otimes n})^G)$.

Let G be a simple, simply-connected algebraic group. Let T be a maximal torus in G and B be a Borel subgroup of G containing T. Let W be the Weyl group of G with respect to T and Q be the root lattice of G. Let λ be a dominant weight of G and let P_λ be the parabolic subgroup of G associated to λ which is by definition the subgroup of G generated by the Borel subgroup B and the isotropy group W_λ of λ in W. Let $\mathcal{L}_\lambda$ be the homogeneous ample line bundle on G/P_λ associated to λ. The T-linearization of $\mathcal{L}_\lambda$ is given by restricting the action of G on $\mathcal{L}_\lambda$ to T. Then the following theorem due to Shrawan Kumar describes which line bundles descend to the GIT quotient $T \backslash \backslash (G/P_\lambda)_T^{ss}(\mathcal{L}_\lambda)$ (see [18, Theorem 3.10]). We note that Kumar's result in [18] is more general than what is presented here.

Theorem 2.1. *With all the notations as above, the line bundle $\mathcal{L}_\lambda$ descends to a line bundle on the GIT quotient $T \backslash \backslash (G/P_\lambda)_T^{ss}(\mathcal{L}_\lambda)$ if and only if λ is of the following form depending upon the type G:*

1. *G of type $A_n (n \geq 1)$: $\lambda \in Q$,*
2. *G of type B_2: $\lambda \in \mathbb{Z}\alpha_1 + 2\mathbb{Z}\alpha_2 = 2X(T)$,*
3. *G of type $B_n (n \geq 3)$: $\lambda \in 2Q$.*

3. Some preliminaries of Standard Monomial Theory

Let $\{e_1, e_2, \ldots, e_n\}$ be the standard basis of $\mathbb{C}^n$. Let $I_{r,n} = \{(i_1, i_2, \ldots, i_r) | 1 \leq i_1 < \cdots < i_r \leq n\}$. The set $\{e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_r} | (i_1, i_2, \ldots, i_r) \in I_{r,n})\}$ forms a basis of $\wedge^r \mathbb{C}^n$. We denote by $\{p_{i_1, i_2, \ldots, i_r}\}$ the dual basis of the basis $\{e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_r}\}$; the $p_{i_1, i_2, \ldots, i_r}$ are called the Plücker coordinates of $\mathbb{P}(\wedge^r \mathbb{C}^n)$.

The Grassmannian $G_{r,n} \subseteq \mathbb{P}(\wedge^r \mathbb{C}^n)$ is precisely the zero set of the following well known Plücker relations:

$$\sum_{h=1}^{r+1} (-1)^h p_{i_1, i_2, \ldots, i_{r-1} j_h} p_{j_1, \ldots, \hat{j}_h, \ldots, j_{r+1}},$$

where $\{i_1, \ldots, i_{r-1}\}$ and $\{j_1, \ldots, j_{r+1}\}$ are two subsets of $\{1, 2, \ldots, n\}$.

3.1. SL_n-standard Young tableau

In this subsection we recall some basic facts about standard Young tableau for generalized flag varieties (see [19, pg. 216]).

Let $G = SL_n$ and $\lambda = \Sigma_{i=1}^{n-1} a_i \varpi_i$, $a_i \in \mathbb{Z}^+$ be a dominant weight. To λ we associate a Young diagram (denoted by Γ) with λ_i number of boxes in the i-th column, where $\lambda_i := a_i + \ldots + a_{n-1}$, $1 \leq i \leq n-1$.

A Young diagram Γ associated to a dominant weight λ is said to be a Young tableau if the diagram is filled with integers $1, 2, \ldots, n$. We also denote this Young tableau by Γ. The Young tableau is said to be standard if it is strictly increasing in the row and non-decreasing in column.

Given a Young tableau Γ, let $\tau = \{i_1, i_2, \ldots, i_d\}$ be a typical row in Γ, where $1 \leq i_1 < \cdots < i_d \leq n$, for some $1 \leq d \leq n-1$. To the row τ, we associate the Plücker coordinate $p_{i_1, i_2, \ldots, i_d}$. We set $p_\Gamma = \prod_\tau p_\tau$, where the product is taken over all the rows of Γ. We say that p_Γ is a standard monomial on G/P_λ if Γ is standard, where P_λ is the parabolic subgroup of G associated to the weight λ.

Now we recall the definition of weight of a standard Young tableau Γ (see [22, Section 2]). For a positive integer $1 \leq i \leq n$, we denote by $c_\Gamma(i)$, the number of boxes of Γ containing the integer i. Let $\epsilon_i : T \to \mathbb{G}_m$ be the character defined as $\epsilon_i(diag(t_1, \ldots, t_n)) = t_i$. We define the weight of Γ as

$$wt(\Gamma) := c_\Gamma(1)\epsilon_1 + \cdots + c_\Gamma(n)\epsilon_n.$$

We have the following lemma about T-invariant monomials in $H^0(G/P_\lambda, \mathcal{L}_\lambda)$.

Lemma 3.1. *A monomial $p_\Gamma \in H^0(G/P_\lambda, \mathcal{L}_\lambda)$ is T-invariant if and only if all the entries in Γ appear equal number of times.*

Proof. Recall that the action of T on $H^0(G/P_\lambda, \mathcal{L}_\lambda)$ is given by

$$(t_1, \ldots, t_n) \cdot p_{i_1, i_2, \ldots, i_r} = (t_{i_1} \cdots t_{i_r})^{-1} p_{i_1, i_2, \ldots, i_r}.$$

Since $p_{i_1, \ldots, i_r}$ is the dual of $e_{i_1} \wedge \cdots \wedge e_{i_r}$, the weight of $p_{i_1, \ldots, i_r}$ is $-(\epsilon_{i_1} + \cdots + \epsilon_{i_r})$. Thus the weight of p_Γ is $-wt(\Gamma)$. Therefore, we see that p_Γ is T invariant if and only if the weight of Γ is zero. Since the weight of Γ is $\sum_{i=1}^n c_\Gamma(i)\epsilon_i$ and $\sum_{i=1}^n \epsilon_i = 0$, we conclude that p_Γ is T-invariant if and only if $c_\Gamma(i) = c_\Gamma(j)$ for all $1 \leq i, j \leq n$. This proves the lemma. $\square$

3.2. $Spin_{2n+1}$-standard Young tableau

In this subsection we recall some basic facts about standard Young tableau for G, where $G = Spin_{2n+1}$. (see the Appendix in [22]).

Let $\lambda = \Sigma_{i=1}^n a_i \varpi_i$, $a_i \in \mathbb{Z}^+$ be a dominant weight. Define $p_i = \Sigma_{j=i}^{n-1} 2a_j + a_n$, for $1 \leq i \leq n$.

To λ we associate a Young diagram (denoted by Γ) of shape $p(\lambda) = (p_1, p_2, \ldots)$ with $p_1 \geq p_2 \geq \ldots$ consists of p_1 boxes in the first column, p_2 in the second column etc.

Let $r = (i_1, \ldots, i_t)$ be a row of length $t \leq n$ with entries $i_j \leq 2n$. For $i = 1, \ldots, n$ denote by $s_i(r)$ the row defined as follows:

If $i < n$ and $i+1$ and $2n+1-i$ are entries of the row r, then $s_i(r)$ is the row obtained from r by replacing the entry $i+1$ by i and the entry $2n+1-i$ by $2n-i$. Else we set $s_i(r) := r$. If $i = n$ and $n+1$ is an entry of the row r, then denote by $s_i(r)$ the row obtained from r by replacing the entry $n+1$ by n. Else we set $s_n(r) := r$.

We say that a pair of rows (r, r') are admissible if $r = r'$ or there exists a sequence of different rows $(r_0, r_1, \ldots, r_l)$ such that $r_0 = r$, $r_l = r'$ and $s_{i_k}(r_{k-1}) = r_k$ for $k = 1, 2, \ldots, l$ for some integers $i_1, \ldots, i_l \in \{1, 2, \ldots n\}$.

A Young diagram Γ of shape p is said to be a Young tableau (also denoted by Γ) if the diagram is filled with positive integers such that

1. the entries are less than or equal to $2n$,
2. i and $2n+1-i$ do not occur in the same row, for all $1 \le i \le n$ and
3. For all $i = 1, \ldots, \bar{p_1}$, the pair of rows (r_{2i}, r_{2i-1}) are admissible, where $\bar{p_1} = \frac{p_1 - a_n}{2}$.

The Young tableau is said to be standard if it is strictly increasing in the row and non-decreasing in column. If i is a positive integer and Γ is a given Young tableau then we denote by $c_\Gamma(i)$, the number of boxes of Γ containing the integer i. We define the weight of the young tableau Γ as

$$wt(\Gamma) := \frac{1}{2}((c_\Gamma(1) - c_\Gamma(2n))\epsilon_1 + \cdots + (c_\Gamma(n) - c_\Gamma(n+1))\epsilon_n).$$

Let P_λ be the parabolic subgroup of G associated to λ. Then the T-eigenvectors of $H^0(G/P_\lambda, \mathcal{L}_\lambda)$ are denoted by p_Γ which are indexed by the Young tableau Γ of shape $p(\lambda)$ and the weight of p_Γ is defined to be $wt(\Gamma)$. We say that p_Γ is a standard monomial if Γ is standard.

Lemma 3.2. *A monomial $p_\Gamma \in H^0(G/P_\lambda, \mathcal{L}_\lambda)$ is T-invariant if and only if $c_\Gamma(t) = c_\Gamma(2n+1-t)$, for all $1 \le t \le 2n$.*

Proof. A monomial $p_\Gamma \in H^0(G/P_\lambda, \mathcal{L}_\lambda)$ is T-invariant if and only if the weight of Γ is zero. Recall that weight of a Young tableau Γ is given by $\frac{1}{2} \sum_{j=1}^{n} (c_\Gamma(j) - c_\Gamma(2n+1-j))\epsilon_j$. Thus, p_Γ is T-invariant if and only if $c_\Gamma(t) = c_\Gamma(2n+1-t)$, for all $1 \le t \le 2n$. $\square$

The main theorem of the Standard Monomial Theory for any classical group is the following (see [21], [22]):

Theorem 3.3. *Let G be a simple, simply connected algebraic group and P_λ be the parabolic subgroup of G associated to a dominant weight λ. Then the standard monomials p_Γ form a basis of $H^0(G/P_\lambda, \mathcal{L}_\lambda^{\otimes m})$ as a vector space, where Γ is a standard Young tableau associated to the weight $m\lambda$.*

We also note that the above standard monomial basis of $H^0(G/P_\lambda, \mathcal{L}_\lambda^{\otimes m})$ is compatible when restricting to Schubert and Richardson varieties. For a precise statement see Theorem 16 of [20].

4. Some preliminaries of graph theory

We follow [1] for the preliminary definitions in graph theory.

Let $\mathcal{G}$ be a graph which is represented by the pair $(V(\mathcal{G}), E(\mathcal{G}))$, where $V(\mathcal{G})$ denotes the set of vertices and $E(\mathcal{G})$ denotes the set of edges respectively. A graph having loops and multiple edges is called a **general graph**. A graph having no loops but having multiple edges is called a **multigraph**. A graph without loops and at most one edge between any two vertices is called a **simple graph**. The **degree** of a vertex in a graph

is the number of edges connected to the vertex with loops counted twice. A graph $\mathcal{G}$ is called **k-regular** if each vertex of $V(\mathcal{G})$ is of degree k.

A **spanning subgraph** of $\mathcal{G}$ is a subgraph of $\mathcal{G}$ which contains every vertex of $\mathcal{G}$. For a positive integer k, a **k-factor** of $\mathcal{G}$ is a spanning subgraph of $\mathcal{G}$ that is k-regular. A graph $\mathcal{G}$ is said to be **k-factorable** if it has a k-factor.

A **walk** in a graph is defined as a sequence of alternating vertices and edges $v_0, e_1, v_1, e_2, \ldots, v_{k-1}, e_k, v_k$, where $e_i = (v_{i-1}, v_i)$ is the edge between v_{i-1} and v_i. The **length** of this walk is k. A walk that passes through every one of its vertices exactly once is called a **path**. Thus, by an even length path we mean k is even and by an odd length path we mean k is odd. If the initial vertex v_0 and the terminal vertex v_k of a walk are the same, then we say that such a walk is closed. A **cycle** is a closed path i.e. initial and terminal vertices of the path are same.

A graph $\mathcal{G}$ is called a **bipartite graph** if $V(\mathcal{G})$ is partitioned into two disjoint sets A and B such that every edge of $\mathcal{G}$ joins a vertex of A to a vertex of B.

Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two graphs where $V(\mathcal{G}_1)$ is same as $V(\mathcal{G}_2)$. Then we define $\mathcal{G}_1 \circ \mathcal{G}_2 = (V(\mathcal{G}_1), E(\mathcal{G}_1) \circ E(\mathcal{G}_2))$, where $E(\mathcal{G}_1) \circ E(\mathcal{G}_2) = \{e \mid e \in E(\mathcal{G}_1) \text{ or } e \in E(\mathcal{G}_2)\}$ and $\mathcal{G}_1 \backslash \mathcal{G}_2 = (V(\mathcal{G}_1), E(\mathcal{G}_1) \backslash E(\mathcal{G}_2))$, where $E(\mathcal{G}_1) \backslash E(\mathcal{G}_2) = \{e \mid e \in E(\mathcal{G}_1) \text{ and } e \notin E(\mathcal{G}_2)\}$.

We recall the following two results which will be used in the proof of the main theorem.

Theorem 4.1 *(Petersen's 2-factor theorem).* [1, *Theorem* 3.1, *pg.* 70] *For every integer $r \geq 1$, every $2r$-regular general graph is 2-factorable. More generally, for every integer k, $1 \leq k \leq r$, every $2r$-regular general graph has a $2k$-regular factor.*

Theorem 4.2. [1, *Theorem* 2.2, *pg.* 18] *Every regular bipartite multigraph is 1-factorable, in particular, it has a 1-factor.*

In Section 6, our definition of 'degree of a vertex' differs from 'degree of a vertex' in [1]. The difference is because of the number of degrees contributed by a loop - in our case, a loop is counted once, however in [1], it is counted twice. Since we will be using the results of [1] directly, we make the following remark.

Remark 4.3. In [1], a general graph means a graph with multiple edges and loops where one loop contributes with degree 2 to a vertex incident to it. In our case, one loop contributes with degree 1 to the vertex incident to it. Consider a graph $\mathcal{G}$ with the vertex set $\{v_i : 1 \leq i \leq n\}$, with degree defined as in our case. If $\mathcal{G}$ has an even number of loops then the same number of loops at vertex v_i and vertex v_j can be paired up and joined together to get edges between v_i and v_j. Doing this procedure will result in an even number of loops remaining at a vertex. Now, any two loops at this vertex can be joined together to get a new loop. Now this loop contributes with degree 2 to the vertex. This will result in a graph in [1], without changing the degree of any vertices. For example, see Fig. 1.

Here the labels on the edges are their multiplicities. In Graph (a), there are four loops at the vertex v_5 and ten loops at the vertex v_6. Now we pair up four loops at vertex v_5 with four loops at vertex v_6 and this results in four edges between vertex v_5 and vertex v_6 (the dotted lines in Graph (b)). After doing this the remaining number of loops at vertex v_6 is six. So we pair up two loops together to get a new loop and so this results in three loops at the vertex v_6 (the dotted loops in Graph (b)), each of which contributes with degree 2 to the vertex v_6.

Note that the notion of a k-factor is preserved in the modification of the graph with respect to the different notions of degree.
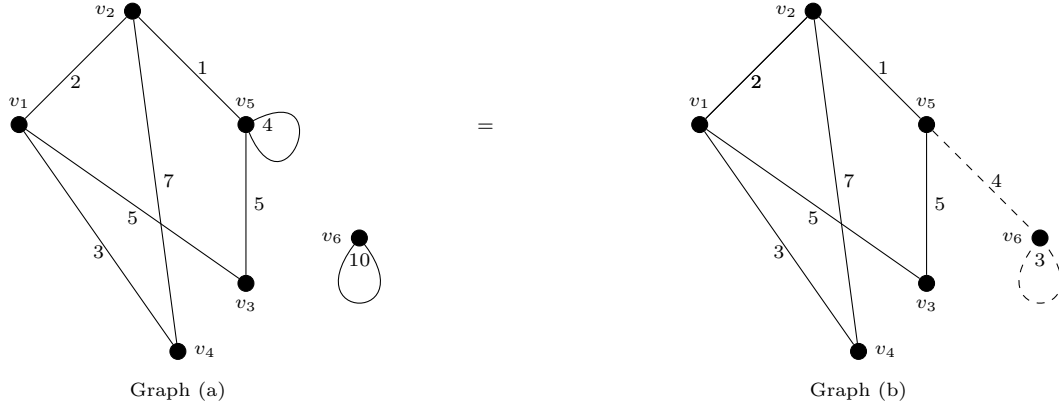
Fig. 1. In Graph (a) the loops at v_5 and v_6 are joined to get Graph (b).

5. Isomorphic torus GIT quotients

Let $G = SL_n(\mathbb{C})$ and T be a maximal torus of G. Let $\mathcal{L}_{\varpi_r}$ and $\mathcal{L}_{\varpi_{n-r}}$ be the line bundles associated to the fundamental weights ϖ_r and ϖ_{n-r} respectively. The projective varieties $G_{r,n}$ and $G_{n-r,n}$ are isomorphic. In the following proposition we show that their torus quotients are also isomorphic.

Proposition 5.1. *The GIT quotients* $T\backslash\backslash(G_{r,n})(\mathcal{L}_{n\varpi_r})$ *and* $T\backslash\backslash(G_{n-r,n})(\mathcal{L}_{n\varpi_{n-r}})$ *are isomorphic.*

Proof. Note that $n\varpi_r$ and $n\varpi_{n-r}$ are in the root lattice Q. So by [18, Theorem 3.10] the line bundle $\mathcal{L}_{n\varpi_r}$ (resp. $\mathcal{L}_{n\varpi_{n-r}}$) descends to the quotient $T\backslash\backslash(G_{r,n})_T^{ss}(\mathcal{L}_{\varpi_r})$ (resp. $T\backslash\backslash(G_{n-r,n})_T^{ss}(\mathcal{L}_{\varpi_{n-r}})$).

Let P_r and P_{n-r} be the maximal parabolic subgroups of G corresponding to the simple roots α_r and α_{n-r} respectively. Let $\mathcal{P}_r$ (resp. $\mathcal{P}_{n-r}$) denote the conjugacy class of P_r (resp. P_{n-r}) with respect to the conjugation action of G. Then there is an G-equivariant isomorphism between $G_{r,n}$ (resp. $G_{n-r,n}$) and the variety $\mathcal{P}_r$ (resp. $\mathcal{P}_{n-r}$).

There exists an outer automorphism $\phi : G \to G$ that sends P_r to P_{n-r}. Note that the outer automorphism comes from the non-trivial diagram automorphism of the Dynkin diagram of G. Hence the induced map $\phi_r : \mathcal{P}_r \to \mathcal{P}_{n-r}$, $H \mapsto \phi(H)$ is an isomorphism. This map ϕ_r is not G-equivariant but the actions of G on $\mathcal{P}_r$ and $\mathcal{P}_{n-r}$ are intertwined by ϕ. That is $\phi(gHg^{-1}) = \phi(g)\phi(H)\phi(g)^{-1}$. Note that $\phi(T) = T$.

Let $q : (\mathcal{P}_r)_T^{ss}(\mathcal{L}_{n\varpi_r}) \to T\backslash\backslash(\mathcal{P}_r)_T^{ss}(\mathcal{L}_{n\varpi_r})$ and $q' : (\mathcal{P}_{n-r})_T^{ss}(\mathcal{L}_{n\varpi_{n-r}}) \to T\backslash\backslash(\mathcal{P}_{n-r})_T^{ss}(\mathcal{L}_{n\varpi_{n-r}})$ be the quotient morphisms. Since $\phi_r^*(\mathcal{L}_{n\varpi_{n-r}}) = \mathcal{L}_{n\varpi_r}$ the map ϕ_r restricts to an isomorphism (we still call it ϕ_r)

$$(\mathcal{P}_r)_T^{ss}(\mathcal{L}_{n\varpi_r}) \to (\mathcal{P}_{n-r})_T^{ss}(\mathcal{L}_{n\varpi_{n-r}}).$$

Then the map $q' \circ \phi_r : (\mathcal{P}_r)_T^{ss}(\mathcal{L}_{n\varpi_r}) \to T\backslash\backslash(\mathcal{P}_{n-r})_T^{ss}(\mathcal{L}_{n\varpi_{n-r}})$ is a morphism. This map is also T-invariant.

So there exists a unique map $\psi : T\backslash\backslash(\mathcal{P}_r)_T^{ss}(\mathcal{L}_{n\varpi_r}) \to T\backslash\backslash(\mathcal{P}_{n-r})_T^{ss}(\mathcal{L}_{n\varpi_{n-r}})$ such that $\psi \circ q = q' \circ \phi_r$ and it follows that ψ is an isomorphism. $\square$

Remark. From the proof of the above theorem we conclude that the polarised varieties $(T\backslash\backslash(G_{r,n})_T^{ss}(\mathcal{L}_{n\varpi_r}), \mathcal{L}_{n\varpi_r})$ and $(T\backslash\backslash(G_{n-r,n})_T^{ss}(\mathcal{L}_{n\varpi_{n-r}}), \mathcal{L}_{n\varpi_{n-r}})$ are isomorphic as $\phi_r^*(\mathcal{L}_{n\varpi_{n-r}}) = \mathcal{L}_{n\varpi_r}$.

6. Projective normality of the torus quotient of Grassmannians

For the fundamental weight ϖ_r, $n\varpi_r \in Q$. So the line bundle $\mathcal{L}_{n\varpi_r}$ descends to the quotient $T\backslash\backslash(G_{r,n})_T^{ss}(\mathcal{L}_{n\varpi_r})$ (see [18]). In this section we prove that the quotient $T\backslash\backslash(G_{2,n})_T^{ss}(\mathcal{L}_{n\varpi_2})$ is projectively normal with respect to the descent of $\mathcal{L}_{n\varpi_2}$ using standard monomial theory and some graph

theoretic techniques and we give a degree bound of the generators of the homogeneous coordinate ring of $T\backslash\backslash(G_{3,6})_T^{ss}(\mathcal{L}_{2\varpi_3})$.

Theorem 6.1. *The GIT quotient $T\backslash\backslash(G_{r,n})_T^{ss}(\mathcal{L}_{n\varpi_r})$ is projectively normal with respect to the descent of $\mathcal{L}_{n\varpi_r}$ if $r = 1, 2, n-2, n-1$.*

Proof. For $r = 1$, $G_{r,n} \cong \mathbb{P}^{n-1}$ and hence the quotient $T\backslash\backslash\mathbb{P}^{n-1}(\mathcal{O}(n))$ is projectively normal.

Let $r = 2$. We have

$$T\backslash\backslash(G_{2,n})_T^{ss}(\mathcal{L}_{n\varpi_2}) = Proj(\oplus_{k\in\mathbb{Z}_{\geq 0}}H^0(G_{2,n}, \mathcal{L}_{n\varpi_2}^{\otimes k})^T) = Proj(\oplus_{k\in\mathbb{Z}_{\geq 0}}R_k),$$

where $R_k := H^0(G_{2,n}, \mathcal{L}_{n\varpi_2}^{\otimes k})^T$. Let $R := \oplus_{k\in\mathbb{Z}_{\geq 0}}R_k$. The $\mathbb{C}$-algebra R is normal since $\oplus_{k\in\mathbb{Z}_{\geq 0}}H^0(G_{2,n}, \mathcal{L}_{n\varpi_2}^{\otimes k})$ is normal. Hence it is enough to prove that R is generated by R_1 as a $\mathbb{C}$-algebra.

As a vector space the T-invariant standard monomials in Plücker coordinates of the form of $\prod_{i<j} p_{ij}^{m_{ij}}$ form a $\mathbb{C}$-basis of R_k, where $1 \leq i, j \leq n$. Note that since $\prod_{i<j} p_{ij}^{m_{ij}} \in R_k$ is a T-invariant standard monomial of degree k, by Lemma 3.1 we have $\sum_{j>i} m_{i,j} + \sum_{j<i} m_{j,i} = 2k$, for all $1 \leq i \leq n$ and $\sum_{1\leq i<j\leq n} m_{ij} = nk$.

Given a standard monomial $M = \prod_{i<j} p_{ij}^{m_{ij}}$ in Plücker coordinates we associate a multigraph as follows. For each $1 \leq i \leq n$ we associate a vertex v_i and for each p_{ij} appearing in M we associate an edge joining the vertex v_i to the vertex v_j. Similarly using the reverse process, from every multigraph, we can associate a monomial in Plücker coordinates. If moreover M is T-invariant then each of the indices $1 \leq i \leq n$ appears exactly $2k$ times in the monomial M. So each vertex in the graph is connected to exactly $2k$ number of edges. Hence it is a $2k$-regular graph.

Using Petersen's 2-factor theorem this graph can be decomposed into k line-disjoint 2-factors. Each 2-factor sub-graph is associated to a sub-tableau of the original tableau where each integer in $\{1, 2, \ldots, n\}$ appears exactly twice. The sub-tableau is standard as the original tableau was standard. So the standard monomials associated to the 2-factor sub-graphs lie in R_1. So by induction we conclude that each standard monomial in R_k can be written as a product of k standard monomials in R_1. So R is generated by R_1 as an algebra and hence the GIT quotient $T\backslash\backslash(G_{2,n})_T^{ss}(\mathcal{L}_{n\varpi_2})$ is projectively normal with respect to the descent of the line bundle $\mathcal{L}_{n\varpi_2}$.

For $r = n - 2$ and $n - 1$, the proof follows from Proposition 5.1. $\square$

Corollary 6.2. *The GIT quotient of a Schubert variety and a Richardson variety in $G_{2,n}$ and $G_{n-2,n}$ by a maximal torus T of SL_n is projectively normal with respect to the descent of the line bundles $\mathcal{L}_{n\varpi_2}$ and $\mathcal{L}_{n\varpi_{n-2}}$ respectively.*

Proof. Let X_w be a Schubert variety in $G_{2,n}$, $w \in W^{P_{\alpha_2}}$. Since T is linearly reductive, the restriction map $\phi : H^0(G_{2,n}, \mathcal{L}_{n\varpi_2}^{\otimes k})^T \to H^0(X_w, \mathcal{L}_{n\varpi_2}^{\otimes k})^T$ such that $f \mapsto f|_{X_w}$ is surjective. So by Theorem 6.1, $H^0(X_w, \mathcal{L}_{n\varpi_2})^T$ is generated by $H^0(X_w, \mathcal{L}_{n\varpi_2})^T$. Since $(X_w)_T^{ss}(\mathcal{L}_{n\varpi_2})$ is normal, $T\backslash\backslash(X_w)_T^{ss}(\mathcal{L}_{n\varpi_2})$ is projectively normal.

Let X_w^v be a Richardson variety in $G_{2,n}$, $v, w \in W^{P_{\alpha_2}}$. By [3, Proposition 1], the map $H^0(X_w, \mathcal{L}_{n\varpi_2}^{\otimes k}) \to H^0(X_w^v, \mathcal{L}_{n\varpi_2}^{\otimes k})$ is surjective. Since T is linearly reductive, the map $\phi : H^0(X_w, \mathcal{L}_{n\varpi_2}^{\otimes k})^T \to H^0(X_w^v, \mathcal{L}_{n\varpi_2}^{\otimes k})^T$ surjective. Since the quotient $T\backslash\backslash(X_w)_T^{ss}(\mathcal{L}_{n\varpi_2})$ is projectively normal and $(X_w^v)_T^{ss}(\mathcal{L}_{n\varpi_2})$ is normal, the quotient $T\backslash\backslash(X_w^v)_T^{ss}(\mathcal{L}_{n\varpi_2})$ is projectively normal.

The proof of the projective normality of the GIT quotient of a Schubert variety and a Richardson variety in $G_{n-2,n}$ is similar. $\square$

For $r \geq 3$, the combinatorics of the standard monomials in $\oplus_{k\in\mathbb{Z}_{\geq 0}}(H^0(G_{r,n}, \mathcal{L}_r^{\otimes k})^T)$ is complicated. So we restrict our case to $n = 6$. Again $\mathcal{L}_{\frac{6}{gcd(6,r)}\varpi_r}$ is the smallest line bundle on $G_{r,6}$ which descends to the quotient $T\backslash\backslash(G_{r,6})_T^{ss}(\mathcal{L}_{\frac{6}{gcd(6,r)}\varpi_r})$.

For $r = 1, 2, 4$ and 5 the quotient $T\backslash\backslash(G_{r,6})^{ss}_T(\mathcal{L}_{\frac{6}{gcd(6,r)}\varpi_r})$ is projectively normal with respect to the descent of the line bundle $\mathcal{L}_{\frac{6}{gcd(6,r)}\varpi_r}$. For $r = 1$, $G_{1,6} \cong \mathbb{P}^5$ and hence the quotient $T\backslash\backslash(\mathbb{P}^5)^{ss}_T(\mathcal{O}(6))$ is projectively normal. For $r = 2$, $T\backslash\backslash(G_{2,6})^{ss}_T(\mathcal{L}_{3\varpi_2})$ is projectively normal as proved in [10]. For $r = 4$ and 5 the quotient $T\backslash\backslash(G_{r,6})^{ss}_T(\mathcal{L}_{\frac{6}{gcd(6,r)}\varpi_r})$ is projectively normal by Proposition 5.1.

In the following theorem we give a degree bound of the generators of the homogeneous coordinate ring of the quotient $T\backslash\backslash(G_{3,6})^{ss}_T(\mathcal{L}_{2\varpi_3})$.

Theorem 6.3. *The homogeneous coordinate ring of the quotient $T\backslash\backslash(G_{3,6})^{ss}_T(\mathcal{L}_{2\varpi_3})$ is generated by elements of degree at most* 2.

Proof. We have

$$T\backslash\backslash(G_{3,6})^{ss}_T(\mathcal{L}_{2\varpi_3}) = Proj(\oplus_{k\in\mathbb{Z}_{\geq 0}}H^0(G_{3,6}, \mathcal{L}^{\otimes k}_{2\varpi_3})^T) = Proj(\oplus_{k\in\mathbb{Z}_{\geq 0}}R_k),$$

where $R_k = H^0(G_{3,6}, \mathcal{L}^{\otimes k}_{2\varpi_3})^T$. Let M be a standard monomial in Plücker coordinates in R_k. Then M is associated to a $2k \times 3$ tableau having each of the integers from 1 to 6 appearing exactly k times with strictly increasing rows and non-decreasing columns. Let Row_i denote the ith row of the tableau and Col_j denote the jth column of the tableau, where $1 \leq i \leq 2k$ and $1 \leq j \leq 3$. Let $E_{i,j}$ be the (i,j)-th entry of the tableau and $N_{t,j} = \#\{i|E_{i,j} = t\}$. Clearly,

$$\sum_t N_{t,j} = 2k \quad \text{and} \quad \sum_j N_{t,j} = k. \tag{6.1}$$

Note that $E_{i,1} = 1$ for all $1 \leq i \leq k$ and $E_{i,3} = 6$ for all $k + 1 \leq i \leq 2k$.

If $E_{1,2} = 4$ then $N_{t,1} = k$ for $1 \leq t \leq 3$, a contradiction. Similarly $E_{1,2}$ cannot be 5. So, Row_1 can be one of the elements from the set $\{(1,2,3), (1,2,4), (1,2,5), (1,3,4), (1,3,5)\}$.

If $\text{Row}_1 = (1,3,5)$ then we have $N_{2,1} = k$ and $E_{i,1} = 2$ for all $k + 1 \leq i \leq 2k$. In particular, we have $E_{2k,1} = 2$. Since $E_{1,3} = 5$, we have $N_{4,2} = k$ and $N_{3,2} = k$. So we have $E_{2k,2} = 4$. Hence we conclude that $\text{Row}_{2k} = (2,4,6)$. Then $p_{135}p_{246} \in R_1$ and divides M. So by induction we are done.

If $\text{Row}_1 = (1,3,4)$ then we have $E_{2k,1} = 2$. Since $E_{1,3} = 4$ we have $N_{5,2} \geq 1$. So $E_{2k,2} = 5$. Hence we conclude that $\text{Row}_{2k} = (2,5,6)$. Then $p_{134}p_{256} \in R_1$ and is a factor of M.

If $\text{Row}_1 = (1,2,5)$ then $E_{i,3} = 5$, for all $1 \leq i \leq k$. So $N_{5,2} = 0$ and $E_{2k,2} = 4$ and hence $E_{2k,1} = 2$ or 3. Since $N_{2,2} \geq 1$, we have $N_{2,1} \leq k - 1$ and hence $E_{2k,1} = 3$. So we conclude that $\text{Row}_{2k} = (3,4,6)$. Then $p_{125}p_{346} \in R_1$ and is a factor of M.

We are now left with two cases, either $\text{Row}_1 = (1,2,3)$ or $\text{Row}_1 = (1,2,4)$

Case - 1 $\text{Row}_1 = (1,2,4)$

Since $E_{1,3} = 4$ we have $N_{5,3} < k$. Since $N_{5,1} = 0$ it follows that $N_{5,2} \geq 1$ and hence, $E_{2k,2} = 5$. If $N_{4,1} = 0$ then $N_{3,1} \geq 1$. It follows that $\text{Row}_{2k} = (3,5,6)$. So the monomial $p_{124}p_{356} \in R_1$ and is a factor of M. If $N_{4,1} \geq 1$ then $E_{2k,1} = 4$ and hence $\text{Row}_{2k} = (4,5,6)$. We claim that $\text{Row}_k = (1,3,5)$.

(a) If $E_{k,2} = 2$ then we have $E_{i,2} = 2$ for all $1 \leq i \leq k$. Since $E_{1,3} = 4$ we have $N_{3,3} = 0$. Since $\text{Row}_{2k} = (4,5,6)$ we have $N_{3,1} + N_{3,2} < k$, a contradiction.

(b) If $E_{k,2} = 4$, then $(N_{4,1} + N_{4,2} + N_{4,3}) + (N_{5,2} + N_{5,3}) \geq 2k + 2$, which is a contradiction.

(c) For a similar reason we cannot have $E_{k,2} = 5$.

Hence, $E_{k,2} = 3$.

If $E_{k,3} = 4$ then $E_{i,3} = 4$ for all $1 \leq i \leq k$. So, $N_{4,1} + N_{4,3} \geq k + 1$, a contradiction. So we conclude that $\text{Row}_k = (1,3,5)$, the claim is proved.

Now we consider the entries $E_{i,2}$, where $1 \leq i \leq k$. Since $E_{1,2} = 2$ and $E_{k,2} = 3$ we have $E_{i,2} = 2$ or 3 for $1 \leq i \leq k$. Let $m_1 = \#\{i : E_{i,2} = 2, 1 \leq i \leq k\}$ and $m_2 = \#\{i : E_{i,2} = 3, 1 \leq i \leq k\}$. Then $m_1, m_2 \geq 1$ and $m_1 + m_2 = k$.

Subcase - 1. $m_1 = m_2 = \frac{k}{2}$.

(a) If $N_{4,3} = \frac{k}{2}$ then $\text{Row}_i = (1,2,4)$ for all $1 \leq i \leq \frac{k}{2}$ and $\text{Row}_i = (1,3,5)$ for all $\frac{k}{2} + 1 \leq i \leq k$. Then the monomial M is $p_{124}^{\frac{k}{2}} p_{135}^{\frac{k}{2}} p_{236}^q p_{246}^{\frac{k}{2}-q} p_{356}^{\frac{k}{2}-q} p_{456}^q$ with $q \geq 1$.

If $q < \frac{k}{2}$ then M has a factor $p_{124}p_{356} \in R_1$.

If $q = \frac{k}{2}$ then the monomial M is $(p_{124}p_{135}p_{236}p_{456})^{\frac{k}{2}}$. Then $p_{124}p_{135}p_{236}p_{456} \in R_2$ and is a factor of M.

(b) If $N_{4,3} < \frac{k}{2}$ then $N_{5,3} > \frac{k}{2}$ and so $E_{\frac{k}{2},3} = 5$. Hence, $\text{Row}_{\frac{k}{2}} = (1,2,5)$. Since $N_{5,3} > \frac{k}{2}$ we have $N_{5,2} < \frac{k}{2}$ and since $N_{2,1} = \frac{k}{2}$ we have $E_{i,1} = 2$ for all $k+1 \leq i \leq \frac{3k}{2}$ and $2 \notin \text{Row}_{\frac{3k}{2}+1}$. Since $N_{5,2} < \frac{k}{2}$ we have $5 \notin E_{\frac{3k}{2}+1,2}$ and hence $\text{Row}_{\frac{3k}{2}+1} = (3,4,6)$. So the monomial $p_{125}p_{346} \in R_1$ and is a factor of M.

(c) If $N_{4,3} > \frac{k}{2}$ then $\text{Row}_{\frac{k}{2}+1} = (1,3,4)$. Now using a similar argument as (b) we get $p_{134}p_{256} \in R_1$ and is a factor of M.

Subcase - 2. Let $m_1 \neq m_2$.

Let $m_1 > m_2$. Note that $m_1 > \frac{k}{2}$.

(a) If $N_{4,3} = m_1$ then $\text{Row}_i = (1,2,4)$ for all $1 \leq i \leq m_1$ and $\text{Row}_i = (1,3,5)$ for all $m_1 + 1 \leq i \leq k$ i.e. $N_{5,3} = m_2$, $N_{5,2} = m_1$ and $N_{2,1} = m_2$. Hence, $N_{3,2} < 2m_2 < k$ and it follows that $N_{3,1} > 1$. So $\text{Row}_{k+m_2+1} = (3,5,6)$ and hence the monomial $p_{124}p_{356} \in R_1$ and is a factor of M.

(b) If $N_{4,3} > m_1$ then $\text{Row}_{m_1+1} = (1,3,4)$ and $N_{5,3} < m_2$. Hence $N_{5,2} > m_1$ and $\text{Row}_{k+m_2} = (2,5,6)$. So the monomial $p_{134}p_{256} \in R_1$ and is a factor of M.

(c) If $N_{4,3} < m_1$ then using a similar argument as (b) we get $p_{125}p_{346} \in R_1$ and is a factor of M.

The proof for the case $m_1 < m_2$ is similar.

Case - 2 $\text{Row}_1 = (1,2,3)$

In this case since $E_{1,3} = 3$ we have $N_{5,3} < k$. Further since $N_{5,1} = 0$ we have $N_{5,2} \geq 1$. Therefore, we have $E_{2k,2} = 5$. Since $N_{2,2} > 1$ we have $N_{2,1} < k$. Hence, $E_{2k,1} = 3$ or 4.

If $E_{2k,1} = 4$ then $\text{Row}_{2k} = (4,5,6)$ and so the monomial $p_{123}p_{456} \in R_1$ and is a factor of M.

If $E_{2k,1} = 3$ then $\text{Row}_{2k} = (3,5,6)$. We claim that $\text{Row}_k = (1,4,5)$.

(a) If $E_{k,2} = 2$ then we have $E_{i,2} = 2$ for all $1 \leq i \leq k$. Then $E_{i,1} = 3$ for all $k+1 \leq i \leq 2k$ and since $E_{1,3} = 3$ we have $N_{3,1} + N_{3,3} > k$, a contradiction.

(b) If $E_{k,2} = 3$ then $(N_{3,1} + N_{3,2} + N_{3,3}) + (N_{2,1} + N_{2,2}) \geq 2k+1$, a contradiction.

(c) If $E_{k,2} = 5$ then $E_{k,3} = 6$, a contradiction.

Hence, $E_{k,2} = 4$ and we conclude that $\text{Row}_k = (1,4,5)$.

Now we claim that $\text{Row}_{k+1} = (2,4,6)$.

Since $N_{3,3} > 1$ we have $N_{3,1} < k$ and hence, $E_{k+1,1} = 2$. Since $E_{k,2} = 4$ we have $E_{k+1,2} = 4$ or 5. If $E_{k+1,2} = 5$ then $N_{5,2} + N_{5,3} > k$, a contradiction. Hence, $E_{k+1,2} = 4$. Therefore, $\text{Row}_{k+1} = (2,4,6)$. So the monomial $p_{123}p_{145}p_{246}p_{356} \in R_2$ and is a factor of M.

So by induction we conclude that M is generated by the elements of degree at most 2 and hence the homogeneous coordinate ring of the quotient $T \backslash\backslash (G_{3,6})_T^{ss}(\mathcal{L}_{2\varpi_3})$ is generated by elements of degree at most 2. $\quad \square$

7. Torus quotient of partial flag varieties

Let $G = SL_n$ and ϖ_1, ϖ_2 be the fundamental weights associated to the simple roots α_1 and α_2 respectively. Let $P = P_{\alpha_1} \cap P_{\alpha_2}$. Since $n(r_1\varpi_1 + r_2\varpi_2) \in Q$ for $r_1, r_2 \in \mathbb{N}$, the line bundle $\mathcal{L}_{n(r_1\varpi_1 + r_2\varpi_2)}$ descends to the quotient $T \backslash\backslash (G/P)(\mathcal{L}_{n(r_1\varpi_1 + r_2\varpi_2)})$ (see [18]). In this section we prove that the quotient $T \backslash\backslash (G/P)_T^{ss}(\mathcal{L}_{n(r_1\varpi_1 + r_2\varpi_2)})$ is projectively normal with respect to the descent of the line bundle $\mathcal{L}_{n(r_1\varpi_1 + r_2\varpi_2)}$.

Remark 7.1. Any 2-regular graph is a disjoint union of (i) even cycles, (ii) odd cycles, (iii) even length paths starting with a loop and ending with a loop, (iv) odd length paths starting with a loop and ending with a loop, and (v) vertices with two loops.

Proof. Recall that in our case a loop contributes with degree 1 to a vertex. It is well known that a 2-regular connected simple graph is a cycle [1, pg. 83]. If the graph is not simple then it may have loops and multiple edges. If it has multiple edges then at least two of the vertices are connected by two edges, hence, it is a 2-cycle. If the graph has a loop at a vertex v then either v has another loop around it or it is connected to another vertex w by an edge. In the later case w may have another loop around it or connected to another vertex u by an edge. In the former case the graph is an odd length path starting with a loop and ending with a loop and continuing this process we get either an even length path starting with a loop and ending with a loop or an odd length path starting with a loop and ending with a loop. $\square$

Now we are in a position to state and prove the main theorem of this section.

Theorem 7.2. *Let $G = SL_n$ and $P = P_{\alpha_1} \cap P_{\alpha_2}$, $\varpi = r_1\varpi_1 + r_2\varpi_2$. The GIT quotient $T\backslash\backslash(G/P)_T^{ss}(\mathcal{L}_{n\varpi})$ is projectively normal with respect to the descent of the line bundle $\mathcal{L}_{n\varpi}$.*

Proof. Note that $T\backslash\backslash(G/P)_T^{ss}(\mathcal{L}_{n\varpi}) = Proj(\oplus_{k\in\mathbb{Z}_{\geq 0}} H^0(G/P, \mathcal{L}_{n\varpi}^{\otimes k})^T) = Proj(\oplus_{k\in\mathbb{Z}_{\geq 0}} R_k)$, where $R_k = H^0(G/P, \mathcal{L}_{n\varpi}^{\otimes k})^T$. The algebra $R = \oplus_{k\in\mathbb{Z}_{\geq 0}} R_k$ is normal. Here we use induction on k to prove that R is generated by R_1 as a $\mathbb{C}$-algebra. We set $s = r_1 + 2r_2$, $l_1 = n(r_1 + r_2)$ and $l_2 = nr_2$.

Let $f = \prod_{t=1}^{kl_2} p_{i_t j_t} \prod_{t=kl_2+1}^{kl_1} p_{m_t} \in R_k$ be a T-invariant standard monomial in the Plücker coordinates. We associate a graph $\mathcal{G}_f$ corresponding to f as follows:

(a) for each integer $1 \leq i \leq n$, associate a vertex v_i,
(b) for each p_{ij} appearing in f, associate an edge between v_i and v_j, and
(c) for each p_k appearing in f, associate a loop at the vertex v_k.

Similarly, using the reverse process, we can associate a monomial $f_\mathcal{G}$ in Plücker coordinates with a graph $\mathcal{G}$.

For $f \in R_k$, the associated graph $\mathcal{G}_f$ has total $k(l_1 - l_2) = knr_1$ number of loops. Since $f \in R_k$, it is T invariant and so each of the indices $1 \leq i \leq n$ appears exactly ks-times in the monomial f. This results in all the vertices of $\mathcal{G}_f$ having the same degree. Thus $\mathcal{G}_f$ is ks-regular.

Here we introduce some operations on the graphs which are induced by the operations on the monomials corresponding to the graphs:

$$\begin{aligned}
\mathcal{G} &= \mathcal{G}_1 + \mathcal{G}_2 \quad \text{if } f_\mathcal{G} = f_{\mathcal{G}_1} + f_{\mathcal{G}_2}, \\
\mathcal{G} &= \mathcal{G}_1 - \mathcal{G}_2 \quad \text{if } f_\mathcal{G} = f_{\mathcal{G}_1} - f_{\mathcal{G}_2}, \\
\mathcal{G} &= \mathcal{G}_1 \circ \mathcal{G}_2 \quad \text{if } f_\mathcal{G} = f_{\mathcal{G}_1}.f_{\mathcal{G}_2}, \\
\mathcal{G} &= \mathcal{G}_1 \backslash \mathcal{G}_2 \quad \text{if } f_\mathcal{G} = f_{\mathcal{G}_1}/f_{\mathcal{G}_2},
\end{aligned}$$

where $f_\mathcal{G}$ and $f_{\mathcal{G}_i}$ are the monomials associated to the graphs $\mathcal{G}$ and $\mathcal{G}_i$ respectively, for $i = 1, 2$.

We proceed case by case and in each case we first show that $\mathcal{G}_f$ is a linear combination of ks-regular graphs and from each summand we get a s-factor.

Case 1: r_1 is even.

In this case s is even and the number of loops in the graph is even. So by Theorem 4.1 and Remark 4.3, $\mathcal{G}_f$ has a s-factor.

Case 2: r_1 is odd.

In this case s is odd. We consider two cases, k is even and k is odd.

<u>k is even.</u>

In this subcase the number of loops in the graph is even. So by Theorem 4.1, $\mathcal{G}_f$ can be factored into $\frac{ks}{2}$ number of 2-factors. We make the following claim.
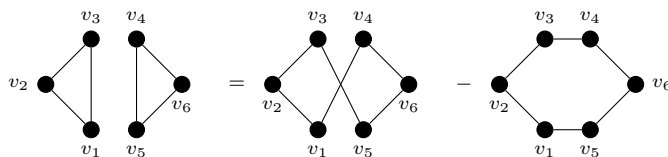
Fig. 2. Plücker relation on the edges (v_1, v_3) and (v_4, v_5).

Claim 1. *One of the 2-factors can be written as a linear combination of 2-regular graphs such that from each of the summands we can extract a 1-factor.*

Since $\mathcal{G}_f$ had at least two loops, by Remark 7.1, one of the 2-factors also has at least two loops. Denote this particular 2-factor (with at least two loops) by $\mathcal{G}_{f^{(2)}}$. We denote other 2-factors by $\mathcal{G}_{f^1}, \mathcal{G}_{f^2}, \ldots, \mathcal{G}_{f^{\frac{ks}{2}-1}}$.

Since $\mathcal{G}_{f^{(2)}}$ is a 2-regular graph, using Remark 7.1, $\mathcal{G}_{f^{(2)}}$ is a disjoint union of (i) even cycles, (ii) odd cycles, (iii) even length paths starting with a loop and ending with a loop, (iv) odd length paths starting with a loop and ending with a loop, and (v) vertices with two loops.

Now, to get the graph free of odd cycles we merge two odd cycles together by taking one edge from each and apply Plücker relations on them.

In the following example we use the Plücker relation $p_{13}p_{45} = p_{14}p_{35} - p_{15}p_{34}$ on the edges (v_1, v_3) and (v_4, v_5) to merge two odd cycles (see Fig. 2).

Repeating this process we write $\mathcal{G}_{f^{(2)}} = \sum_{i=1}^{p} a_i \mathcal{G}_{f_i^{(2)}}$, where $a_i \in \mathbb{Z}$ and each $\mathcal{G}_{f_i^{(2)}}$ is a 2-regular graph which is a disjoint union of (i) even cycles, (ii) even length paths starting with a loop and ending with a loop, (iii) odd length paths starting with a loop and ending with a loop, (iv) vertices with two loops and (v) **possibly one odd cycle**.

(a) Suppose $\mathcal{G}_{f_i^{(2)}}$ has no odd cycle. We can extract a 1-factor from $\mathcal{G}_{f_i^{(2)}}$ in the following ways:

If $\mathcal{G}_{f_i^{(2)}}$ has an even cycle as a component it can be factored into two 1-factors by taking every alternate edge.

If $\mathcal{G}_{f_i^{(2)}}$ has an even length path starting with a loop and ending with a loop as a component we pick a loop and every alternate edge to get a 1-factor.

If $\mathcal{G}_{f_i^{(2)}}$ has an odd length path stating with a loop and ending with a loop as a component we pick up the two loops and every alternate edge to get a 1-factor.

If $\mathcal{G}_{f_i^{(2)}}$ has a vertex with two loops as a component we take one loop from it.

(b) Suppose $\mathcal{G}_{f_i^{(2)}}$ has an odd cycle. Since $\mathcal{G}_{f^{(2)}}$ has at least two loops, $\mathcal{G}_{f_i^{(2)}}$ will also have at least two loops. To get the graph free of the odd cycle we choose an edge (v_i, v_j) in the odd cycle, and a loop (v_k, v_k) (w.l.o.g. $\{i, j, k : i < j < k\}$) and apply the Plücker relation $p_{ij}p_k = p_{ik}p_j - p_{jk}p_i$.

We may take an odd cycle and one of the components of the following types to apply Plücker relation:

a. Vertex with two loops.
b. Even length path starting with a loop and ending with a loop.
c. Odd length path starting with a loop and ending with a loop.

In the following examples the Plücker relation $p_{13}p_4 = p_{14}p_3 - p_{34}p_1$ is applied on the edge (v_1, v_3) and the loop (v_4, v_4) (see Fig. 3).

After doing this we write $\mathcal{G}_{f_i^{(2)}} = \sum_{k=1}^{m_i} b_{i_k} \mathcal{G}_{f_{i_k}^{(2)}}$, where $b_{i_k} \in \mathbb{Z}$ and each $\mathcal{G}_{f_{i_k}^{(2)}}$ is a 2-regular graph which is a disjoint union of (i) even cycles, (ii) even length paths starting with a loop and ending with a loop, (iii) odd length paths starting with a loop and ending with a loop, (iv) vertices with two loops. So, from each of the components of $\mathcal{G}_{f_{i_k}^{(2)}}$ we can extract a 1-factor $\mathcal{G}_{f_{i_k,1}^{(2)}}$ as explained in (a).
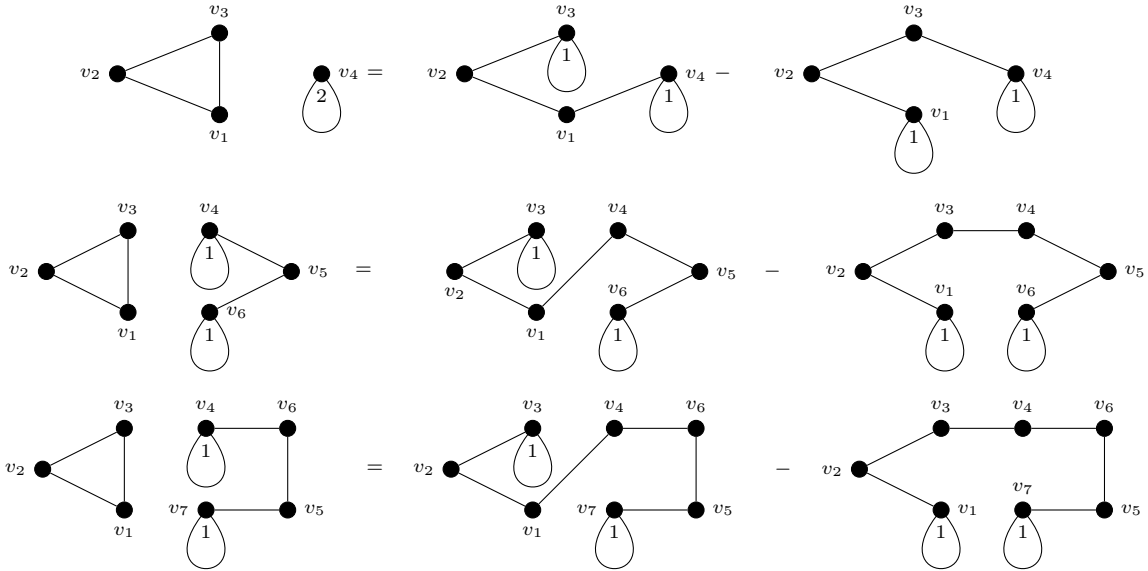
Fig. 3. Plücker relation on the edge (v_1, v_3) and the loop (v_4, v_4).

We have $\mathcal{G}_f = (\mathcal{G}_{f^1} \circ \mathcal{G}_{f^2} \circ \cdots \circ \mathcal{G}_{f^{\frac{ks}{2}-1}}) \circ \mathcal{G}_{f^{(2)}} = (\mathcal{G}_{f^1} \circ \mathcal{G}_{f^2} \circ \cdots \circ \mathcal{G}_{f^{\frac{ks}{2}-1}}) \circ \sum_{i=1}^{p} \sum_{k=1}^{m_i} a_i b_{i_k} \mathcal{G}_{f_{i_k}^{(2)}} = \sum_{i=1}^{p} \sum_{k=1}^{m_i} a_i b_{i_k} (\mathcal{G}_{f^1} \circ \mathcal{G}_{f^2} \circ \cdots \circ \mathcal{G}_{f^{\frac{ks}{2}-1}}) \circ \mathcal{G}_{f_{i_k}^{(2)}} = \sum_{i=1}^{p} \sum_{k=1}^{m_i} a_i b_{i_k} \mathcal{G}_{f_{i_k}}{}''$, where each $\mathcal{G}_{f_{i_k}}{}''$ is a ks-regular graph and for each $\mathcal{G}_{f_{i_k}}{}''$ we get a s-factor by combining $\mathcal{G}_{f^1} \circ \mathcal{G}_{f^2} \circ \cdots \circ \mathcal{G}_{f^{\frac{s-1}{2}}}$ with the 1-factor $\mathcal{G}_{f_{i_k,1}^{(2)}}$ which corresponds to the monomial $f^1 \cdot f^2 \cdots f^{\frac{s-1}{2}} \cdot f_{i_k,1}^{(2)}$.

k is odd and n is even.

In this case $\mathcal{G}_f$ is ks-regular with an even number of vertices and an even number of loops. We form a new graph $\mathcal{G}_{\tilde{f}}$ by doubling the vertex set: for each vertex v_i we associate two vertices M_i and N_i, i.e., the vertex set of $\mathcal{G}_{\tilde{f}}$ is:

$$\text{Vert}(\mathcal{G}_{\tilde{f}}) = \{M_1, \ldots, M_n, N_1, \ldots, N_n\}.$$

For each edge (v_i, v_j) of $\mathcal{G}$, we associate two edges (M_i, N_j) and (M_j, N_i) in $\mathcal{G}_{\tilde{f}}$. For each loop (v_i, v_i), we associate an edge (M_i, N_i) in $\mathcal{G}_{\tilde{f}}$. Note that $\mathcal{G}_{\tilde{f}}$ is ks-regular and bipartite between M and N. So, by Theorem 4.2, it has a 1-factor, say $\tilde{\Delta}$, in $\mathcal{G}_{\tilde{f}}$.

From $\tilde{\Delta}$ we construct another graph Δ as follows:

(a) Δ has n vertices, denoted by $\{1, 2, \ldots, n\}$.
(b) For each edge (M_i, N_j) in $\tilde{\Delta}$, we associate an edge (i, j) in Δ.
(c) For each edge (M_i, N_i) in $\tilde{\Delta}$, we associate a loop (i, i) in Δ.

Note that the loops are disjoint components in Δ and the remaining graph ($\Delta \backslash \{\text{vertices having loops}\}$) is 2-regular consisting of cycles. However, we may have both (M_i, N_j) and (M_j, N_i) are edges of $\tilde{\Delta}$. This may result in two occurrences of the edge (i, j) in Δ but only one in $\mathcal{G}_f$. So, this type of component is a 2-cycle. Note that $\Delta \backslash \{2\text{-cycles}\}$ is a subgraph of $\mathcal{G}_f$. If we pick 1-factor from each of the 2-cycles then the graph obtained by taking union of $\Delta \backslash \{2\text{-cycles}\}$ and the chosen 1-factors of 2-cycles is a spanning subgraph of $\mathcal{G}_f$ and we denote it by $\mathcal{G}_{f'}$. Now we apply Plücker relations to write $\mathcal{G}_{f'}$ as a linear combination of graphs such that from each of the summands we can extract a 1-factor in the following way and in each case we get a s-factor from each of the summands of $\mathcal{G}_f$.

(1) If $\mathcal{G}_{f'}$ has some loops then we consider two cases:

(a) If $\mathcal{G}_{f'}$ has an even number of odd cycles then $\mathcal{G}_{f'}$ has an even number of loops. Now we use Plücker relations repeatedly to merge two odd cycles into an even cycle and write $\mathcal{G}_{f'} = \sum_i a_i \mathcal{G}_{f'_i}$, $a_i \in \mathbb{Z}$, where $\mathcal{G}_{f'_i}$ is a disjoint union of even cycles, loops and 1-factors of 2-cycles of $\mathcal{G}_{f'}$. Now we can extract a 1-factor $\mathcal{G}_{f'_{i,1}}$ from each $\mathcal{G}_{f'_i}$ as explained in (a) of the case $\underline{k \text{ is even}}$.

$\mathcal{G}_f = (\mathcal{G}_f \backslash \mathcal{G}_{f'}) \circ \mathcal{G}_{f'} = (\mathcal{G}_f \backslash \mathcal{G}_{f'}) \circ \sum_i a_i \mathcal{G}_{f'_i} = \sum_i a_i ((\mathcal{G}_f \backslash \mathcal{G}_{f'}) \circ \mathcal{G}_{f'_i}) = \sum_i a_i \mathcal{G}_{f_i''}$, where each $\mathcal{G}_{f_i''}$ is a ks-regular graph and for each $\mathcal{G}_{f_i''}$ we get a s-factor by combining any $\frac{s-1}{2}$ number of 2-factors of $(\mathcal{G}_f \backslash \mathcal{G}_{f'}) \circ (\mathcal{G}_{f'_i} \backslash \mathcal{G}_{f'_{i,1}})$ (which is a $ks-1$-regular graph with an even number of loops) with the 1-factor $\mathcal{G}_{f'_{i,1}}$.

(b) If $\mathcal{G}_{f'}$ has an odd number of odd cycles then $\mathcal{G}_{f'}$ also has an odd number of loops. Now we use Plücker relations repeatedly to merge two odd cycles into an even cycle and write $\mathcal{G}_{f'} = \sum_i a_i \mathcal{G}_{f'_i}$, $a_i \in \mathbb{Z}$ where $\mathcal{G}_{f'_i}$ is a disjoint union of even cycles, loops, one odd cycle and 1-factors of 2-cycles of $\mathcal{G}_{f'}$.

Since $\mathcal{G}_{f'}$ has at least one loop, each $\mathcal{G}_{f'_i}$ also has at least one loop. So to get $\mathcal{G}_{f'_i}$ free of the odd cycle we apply Plücker relation on an edge of the odd cycle and one of the loops to write $\mathcal{G}_{f'_i} = \sum_k b_{i_k} \mathcal{G}_{f'_{i_k}}$, $b_{i_k} \in \mathbb{Z}$, where $\mathcal{G}_{f'_{i_k}}$ is a disjoint union of even cycles, loops (even in number) and one odd path starting with a loop and 1-factors of 2-cycles of $\mathcal{G}_{f'}$. We now extract a 1-factor from the odd path by taking alternate edges and we extract 1-factors from the other components of the linear combination as explained in (a) of the case $\underline{k \text{ is even}}$. Thus we extract a 1-factor $\mathcal{G}_{f'_{i_k,1}}$ from each $\mathcal{G}_{f'_{i_k}}$.

In this case $\mathcal{G}_f$ can be written as a linear combination of ks-regular graphs and for each of the summands in $\mathcal{G}_f$ we can get a s-factor as explained in (1)(a).

(2) If $\mathcal{G}_{f'}$ does not contain any loop then since the number of vertices is even, there are an even number of odd cycles in $\mathcal{G}_{f'}$. Now we use Plücker relations repeatedly to merge two odd cycles into an even cycle and write $\mathcal{G}_{f'} = \sum_i a_i \mathcal{G}_{f'_i}$, $a_i \in \mathbb{Z}$ where $\mathcal{G}_{f'_i}$ is a disjoint union of even cycles and 1-factors of 2-cycles of $\mathcal{G}_{f'}$. We then extract a 1-factor from each $\mathcal{G}_{f'_i}$ as explained in (a) of the case $\underline{k \text{ is even}}$.

So, $\mathcal{G}_f$ can be written as a linear combination of ks-regular graphs and for each of the summands in $\mathcal{G}_f$ we can get a s-factor as explained in (1)(a).

$\underline{k \text{ **and** } n \text{ **both are odd.**}}$

In this case $\mathcal{G}_f$ is a ks-regular graph with an odd number of loops. As in the case where k is odd and n is even in this case also we get a bipartite graph, with bipartitions M and N, which is 1-factorable. Note that one of the factors contains an odd number of edges of type (M_i, N_i). So the associated graph Δ contains an odd number of loops. Note that $\Delta \backslash \{2\text{-cycles}\}$ is a subgraph of $\mathcal{G}_f$. If we pick a 1-factor from each of the 2-cycles then the graph obtained by taking union of $\Delta \backslash \{2\text{-cycles}\}$ and the chosen 1-factors of 2-cycles is a spanning subgraph of $\mathcal{G}_f$ and we denote it by $\mathcal{G}_{f'}$. Note that $\mathcal{G}_{f'}$ is a disjoint union of an even number of odd cycles, even cycles, an odd number of loops and 1-factors of 2-cycles. Now we apply Plücker relations repeatedly to merge two odd cycles into an even cycle and write $\mathcal{G}_{f'} = \sum a_i \mathcal{G}_{f'_i}$, where $a_i \in \mathbb{Z}$ and each $\mathcal{G}_{f'_i}$ is a disjoint union of even cycles, loops and 1-factors of 2-cycles. Then we can extract a 1-factor $\mathcal{G}_{f'_{i,1}}$ from each $\mathcal{G}_{f'_i}$ as explained in (a) of the case $\underline{k \text{ is even}}$.

In this case also $\mathcal{G}_f$ can be written as a linear combination of ks-regular graphs and for each of the summands in $\mathcal{G}_f$ we can get a s-factor as explained in (1)(a).

Now using the s-factors that we have obtained in each of the above cases we will get a s-factor with nr_1 number of loops with which the associated monomial lies in R_1.

We interchange loops and edges between the s-factor and the $(k-1)s$-factor without interchanging the degree of the vertices so that the monomial associated to the new s-factor lies in R_1. For example, (1) if (v_i, v_i) and (v_j, v_j) are two loops in the s-factor and (v_i, v_j) is an edge in the $(k-1)s$-factor then we can interchange them and (2) if $\{(v_i, v_j), (v_k, v_k), (v_l, v_l)\}$ is a set of an edge and two loops in the s-factor and $\{(v_i, v_k), (v_j, v_l)\}$ is a set of edges in the $(k-1)s$-factor then we can interchange them. We shall do this interchange for all possible loops and edges in the s-factor and the $(k-1)s$-factor.

If interchange between loops and edges is not possible then we use Plücker relation on the factors repeatedly (possibly multiple times) to get a set of graphs where interchange between loops and edges is possible.
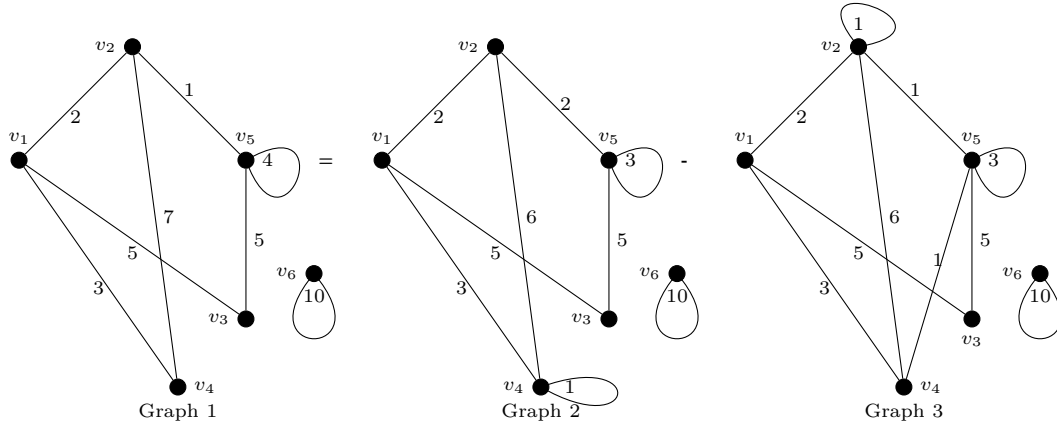
Fig. 4. Plücker relation on the edge (v_2, v_4) and the loop (v_5, v_5).

We use Plücker relations to shift a loop at a vertex to another vertex in such a way that a set of edges and loops can be interchanged with another set of edges. Hence, we always end up with a situation where interchanges between loops and edges are possible. The Plücker relation on the edge (v_i, v_j) and the loop (v_k, v_k) (w.l.o.g. we take $\{i, j, k : i < j < k\}$), is $p_{ij}p_k = p_{ik}p_j - p_{jk}p_i$, and the Plücker relation on two edges (v_i, v_j) and (v_k, v_l) (w.l.o.g. $\{i, j, k, l : i < j < k < l\}$), is $p_{ij}p_{kl} = p_{ik}p_{jl} - p_{il}p_{jk}$. We illustrate this possibility by an example given below (see Fig. 4).

Now using induction on the number of loops we get a s-factor with which the associated monomial lies in R_1.

So we conclude that R is generated by R_1. Hence, the quotient $T\backslash\backslash(G/P)^{ss}_T(\mathcal{L}_{n\varpi})$ is projectively normal with respect to the descent of the line bundle $\mathcal{L}_{n\varpi}$. $\quad\square$

Example. Here we give an example where interchange between loops and edges in the above theorem is not possible. Then we use Plücker relation on the factors to get a set of graphs where interchange between loops and edges is possible. Let us consider $\lambda = 6(\varpi_1 + 2\varpi_2)$ and $k = 3$. So $s = 5$ and $nr_1 = 6$. After using the above procedure, suppose we get a 5-factor with which the associated monomial is

$$p_{12}^2 p_{14}^3 p_{24}^2 p_{25} p_{35}^4 p_{36} p_6^4 \text{ (Graph 4)},$$

and another graph which is a 10-factor, with which the associated monomial is

$$p_{12}^2 p_{13}^5 p_{14}^3 p_{24}^7 p_{25} p_{35}^5 p_5^4 p_6^{10} \text{ (Graph 1)}.$$

Here directly we can not interchange loops and edges between the factors. So we apply Plücker relation on the edge (v_2, v_4) and the loop (v_5, v_5) in Graph 1, and obtain

$$p_{12}^2 p_{13}^5 p_{14}^3 p_{24}^7 p_{25} p_{35}^5 p_5^4 p_6^{10} = p_{12}^2 p_{13}^5 p_{14}^3 p_{24}^6 p_{25}^2 p_{35}^5 p_5^3 p_4 p_6^{10} - p_{12}^2 p_{13}^5 p_{14}^3 p_{24}^6 p_{25} p_{35}^5 p_{45} p_2 p_5^3 p_6^{10}.$$

The graph associated with the monomial $p_{12}^2 p_{13}^5 p_{14}^3 p_{24}^6 p_{25}^2 p_{35}^5 p_5^3 p_4 p_6^{10}$ is in Graph 2, and the graph associated with the monomial $p_{12}^2 p_{13}^5 p_{14}^3 p_{24}^6 p_{25} p_{35}^5 p_{45} p_2 p_5^3 p_6^{10}$ is in Graph 3.

Now we can do the interchange as follows:

(a) Interchange the pair of edges $\{(v_1, v_4), (v_3, v_6)\}$ in Graph 4 with the set $\{(v_1, v_3), (v_4, v_4), (v_6, v_6)\}$ in Graph 2, and obtain the graph in Graph 5 with which the associated monomial $p_{12}^2 p_{13} p_{14}^4 p_{24}^2 p_{25} p_{35}^5 p_4 p_6^5$ lies in R_1 and the graph in Graph 6 with which the associated monomial lies in R_2 (see Fig. 5).
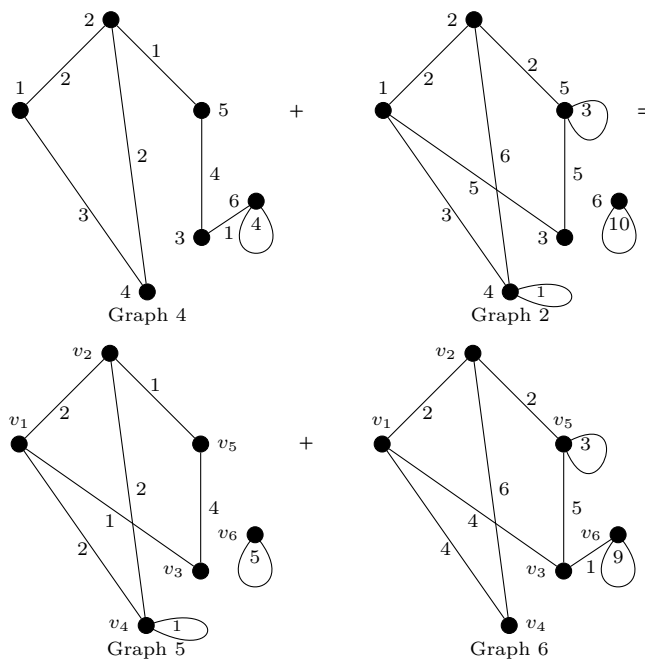
Fig. 5. The pair of edges $\{(v_1, v_4), (v_3, v_6)\}$ in Graph 4 are interchanged with the set $\{(v_1, v_3), (v_4, v_4), (v_6, v_6)\}$ in Graph 2.
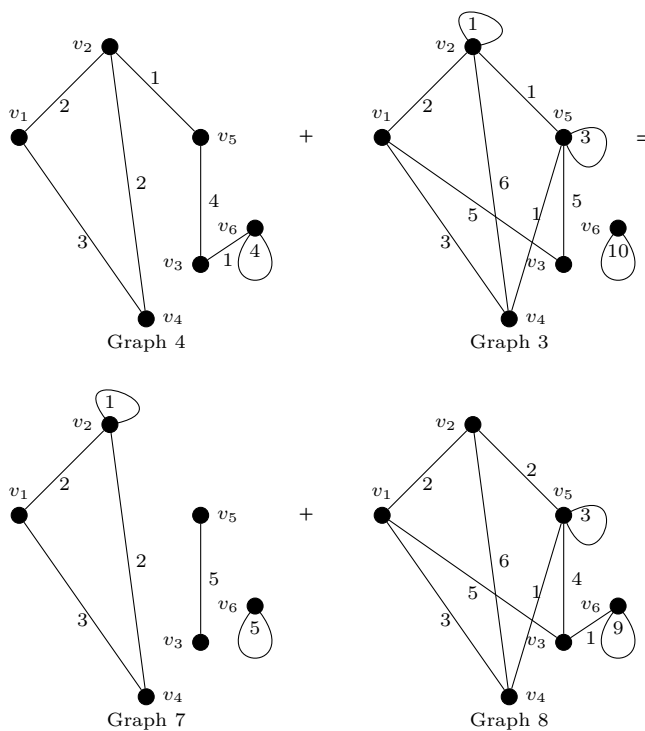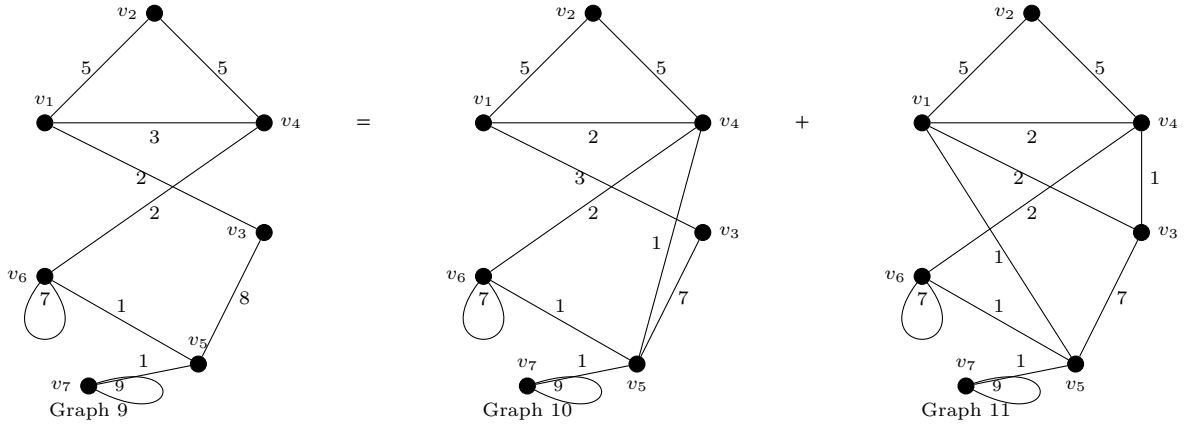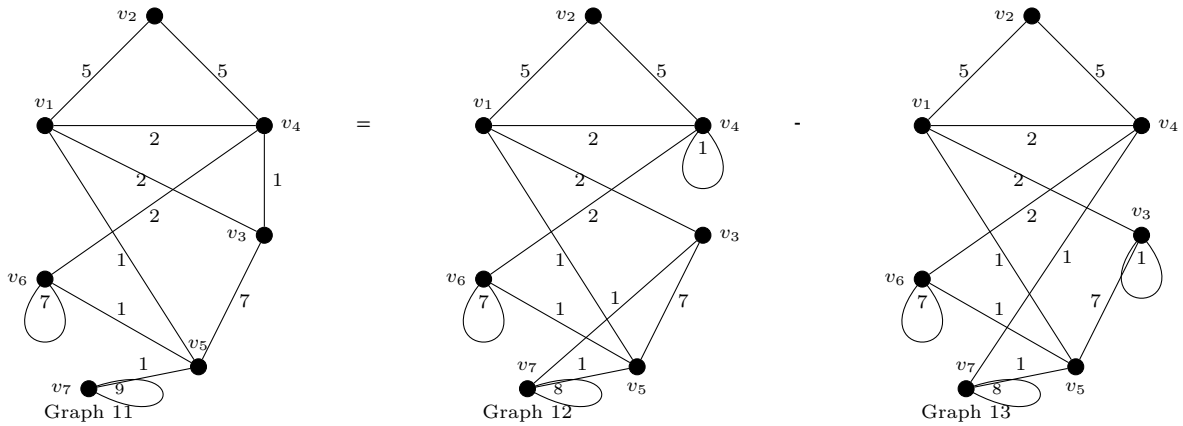

Fig. 6. The pair of edges $\{(v_2, v_5), (v_3, v_6)\}$ in Graph 4 are interchanged with the set $\{(v_3, v_5), (v_2, v_2), (v_6, v_6)\}$ in Graph 3.

(b) Interchange the pair of edges $\{(v_2, v_5), (v_3, v_6)\}$ in Graph 4 with the set $\{(v_3, v_5), (v_2, v_2), (v_6, v_6)\}$ in Graph 3 to obtain the graph in Graph 7 with which the associated monomial $p_{12}^2 p_{14}^3 p_{24}^2 p_{35}^5 p_2 p_6^5$ lies in R_1 and the graph in Fig. 8 with which the associated monomial lies in R_2 (see Fig. 6).

Fig. 7. Plücker relation on the edges (v_1, v_4) and (v_3, v_5).

Fig. 8. Plücker relation on the edge (v_3, v_4) and the loop (v_7, v_7).

Example. Here we give another example with both k and n are odd and where interchange between loops and edges in the above theorem is not possible. Then we use Plücker relation on the factors to get a set of graphs where interchange between loops and edges is possible. Let $\lambda = 7(\varpi_1 + 2\varpi_2)$ and $k = 3$. Here $s = 5$ and $nr_1 = 7$. After using the above procedure, suppose we get a 5-factor with which the associated monomial is $p_{12}^3 p_{13}^2 p_{24}^2 p_{35}^3 p_{46}^3 p_{57}^2 p_6^2 p_7^3$ (Graph 14) and another graph which is a 10-factor, with which the associated monomial is $p_{12}^5 p_{13}^2 p_{14}^3 p_{24}^5 p_{35}^8 p_{46}^2 p_{56} p_{57} p_6^7 p_7^9$ (Graph 9). Here directly we can not interchange loops and edges between the factors. We apply Plücker relation on the edges (v_1, v_4) and (v_3, v_5) in Graph 9 (see Fig. 7) and obtain $p_{12}^5 p_{13}^2 p_{14}^3 p_{24}^5 p_{35}^8 p_{46}^2 p_{56} p_{57} p_6^7 p_7^9 = p_{12}^5 p_{13}^3 p_{14}^2 p_{24}^5 p_{35}^7 p_{45} p_{46}^2 p_{56} p_{57} p_6^7 p_7^9 + p_{12}^5 p_{13}^2 p_{14}^2 p_{15} p_{24}^5 p_{34} p_{35}^7 p_{46}^2 p_{56} p_{57} p_6^7 p_7^9$. The graph associated with the monomial $p_{12}^5 p_{13}^3 p_{14}^2 p_{24}^5 p_{35}^7 p_{45} p_{46}^2 p_{56} p_{57} p_6^7 p_7^9$ is in Graph 10 and the graph associated with the monomial $p_{12}^5 p_{13}^2 p_{14}^2 p_{15} p_{24}^5 p_{34} p_{35}^7 p_{46}^2 p_{56} p_{57} p_6^7 p_7^9$ is in Graph 11. Again we apply Plücker relation on the edge (v_3, v_4) and the loop (v_7, v_7) in Graph 11 (see Fig. 8) and obtain $p_{12}^5 p_{13}^2 p_{14}^2 p_{15} p_{24}^5 p_{34} p_{35}^7 p_{46}^2 p_{56} p_{57} p_6^7 p_7^9 = p_{12}^5 p_{13}^2 p_{14}^2 p_{15} p_{24}^5 p_{37} p_{35}^7 p_{46}^2 p_{56} p_{57} p_4 p_6^7 p_7^8 - p_{12}^5 p_{13}^2 p_{14}^2 p_{15} p_{24}^5 p_{35}^7 p_{46}^2 p_{47} p_{56} p_{57} p_3 p_6^7 p_7^8$. The graph associated with the monomials $p_{12}^5 p_{13}^2 p_{14}^2 p_{15} p_{24}^5 p_{37} p_{35}^7 p_{46}^2 p_{56} p_{57} p_4 p_6^7 p_7^8$ and $p_{12}^5 p_{13}^2 p_{14}^2 p_{15} p_{24}^5 p_{35}^7 p_{46}^2 p_{47} p_{56} p_{57} p_3 p_6^7 p_7^8$ are in Graphs 12 and 13 respectively.
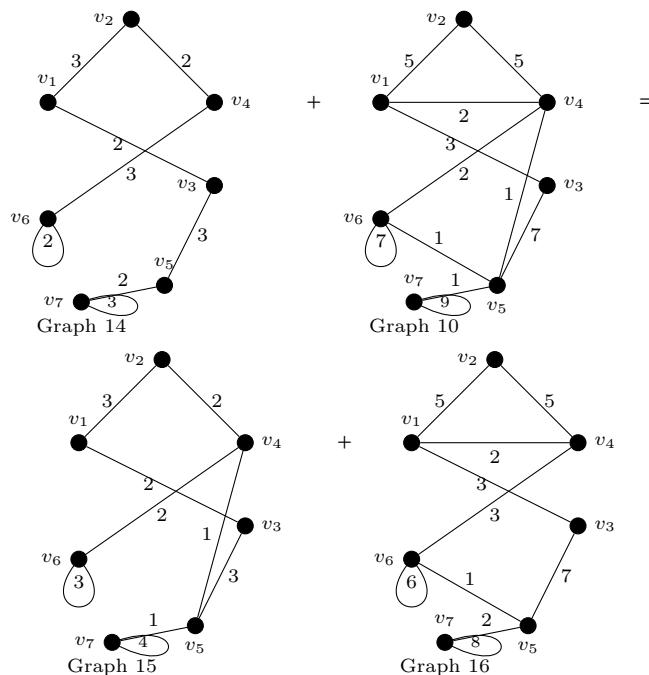
Now we do the following

Fig. 9. The pair of edges $\{(v_4, v_6), (v_5, v_7)\}$ in Graph 14 are interchanged with the set $\{(v_4, v_5), (v_6, v_6), (v_7, v_7)\}$ in Graph 10.

(a) Interchange the pair of edges $\{(v_4, v_6), (v_5, v_7)\}$ in Graph 14 with the set $\{(v_4, v_5), (v_6, v_6), (v_7, v_7)\}$ in Graph 10 and obtain the graph in Graph 15 with which the associated monomial $p_{12}^3 p_{13}^2 p_{24}^2 p_{35}^3 p_{45} p_{46}^2 p_{57} p_6^3 p_7^4$ lies in R_1 and the graph in Graph 16 with which the associated monomial lies in R_2 (see Fig. 9).

(b) Interchange the edge $\{(v_4, v_6)\}$ in Graph 14 with the set $\{(v_4, v_4), (v_6, v_6)\}$ in Graph 12 to obtain the graph in Graph 17 with which the associated monomial $p_{12}^3 p_{13}^2 p_{24}^2 p_{35}^3 p_{46}^2 p_{57}^2 p_4 p_6^3 p_7^3$ lies in R_1 and the graph in Graph 18 with which the associated monomial lies in R_2 (see Fig. 10).

(c) Interchange the pair of edges $\{(v_1, v_3), (v_5, v_7)\}$ in Graph 14 with the set $\{(v_1, v_5), (v_3, v_3), (v_7, v_7)\}$ in Graph 13 to obtain the graph in Graph 19 with which the associated monomial $p_{12}^3 p_{13} p_{15} p_{24}^2 p_{35}^3 p_{46}^3 p_{57} p_3 p_6^2 p_7^4$ lies in R_1 and the graph in Graph 20 with which the associated monomial lies in R_2 (see Fig. 11).

Corollary 7.3. *The GIT quotient of a Schubert variety and a Richardson variety in $SL_n/(P_{\alpha_1} \cap P_{\alpha_2})$ by a maximal torus T is projectively normal with respect to the descent of the line bundle $\mathcal{L}_{n(r_1 \varpi_1 + r_2 \varpi_2)}$.*

Proof. The proof is same as the proof of Corollary 6.2. □

8. $Spin_5$ and $Spin_7$

In this section for $G = Spin_5$ or $Spin_7$ and for a maximal parabolic subgroup P we study projective normality of the quotient $T \backslash\backslash (G/P)$ with respect to the descent of a suitable line bundle on G/P.

Notation. We denote a Young tableau Γ with rows $\mathrm{Row}_1, \mathrm{Row}_2, \ldots, \mathrm{Row}_n$ by $\Gamma = (\mathrm{Row}_1, \mathrm{Row}_2, \ldots, \mathrm{Row}_n)$.

8.1. $Spin_5$

Let $G = Spin_5$. Let ϖ_1 and ϖ_2 be the fundamental weights associated to the simple roots α_1 and α_2 respectively. Since $2\varpi_1 \in 2Q$ and $2\varpi_2 \in \mathbb{Z}\alpha_1 + \mathbb{Z}2\alpha_2$, by Theorem 2.1, the line bundles $\mathcal{L}_{2\varpi_1}$ and $\mathcal{L}_{2\varpi_2}$ descend to the quotients $T\backslash\backslash(G/P_{\alpha_1})_T^{ss}(\mathcal{L}_{2\varpi_1})$ and $T\backslash\backslash(G/P_{\alpha_2})_T^{ss}(\mathcal{L}_{2\varpi_2})$ respectively. We have,
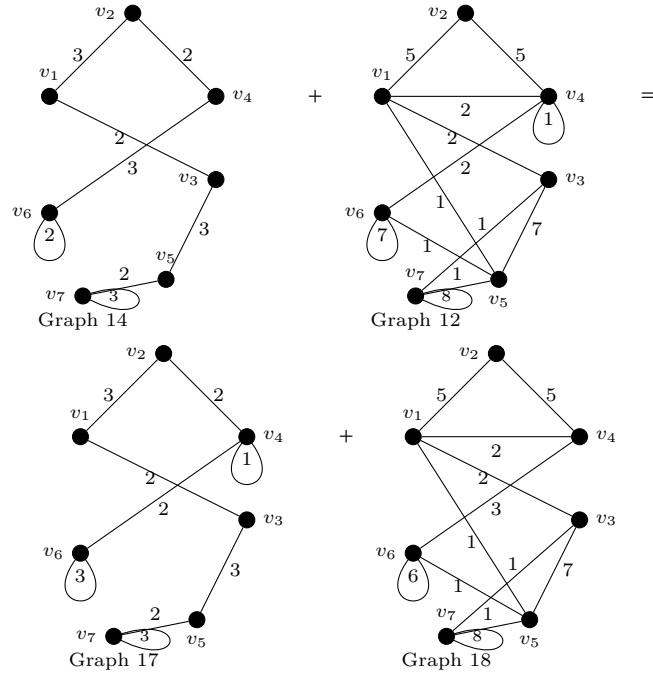
Fig. 10. The edge $\{(v_4, v_6)\}$ in Graph 14 interchanged with the set $\{(v_4, v_4), (v_6, v_6)\}$ in Graph 12.
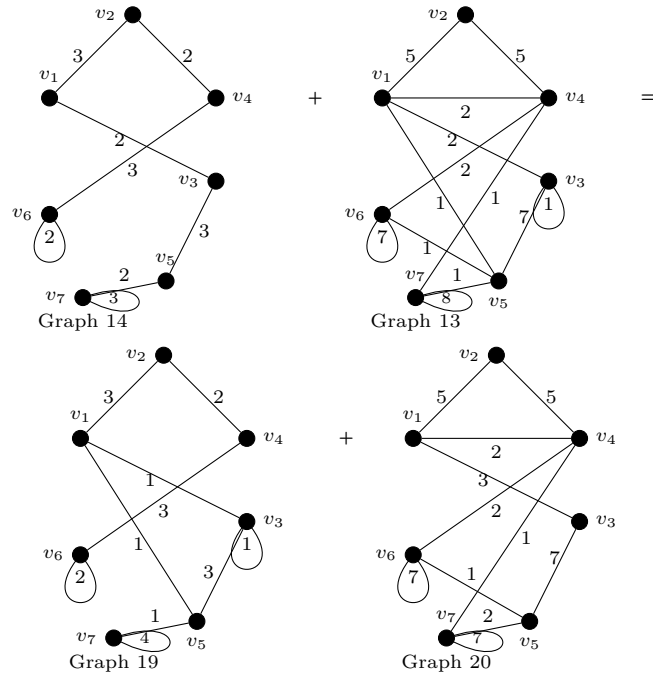

Fig. 11. The pair of edges $\{(v_1, v_3), (v_5, v_7)\}$ in Graph 14 are interchanged with the set $\{(v_1, v_5), (v_3, v_3), (v_7, v_7)\}$ in Graph 13.

$$T\backslash\backslash(G/P_{\alpha_1})^{ss}_T(\mathcal{L}_{2\varpi_1}) \cong Proj(\oplus H^0(G/P_{\alpha_1}, \mathcal{L}^{\otimes k}_{2\varpi_1})^T) \cong Proj(\oplus_{k \in \mathbb{Z}_{\geq 0}} R_k),$$

where $R_k := H^0(G/P_{\alpha_1}, \mathcal{L}^{\otimes k}_{2\varpi_1})^T$. By Theorem 3.2.4 the standard monomials p_Γ form a basis of R_k, where Γ is a standard Young tableau associated to the weight $2k\varpi_1$. The standard monomials in R_k are of the form p_Γ, where

$$\Gamma = (\underbrace{(1),\dots,(1)}_{2q}, \underbrace{(2),\dots,(2)}_{2k-2q}, \underbrace{(3),\dots,(3)}_{2k-2q}, \underbrace{(4),\dots,(4)}_{2q}),$$

$0 \le q \le k$. So, the homogeneous coordinate ring of the quotient $T\backslash\backslash(G/P_{\alpha_1})_T^{ss}(\mathcal{L}_{2\varpi_1})$ is generated by p_{Γ_1} and p_{Γ_2}, where $\Gamma_1 = ((1),(1),(4),(4))$ and $\Gamma_2 = ((2),(2),(3),(3))$ as an algebra. Since $T\backslash\backslash(G/P_{\alpha_1})_T^{ss}(\mathcal{L}_{2\varpi_1})$ is normal, it is projectively normal. In fact, in this case $T\backslash\backslash(G/P_{\alpha_1})_T^{ss}(\mathcal{L}_{2\varpi_1}) \cong \mathbb{P}^1$.

For the quotient $T\backslash\backslash(G/P_{\alpha_2})_T^{ss}(\mathcal{L}_{2\varpi_2})$, the standard monomials in R_k are of the form p_Γ, $\Gamma = (\underbrace{(1,2),\dots,(1,2)}_{q}, \underbrace{(1,3),\dots,(1,3)}_{k-q}, \underbrace{(2,4),\dots,(2,4)}_{k-q}, \underbrace{(3,4),\dots,(3,4)}_{q})$, $0 \le q \le k$. So, the homogeneous co-ordinate ring of the quotient $T\backslash\backslash(G/P_{\alpha_1})$ is generated by p_{Γ_1} and p_{Γ_2}, where $\Gamma_1 = ((1,2),(3,4))$ and $\Gamma_2 = ((1,3),(2,4))$ as an algebra. Since the quotient $T\backslash\backslash(G/P_{\alpha_2})$ is normal, it is projectively normal. In this case also $T\backslash\backslash(G/P_{\alpha_2})_T^{ss}(\mathcal{L}_{2\varpi_2}) \cong \mathbb{P}^1$.

8.2. Spin$_7$

Let $G = Spin_7$ and P_{α_i} be the maximal parabolic subgroup associated to α_i, $1 \le i \le 3$. In this case the line bundle $\mathcal{L}_{2\varpi_i}$ descends to the quotient $T\backslash\backslash(G/P_{\alpha_i})_T^{ss}(\mathcal{L}_{2\varpi_i})$ for $1 \le i \le 2$ whereas $\mathcal{L}_{4\varpi_3}$ descends to the quotient $T\backslash\backslash(G/P_{\alpha_3})_T^{ss}(\mathcal{L}_{4\varpi_3})$.

We show that $T\backslash\backslash(G/P_{\alpha_1})_T^{ss}(\mathcal{L}_{2\varpi_1})$ and $T\backslash\backslash(G/P_{\alpha_3})_T^{ss}(\mathcal{L}_{4\varpi_3})$ are projectively normal with respect to the descent of the line bundles $\mathcal{L}_{2\varpi_1}$ and $\mathcal{L}_{4\varpi_3}$ respectively whereas we give a degree bound of the generators of the homogeneous coordinate ring of $T\backslash\backslash(G/P_{\alpha_2})_T^{ss}(\mathcal{L}_{2\varpi_2})$.

We have $T\backslash\backslash(G/P_{\alpha_1})_T^{ss}(\mathcal{L}_{2\varpi_1}) \cong Proj(\oplus_{k\in\mathbb{Z}_{\ge 0}} R_k)$, where $R_k := H^0(G/P_{\alpha_1}, \mathcal{L}_{2\varpi_1}^{\otimes k})^T$. The standard monomials p_Γ form a basis of R_k, where Γ is a standard Young tableau associated to the weight $2k\varpi_1$. The standard monomials in R_k are of the form p_Γ, where

$$\Gamma = (\underbrace{(1),\dots,(1)}_{k_1}, \underbrace{(2),\dots,(2)}_{k_2}, \underbrace{(3),\dots,(3)}_{2k-k_1-k_2}, \underbrace{(4),\dots,(4)}_{2k-k_1-k_2}, \underbrace{(5),\dots,(5)}_{k_2}, \underbrace{(6),\dots,(6)}_{k_1}),$$

where $0 \le k_1 + k_2 \le 2k$. So the homogeneous coordinate ring of the GIT quotient $T\backslash\backslash(G/P_{\alpha_1})_T^{ss}(\mathcal{L}_{2\varpi_1})$ is generated by $p_{\Gamma_1}, p_{\Gamma_2}$ and p_{Γ_3} as an algebra, where

$$\Gamma_1 = ((1),(1),(6),(6)), \ \Gamma_2 = ((2),(2),(5),(5)) \text{ and } \Gamma_3 = ((3),(3),(4),(4)).$$

Since the quotient $T\backslash\backslash(G/P_{\alpha_1})_T^{ss}(\mathcal{L}_{2\varpi_1})$ is normal so it is projectively normal. In fact, in this case $T\backslash\backslash(G/P_{\alpha_1})_T^{ss}(\mathcal{L}_{2\varpi_1}) \cong \mathbb{P}^2$.

In the following theorem we give a degree bound of the generators of the homogeneous coordinate ring of the quotient $T\backslash\backslash(G/P_{\alpha_2})_T^{ss}(\mathcal{L}_{2\varpi_2})$.

Theorem 8.1. *The homogeneous coordinate ring of the quotient $T\backslash\backslash(G/P_{\alpha_2})_T^{ss}(\mathcal{L}_{2\varpi_2})$ is generated by elements of degree at most* 3.

Proof. We have

$$T\backslash\backslash(G/P_{\alpha_2})_T^{ss}(\mathcal{L}_{2\varpi_2}) = Proj(\oplus_{k\in\mathbb{Z}_{\ge 0}} H^0(G/P_{\alpha_2}, \mathcal{L}_{2\varpi_2}^{\otimes k})^T) = Proj(\oplus_{k\in\mathbb{Z}_{\ge 0}} R_k),$$

where $R_k := H^0(G/P_{\alpha_2}, \mathcal{L}_{2\varpi_2}^{\otimes k})^T$. Let $f \in R_k$ be a standard monomial.

We claim that $f = f_1.f_2$ where f_1 is in R_1 or R_2 or R_3.

From the discussion in Section 3.2, the Young diagram associated to f has the shape $p = (p_1, p_2) = (4k, 4k)$. So the Young tableau Γ associated to this Young diagram has $4k$ rows and 2 columns with strictly increasing rows and non-decreasing columns. Since f is T-invariant, by Lemma 3.2 we have,

$$c_\Gamma(t) = c_\Gamma(7 - t) \text{ for all } 1 \le t \le 6, \tag{8.1}$$

where $c_\Gamma(t) = \#\{t | t \in \Gamma\}$. Also from the discussion in Section 3.2, we have $(\text{Row}_{2i}, \text{Row}_{2i-1})$ is an admissible pair for all $1 \le i \le 2k$, where Row_i denotes the i-th row of the tableau for all $1 \le i \le 4k$. We also have if $t \in \text{Row}_i$ then $7 - t \notin \text{Row}_i$, for all $1 \le t \le 6$ and for any $i, 1 \le i \le 4k$.

Let Col_j denote the j-th column where $1 \le j \le 2$. Let $E_{i,j}$ be the (i, j)-th entry of the tableau Γ and $N_{t,j} = \#\{i | E_{i,j} = t\}$.

Since $(\text{Row}_{2i}, \text{Row}_{2i-1})$ is admissible, either $\text{Row}_{2i-1} = \text{Row}_{2i}$ or $(\text{Row}_{2i-1}, \text{Row}_{2i}) \in \{((1, 3), (1, 4)), ((1, 5), (2, 6)), ((2, 3), (2, 4)), ((2, 4), (3, 5)), ((2, 3), (3, 5)), ((2, 4), (4, 5)), ((2, 3), (4, 5)), ((3, 5), (4, 5)), ((3, 6), (4, 6))\}$.

We consider Row_1. If $E_{1,1} = 3$ then $E_{1,2} \ne 4$ and so $E_{1,2} = 5$ or 6, a contradiction to Eq. (8.1). By a similar reason, $E_{1,1}$ can not be $4, 5$ or 6. So $\text{Row}_1 \in \{(1, 2), (1, 3), (1, 4), (1, 5), (2, 3), (2, 4)\}$.

(a) Let $\text{Row}_1 = (2, 4)$. Since $(\text{Row}_2, \text{Row}_1)$ is admissible, we have $\text{Row}_2 = (2, 4), (3, 5)$, or $(4, 5)$. If $\text{Row}_2 = (4, 5)$ then 5 or 6 has to appear in one of the rows below, which is a contradiction to (8.1). By the same reason $\text{Row}_2 \ne (3, 5)$. If $\text{Row}_2 = (2, 4)$ then $(\text{Row}_{4k-1}, \text{Row}_{4k}) \in \{((3, 5), (3, 5)), ((3, 5), (4, 5)), ((4, 5), (4, 5))\}$.

If $(\text{Row}_{4k-1}, \text{Row}_{4k}) = ((4, 5), (4, 5))$ then $c_\Gamma(4) + c_\Gamma(5) \ge 4k + 2$ and hence $c_\Gamma(2) + c_\Gamma(3) \le 4k - 2$, a contradiction. By a similar reason $(\text{Row}_{4k-1}, \text{Row}_{4k})$ can not be $((3, 5), (4, 5))$. If $(\text{Row}_{4k-1}, \text{Row}_{4k}) = ((3, 5), (3, 5))$ then $p_\Omega \in R_1$, where $\Omega = ((2, 4), (2, 4), (3, 5), (3, 5))$ and is a factor of f.

(b) If $\text{Row}_1 = (2, 3)$ then Row_2 is either $(2, 3)$ or $(2, 4)$ or $(3, 5)$. If $\text{Row}_2 = (3, 5)$ then 5 or 6 has to appear in one of the rows below, which is a contradiction to (8.1) and if $\text{Row}_2 = (2, 3)$ then by a similar argument as above, $(\text{Row}_{4k-1}, \text{Row}_{4k}) = ((4, 5), (4, 5))$. Then $p_\Omega \in R_1$, where $\Omega = ((2, 3), (2, 3), (4, 5), (4, 5))$ and is a factor of f.

Similarly if $\text{Row}_2 = (2, 4)$ then we have $p_\Omega \in R_1$, where $\Omega = ((2, 3), (2, 4), (3, 5), (4, 5))$ and is a factor of f.

(c) If $\text{Row}_1 = (1, 5)$ then $\text{Row}_2 = (1, 5)$. Then $(\text{Row}_{4k-1}, \text{Row}_{4k}) \in \{((2, 6), (2, 6)), ((3, 6), (3, 6)), ((3, 6), (4, 6)), ((4, 6), (4, 6)), ((5, 6), (5, 6))\}$. By a similar argument as above $(\text{Row}_{4k-1}, \text{Row}_{4k})$ can not be any other pair except $((2, 6), (2, 6))$. Then $p_\Omega \in R_1$, where $\Omega = ((1, 5), (1, 5), (2, 6), (2, 6))$ and is a factor of f.

(d) If $\text{Row}_1 = (1, 4)$ then $\text{Row}_2 = (1, 4)$. By a similar argument as above $(\text{Row}_{4k-1}, \text{Row}_{4k})$ has to be $((3, 6), (3, 6))$. Then $p_\Omega \in R_1$ and is a factor of f, where $\Omega = ((1, 4), (1, 4), (3, 6), (3, 6))$.

(e) If $\text{Row}_1 = (1, 2)$ then $\text{Row}_2 = (1, 2)$ and $(\text{Row}_{4k-1}, \text{Row}_{4k}) \in \{((2, 6), (2, 6)), ((3, 6), (3, 6)), ((3, 6), (4, 6)), ((4, 6), (4, 6)), ((5, 6), (5, 6))\}$. By a similar argument as above $(\text{Row}_{4k-1}, \text{Row}_{4k})$ is either $((5, 6), (5, 6))$ or $((4, 6), (4, 6))$. If $(\text{Row}_{4k-1}, \text{Row}_{4k}) = ((5, 6), (5, 6))$ then $p_\Omega \in R_1$ and is a factor of f, where $\Omega = ((1, 2), (1, 2), (5, 6), (5, 6))$.

Now assume that $(\text{Row}_{4k-1}, \text{Row}_{4k}) = ((4, 6), (4, 6))$. Let $N_{1,1} = N_{6,2} = m_1$. By the admissibility property m_1 is even.

Case 1: Let $m_1 = 2$. Note that Row_i has either 2 or 5 as an entry for all $3 \le i \le 4k - 2$. Since $E_{1,2} = E_{2,2} = 2$ we have $c_\Gamma(2) = c_\Gamma(5) = 2k - 1$. Similarly, we have $c_\Gamma(3) = c_\Gamma(4) = 2k - 1$. Since $E_{4k,1} = 4$ we have $E_{i,2} = 5$, for all $2k \le i \le 4k - 2$. Also since $c_\Gamma(4) = 2k - 1$ we have $\text{Row}_{2k} = \text{Row}_{2k+1} = (3, 5)$. Since the pairs $(\text{Row}_{2k}, \text{Row}_{2k-1})$ and $(\text{Row}_{2k+2}, \text{Row}_{2k+1})$ are admissible we have $\text{Row}_{2k-1} = (2, 3)$ or $(2, 4)$. If $\text{Row}_{2k-1} = (2, 3)$ then $c_\Gamma(1) + c_\Gamma(2) + c_\Gamma(3) \ge (2k+1) + (2k-1) = 4k$. Since $c_\Gamma(1) + c_\Gamma(2) + c_\Gamma(3) = 4k$ we have $\text{Row}_{2k+2} = (4, 5)$. Then $p_\Omega \in R_2$ and is a factor of f, where $\Omega = ((1, 2), (1, 2), (2, 3), (3, 5), (3, 5), (4, 5), (4, 6), (4, 6))$. For a similar reason if $\text{Row}_{2k-1} = (2, 4)$ then $\text{Row}_{2k+2} = (3, 5)$. Then $p_\Omega \in R_2$ and is a factor of f, where $\Omega = ((1, 2), (1, 2), (2, 4), (3, 5), (3, 5), (3, 5), (4, 6), (4, 6))$.

Case 2: Let $m_1 = 2k$. We have $E_{i,1} = 1$ for all $1 \le i \le 2k$ and $E_{i,2} = 6$ for all $2k + 1 \le i \le 4k$. Note that $E_{2k-1,2} = E_{2k,2} = 5$. Since $c_\Gamma(1) + c_\Gamma(2) + c_\Gamma(4) = 4k$ we have $E_{i,1} = E_{i+1,1} = 3$, for some $i, 2k + 1 \le i \le 4k$. Then $p_\Omega \in R_2$ and is a factor of f, where $\Omega = ((1, 2), (1, 2), (1, 5), (1, 5), (3, 6), (3, 6), (4, 6), (4, 6))$.

Case 3: Let $4 \leq m_1 \leq 2k - 2$. Note that $k \geq 3$. Since $E_{1,2} = E_{2,2} = 2$ and Row_i contains either 2 or 5 as an entry for $m_1 + 1 \leq i \leq 4k - m_1$ we have $N_{5,2} \geq 2k - m_1 + 1$. Hence, $\text{Row}_{2k} = \text{Row}_{2k+1} = (3, 5)$. Let $N_{2,1} = l$.

(i) If $l = 0$ then $E_{i,2} = 5$ for all $m_1 + 1 \leq i \leq 4k - m_1$ and hence $N_{2,2} \geq 4$. So we have $\text{Row}_3 = \text{Row}_4 = (1, 2)$. Since $c_\Gamma(4) \leq 2k - 1$ we have $\text{Row}_{2k-1} = (3, 5)$. Since $(\text{Row}_{2k+2}, \text{Row}_{2k+1})$ is admissible we have Row_{2k+2} is either $(3, 5)$ or $(4, 5)$. If $\text{Row}_{2k+2} = (4, 5)$ then $c_\Gamma(4) = 2k - 1$ whereas $c_\Gamma(3) \leq (2k-4)+1 = 2k-3$, a contradiction. Hence $\text{Row}_{2k+2} = (3, 5)$.

We claim that $N_{4,1} \geq 4$. If not then $N_{4,1} \leq 3$. In this case $c_\Gamma(3) \geq 2k - 3 + 2 = 2k - 1$ whereas $c_\Gamma(4) \leq 2k - 6 + 3 = 2k - 3$, a contradiction. Hence, $N_{4,1} \geq 4$. So we have $\text{Row}_{4k-3} = \text{Row}_{4k-2} = (4, 6)$. Then $p_\Omega \in R_3$ and is a factor of f, where

$$\Omega = ((1, 2), (1, 2), (1, 2), (1, 2), (3, 5), (3, 5), (3, 5), (3, 5), (4, 6), (4, 6), (4, 6), (4, 6)).$$

(ii) If $l = 1$ then $\text{Row}_{m_1+1} = (2, 4)$. Since $(\text{Row}_{m_1+2}, \text{Row}_{m_1+1})$ is admissible and $c_\Gamma(4) \leq 2k - 1$ we have $\text{Row}_{m_1+2} = (3, 5)$. Again since $c_\Gamma(4) \leq 2k - 1$ we have $\text{Row}_{2k+1} = \text{Row}_{2k+2} = (3, 5)$. Then $p_\Omega \in R_2$ and is a factor of f, where

$$\Omega = ((1, 2), (1, 2), (2, 4), (3, 5), (3, 5), (3, 5), (4, 6), (4, 6)).$$

(iii) If $l \geq 2$ then the rows of Γ containing 2 as the first entry are either $(2, 3)$ or $(2, 4)$. If Γ has at least two rows equal to $(2, 4)$ then $p_\Omega \in R_1$ and is a factor of f, where $\Omega = ((2, 4), (2, 4), (3, 5), (3, 5))$. If Γ has exactly one row equal to $(2, 4)$ then since $c_\Gamma(4) \leq 2k - 1$ we have $\text{Row}_{2k+2} = (3, 5)$. So $p_\Omega \in R_2$ and is a factor of f, where $\Omega = ((1, 2), (1, 2), (2, 4), (3, 5), (3, 5), (3, 5), (4, 6), (4, 6))$.

Now suppose that none of the rows of Γ is $(2, 4)$. Since $\text{Row}_{2k} = (3, 5)$ and $(\text{Row}_{2k}, \text{Row}_{2k-1})$ is admissible we have $\text{Row}_{2k-1} = (2, 3)$ or $(3, 5)$. If $\text{Row}_{2k-1} = (2, 3)$ then $E_{i,1} = 2$ for all $m_1 + 1 \leq i \leq 2k - 1$ and $\text{Row}_i = (2, 3)$ for all $m_1 + 1 \leq i \leq 2k - 1$. Since $c_\Gamma(4) + c_\Gamma(5) + c_\Gamma(6) = 4k$ and $c_\Gamma(5) + c_\Gamma(6) = 2k + 1$ we have $c_\Gamma(4) = 2k - 1$. Hence, $E_{i,1} = 4$, for all $2k + 2 \leq i \leq 4k$. In particular, $\text{Row}_{2k+2} = (4, 5)$. So $p_\Omega \in R_2$ and is a factor of f, where $\Omega = ((1, 2), (1, 2), (2, 3), (3, 5), (3, 5), (4, 5), (4, 6), (4, 6))$.

If $\text{Row}_{2k-1} = (3, 5)$ then $N_{5,2} \geq 2k - m_1 + 2$. Then $N_{2,1} \leq 2k - m_1 - 2$ and so $N_{2,2} \geq 4$. Hence, $\text{Row}_3 = \text{Row}_4 = (1, 2)$. We claim that $\text{Row}_{4k-3} = \text{Row}_{4k-2} = (4, 6)$. If not then $c_\Gamma(4) \leq 3$ whereas $c_\Gamma(3) \geq 2k - 3 + 2 = 2k - 1$, a contradiction to $k \geq 3$. Since $(\text{Row}_{4k-m_1}, \text{Row}_{4k-m_1-1})$ is admissible we have $(\text{Row}_{4k-m_1-1}, \text{Row}_{4k-m_1}) \in \{((3, 5), (3, 5)), ((3, 5), (4, 5)), ((4, 5), (4, 5))\}$.

If $(\text{Row}_{4k-m_1-1}, \text{Row}_{4k-m_1}) = ((4, 5), (4, 5))$ then $p_\Omega \in R_1$ and is a factor of f, where $\Omega = ((2, 3), (2, 3), (4, 5), (4, 5))$.

If $(\text{Row}_{4k-m_1-1}, \text{Row}_{4k-m_1}) = ((3, 5), (3, 5))$ then $\text{Row}_{2k+2} = (3, 5)$ and in this case $p_\Omega \in R_3$ and is a factor of f, where

$$\Omega = ((1, 2), (1, 2), (1, 2), (1, 2), (3, 5), (3, 5), (3, 5), (3, 5), (4, 6), (4, 6), (4, 6), (4, 6)).$$

If $(\text{Row}_{4k-m_1-1}, \text{Row}_{4k-m_1}) = ((3, 5), (4, 5))$ then for $m_1 = 2k - 2$ we have $c_\Gamma(4) = 2k - 1$ whereas $c_\Gamma(3) \leq (2k-4)+1 = 2k-3$, a contradiction and for $m_1 \leq 2k-4$ we have $\text{Row}_{2k+2} = (3, 5)$. Then $p_\Omega \in R_3$ and is a factor of f, where

$$\Omega = ((1, 2), (1, 2), (1, 2), (1, 2), (3, 5), (3, 5), (3, 5), (3, 5), (4, 6), (4, 6), (4, 6), (4, 6)).$$

(f) If $\text{Row}_1 = (1, 3)$ then $\text{Row}_2 = (1, 3)$ or $(1, 4)$.

If $\text{Row}_2 = (1, 4)$ then $(\text{Row}_{4k-1}, \text{Row}_{4k}) \in \{((2, 6), (2, 6)), ((3, 6), (3, 6)), ((3, 6), (4, 6)), ((4, 6), (4, 6)), ((5, 6), (5, 6))\}$. By a similar argument as above $(\text{Row}_{4k-1}, \text{Row}_{4k}) = ((3, 6), (4, 6))$. Then $p_\Omega \in R_1$ and is a factor of f, where $\Omega = ((1, 3), (1, 4), (3, 6), (4, 6))$.

If $\text{Row}_2 = (1, 3)$ then $(\text{Row}_{4k-1}, \text{Row}_{4k}) \in \{((2, 6), (2, 6)), ((3, 6), (3, 6)), ((3, 6), (4, 6)), ((4, 6), (4, 6)),$ $((5, 6), (5, 6))\}$. By a similar argument as above $(\text{Row}_{4k-1}, \text{Row}_{4k})$ is either $((4, 6), (4, 6))$ or $((5, 6), (5, 6))$. If $(\text{Row}_{4k-1}, \text{Row}_{4k}) = ((4, 6), (4, 6))$ then $p_\Omega \in R_1$ and is a factor of f, where $\Omega = ((1, 3), (1, 3), (4, 6), (4, 6))$.

Now assume $(\text{Row}_{4k-1}, \text{Row}_{4k}) = ((5, 6), (5, 6))$. If we flip over the Young tableau and replace k by $7 - k$ then this case is reduced to the case (e) where $(\text{Row}_1, \text{Row}_2) = ((1, 2), (1, 2))$ and $(\text{Row}_{4k-1}, \text{Row}_{4k}) = ((4, 6), (4, 6))$.

So by induction we conclude that f is generated by the elements of degree at most 3. $\square$

Theorem 8.2. *The quotient $T \backslash\backslash (G/P_{\alpha_3})_T^{ss}(\mathcal{L}_{4\varpi_3})$ is projectively normal with respect to the descent of the line bundle $\mathcal{L}_{4\varpi_3}$.*

Proof. We have

$$T \backslash\backslash (G/P_{\alpha_3})_T^{ss}(\mathcal{L}_{4\varpi_3}) = Proj(\oplus_{k \in \mathbb{Z}_{\geq 0}} H^0(G/P_{\alpha_3}, \mathcal{L}_{4\varpi_3}^{\otimes k})^T) = Proj(\oplus_{k \in \mathbb{Z}_{\geq 0}} R_k),$$

where $R_k := H^0(G/P_{\alpha_3}, \mathcal{L}_{4\varpi_3}^{\otimes k})^T$. Since the quotient $T \backslash\backslash (G/P_{\alpha_3})_T^{ss}(\mathcal{L}_{4\varpi_3})$ is normal, in order to show that it is projectively normal we show that R_k is generated by R_1. Let $f \in R_k$ be a standard monomial. The Young diagram associated to f has the shape $p = (p_1, p_2, p_3) = (4k, 4k, 4k)$. So the Young tableau Γ associated to this Young diagram has $4k$ rows and 3 columns with strictly increasing rows and non-decreasing columns. Since f is T-invariant, by Lemma 3.2 we have,

$$c_\Gamma(t) = c_\Gamma(7 - t) \text{ for all } 1 \leq t \leq 6. \tag{8.2}$$

Since $\bar{p}_1 = 0$, the admissibility condition is empty. We also have if $t \in \text{Row}_i$ then $7 - t \notin \text{Row}_i$, for all $1 \leq t \leq 6$ and for any $i, 1 \leq i \leq 4k$, where Row_i denotes the ith row of the tableau. For $1 \leq t \leq 6$ all the rows of Γ contain either t or $7 - t$. So $c_\Gamma(t) = 2k$ for all $1 \leq t \leq 6$.

Let Col_j denote the jth column of the tableau. Let $E_{i,j}$ be the (i, j)-th entry of the tableau and $N_{t,j} = \#\{i | E_{i,j} = t\}$.

Note that $E_{i,1} = 1$ for all $1 \leq i \leq 2k$ and $E_{i,3} = 6$ for all $2k + 1 \leq i \leq 4k$.

If $E_{1,2} = 4$ or 5 then 1, 2 and 3 appear $2k$ times each in the first column, a contradiction.

So, $\text{Row}_1 \in \{(1, 2, 3), (1, 2, 4), (1, 3, 5)\}$.

Case - 1 $\text{Row}_1 = (1, 3, 5)$

In this case we have $E_{i,1} = 2$ for all $2k + 1 \leq i \leq 4k$, $E_{i,2} = 3$ for all $1 \leq i \leq 2k$, $E_{i,3} = 5$ for all $1 \leq i \leq 2k$ and $E_{i,2} = 4$ for all $2k + 1 \leq i \leq 4k$. So we conclude that, $\text{Row}_i = (2, 4, 6)$ for all $2k + 1 \leq i \leq 4k$ and $\text{Row}_i = (1, 3, 5)$ for all $1 \leq i \leq 2k$. Then $p_\Omega \in R_1$ and divides f, where $\Omega = ((1, 3, 5), (1, 3, 5), (2, 4, 6), (2, 4, 6))$. So by induction we conclude that f belongs to the subalgebra generated by R_1.

Case - 2 $\text{Row}_1 = (1, 2, 4)$

In this case we have $E_{4k,2} = 5$ and $E_{4k,1}$ is either 3 or 4.

(a) If $E_{4k,1} = 3$ then $\text{Row}_{4k} = (3, 5, 6)$. In this case $E_{2k,3}$ is either 4 or 5.

If $E_{2k,3} = 4$ then $E_{2k,2} = 2$ and hence, $E_{i,2} = 2$ for all $1 \leq i \leq 2k$ and $E_{i,3} = 4$ for all $1 \leq i \leq 2k$. So we conclude that $E_{i,1} = 3$ for all $2k + 1 \leq i \leq 4k$ and $E_{i,3} = 5$ for all $2k + 1 \leq i \leq 4k$. Then $p_\Omega \in R_1$ and is a factor of f, where $\Omega = ((1, 2, 4), (1, 2, 4), (3, 5, 6), (3, 5, 6))$.

If $E_{2k,3} = 5$ then $E_{2k,2}$ is either 3 or 4. If $E_{2k,2} = 3$ then $E_{2k+1,1} = 2$ and so $E_{2k+1,2}$ is either 3 or 4. If $E_{2k+1,2} = 4$ then $p_\Omega \in R_1$ and is a factor of f, where $\Omega = ((1, 2, 4), (1, 3, 5), (2, 4, 6), (3, 5, 6))$. If $E_{2k+1,2} = 3$ then $c_\Gamma(2) + c_\Gamma(3) \geq 4k + 1$, a contradiction.

(b) If $E_{4k,1} = 4$ then $E_{2k,3} = 5$ and so $E_{2k,2}$ is either 3 or 4. If $E_{2k,2} = 3$ then $E_{2k+1,1} = 2$ and so $E_{2k+1,2}$ is either 3 or 4. If $E_{2k+1,2} = 4$ then $c_\Gamma(4) + c_\Gamma(5) \geq 4k + 1$, a contradiction. If $E_{2k+1,2} = 3$ then $p_\Omega \in R_1$ and is a factor of f, where $\Omega = ((1, 2, 4), (1, 3, 5), (2, 3, 6), (4, 5, 6))$.

Case - 3 $\text{Row}_1 = (1, 2, 3)$

In this case $E_{4k,2} = 5$ and $E_{4k,1}$ is either 3 or 4.

(a) If $E_{4k,1} = 4$ then $E_{2k,3}$ is either 3 or 4 or 5.

If $E_{2k,3} = 3$ then $E_{i,3} = 3$ for all $1 \leq i \leq 2k$ and $E_{i,2} = 2$ for all $1 \leq i \leq 2k$. Hence $E_{i,1} = 4$ for all $2k + 1 \leq i \leq 4k$ and $E_{i,2} = 5$ for all $2k + 1 \leq i \leq 4k$. Then $p_\Omega \in R_1$ and is a factor of f, where $\Omega = ((1, 2, 3), (1, 2, 3), (4, 5, 6), (4, 5, 6))$.

If $E_{2k,3} = 4$ then $E_{2k,2} = 2$, $E_{i,2} = 5$ for all $2k + 1 \leq i \leq 4k$ and $\text{Row}_{2k+1} = (3, 5, 6)$. Then $p_\Omega \in R_1$ and is a factor of f, where $\Omega = ((1, 2, 3), (1, 2, 4), (3, 5, 6), (4, 5, 6))$.

If $E_{2k,3} = 5$ then $E_{2k,2}$ is either 3 or 4.

If $E_{2k,2} = 3$ then $E_{2k+1,1} = 2$ and so $E_{2k+1,2}$ is either 3 or 4. If $E_{2k+1,2} = 4$ then $p_\Omega \in R_1$ and is a factor of f, where $\Omega = ((1, 2, 3), (1, 3, 5), (2, 4, 6), (4, 5, 6))$. If $E_{2k+1,2} = 3$ then for all $2 \leq i \leq 2k + 1$ we have $E_{i,2}$ either 2 or 3. We claim that Γ will either have a row $(1, 2, 4)$ or a row $(2, 4, 6)$. If not then 2 and 3 appear in all the rows of Γ containing 3, which is a contradiction, since $\text{Row}_{2k} = (1, 3, 5)$. If $(1, 2, 4)$ is a row of Γ then p_{Ω_1} is a factor of f and if $(2, 4, 6)$ is a row then p_{Ω_2} is a factor of f, where $\Omega_1 = ((1, 2, 4), (1, 3, 5), (2, 3, 6), (4, 5, 6))$ and $\Omega_2 = ((1, 2, 3), (1, 3, 5), (2, 4, 6), (4, 5, 6))$.

If $E_{2k,2} = 4$ then $E_{2k+1,1} = 2$ and $E_{2k+1,2} = 4$. So $\text{Row}_{2k+1} = (2, 4, 6)$. Since $E_{2k,3} = 5$ we have $N_{3,1} \geq 1$. So if $E_{q,1} = 3$ for some $2k + 2 \leq q \leq 4k - 2$ we have $\text{Row}_q = (3, 5, 6)$. Then $p_\Omega \in R_1$ and is a factor of f, where $\Omega = ((1, 2, 3), (1, 4, 5), (2, 4, 6), (3, 5, 6))$.

(b) If $E_{4k,1} = 3$ then $E_{2k+1,1} = 2$ and $E_{2k,3}$ is either 4 or 5. If $E_{2k,3} = 4$ then $E_{i,2} = 5$ for all $2k + 1 \leq i \leq 4k$. Hence, $c_\Gamma(4) < 2k$, a contradiction. If $E_{2k,3} = 5$ then $E_{2k,2}$ is either 3 or 4. If $E_{2k,2} = 3$ then $c_\Gamma(2) + c_\Gamma(3) \geq 4k + 1$, a contradiction. If $E_{2k,2} = 4$ and in this case we have $E_{2k+1,2} = 4$. Then $p_\Omega \in R_1$ and is a factor of f, where $\Omega = ((1, 2, 3), (1, 4, 5), (2, 4, 6), (3, 5, 6))$.

So by induction we conclude that f belongs to the subalgebra generated by R_1 and hence the quotient is projectively normal. $\quad \square$

Acknowledgements

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