



# Contravariant forms and extremal projectors

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## ABSTRACT

Tensor product of irreducible modules of highest weight over a semi-simple quantum group is completely reducible if and only if a natural contravariant form is non-degenerate when restricted to the span of singular vectors. We express this restriction through the extremal projector of the quantum group providing a computationally feasible criterion for complete reducibility of tensor products.

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## 1. Introduction

This is a continuation of [20] where we gave a complete reducibility criterion for a tensor product  $V \otimes Z$  of two irreducible modules of highest weight over the (classical or) quantum universal enveloping algebra  $U_q(\mathfrak{g})$  of a semi-simple Lie algebra  $\mathfrak{g}$ . It is formulated in terms of a contravariant symmetric bilinear form on  $V \otimes Z$ , which is the product of the contravariant forms on the tensor factors, relative to the Chevalley involution  $\omega: U_q(\mathfrak{g}) \rightarrow U_q(\mathfrak{g})$ . Specifically,  $V \otimes Z$  is completely reducible if and only if the form is non-degenerate when restricted to the span  $(V \otimes Z)^+ \subset V \otimes Z$  of singular (extremal) vectors. In this paper, we develop an efficient computational method for practical use of that criterion. It reveals a close relation of the form with extremal projector [2,18], which was pointed out in some special cases in [20].

The subspace  $(V \otimes Z)^+$  supports a trivial representation of a subalgebra  $U_q(\mathfrak{g}_+) \subset U_q(\mathfrak{g})$ , the quantized universal enveloping algebra of the nilpotent Lie subalgebra  $\mathfrak{g}_+$  spanned by the positive root spaces. Our approach is a version of the method of “ $Z$ -invariants” (here  $Z$  is the maximal unipotent subgroup in an algebraic group with Lie algebra  $\mathfrak{g}$ ) of [26], with a stipulation that tensor products of irreducible modules

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are not necessarily completely reducible if one of them is infinite dimensional. The question of complete reducibility of such tensor products is challenging in general and it is of our prime interest, while the special case of finite dimensional modules also plays a role in our approach albeit in the converse direction: complete reducibility, that is due to the Weyl theorem [11] and its  $q$ -counterpart [12], implies non-degeneracy of linear operators involved.

The space  $(V \otimes Z)^+$  is embedded in  $V \otimes Z$  in a complicated way, so we parameterize it with a certain subspace in one of the tensor factors, e.g. in  $V$  (the choice of a factor is a matter of convenience). There is a linear isomorphism between  $(V \otimes Z)^+$  and  $\text{Hom}_{U_q(\mathfrak{g}_+)}(*Z, V)$ , where  $*Z$  is a restricted (right) dual to  $Z$ . At the same time, there is a module of lowest weight  $Z'$  and an invariant pairing  $Z \otimes Z' \rightarrow \mathbb{C}$  giving rise to a homomorphism  $Z' \rightarrow *Z$ . This pairing is unique up to a scalar multiplier and non-degenerate if and only if  $Z$  (and therefore  $Z'$ ) is irreducible; then the homomorphism  $Z' \rightarrow *Z$  becomes an isomorphism.

For irreducible  $Z$ , the set  $\text{Hom}_{U_q(\mathfrak{g}_+)}(*Z, V)$  can be characterized as the kernel  $V_Z^+ \subset V$  of the left ideal  $I_Z^+ \subset U_q(\mathfrak{g}_+)$  annihilating the lowest vector in  $Z'$ . This establishes a linear bijection between  $(V \otimes Z)^+$  and  $V_Z^+$ . We consider pull-back of the contravariant form from  $(V \otimes Z)^+$  to  $V_Z^+$  which we call extremal twist. Regarded as a linear map from  $V_Z^+$  to its dual vector space  $V/\omega(I_Z^+)V$ , it relates two natural constructions of singular vectors in  $V \otimes Z$ : via the inverse invariant form on  $Z' \otimes Z$  and via the extremal projector.

The extremal twist can be obtained as a representation of a universal element  $\Theta_Z$  in a certain extension of  $U_q(\mathfrak{g})$ , which itself can be expressed through the inverse invariant form lifted from  $Z' \otimes Z$  to  $U_q(\mathfrak{g}_+) \otimes U_q(\mathfrak{g}_-)$ . The element  $\Theta_Z$  appeared before in the theory of dynamical twist, for  $Z$  a parabolic Verma module relative to a Levi subalgebra  $\mathfrak{k} \subset \mathfrak{g}$ , cf. [9,15].

When  $\mathfrak{k}$  is the Cartan subalgebra  $\mathfrak{h} \subset \mathfrak{g}$  and  $Z$  is an ordinary Verma module, the inverse element  $\Theta_Z^{-1}$  participated in construction of dynamical Weyl group in [9]. It equals the shifted extremal projector  $p_{\mathfrak{g}}(\zeta)$  of  $U_q(\mathfrak{g})$ , by the highest weight  $\zeta$  of  $Z$ . We extend that relation to all irreducible  $Z$  of highest weight, provided certain regularity assumptions on the operator  $p_{\mathfrak{g}}(\zeta)$  as a rational trigonometric function of  $\zeta$  are fulfilled. This finding reduces the problem of semi-simplicity of tensor products to computing the determinant of  $p_{\mathfrak{g}}(\zeta)$ . The shifted extremal projector is naturally interpreted as the universal inverse of the contravariant form transferred from  $(\cdot \otimes Z)^+$  to  $\text{Hom}_{U_q(\mathfrak{g}_+)}(Z', \cdot)$ . In this incarnation, as a map between invariants of a left ideal  $J$  and its dual space of coinvariants of the right ideal  $\omega(J)$ , the shifted projector was considered in [19] in relation with representations of Yangians.

As an example, we consider a parabolic Verma module  $Z$  relative to a Levi subalgebra  $U_q(\mathfrak{k}) \subset U_q(\mathfrak{g})$ . Such a module is parabolically induced from a finite dimensional  $U_q(\mathfrak{k})$ -module  $X$  of highest weight  $\zeta$ . The factor  $p_{\mathfrak{k}}(\zeta)$  entering  $p_{\mathfrak{g}}(\zeta)$  is invertible on the subspace of concern for every finite dimensional module  $V$ . Then  $p_{\mathfrak{g}}(\zeta)$  essentially reduces to a product  $p_{\mathfrak{g}/\mathfrak{k}}(\zeta)$  of shifted  $\mathfrak{sl}(2)$ -projectors over the roots from  $R_{\mathfrak{g}}^+ - R_{\mathfrak{k}}^+$ . The universal extremal twist  $\Theta_Z$  coincides with  $p_{\mathfrak{g}/\mathfrak{k}}^{-1}(\zeta)$  up to an invertible factor which degenerates to 1 for scalar  $X$ . Poles of  $p_{\mathfrak{g}/\mathfrak{k}}^{-1}(\zeta)$  correspond to reducible  $Z$ .

Extremal twist relates the inverse invariant form on a module of highest weight with extremal projector. This relation allows us to give an alternative expression of an equivariant star product on conjugacy classes with Levi stabilizer in terms of extremal projector. Earlier results were using dynamical twist or, equivalently, the inverse invariant form [1,6–8,16,17], whose explicit expression is unknown for a general parabolic Verma module. Our presentation of the star product benefits from explicitly formulated extremal projectors.

As another application, we compute the extremal twist in the case when  $Z$  is the base module for a quantum sphere  $S^{2n}$ , [22], and thereby prove that all tensor products  $V \otimes Z$  with finite dimensional quasi-classical  $U_q(\mathfrak{so}(2n+1))$ -modules  $V$  are completely reducible.

## 2. Quantized universal enveloping algebras

Suppose that  $\mathfrak{g}$  is a semi-simple complex Lie algebra and  $\mathfrak{h} \subset \mathfrak{g}$  its Cartan subalgebra. Fix a triangular decomposition  $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{h} \oplus \mathfrak{g}_+$  with nilpotent Lie subalgebras  $\mathfrak{g}_{\pm}$ . Denote by  $R$  the root system of  $\mathfrak{g}$ , and

by  $R^+$  the subset of positive roots with basis  $\Pi \subset R^+$ . Choose an inner product  $(\cdot, \cdot)$  on  $\mathfrak{h}$  as a multiple of the restricted Killing form and transfer it to  $\mathfrak{h}^*$  by duality. For each  $\lambda \in \mathfrak{h}^*$  denote by  $h_\lambda$  a unique element of  $\mathfrak{h}$  such that  $\mu(h_\lambda) = (\mu, \lambda)$ , for all  $\mu \in \mathfrak{h}^*$ . Set  $\lambda^\vee = \frac{2}{(\lambda, \lambda)} \lambda$  for non-zero  $\lambda \in \mathfrak{h}^*$ .

By  $U_q(\mathfrak{g})$  we understand the standard quantum group, cf. [4,3,12]. It is a  $\mathbb{C}$ -algebra with the set of generators  $e_\alpha$ ,  $f_\alpha$ , and  $q^{\pm h_\alpha}$  labeled with  $\alpha \in \Pi$  obeying

$$q^{h_\alpha} e_\beta = q^{(\alpha, \beta)} e_\beta q^{h_\alpha}, \quad [e_\alpha, f_\beta] = \delta_{\alpha, \beta} \frac{q^{h_\alpha} - q^{-h_\alpha}}{q_\alpha - q_\alpha^{-1}}, \quad q^{h_\alpha} f_\beta = q^{-(\alpha, \beta)} f_\beta q^{h_\alpha}, \quad \alpha, \beta \in \Pi,$$

where  $q_\alpha = q^{\frac{(\alpha, \alpha)}{2}}$  and  $q^{h_\alpha} q^{-h_\alpha} = 1 = q^{-h_\alpha} q^{h_\alpha}$ . The elements  $e_\alpha$  and  $e_{-\alpha} = f_\alpha$  satisfy the quantized Serre relations

$$\sum_{k=0}^{1-a_{\alpha\beta}} (-1)^k \binom{1-a_{\alpha\beta}}{k}_{q_\alpha} e_{\pm\alpha}^k e_{\pm\beta} e_{\pm\alpha}^{1-a_{\alpha\beta}-k}, \quad \alpha \neq \beta.$$

We use the notation  $a_{\alpha\beta} = (\alpha^\vee, \beta)$  for the entries of the Cartan matrix and set  $\binom{m}{n}_q = \frac{[m]_q!}{[n]_q! [m-n]_q!}$ , where  $[m]_q! = [1]_q \cdots [m]_q$ , with  $[0]_q! = 1$ . Here and throughout the paper we write  $[z]_q = \frac{q^z - q^{-z}}{q - q^{-1}}$  for  $z \in \mathfrak{h} + \mathbb{C}$ . The complex parameter  $q \neq 0$  is assumed not a root of unity.

We choose a Hopf algebra structure on  $U_q(\mathfrak{g})$  by setting comultiplication on the generators as

$$\Delta(f_\alpha) = f_\alpha \otimes 1 + q^{-h_\alpha} \otimes f_\alpha, \quad \Delta(q^{h_\alpha}) = q^{h_\alpha} \otimes q^{h_\alpha}, \quad \Delta(e_\alpha) = e_\alpha \otimes q^{h_\alpha} + 1 \otimes e_\alpha. \quad (2.1)$$

The antipode  $\gamma$  acts by the assignment  $\gamma(f_\alpha) = -q^{h_\alpha} f_\alpha$ ,  $\gamma(q^{h_\alpha}) = q^{-h_\alpha}$ ,  $\gamma(e_\alpha) = -e_\alpha q^{-h_\alpha}$  extended as an anti-algebra automorphism to entire  $U_q(\mathfrak{g})$ . The counit homomorphism  $\epsilon: U_q(\mathfrak{g}) \rightarrow \mathbb{C}$  returns on the generators  $\epsilon(e_\alpha) = \epsilon(f_\alpha) = 0$ , and  $\epsilon(q^{h_\alpha}) = 1$ .

We will use a Sweedler notation with suppressed summation for the coproducts, e.g.  $\Delta(x) = x^{(1)} \otimes x^{(2)}$  for  $x \in U_q(\mathfrak{g})$ . For general tensors, like a universal R-matrix  $\mathcal{R} \in U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ , we will adopt a similar convention marking their tensor factors without summation, e.g.  $\mathcal{R} = \mathcal{R}_1 \otimes \mathcal{R}_2$ . This trick is broadly used in the quantum groups literature for calculations with tensor factors when the involved operations are linear.

Denote by  $U_q(\mathfrak{h})$ ,  $U_q(\mathfrak{g}_+)$ ,  $U_q(\mathfrak{g}_-)$  the subalgebras in  $U_q(\mathfrak{g})$  generated by, respectively,  $\{q^{\pm h_\alpha}\}_{\alpha \in \Pi}$ ,  $\{e_\alpha\}_{\alpha \in \Pi}$ , and  $\{f_\alpha\}_{\alpha \in \Pi}$ . The algebra  $U_q(\mathfrak{g})$  is a free  $U_q(\mathfrak{g}_-) - U_q(\mathfrak{g}_+)$ -bimodule generated by  $U_q(\mathfrak{h})$  and it features a triangular factorization  $U_q(\mathfrak{g}) = U_q(\mathfrak{g}_-) U_q(\mathfrak{h}) U_q(\mathfrak{g}_+)$ , as in the classical case  $q \rightarrow 1$ .

Quantum Borel subgroups are defined as  $U_q(\mathfrak{b}_\pm) = U_q(\mathfrak{g}_\pm) U_q(\mathfrak{h})$ ; they are Hopf subalgebras in  $U_q(\mathfrak{g})$ .

We will need the following involutive maps on  $U_q(\mathfrak{g})$ . The assignment

$$\sigma: e_\alpha \mapsto f_\alpha, \quad \sigma: f_\alpha \mapsto e_\alpha, \quad \sigma: q^{h_\alpha} \mapsto q^{-h_\alpha} \quad (2.2)$$

extends to an algebra automorphism of  $U_q(\mathfrak{g})$  and coalgebra anti-automorphism. The involution  $\omega = \gamma^{-1} \circ \sigma = \sigma \circ \gamma$  preserves the comultiplication but flips the multiplication.

All  $U_q(\mathfrak{g})$ -modules are assumed left and diagonalizable over  $U_q(\mathfrak{h})$ . Given a module  $V$ , we write  $V[\lambda]$  for its subspace of weight  $\lambda \in \mathfrak{h}^*$ , i.e. the set of vectors  $v \in V$  satisfying  $q^{h_\alpha} v = q^{(\lambda, \alpha)} v$  for all  $\alpha \in \Pi$ . This notation also applies to any  $U_q(\mathfrak{h})$ -module. We denote by  $\Lambda(V) \subset \mathfrak{h}^*$  the set of weights of  $V$ , i.e.  $\lambda \in \Lambda(V)$  if  $V[\lambda] \neq \{0\}$ .

The integral weight lattice of  $\mathfrak{g}$  is denoted by  $\Lambda \subset \mathfrak{h}^*$ . It is an Abelian group generated by fundamental weights  $\pi_\alpha$  satisfying  $(\pi_\alpha, \beta^\vee) = \delta_{\alpha, \beta}$  for  $\alpha, \beta \in \Pi$ . The semigroup of dominant weights is denoted by  $\Lambda^+$ . For every  $U_q(\mathfrak{h})$ -module  $V$  we set  $\Lambda^+(V) = \Lambda^+ \cap \Lambda(V)$ .

A  $U_q(\mathfrak{g})$ -module  $*V$  is called right dual to a module  $V$  if there is a non-degenerate  $U_q(\mathfrak{g})$ -equivariant pairing  $V \otimes *V \rightarrow \mathbb{C}$ . Left dual  $V^*$  is defined similarly, through a pairing  $V^* \otimes V \rightarrow \mathbb{C}$ . Although left and

right duals are different, they are isomorphic. The intertwiner is delivered by the squared antipode, which is an internal automorphism of  $U_q(\mathfrak{g})$ .

### 2.1. Contravariant form on $V \otimes Z$ and extremal twist

In this section we recall a criterion for a tensor product  $V \otimes Z$  to be completely reducible, following [20].

Recall that a  $U_q(\mathfrak{g})$ -module  $Z$  is said to be of highest weight  $\zeta$ , if it is generated by a vector  $1_Z \in Z[\zeta]$  of weight  $\zeta$  that is annihilated by all  $e_\alpha$ ,  $\alpha \in \Pi$ . It is called Verma module if it is free over  $U_q(\mathfrak{g}_-)$ . Similarly one defines modules of lowest weight, replacing  $e_\alpha$  with  $f_\alpha$ .

A symmetric bilinear form  $\langle \cdot, \cdot \rangle$  on a module  $Z$  is called contravariant with respect to involution  $\omega$  if  $\langle xz, w \rangle = \langle z, \omega(x)w \rangle$  for all  $z, w \in Z$  and all  $x \in U_q(\mathfrak{g})$ . It is known that every module of highest weight has a unique, up to a scalar multiplier, contravariant form, which is non-degenerate if and only if the module is irreducible. Here we recall its construction.

Let  $\wp: U_q(\mathfrak{g}) \rightarrow U_q(\mathfrak{h})$  denote the projection along the sum  $\mathfrak{g}_-U_q(\mathfrak{g}) + U_q(\mathfrak{g})\mathfrak{g}_+$  of right and left ideals generated by respectively negative and positive simple root vectors. This projection is facilitated by the triangular decomposition  $U_q(\mathfrak{g}) = U_q(\mathfrak{g}_-)U_q(\mathfrak{h})U_q(\mathfrak{g}_+)$ . If  $Z$  is the Verma module with highest weight  $\zeta$  and the highest vector  $1_Z$ , then the form is defined by  $\langle x1_Z, y1_Z \rangle = \zeta(\wp(\omega(x)y))$  for all  $x, y \in U_q(\mathfrak{g})$ . Its kernel is the maximal proper submodule, therefore the form descends to any quotient module.

Suppose that  $X$  is a module of lowest weight  $\xi$  and  $Z$  is a module of highest weight  $\zeta$ . We extend the tensor product  $X \otimes Z$  to a bigger space  $X \hat{\otimes} Z$  as follows. For  $\beta \in \mathbb{Z}\Pi$ , we define  $(X \hat{\otimes} Z)[\xi + \zeta + \beta]$  as the vector space of formal sums of tensors from  $X[\mu + \xi] \otimes Z[-\nu + \zeta]$  over  $\mu, \nu \in \mathbb{Z}_+\Pi$  subject to  $\mu - \nu = \beta$ . Then  $X \hat{\otimes} Z$  consists of finite linear combinations of elements from  $(X \hat{\otimes} Z)[\xi + \zeta + \beta]$  with  $\beta \in \mathbb{Z}\Pi$ . It is easy to see that the  $U_q(\mathfrak{g})$ -action on  $X \otimes Z$  extends to an action on  $X \hat{\otimes} Z$ . We apply such extension to tensor products of any diagonalizable  $U_q(\mathfrak{h})$ -modules with finite dimensional weight spaces whose weights are bounded from below and, respectively, from above.

Given a module  $Z$  of highest weight define its opposite module  $Z'$  of lowest weight as follows. Take  $Z$  for the underlying vector space and the composition  $U_q(\mathfrak{g}) \xrightarrow{\sigma} U_q(\mathfrak{g}) \rightarrow \text{End}(Z)$  for the representation homomorphism. Clearly  $Z'$  is of lowest weight that is negative highest weight of  $Z$  because the involution  $\sigma$  flips the algebras  $U_q(\mathfrak{g}_\pm)$  and inverts  $q^{h_\alpha}$ .

The contravariant form on  $Z$  is equivalent to an invariant pairing  $Z \otimes Z' \rightarrow \mathbb{C}$ , which we call, abusing terminology, the invariant form on  $Z$ . The two forms are related by a linear isomorphism  $\text{id} \otimes \sigma_Z: Z \otimes Z \rightarrow Z \otimes Z'$ , where  $\sigma_Z(f1_Z) = \sigma(f)1_{Z'}$  for  $f \in U_q(\mathfrak{g}_-)$  and  $1_{Z'}$  is the lowest vector in  $Z'$ .

We also consider a restricted (right) dual  ${}^*Z$ , defined on the dual  $U_q(\mathfrak{h})$ -module to  $Z$  as the underlying vector space. Since all weight spaces in  $Z$  are finite dimensional, the  $U_q(\mathfrak{g})$ -action on  ${}^*Z$  can be defined by duality:  $(z, x{}^*z) = (\gamma^{-1}(x)z, {}^*z)$ , for all  $z \in Z$ ,  ${}^*z \in {}^*Z$ , and  $x \in U_q(\mathfrak{g})$ . Thanks to invariant pairing, there is a homomorphism  $Z' \rightarrow {}^*Z$ .

If the invariant form on  $Z$  is non-degenerate, then the map  $Z' \rightarrow {}^*Z$  is an isomorphism, and there exists a  $U_q(\mathfrak{g})$ -invariant element (the inverse form)  $\mathcal{S} \in Z' \hat{\otimes} Z$ . The converse is also true, as stated by the following proposition.

**Proposition 2.1.** *Suppose there exists a  $U_q(\mathfrak{b}_+)$ -invariant element  $\mathcal{S} \in Z' \hat{\otimes} Z$  such that  $\mathcal{S}_1 \langle 1_Z, \mathcal{S}_2 \rangle = 1_{Z'}$ . Then  $Z$  is irreducible, and  $\mathcal{S}$  is the inverse  $U_q(\mathfrak{g})$ -invariant form.*

**Proof.** For any  $f \in U_q(\mathfrak{g}_-)$ , one has  $\omega(f) = (\gamma^{-1} \circ \sigma)(f) \in U_q(\mathfrak{b}_+)$ . Then

$$\mathcal{S}_1 \langle \mathcal{S}_2, f1_Z \rangle = \mathcal{S}_1 \langle (\gamma^{-1} \circ \sigma)(f) \mathcal{S}_2, 1_Z \rangle = \sigma(f) \mathcal{S}_1 \langle \mathcal{S}_2, 1_Z \rangle = \sigma(f)1_{Z'} = \sigma_Z(f1_Z).$$

In other words, the map  $z \mapsto \sigma_Z^{-1}(\mathcal{S}_1)\langle \mathcal{S}_2, z \rangle$  is identical on  $Z$ . Therefore the contravariant form is non-degenerate, and  $\langle z \otimes w, (\sigma_Z^{-1} \otimes \text{id})(\mathcal{S}) \rangle = \langle z, w \rangle$  for all  $z, w \in Z$  as required. Thus,  $(\sigma_Z^{-1} \otimes \text{id})(\mathcal{S})$  is the inverse contravariant form and hence  $\mathcal{S}$  is the inverse invariant form.  $\square$

Suppose that  $Z$  is irreducible and let  $V$  be another irreducible module of highest weight. Denote by  $(V \otimes Z)^+$  the span of singular vectors in  $V \otimes Z$ , i.e. the space of  $U_q(\mathfrak{g}_+)$ -invariants. Define canonical contravariant symmetric bilinear form on  $V \otimes Z$  as the product of contravariant forms on  $V$  and  $Z$ .

**Theorem 2.2** ([20]). *The tensor product  $V \otimes Z$  is completely reducible if and only if the canonical form is non-degenerate when restricted to  $(V \otimes Z)^+$ .*

Note that the form is non-degenerate on the entire  $V \otimes Z$  but may not be so on  $(V \otimes Z)^+$ .

Remark that Theorem 2.2 is proved for simple  $\mathfrak{g}$  but obviously holds for reductive  $\mathfrak{g}$ . It can also be generalized for completely reducible tensor factors in a straightforward way.

In order to compute the restricted form, we use an isomorphism between  $(V \otimes Z)^+$  and the vector space  $\text{Hom}_{U_q(\mathfrak{g}_+)}(*Z, V)$ . For irreducible  $Z$ , the module  $*Z \simeq Z'$  is cyclically generated by its lowest vector,  $1_{Z'}$ . Let  $I_Z^+ \subset U_q(\mathfrak{g}_+)$  denote the left ideal annihilating  $1_{Z'}$ , so that  $Z' \simeq U_q(\mathfrak{g}_+)/I_Z^+$ . Then  $\text{Hom}_{U_q(\mathfrak{g}_+)}(*Z, V)$  is naturally identified with the kernel  $V_Z^+$  of  $I_Z^+$  in  $V$ . The isomorphism  $(V \otimes Z)^+ \rightarrow V_Z^+$  is delivered by restriction of the linear map  $\bar{\delta}: V \otimes Z \rightarrow V$ ,  $v \otimes z \mapsto v\langle 1_Z, z \rangle$ . We denote by  $\delta$  the inverse isomorphism.

The annihilator  $V_Z^\perp$  of  $V_Z^+$  with respect to the contravariant form coincides with  $\omega(I_Z^+)V$ . Indeed, the latter is a  $U_q(\mathfrak{h})$ -invariant subspace in  $V_Z^\perp$ , and  $V_Z^+$  is also  $U_q(\mathfrak{h})$ -invariant. Since every weight subspace in  $V$  is finite dimensional, different weight subspaces are orthogonal, and the form is non-degenerate,  $\omega(I_Z^+)V$  exhausts all of  $V_Z^\perp$ .

**Definition 2.3.** Let  $Z_1, Z_2$  be a pair of vector spaces and  $\phi: Z_1 \rightarrow Z_2$  a linear surjection. A linear map  $\psi: Z_2 \rightarrow Z_1$  is called section of  $\phi$  if  $\phi \circ \psi = \text{id}_{Z_2}$ . For any  $z \in Z_2$  we call  $\psi(z) \in Z_1$  a lift of  $z$  by  $\psi$ .

Regard  $Z$  as a cyclic  $U_q(\mathfrak{g}_-)$ -module and its right dual module  $*Z$  as one over  $U_q(\mathfrak{g}_+)$ . Denote by  $\mathcal{F} \in U_q(\mathfrak{g}_+) \hat{\otimes} U_q(\mathfrak{g}_-)$  a lift of  $\mathcal{S} \in *Z \hat{\otimes} Z$  by a  $U_q(\mathfrak{h})$ -invariant linear section of the  $U_q(\mathfrak{g}_+) \otimes U_q(\mathfrak{g}_-)$ -module epimorphism  $U_q(\mathfrak{g}_+) \hat{\otimes} U_q(\mathfrak{g}_-) \rightarrow *Z \hat{\otimes} Z$ . The element  $\Theta_Z = \gamma^{-1}(\mathcal{F}_2)\mathcal{F}_1$  (Sweedler notation) belongs to a certain extension of  $U_q(\mathfrak{g})$  and gives rise to a linear map  $\theta_{V,Z}: V_Z^+ \rightarrow V/V_Z^\perp$  by

$$\langle \theta_{V,Z}(v), w \rangle = \langle \Theta_Z v, w \rangle,$$

which is independent of the choice of lift  $\mathcal{F}$  for  $\mathcal{S}$ .

**Proposition 2.4** ([20]). *The form  $\langle \theta_{V,Z}(\cdot), \cdot \rangle$  is the pullback of the canonical form under the isomorphism  $\delta: V_Z^+ \rightarrow (V \otimes Z)^+$ .*

The coset space  $V/V_Z^\perp$  can be identified with a  $U_q(\mathfrak{h})$ -invariant subspace  ${}^+V_Z \subset V$  that is transversal to  $V_Z^\perp$ . Consider the sum  $\widehat{V_Z^+} \subset V$  of weight subspaces over weights in  $\Lambda(V_Z^+)$  and take for  ${}^+V_Z$  any  $U_q(\mathfrak{h})$ -invariant subspace in  $\widehat{V_Z^+}$  that is complementary to  $\widehat{V_Z^+} \cap V_Z^\perp$ . Since the contravariant form on  $V$  is non-degenerate, it induces a non-degenerate pairing  $V_Z^+ \otimes {}^+V_Z \rightarrow \mathbb{C}$ .

If the form is non-degenerate when restricted to  $V_Z^+$ , then  $V = V_Z^+ \oplus V_Z^\perp$ , and one can take  ${}^+V_Z = V_Z^+$ . Then  $\theta_{V,Z}$  becomes an operator from  $\text{End}(V_Z^+)$ .

## 2.2. Braid group action on $U_q(\mathfrak{g})$ and a Cartan-Weyl basis

The algebra  $U_q(\mathfrak{g})$  admits a Poincaré-Birkhoff-Witt (PBW)-like basis of ordered monomials in “root vectors”, which are constructed from the generators via an action of the braid group, [3], Ch. 8.1. We will use such a basis in the next sections to express extremal projectors of  $U_q(\mathfrak{g})$ .

Define  $m_{\alpha\beta} = 2, 3, 4, 6$  for  $\alpha, \beta \in \Pi$  if the entries of the Cartan matrix satisfy, respectively,  $a_{\alpha\beta}a_{\beta\alpha} = 0, 1, 2, 3$ . The braid group  $\mathcal{B}_{\mathfrak{g}}$  associated with  $\mathfrak{g}$  is generated by  $T_{\alpha}$ ,  $\alpha \in \Pi$ , subject to the relations  $(T_{\alpha}T_{\beta})^{m_{\alpha\beta}} = (T_{\beta}T_{\alpha})^{m_{\alpha\beta}}$ ,  $\alpha \neq \beta$ .

The group  $\mathcal{B}_{\mathfrak{g}}$  admits a homomorphism onto the Weyl group  $\mathcal{W}$  of the root system  $R$  with the kernel generated by the relations  $T_{\alpha}^2 = 1$ ,  $\alpha \in \Pi$ , and sending  $T_{\alpha}$  to the simple reflections  $\sigma_{\alpha} \in \mathcal{W}$ . The length  $\ell(T)$  of an element of  $T \in \mathcal{B}_{\mathfrak{g}}$  is defined as the minimal number of generators in a presentation of  $T$ , which is called a reduced decomposition of  $T$ . The length of an element of a Weyl group is defined similarly, as the number of simple reflections in a reduced decomposition. There is a length preserving section of the epimorphism  $\mathcal{B}_{\mathfrak{g}} \rightarrow \mathcal{W}$ , which is a map of sets.

The  $T_{\alpha}$ -action on the generators of the quantum group is defined by

$$\begin{aligned} T_{\alpha}(f_{\alpha}) &= -q^{-h_{\alpha}}e_{\alpha}, \quad T_{\alpha}(q^{h_{\beta}}) = q^{h_{\beta}-a_{\alpha\beta}h_{\alpha}}, \quad T_{\alpha}(e_{\alpha}) = -f_{\alpha}q^{h_{\alpha}}, \\ T_{\alpha}(e_{\beta}) &= \sum_{k=0}^{-a_{\alpha\beta}} \frac{(-1)^{a_{\alpha\beta}-k}q^{-k}}{[k]_q![-a_{\alpha\beta}-k]_q!} e_{\alpha}^{-a_{\alpha\beta}-k} e_{\beta} e_{\alpha}^k, \quad \alpha \neq \beta, \\ T_{\alpha}(f_{\beta}) &= \sum_{k=0}^{-a_{\alpha\beta}} \frac{(-1)^{a_{\alpha\beta}-k}q^k}{[k]_q![-a_{\alpha\beta}-k]_q!} f_{\alpha}^k f_{\beta} f_{\alpha}^{-a_{\alpha\beta}-k}, \quad \alpha \neq \beta. \end{aligned}$$

It extends to an algebra automorphism of  $U_q(\mathfrak{g})$ . The operators  $\{T_{\alpha}\}_{\alpha \in \Pi}$  amount to an action of  $\mathcal{B}_{\mathfrak{g}}$  on  $U_q(\mathfrak{g})$ .

**Proposition 2.5** ([3], Prop. 8.1.6). *Let  $w \in \mathcal{W}$  be such that  $\beta = w(\alpha) \in \Pi$  for some simple root  $\alpha$ . Then  $T_w(e_{\alpha}) = e_{\beta}$  and  $T_w(f_{\alpha}) = f_{\beta}$ .*

Let  $\sigma_{i_1} \dots \sigma_{i_N}$ , where  $\sigma_i = \sigma_{\alpha_i}$  and  $N = \#R^+$ , be a reduced decomposition of the longest element of  $\mathcal{W}$ . Define a sequence of positive roots by

$$\mu^1 = \alpha_{i_1}, \quad \mu^2 = \sigma_1(\alpha_{i_2}), \quad \dots, \quad \mu^N = \sigma_1 \dots \sigma_{N-1}(\alpha_{i_N}).$$

This sequence induces a total ordering on  $R^+$  such that any root of the form  $\alpha + \beta$  with  $\alpha, \beta \in R^+$  is between  $\alpha$  and  $\beta$ . Such orderings are called normal and they are in bijection with reduced decompositions of the longest element.

Any subset  $\tilde{\Pi} \subset \Pi$  generates a root subsystem  $\tilde{R} \subset R$ . There is a normal ordering where all roots from  $\tilde{R}^+$  are on the right of the roots from  $R^+ \setminus \tilde{R}^+$ , see e.g. [25], Exercise 1.7.10. We will use this fact in Section 4 in relation with Levi subalgebras in  $\mathfrak{g}$ .

A Cartan-Weyl basis in  $U_q(\mathfrak{g})$  depends on a normal ordering and is defined as follows. The root  $\mu^1$  is simple, so  $e_{\mu^1}$  and  $f_{\mu^1}$  are the corresponding Chevalley generators. For  $k > 1$  set

$$e_{\mu^k} = T_{\alpha_{i_1}} \circ \dots \circ T_{\alpha_{i_{k-1}}}(e_{\alpha_{i_k}}), \quad f_{\mu^k} = T_{\alpha_{i_1}} \circ \dots \circ T_{\alpha_{i_{k-1}}}(f_{\alpha_{i_k}}).$$

Proposition 2.5 guarantees that all simple root generators are included in this set. It is known that normally ordered monomials in these root vectors deliver a PBW-type basis in  $U_q(\mathfrak{g}_{\pm})$  when  $q$  is not a root of unity.

Regarding  $U_q(\mathfrak{g})$  as a  $\mathbb{C}[q, q^{-1}]$ -algebra, define an anti-automorphism  $\tilde{\omega}$  setting it on generators as

$$\tilde{\omega}: e_\alpha \mapsto f_\alpha, \quad \tilde{\omega}: f_\alpha \mapsto e_\alpha, \quad \tilde{\omega}: q^{h_\alpha} \mapsto q^{-h_\alpha}, \quad \tilde{\omega}: q \mapsto q^{-1}. \quad (2.3)$$

It commutes with the action of  $\mathcal{B}_{\mathfrak{g}}$  on  $U_q(\mathfrak{g})$ .

### 2.3. Properties of the Cartan-Weyl basis

For each  $\mu \in R^+$ , one has  $[e_\mu, f_\mu] = a_\mu \frac{q^{h_\mu} - q^{-h_\mu}}{q_\mu - q_\mu^{-1}}$  for some  $a_\mu \in \mathbb{C}^\times$ . This relation facilitates an embedding  $\iota_\mu: U_q(\mathfrak{sl}(2)) \rightarrow U_q(\mathfrak{g})$  determined by the assignment

$$f \mapsto \frac{1}{a_\mu} f_\mu, \quad e \mapsto e_\mu, \quad q^h \mapsto q^{h_\mu}, \quad q \mapsto q_\mu,$$

where  $e, f, q^h$  are the standard generators of  $U_q(\mathfrak{sl}(2))$  satisfying  $q^h e q^{-h} = q^2 e$ ,  $q^h f q^{-h} = q^{-2} f$ , and  $[e, f] = [h]_q$ . We denote by  $U_q(\mathfrak{g}^\mu)$  the image of  $\iota_\mu$  under this embedding.

For  $\alpha, \beta \in R^+$  such that  $\alpha < \beta$ , denote by  $U_{\alpha, \beta}^+$  a right  $U_q(\mathfrak{h})$ -submodule in  $U_q(\mathfrak{g})$  under the multiplication generated by the monomials  $e_{\mu^i}^{k_i} \dots e_{\mu^j}^{k_j}$  with  $\alpha \leq \mu^i < \dots < \mu^j \leq \beta$  and  $\sum_s k_s > 0$ . We set  $U_{\alpha, \beta}^- = \tilde{\omega}(U_{\alpha, \beta}^+)$  and denote  $U_{\alpha \leq}^\pm = U_{\alpha, \mu^\infty}^\pm$  and  $U_{\leq \alpha}^\pm = U_{\mu^1, \alpha}^\pm$ . We will also use the obvious notation  $U_{< \alpha}^\pm$  and  $U_{\alpha <}^\pm$  involving only the root vectors starting with the roots next to  $\alpha$ .

Given two vector subspaces in  $A, B \subset U_q(\mathfrak{g})$  we denote  $A \bullet B = A + B + AB$ , where the last term is the linear span of products of elements from  $A$  and  $B$ .

**Proposition 2.6.** *The  $U_q(\mathfrak{h})$ -modules  $U_{\leq \alpha}^\pm$ ,  $U_{\alpha \leq}^\pm$ , and  $U_{\leq \alpha}^- \bullet U_{\beta \leq}^+$ ,  $U_{\beta \leq}^- \bullet U_{\leq \alpha}^+$  with  $\alpha < \beta$  are associative (non-unital) subalgebras in  $U_q(\mathfrak{g})$ .*

**Proof.** Set  $\mu^i = \alpha$ ,  $\mu^j = \beta$  with  $i < j$  and put  $\alpha' = \mu^{i+1}$ ,  $\beta' = \mu^{j-1}$ , so that  $\alpha < \alpha' \leq \beta' < \beta$ . Then the following relations hold, [18]:

$$[e_\alpha, e_\beta]_{q^{(\alpha, \beta)}} \in U_{\alpha', \beta'}^+, \quad [f_\beta, f_\alpha]_{q^{-(\alpha, \beta)}} \in U_{\alpha', \beta'}^-, \quad [e_\beta, f_\alpha] \in U_{< \alpha}^- \bullet U_{\beta <}^+, \quad [e_\alpha, f_\beta] \in U_{\beta <}^- \bullet U_{< \alpha}^+. \quad (2.4)$$

Note that the second and fourth inclusions are obtained from the first and third by applying the automorphism  $\tilde{\omega}$ , which flips  $U_{\mu, \nu}^+$  and  $U_{\mu, \nu}^-$ . Now the proof is straightforward.  $\square$

Note that algebras from Proposition 2.6 have zero subspace of  $U_q(\mathfrak{h})$ -invariants under the adjoint action.

**Proposition 2.7.** *For all  $\mu \in R^+$ ,*

$$\Delta(e_\mu) \in e_\mu \otimes q^{h_\mu} + 1 \otimes e_\mu + U_{\mu <}^+ \otimes U_{< \mu}^+, \quad \Delta(f_\mu) \in f_\mu \otimes 1 + q^{-h_\mu} \otimes f_\mu + U_{\mu <}^- \otimes U_{\mu <}^-.$$

**Proof.** There is an invertible element  $\tilde{\mathcal{R}}_\mu \in 1 \otimes 1 + U_{\mu <}^- \hat{\otimes} U_{< \mu}^+$  such that the coproduct  $\Delta(e_\mu)$  can be expressed as

$$\Delta(e_\mu) = \tilde{\mathcal{R}}_{\mu <} (e_\mu \otimes q^{h_\mu} + 1 \otimes e_\mu) \tilde{\mathcal{R}}_{\mu <}^{-1}.$$

This can be found in [18], Proposition 8.3 (for a twisted coproduct as compared to (2.1), so our  $\tilde{\mathcal{R}}$  is flipped). As  $\Delta(e_\mu) \in U_q(\mathfrak{g}_+) \otimes U_q(\mathfrak{b}_+)$ , it suffices to evaluate both sides of this equality on the tensor product of the right “universal Verma modules”, i.e. the quotients of  $U_q(\mathfrak{g})$  by the right ideal  $J$  generated by  $f_\alpha$ ,  $\alpha \in \Pi$ :

$$\Delta(e_\mu) = (e_\mu \otimes q^{h_\mu}) \tilde{\mathcal{R}}_{\mu <}^{-1} + 1 \otimes e_\mu \pmod{J \otimes J}.$$



Pushing the left leg of  $\tilde{\mathcal{R}}_{<\mu}^{-1}$  to the left with the use of the third inclusion from (2.4) one proves the left equality. The right one is obtained by applying  $\tilde{\omega}$ , which flips the comultiplication.  $\square$

We derive the following consequence with regard to the projection  $\wp: U_q(\mathfrak{g}) \rightarrow U_q(\mathfrak{h})$ .

**Proposition 2.8.** *For any ordered sequence of positive roots  $\mu^i < \dots < \mu^k$  the map  $\wp$  annihilates  $U_{\mu^i \leq}^- \bullet U_{\leq \mu^i}^+ \bullet \dots \bullet U_{\mu^k \leq}^- \bullet U_{\leq \mu^k}^+$ .*

**Proof.** The statement follows from the inclusion

$$U_{i \leq}^- \bullet U_{\leq i}^+ \bullet \dots \bullet U_{k \leq}^- \bullet U_{\leq k}^+ \subset U_{i \leq}^- \bullet U_{\leq k}^+, \quad (2.5)$$

where we write  $U_{i \leq}^- = U_{\mu^i \leq}^-$  and  $U_{\leq i}^+ = U_{\leq \mu^i}^+$ . It is clearly true for  $k = i$ . Suppose it is proved for  $k \geq i$ . Then

$$U_{i \leq}^- \bullet U_{\leq i}^+ \bullet \dots \bullet U_{k+1 \leq}^- \bullet U_{\leq k+1}^+ \subset U_{i \leq}^- \bullet U_{\leq k}^+ \bullet U_{k+1 \leq}^- \bullet U_{\leq k+1}^+ \subset U_{i \leq}^- \bullet U_{k+1 \leq}^- \bullet U_{\leq k}^+ \bullet U_{\leq k+1}^+.$$

The right inclusion is due to Proposition (2.6). The result is contained in  $U_{i \leq}^- \bullet U_{\leq k+1}^+$ , again by Proposition (2.6). Induction on  $k$  completes the proof.  $\square$

### 3. Extremal twist and extremal projector

We start with the case of  $\mathfrak{g} = \mathfrak{sl}(2)$  and normalize the inner product so that  $(\alpha, \alpha) = 2$  for its only positive root  $\alpha$ . Set  $e = e_\alpha$ ,  $f = f_\alpha$ , and  $q^{\pm h} = q^{\pm h_\alpha}$  to be the standard generators of  $U_q(\mathfrak{g})$ . Extend  $U_q(\mathfrak{g})$  to  $\hat{U}_q(\mathfrak{g})$  by including infinite sums of elements from  $\mathbb{C}[f]\mathbb{C}[e]$  of same weights with coefficients in the field of fractions  $\mathbb{C}(q^{\pm h})$ . Similar extension works for general semi-simple  $\mathfrak{g}$  making  $\hat{U}_q(\mathfrak{g})$  an associative algebra, see e.g. [18].

Define an element  $p(t) \in \hat{U}_q(\mathfrak{sl}(2))$  depending on a complex parameter  $t$  by

$$p(t) = \sum_{k=0}^{\infty} f^k e^k \frac{(-1)^k q^{k(t-1)}}{[k]_q! \prod_{i=1}^k [h+t+i]_q} \in \hat{U}_q(\mathfrak{g}). \quad (3.6)$$

It is  $U_q(\mathfrak{h})$ -invariant and stable under the involution  $\omega$ .

For every module  $V$  with locally nilpotent action of  $e$ , the function  $t \mapsto p(t)$  is a rational trigonometric endomorphism of every weight space. On a module of highest weight  $\lambda$ , it acts by

$$p(t)v = c \prod_{k=1}^l \frac{[t-k]_q}{[t+\xi(h)+k]_q} v, \quad (3.7)$$

where  $v$  is a vector of weight  $\xi = \lambda - l\alpha$  and  $c = q^{-l\xi(h)+2l^2+l(l-1)} \neq 0$ .

For general  $\mathfrak{g}$  and  $\mu \in \mathbb{R}^+$  let  $p_\mu(t)$  denote the image of  $p(t)$  in  $\hat{U}_q(\mathfrak{g})$  under the embedding  $\iota_\mu: \hat{U}_q(\mathfrak{sl}(2)) \rightarrow \hat{U}_q(\mathfrak{g})$ . Put  $\lambda_i = 2 \frac{(\lambda, \mu^i)}{(\mu^i, \mu^i)} \in \mathbb{C}$  for  $\lambda \in \mathfrak{h}^*$  and  $\mu^i \in \mathbb{R}^+$ . Define

$$p_{\mathfrak{g}}(\lambda) = p_{\mu^1}(\rho_1 + \lambda_1) \cdots p_{\mu^N}(\rho_N + \lambda_N), \quad (3.8)$$

assuming  $\{\mu^i\}_{i=1}^N$  normally ordered. It is independent of a normal ordering and turns to the extremal projector  $p_{\mathfrak{g}}$  at  $\lambda = 0$ , [2,18], which is the only element of  $\hat{U}_q(\mathfrak{g})$  satisfying

$$p_{\mathfrak{g}}^2 = p_{\mathfrak{g}}, \quad e_\alpha p_{\mathfrak{g}} = 0 = p_{\mathfrak{g}} f_\alpha, \quad \forall \alpha \in \Pi.$$



Uniqueness implies that  $p_{\mathfrak{g}}$  is  $\omega$ -invariant.

Let  $V$  and  $W$  be vector spaces. Suppose that  $\mathbb{C}^k \ni \lambda \mapsto F(\lambda) \in \text{Hom}(W, V)$  is a rational trigonometric function. We say that  $F(\lambda_0)$  admits a regularization if there is  $\eta \in \mathbb{C}^k$  such that the function  $\mathbb{C} \ni t \mapsto F(\lambda_0 + t\eta)$ , is regular at  $t = 0$ . If its value is independent of  $\eta$ , then we say that  $F(\lambda_0)$  is well defined.

Furthermore, we say that the extremal projector  $p_{\mathfrak{g}}$  admits a regularization on a subspace  $W$  of a  $U_q(\mathfrak{g})$ -module  $V$  if so does  $p_{\mathfrak{g}}(\lambda)$  at  $\lambda = 0$  and the image of the regularized map  $W \rightarrow V$  is in  $U_q(\mathfrak{g}_+)$ -invariants.

Suppose that  $V$  and  $Z$  are modules of highest weights  $\nu$  and, respectively,  $\zeta$ . Let  $1_V \in V$ ,  $1_Z \in Z$  be their highest vectors. In the following proposition, we do not assume  $V$  and  $Z$  irreducible (we will need this generalization in Section 4). By  $I_Z^+$  we mean the left ideal in  $U_q(\mathfrak{g}_+)$  annihilating the lowest vector in  $Z'$  and by  $V_Z^+ \subset V$  the kernel of  $I_Z^+$ . This is consistent with the case when  $Z$  is irreducible.

**Proposition 3.1.** *Suppose that  $W \subset V$  is a  $U_q(\mathfrak{h})$ -invariant vector subspace such that  $p_{\mathfrak{g}}: W \otimes 1_Z \rightarrow (V \otimes Z)^+$  admits a regularization. Then  $p_{\mathfrak{g}}(\zeta): W \rightarrow V_Z^+$  admits a regularization too. Furthermore,*

$$p_{\mathfrak{g}}(\zeta)(w) = \bar{\delta} \circ p_{\mathfrak{g}}(w \otimes 1_Z), \quad \forall w \in W. \quad (3.9)$$

**Proof.** By Proposition 2.7, for all  $\alpha \in \mathbb{R}^+$  and all  $n \in \mathbb{Z}_+$ , the coproducts satisfy

$$\Delta(f_{\alpha}^n) = f_{\alpha}^n \otimes 1 \mod U_q(\mathfrak{g}) \otimes U_{\alpha \leq}^-, \quad \Delta(e_{\alpha}^n) = e_{\alpha}^n \otimes q^{nh_{\alpha}} \mod U_q(\mathfrak{g}) \otimes U_{\alpha \leq}^+.$$

With  $\eta \in \mathfrak{h}^*$ ,  $t \in \mathbb{C}$ , and  $w \in W$ , evaluation of  $\bar{\delta}(p_{\mathfrak{g}}(w \otimes 1_Z)) = p_{\mathfrak{g}}^{(1)}(t\eta)w \otimes \langle 1_Z, p_{\mathfrak{g}}^{(2)}(t\eta)1_Z \rangle$  (we use the Sweedler notation for the coproduct) reduces to the replacement

$$\Delta(q^{h_{\alpha}}) \rightarrow q^{h_{\alpha}} \otimes q^{h_{\alpha}}, \quad \Delta(f_{\alpha}) \rightarrow f_{\alpha} \otimes 1, \quad \Delta(e_{\alpha}) \rightarrow e_{\alpha} \otimes q^{h_{\alpha}}$$

in  $\Delta(p_{\mathfrak{g}}(t\eta))$  for each  $\alpha \in \mathbb{R}^+$ , because the remainder vanishes in view of Proposition 2.8. This calculation returns  $p_{\mathfrak{g}}(\zeta + t\eta)(w)$ , which forces (3.9).

To show that  $p_{\mathfrak{g}}(\zeta)V \subset V_Z^+$ , pick up  $x \in I_Z^+$  and apply it to  $\bar{\delta} \circ p_{\mathfrak{g}}(w \otimes 1_Z)$ . Since  $p_{\mathfrak{g}}(w \otimes 1_Z)$  is  $x$ -invariant, we have

$$xp_{\mathfrak{g}}^{(1)}(w) \langle 1_Z, p_{\mathfrak{g}}^{(2)}1_Z \rangle = p_{\mathfrak{g}}^{(1)}(w) \langle 1_Z, \gamma^{-1}(x)p_{\mathfrak{g}}^{(2)}1_Z \rangle = p_{\mathfrak{g}}^{(1)}(w) \langle \omega \circ \gamma^{-1}(x)1_Z, p_{\mathfrak{g}}^{(2)}1_Z \rangle = 0$$

because  $\omega \circ \gamma^{-1} = \sigma$  and  $\sigma(x) \in I_Z^-$ , the annihilator of  $1_Z$ .  $\square$

From now to the end of the section we assume that the map

$$\pi: v \mapsto p_{\mathfrak{g}}(v \otimes 1_Z) \in (V \otimes Z)^+ \quad (3.10)$$

admits a regularization on  ${}^+V_Z$ . Then we have a map

$$\bar{\theta}_{V,Z} = \bar{\delta} \circ \pi: {}^+V_Z \rightarrow V_Z^+. \quad (3.11)$$

It is an immediate corollary of Proposition 3.1 that  $\bar{\theta}_{V,Z}$  defines a symmetric bilinear form  $\langle \bar{\theta}_{V,Z}(\cdot), \cdot \rangle$  on  ${}^+V_Z$ , which is the pull-back of the canonical form on  $(V \otimes Z)^+$  under (3.10). We will prove that this form is essentially the inverse to the form determined by  $\theta_{V,Z}$ . In a special context, it was considered by Khoroshkin and Nazarov, [19], Proposition 2.3.

**Theorem 3.2.** *Suppose that  $V$  and  $Z$  are irreducible. Then the bilinear forms  $\langle \theta_{V,Z}(\cdot), \cdot \rangle$  and  $\langle \bar{\theta}_{V,Z}(\cdot), \cdot \rangle$  are non-degenerate simultaneously. In that case, they are inverse to each other.*

**Proof.** Suppose that  $\theta_{V,Z}$  is invertible and compute  $\langle (\delta \circ \bar{\theta}_{V,Z} \circ \theta_{V,Z})(v), \delta(w) \rangle$  for a pair of vectors  $v, w \in V_Z^+$  in two different ways (we put  $u = \delta(w)$  for short below).

i) Applying the definition (3.11) we find the matrix element equal to  $\langle p_{\mathfrak{g}}(\theta_{V,Z}(v) \otimes 1_Z), u \rangle$ . Presenting  $p_{\mathfrak{g}} = 1 + \sum_i \phi_i \psi_i$ , where  $\phi_i \in U_q(\mathfrak{g}_-)$  and  $\psi_i \in U_q(\mathfrak{b}_+)$  carry non-zero weights, we continue with

$$\langle \theta_{V,Z}(v) \otimes 1_Z, u \rangle + \sum_i \langle \phi_i \psi_i(\theta_{V,Z}(v) \otimes 1_Z), u \rangle = \langle \theta_{V,Z}(v) \otimes 1_Z, u \rangle = \langle \theta_{V,Z}(v), w \rangle = \langle v, \theta_{V,Z}(w) \rangle.$$

The sum on the left vanishes because  $\langle \phi_i \psi_i(\dots), u \rangle = \langle \psi_i(\dots), \omega(\phi_i)u \rangle = 0$ .

ii) By Proposition 2.4 the matrix element in question equals  $\langle \bar{\theta}_{V,Z} \circ \theta_{V,Z}(v), \theta_{V,Z}(w) \rangle$ . Since  $\theta_{V,Z}$  is invertible, the image of  $\theta_{V,Z}$  is entire  ${}^+V_Z$ , and  $\bar{\theta}_{V,Z} \circ \theta_{V,Z} = \text{id}$  on  $V_Z^+$ .

Now suppose that  $\bar{\theta}_{V,Z}$  is invertible and evaluate  $\langle \delta \circ \bar{\theta}_{V,Z}(v), \delta \circ \bar{\theta}_{V,Z}(w) \rangle$  for  $v, w \in {}^+V_Z$  in two different ways as follows.

i) On the one hand, it is equal to

$$\langle p_{\mathfrak{g}}(v \otimes 1_Z), p_{\mathfrak{g}}(w \otimes 1_Z) \rangle = \langle v \otimes 1_Z, \omega(p_{\mathfrak{g}}) \circ p_{\mathfrak{g}}(w \otimes 1_Z) \rangle = \langle v, \bar{\delta}(p_{\mathfrak{g}}(w \otimes 1_Z)) \rangle = \langle v, \bar{\theta}_{V,Z}(w) \rangle.$$

ii) On the other hand, it equals  $\langle \theta_{V,Z} \circ \bar{\theta}_{V,Z}(v), \bar{\theta}_{V,Z}(w) \rangle$  by Proposition 2.4. Since the image of  $\bar{\theta}_{V,Z}$  is  $V_Z^+$ , one arrives at  $\theta_{V,Z} \circ \bar{\theta}_{V,Z} = \text{id}$  on  ${}^+V_Z$ .  $\square$

It follows that the regularization  $\lim_{t \rightarrow 0} p_{\mathfrak{g}}(t\eta)|_{V_Z^+ \otimes 1_Z}$  is independent of  $\eta$  if the contravariant form is non-degenerate on  $(V \otimes Z)^+$ .

### 3.1. On regularization of extremal projector

Regularization of the extremal projector is crucial for application of Theorem 3.2 to calculation of the extremal twist. In this section we point out some facts of practical use.

It is natural to employ decomposition of  $p_{\mathfrak{g}}(\lambda)$  to a product of the root factors (3.8) and process them separately.

**Lemma 3.3.** *Let  $V$  be a  $U_q(\mathfrak{g})$ -module and put  $W = V[\mu]$  for some weight  $\mu \in \Lambda(V)$ . Fix a normal order on  $R^+$  and suppose that  $p_{\alpha}(\rho_{\alpha})$  with  $\rho_{\alpha} = (\rho, \alpha^{\vee})$  are well defined on  $W$  for all  $\alpha \in R^+$ . Then the operator  $p_{\mathfrak{g}}(0) = \prod_{\alpha \in R^+}^< p_{\alpha}(\rho_{\alpha})$  is well defined on  $W$  and independent of the normal order.*

**Proof.** Each factor in  $p_{\mathfrak{g}}(\lambda)$  corresponding to a root  $\alpha \in R^+$  depends on  $\lambda$  through a regular function  $\lambda \mapsto (\lambda, \alpha^{\vee})$  and is well defined once admits a regularization. This implies the assertion.  $\square$

Note that one has to consider the entire weight space for  $W$  because it is *a priori* invariant under all root factors in  $p_{\mathfrak{g}}(\lambda)$ . Clearly the statement holds true for  $W$  a sum of weight spaces.

**Lemma 3.4.** *For any  $r \in \mathbb{C}$  the operator  $p_{\alpha}(r)$ ,  $\alpha \in R^+$ , is well defined on a subspace of weight  $\xi$  satisfying  $(\xi, \alpha^{\vee}) + r \in \mathbb{Z}_+$ , in any  $U_q(\mathfrak{g}_+^{\alpha})$ -locally finite  $U_q(\mathfrak{g}^{\alpha})$ -module.*

**Proof.** Denominators  $\prod_{i=1}^k [(\xi, \alpha^{\vee}) + r + i]_{q_{\alpha}}$  in (3.6) do not vanish on such weight spaces ( $q$  is not a root of unity).  $\square$

Although Lemma 3.4 is rather crude it proves to be useful. We also need more delicate criteria, rather in a more special situation.

**Lemma 3.5.** *Let  $V$  be a finite dimensional  $U_q(\mathfrak{g}^\alpha)$ -module,  $\alpha \in \mathbb{R}^+$ . For any  $r \in \mathbb{N}$  the operator  $p_\alpha(r)$  is well defined on  $V$ .*

**Proof.** We can assume that  $V$  is irreducible. Let  $\mu = \frac{m}{2}\alpha$ ,  $m \in \mathbb{Z}_+$ , be the highest weight of  $V$ , and  $t \in \mathbb{C}$ . The eigenvalue of  $p_\alpha(t+r)$  on the subspace of weight  $\mu - l\alpha$  with  $0 \leq l \leq m$  is proportional to  $\frac{\prod_{k=1}^l [t+r-k]_{q_\alpha}}{\prod_{k=0}^{l-1} [t+r+m-l-k]_{q_\alpha}}$ , cf. (3.7). As  $t \rightarrow 0$ , the denominator may have the only vanishing factor that corresponds to non-negative  $k = m+r-l \leq l-1$ . But it is canceled by a factor in the numerator unless  $r > l$  which contradicts the previous inequality in view of  $l \leq m$ .  $\square$

As a consequence, we obtain the following important special case.

**Proposition 3.6.** *The extremal projector  $p_{\mathfrak{g}}$  is well defined on every dominant weight space of a locally finite  $U_q(\mathfrak{g})$ -module  $V$ .*

**Proof.** We can assume that  $V$  is irreducible. Pick up  $\alpha \in \mathbb{R}^+$  and set  $\rho_\alpha = (\rho, \alpha^\vee)$ . By Lemma 3.5, the operator  $p_\alpha(\rho_\alpha)$  is well defined on  $V$ . If  $\alpha$  is simple, then  $\rho_\alpha = 1$  and  $p_\alpha(\rho_\alpha)$  projects  $V[\xi]$  to  $\ker e_\alpha$  for all  $\xi \in \Lambda^+(V)$  because  $(\xi, \alpha^\vee) \geq 0$ . This follows from (3.7) for every irreducible  $U_q(\mathfrak{g}^\alpha)$ -submodule in  $V$ .

Thus all root factors in  $p_{\mathfrak{g}}(0)$  are well defined by Lemma 3.5 and so is  $p_{\mathfrak{g}}(0)$ , by Lemma 3.3. For each simple  $\alpha$  choose a normal order such that  $\alpha$  is in the left-most position. Then  $p_{\mathfrak{g}}(0)$  has the factor  $p_\alpha(1)$  on the left that maps all  $V[\xi]$  with  $\xi \in \Lambda^+(V)$  to  $\ker e_\alpha$ . Since  $p_{\mathfrak{g}}(0)$  is independent of the normal order, it takes values in  $U_q(\mathfrak{g}_+)$ -invariants when restricted to dominant weight spaces of  $V$ .  $\square$

Remark that although  $p_{\mathfrak{g}}(0)$  is well defined on every finite dimensional module by Lemmas 3.3 and 3.5, non-dominant weight spaces are not generally killed by  $p_{\mathfrak{g}}(0)$ , so it is not a projector to  $U_q(\mathfrak{g}_+)$ -invariants. That can be seen already for  $\mathfrak{g} = \mathfrak{sl}(2)$  by examining (3.7) on a module  $V$  of  $\dim V > 2$  and taking  $\xi$  such that  $\xi(h) \leq -2$ .

#### 4. The case of parabolic Verma modules

In this section we compute extremal twist for tensor products of finite dimensional and parabolic Verma modules. The key issue is regularization of the operator  $\Delta(p_{\mathfrak{g}})$  restricted to a certain subspace in the tensor product. We address it relaxing the assumption that parabolic modules involved are irreducible. We will later use the regularized extremal projector to prove irreducibility for a dense open set of highest weights, calculate the extremal twist and do analytical extension to cover all irreducible parabolic Verma modules.

##### 4.1. Regularization of extremal projector

Let  $\mathfrak{k} \subset \mathfrak{g}$  be a Levi subalgebra, i.e. a reductive Lie algebra of maximal rank whose basis of simple roots  $\Pi_{\mathfrak{k}}$  is a subset in  $\Pi_{\mathfrak{g}} = \Pi$ . A weight  $\lambda \in \mathfrak{h}^*$  defines a one-dimensional  $U_q(\mathfrak{k})$ -module  $\mathbb{C}_\lambda$  if and only if  $q^{\lambda(h_\alpha)} = \pm 1$  for all  $\alpha \in \Pi_{\mathfrak{k}}$ . We will assume plus as the other cases can be covered by tensoring with appropriate one-dimensional  $U_q(\mathfrak{g})$ -modules, cf. [3]. Such representations are trivial on  $U_q(\mathfrak{g}_\pm)$  and assign  $\pm 1$  to all  $q^{h_\alpha}$ ,  $\alpha \in \Pi_{\mathfrak{g}}$ .

Let  $\mathfrak{c} \subset \mathfrak{h}$  denote the center of  $\mathfrak{k}$  and  $\mathfrak{c}^* \subset \mathfrak{h}^*$  the subset of weights orthogonal to all  $\alpha \in \Pi_{\mathfrak{k}}$ . Identify the Cartan subalgebra of the semi-simple part of  $\mathfrak{k}$  with the orthogonal complement of  $\mathfrak{c}$  in  $\mathfrak{h}$ . Then the weight lattice  $\Lambda_{\mathfrak{k}}$  of  $\mathfrak{k}$  is a subset of  $\nu \in \mathfrak{h}^*$  such that  $(\nu, \alpha^\vee) \in \mathbb{Z}$  for all  $\alpha \in \Pi_{\mathfrak{k}}$  and  $(\nu, \lambda) = 0$  for all  $\lambda \in \mathfrak{c}^*$ . Note with care that  $\Lambda_{\mathfrak{k}}$  is not embedded as a sublattice in  $\Lambda_{\mathfrak{g}}$  this way.

Fix  $\xi \in \Lambda_{\mathfrak{k}}^+$ ,  $\lambda \in \mathfrak{c}^*$  and put  $\zeta = \xi + \lambda$ . Denote by  $X$  the finite dimensional  $U_q(\mathfrak{k})$ -module of highest weight  $\xi$ . Set  $Z$  to be the quotient of the Verma module  $\hat{M}_\zeta$  of highest weight  $\zeta$  with the highest vector  $1_\zeta$  by the

sum of the submodules  $U_q(\mathfrak{g})f_\alpha^{m_\alpha+1}1_\zeta$ , where  $m_\alpha = (\xi, \alpha^\vee) \in \mathbb{Z}_+$  for all  $\alpha \in \Pi_\mathfrak{k}$ . We keep the notation  $1_Z$  for the highest vector in  $Z$ . The module  $Z$  is locally finite over  $U_q(\mathfrak{k})$ , [21], and  $U_q(\mathfrak{k})1_Z \simeq X \otimes \mathbb{C}_\lambda$ .

Denote by  $I_Z^- \subset U_q(\mathfrak{g}_-)$  the annihilator of the highest vector  $1_Z \in Z$ . It is independent of  $\lambda$ , as well as the left ideal  $I_Z^+ = \sigma(I_Z^-) \subset U_q(\mathfrak{g}_+)$  killing the lowest vector in  $Z'$ . We set  $V_Z^+$  to be the kernel of  $I_Z^+$  in  $V$  regardless whether  $Z$  is irreducible or not, as in Proposition 3.1. We define  ${}^+V_Z$  as a  $U_q(\mathfrak{h})$ -invariant subspace in  $V$  that is dual to  $V_Z^+$  with respect to the contravariant form on  $V$ , similarly as we did in Section 2.

The left ideal  $I_V^+ \subset U_q(\mathfrak{g}_+)$  is generated by  $e_\alpha^{m_\alpha+1}$  with  $\alpha \in \Pi_\mathfrak{k}$ . Observe that exactly the same set of elements generates the left ideal in  $U_q(\mathfrak{k}_+)$  annihilating the highest vector in  $X$ . Regarding  $V$  as a  $U_q(\mathfrak{k})$ -module we conclude that  $V_Z^+ = V_X^+$ , where  $V_X^+ \simeq \text{Hom}_{U_q(\mathfrak{k}_+)}(*X, V)$  is parameterizing  $U_q(\mathfrak{k})$ -singular vectors in  $V \otimes X$ . We can also set  ${}^+V_X = {}^+V_Z$ .

Denote by  $\mathfrak{c}_{reg}^*$  the set of weights  $\lambda \in \mathfrak{c}^*$  such that  $(\lambda, \alpha^\vee) \notin \mathbb{Z}$  for all  $\alpha \in R_{\mathfrak{g}/\mathfrak{k}}^+ = R_{\mathfrak{g}}^+ - R_{\mathfrak{k}}^+$ . It is an open subset in  $\mathfrak{c}^*$  in the Euclidean topology. Choose a normal ordering on  $R_{\mathfrak{g}}^+$  such that  $R_{\mathfrak{g}/\mathfrak{k}}^+ < R_{\mathfrak{k}}^+$ . Denote by  $p_\mathfrak{k}(\zeta)$  the shifted extremal projector of  $U_q(\mathfrak{k})$  and by  $p_{\mathfrak{g}/\mathfrak{k}}(\zeta)$  the ordered product

$$p_{\mathfrak{g}/\mathfrak{k}}(\zeta) = \prod_{\mu^i \in R_{\mathfrak{g}/\mathfrak{k}}^+} p_{\mu^i}(\rho_i + \zeta_i).$$

Note that  $p_\mathfrak{k}(\zeta) = p_\mathfrak{k}(\xi)$  is independent of the summand  $\lambda \in \mathfrak{c}^*$ . The factorization  $p_\mathfrak{g}(\zeta) = p_{\mathfrak{g}/\mathfrak{k}}(\zeta)p_\mathfrak{k}(\xi)$  facilitates regularization of  $p_\mathfrak{g}(\zeta)$  on  ${}^+V_Z$  and  $(V \otimes Z)^+$  as explained next.

**Proposition 4.1.** *For each  $\lambda \in \mathfrak{c}_{reg}^*$  the linear maps  ${}^+V_Z \rightarrow V$ ,  $v \mapsto p_\mathfrak{g}(\zeta)v$  and  ${}^+V_Z \rightarrow (V \otimes Z)^+$ ,  $v \mapsto p_\mathfrak{g}(v \otimes 1_Z)$  are well defined. Furthermore,*

$$p_\mathfrak{g}(\zeta)v = p_{\mathfrak{g}/\mathfrak{k}}(\zeta)p_\mathfrak{k}(\xi)v, \quad p_\mathfrak{g}(v \otimes 1_Z) = p_{\mathfrak{g}/\mathfrak{k}}(0)p_\mathfrak{k}(v \otimes 1_Z)$$

for all  $v \in {}^+V_Z$ , where the factor  $p_\mathfrak{k}(\xi): {}^+V_Z \rightarrow V_Z^+$  is invertible.

**Proof.** Let  $W \subset V \otimes Z$  denote the sum of weight spaces with weights from  $\Lambda({}^+V_Z) + \zeta$ . Since  ${}^+V_Z = {}^+V_X$ , these weights are all  $\mathfrak{k}$ -dominant. Applying Proposition 3.6 to a locally finite  $U_q(\mathfrak{k})$ -module  $V \otimes Z$ , we conclude that  $p_\mathfrak{k}$  is well defined on  $W$  and sends it to the space of  $U_q(\mathfrak{k}_+)$ -invariants. Moreover, the operator  $p_\mathfrak{k}(\xi): {}^+V_Z \rightarrow V_Z^+$  is invertible because  $V \otimes X$  is a completely reducible  $U_q(\mathfrak{k})$ -module.

Furthermore, every root factor in  $p_{\mathfrak{g}/\mathfrak{k}}(0)$  is well defined on  $W$  as none of denominators in (3.6) turns zero once  $\lambda \in \mathfrak{c}_{reg}^*$ . Therefore every simple root factor  $p_\alpha(1)$  sends  $W$  to  $\ker e_\alpha$ . Choosing a normal ordering with  $\alpha$  in the left-most position we conclude that  $p_\mathfrak{g}$  maps  $W$  to  $\ker e_\alpha$ , for each  $\alpha \in \Pi_\mathfrak{g}$ . Thus, the projector  $p_\mathfrak{g}$  is well defined on  ${}^+V_Z \otimes 1_Z \subset W$ , and the statement follows from (3.9).  $\square$

Remark that in the special case of  $\xi = 0$  corresponding to a scalar parabolic module  $Z$  one can take  ${}^+V_Z = V_Z^+$ . Then  $p_\mathfrak{k}(\xi)$  is identical on  $V_Z^+$  and drops from the factorization.

#### 4.2. Extremal twist for parabolic Verma modules

In order to calculate the extremal twist for parabolic Verma modules, we first work out a necessary condition to be irreducible that is fulfilled for generic highest weight. Note that complete irreducibility criteria for classical parabolic Verma modules are given in [13]. We do not appeal to deformation arguments but make use of the relation (3.9) between the inverse invariant pairing and extremal projector.

The idea of our approach originates from Proposition 2.1 because  $\mathcal{S}$  can be intuitively viewed as a singular vector in  $Z' \otimes Z$ . However, we cannot directly apply the extremal projector to construct singular vectors in

$Z' \otimes Z$  since weights of  $Z'$  are not bounded from above. Instead, we will approximate  $Z'$  with a sequence of finite dimensional modules  $\{V_\mu\}$  and modify Proposition 2.1 accordingly. We then construct  $\mathcal{S} \in Z' \hat{\otimes} Z$  as a projective limit of singular vectors in  $V_\mu \otimes Z$ .

Suppose that  $u \in V \otimes Z$  is a singular vector such that  $\bar{\delta}(u) = v \in V$  is not zero. Define a linear map  $\psi_v: Z \rightarrow V$  as  $z \mapsto \psi_v(z) = u^1 \langle u^2, z \rangle$ , where we use the Sweedler notation  $u = u^1 \otimes u^2$ . It factors to a composition  $Z \rightarrow {}^*Z \rightarrow V$ , where the first arrow is the contravariant form regarded as a linear map from  $Z$  to its restricted dual  ${}^*Z$ .

**Proposition 4.2.** *The map  $\psi_v: Z \rightarrow V$  is a homomorphism of  $U_q(\mathfrak{g}_-)$ -modules: for any element  $f \in U_q(\mathfrak{g}_-)$  of weight  $-\beta$ ,  $\psi_v(f1_Z) = q^{-(\zeta+\mu, \beta)} \sigma(f)v$ , where  $\mu$  is the weight of  $v$ . In particular,  $v \in V_Z^+$ .*

**Proof.** It is sufficient to prove the equality for  $f$  a monomial in Chevalley generators. For simple  $\beta \in \Pi$  one has

$$(1 \otimes \omega(f_\beta))u = -(1 \otimes q^{-h_\beta} e_\beta)u = -(\gamma(e_\beta) \otimes q^{-h_\beta})u = (e_\beta q^{-h_\beta} \otimes q^{-h_\beta})u = (\sigma(f_\beta) \otimes 1)q^{-h_\beta}u.$$

This implies  $(1 \otimes \omega(f))u = q^{-(\zeta+\mu, \beta)} (\sigma(f) \otimes 1)u$  for all  $\beta$  and all monomial  $f$ . Now the formula for  $\psi_v$  is immediate upon pairing the second tensor leg with  $1_Z$ . Finally, for all  $f \in I_Z^-$  one has  $f1_Z = 0$  and therefore  $\sigma(f)v = 0$ , that is,  $v \in V_Z^+$ .  $\square$

Regard a fundamental weight  $\pi_\alpha$ ,  $\alpha \in \Pi_{\mathfrak{t}}$ , as an element of  $\mathfrak{h}^*$  satisfying  $(\pi_\alpha, \beta^\vee) = \delta_{\alpha\beta}$  for all  $\beta \in \Pi_{\mathfrak{g}}$ . This extends to a homomorphism  $j: \Lambda_{\mathfrak{t}} \rightarrow \Lambda_{\mathfrak{g}}$  of Abelian groups. For fixed  $\xi \in \Lambda_{\mathfrak{t}}^+$  define  $\mathfrak{c}_{\xi, Z}^*$  as the set of integral weights  $j(\xi) + \lambda$  with  $\lambda \in \Lambda_{\mathfrak{g}} \cap \mathfrak{c}^*$ . In other words,  $\mathfrak{c}_{\xi, Z}^*$  is the affine shift by  $j(\xi)$  of the sublattice generated by all  $\pi_\alpha$  with  $\alpha \in \Pi_{\mathfrak{g}/\mathfrak{t}} = \Pi_{\mathfrak{g}} - \Pi_{\mathfrak{t}}$ .

Introduce a partial ordering on  $\mathfrak{c}_{\xi, Z}^*$  by setting  $\nu \prec \mu$  if  $(\nu, \alpha^\vee) < (\mu, \alpha^\vee)$  for all  $\alpha \in \Pi_{\mathfrak{g}/\mathfrak{t}}$ . Let  $\mathfrak{c}_{\xi, Z_+}^* \subset \mathfrak{c}_{\xi, Z}^*$  be the subset of dominant weights.

For  $\mu \in \mathfrak{c}_{\xi, Z_+}^*$  set  $V_\mu$  to be the finite dimensional  $U_q(\mathfrak{g})$ -module of lowest weight  $-\mu$ . Denote by  $J_\mu^+ \supset I_Z^+$  the left ideal in  $U_q(\mathfrak{g}_+)$  annihilating the lowest vector in  $V_\mu$ . It is generated by  $\{e_\alpha^{m_\alpha+1}\}_{\alpha \in \Pi}$  with  $m_\alpha = (\mu, \alpha^\vee) \in \mathbb{Z}_+$ . There is a  $U_q(\mathfrak{g}_+)$ -invariant projection  $\wp_\mu: Z' \rightarrow V_\mu \simeq U_q(\mathfrak{g}_+)/J_\mu^+$ . The following lemma facilitates approximation of  $Z'$  with an increasing sequence of  $V_\mu$ .

**Lemma 4.3.** *For each  $\beta \in \mathbb{Z}_+ \Pi_{\mathfrak{g}}$  such that  $Z[\zeta - \beta] \neq \{0\}$  there exists  $\mu \in \mathfrak{c}_{\xi, Z_+}^*$  such that  $\dim V_\mu[-\mu + \beta] = \dim Z'[-\zeta + \beta]$ .*

**Proof.** Define height  $\text{ht}(\beta) \in \mathbb{Z}_+$  of a weight  $\beta \in \mathbb{Z}_+ \Pi$  as the sum of coordinates in the basis of simple roots. Given a module  $Z'$  of lowest weight  $-\zeta$  define  $\text{ht}(-\zeta + \beta) = \text{ht}(\beta)$  for  $-\zeta + \beta \in \Lambda(Z')$ .

To prove the lemma, it is sufficient to take  $\mu$  with  $m_\alpha > \text{ht}(\beta)$  for all  $\alpha \in \Pi_{\mathfrak{g}/\mathfrak{t}}$ . The kernel of  $\wp_\mu$  is generated by vectors  $e_\alpha^{m_\alpha+1}1_{Z'}$ ,  $\alpha \in \Pi_{\mathfrak{g}/\mathfrak{t}}$ . The subspace of weight  $-\zeta + \beta$  in  $\ker \wp_\mu$  is zero, because heights of all weights in  $\ker \wp_\mu$  are higher than  $m_\alpha > \text{ht}(-\zeta + \beta)$ .  $\square$

Since  $J_\nu^+ \subset J_\mu^+$  for  $\mu \prec \nu$ , the projection  $\wp_\mu$  factorizes as  $\wp_\mu = \wp_{\nu, \mu} \circ \wp_\nu$  with an  $U_q(\mathfrak{g}_+)$ -equivariant projection  $\wp_{\nu, \mu}: V_\nu \rightarrow V_\mu$ . Lemma 4.3 then implies  $\cap_\mu J_\mu^+ = I_Z^+$ , where the intersection is over  $\mu \in \mathfrak{c}_{\xi, Z_+}^*$ , making  $Z'$  a projective limit of  $U_q(\mathfrak{g}_+)$ -modules  $V_\mu$ .

The lowest vector  $v_\mu \in V_\mu$  belongs to  $(V_\mu)_Z^+$ , and Corollary 4.1 implies that a singular vector  $u_\mu = p_{\mathfrak{g}}(p_{\mathfrak{g}}^{-1}(\zeta)v_\mu \otimes 1_Z)$  with  $\bar{\delta}(u_\mu) = v_\mu$  is well defined for  $\lambda \in \mathfrak{c}_{reg}^*$  (it follows from (3.7) that  $p_{\mathfrak{g}/\mathfrak{t}}(\zeta)$  and therefore  $p_{\mathfrak{g}}(\zeta)$  are invertible for such  $\lambda$ ).

**Corollary 4.4.** *The parabolic Verma module  $Z$  is irreducible once  $\lambda \in \mathfrak{c}_{reg}^*$ .*

**Proof.** Let  $\beta \in \mathbb{Z}_+ \Pi_{\mathfrak{g}}$  be such that  $Z[\zeta - \beta] \neq \{0\}$ . Take  $\mu \in \Lambda_{\xi}^+$  sufficiently large so that  $V_{\mu}[-\mu + \beta] \simeq Z'[-\zeta + \beta]$  (that is possible in view of Lemma 4.3). Since  $\lambda \in \mathfrak{c}_{reg}^*$ , the vector  $p_{\mathfrak{g}}(v_{\mu} \otimes 1_Z) \in V_{\mu} \otimes Z$  is singular, and  $v = p_{\mathfrak{g}}(\zeta)v_{\mu} = p_{\mathfrak{g}/\mathfrak{k}}(\zeta)v_{\mu} \neq 0$  is proportional to  $v_{\mu}$ , by Proposition 4.1. The map  $\psi_v$  is a surjection of  $Z[\zeta - \beta]$  onto  $V_{\mu}[-\mu + \beta]$  by Proposition 4.2 and hence a bijection. Therefore the contravariant form is non-degenerate on  $Z[\zeta - \beta]$  and hence on each weight subspace of  $Z$ .  $\square$

**Proposition 4.5.** *Let  $Z$  be an irreducible parabolic Verma module of highest weight  $\zeta = \xi + \lambda \in \Lambda_{\mathfrak{k}}^+ \oplus \mathfrak{c}^*$ . For every finite dimensional module  $V$ , the extremal twist  $\theta_{V,Z}$  is the operator  $p_{\mathfrak{k}}(\xi)^{-1}p_{\mathfrak{g}/\mathfrak{k}}(\zeta)^{-1}$  restricted to  $V_Z^+$ .*

**Proof.** This is true for  $\lambda \in \mathfrak{c}_{reg}^*$  by Corollary 4.4. The operator  $\theta_{V,Z}$  is a rational trigonometric function of  $\lambda \in \mathfrak{c}^*$  coinciding with  $p_{\mathfrak{k}}(\xi)^{-1}p_{\mathfrak{g}/\mathfrak{k}}(\zeta)^{-1}$  on an open subset  $\mathfrak{c}_{reg}^* \subset \mathfrak{c}^*$  and therefore on  $\mathfrak{c}^*$ .  $\square$

As a consequence we conclude that if  $\lambda \in \mathfrak{c}^*$  is a pole of the map  $p_{\mathfrak{g}/\mathfrak{k}}(\xi + \lambda)^{-1}: V_Z^+ \rightarrow {}^+V_Z$  then the module  $Z$  is reducible because the extremal twist is defined for all irreducible  $Z$ .

#### 4.3. Equivariant star product

In this section we give an expression for an equivariant star product on homogeneous spaces with Levi stabilizer subgroup. Such a space can be realized as a conjugacy class with the Poisson structure restricted from the Semenov-Tian-Shansky bracket on the total group [23]. The corresponding star product was constructed in [8] with the help of dynamical twist, which reduces to the inverse contravariant form, [1]. While that solves the problem in principle, an explicit expression of the inverse form for a general parabolic module is unknown. In this section we present an alternative explicit formula for the deformed multiplication in terms of extremal projector. The idea of our approach is close to [14] (for the special case of  $\mathfrak{k} = \mathfrak{h}$ ) and based on relation (3.9).

In this section we assume that  $\xi = 0$  and  $\zeta = \lambda \in \mathfrak{c}_{reg}^*$ . For  $Z$  we take the scalar parabolic Verma module of highest weight  $\lambda$ . Then  $V_Z^+$  is the subspace  $V^{\mathfrak{k}+}$  of  $U_q(\mathfrak{k}_+)$ -invariants in  $V$ . One can check that the contravariant form is non-degenerate when restricted to  $V^{\mathfrak{k}+}$ , so we can set  ${}^+V_Z = V^{\mathfrak{k}+}$ . The projector  $p_{\mathfrak{k}}$  reduces to the identity operator on  $V^{\mathfrak{k}+}$ , which gives  $p_{\mathfrak{g}}(\lambda) = p_{\mathfrak{g}/\mathfrak{k}}(\lambda) \in \text{End}(V^{\mathfrak{k}+})$ .

Let  $\mathcal{A}$  denote the quantized Hopf algebra of polynomial functions on an algebraic group  $G$  with the Lie algebra  $\mathfrak{g}$ , [10]. It is endowed with a non-degenerate Hopf pairing with  $U_q(\mathfrak{g})$ . The quantum group  $U_q(\mathfrak{g})$  acts on  $\mathcal{A}$  by right translations  $x \triangleright a = a^{(1)}(x, a^{(2)})$  (in Sweedler notation) for  $x \in U_q(\mathfrak{g})$  and  $a \in \mathcal{A}$ .

Let  $\mathcal{F} \in U_q(\mathfrak{g}_+) \hat{\otimes} U_q(\mathfrak{g}_-)$  be a lift of the inverse invariant pairing  $Z \otimes Z' \rightarrow \mathbb{C}$ . It defines a bi-differential operator on  $\mathcal{A}$  by

$$\mathcal{A} \otimes \mathcal{A} \xrightarrow{\mathcal{F}} \mathcal{A} \otimes \mathcal{A} \xrightarrow{\cdot} \mathcal{A}, \quad (4.12)$$

where  $\cdot$  is the multiplication on  $\mathcal{A}$ . This operation is parameterized by the highest weight  $\lambda$  and it is known to be associative when restricted to the subspace  $\mathcal{A}^{\mathfrak{k}}$  of  $U_q(\mathfrak{k})$ -invariants in  $\mathcal{A}$ .

Denote by  $\Phi$  the composition map

$$\mathcal{A} \otimes \mathcal{A} \otimes Z \rightarrow \mathcal{A} \otimes \mathcal{A} \xrightarrow{\cdot} \mathcal{A},$$

where the left arrow is the contravariant pairing of the  $Z$ -factor with the highest vector  $1_Z$ . Then (3.9) in combination with regularization of Section 4.1 gives the following formula for the deformed multiplication.

**Proposition 4.6.** *The star-product on  $\mathcal{A}^{\mathfrak{k}}$  restricted from (4.12) is presentable as*

$$f \star g = \Phi \left( p_{\mathfrak{g}/\mathfrak{k}}(0) \left( p_{\mathfrak{g}/\mathfrak{k}}^{-1}(\lambda) f \otimes p_{\mathfrak{g}/\mathfrak{k}}(0) (p_{\mathfrak{g}/\mathfrak{k}}^{-1}(\lambda) g \otimes 1_Z) \right) \right), \quad f, g \in \mathcal{A}^{\mathfrak{k}}, \quad (4.13)$$

where the action of  $U_q(\mathfrak{g})$  on  $\mathcal{A}$  is  $\triangleright$ .

**Proof.** Given a finite dimensional module  $V$  and  $v \in V^{\mathfrak{k}} \subset V^{\mathfrak{k}+}$ , apply  $\delta$  to (3.9) and rewrite it as

$$\mathcal{F}(v \otimes 1_Z) = p_{\mathfrak{g}/\mathfrak{k}}(0) (p_{\mathfrak{g}/\mathfrak{k}}^{-1}(\lambda) v \otimes 1_Z)$$

The right-hand side is  $U_q(\mathfrak{k})$ -invariant and generates a submodule isomorphic to  $Z \subset V \otimes Z$ , so one can iterate this operation with  $w \in W^{\mathfrak{k}}$  for another finite dimensional module  $W$  and get a vector in  $W \otimes V \otimes Z$ . Pairing of the  $Z$ -factor with  $1_Z$  is  $U_q(\mathfrak{k})$ -invariant and yields a tensor  $\mathcal{F}(w \otimes v) \in (W \otimes V)^{\mathfrak{k}}$ .

Now take  $f$  and  $g$  from  $\mathcal{A}^{\mathfrak{k}}$ , which is a direct sum of finite dimensional modules thanks to the Peter-Weyl decomposition. Then

$$\mathcal{F}(f \otimes \mathcal{F}(g \otimes 1_Z)) = p_{\mathfrak{g}/\mathfrak{k}}(0) \left( p_{\mathfrak{g}/\mathfrak{k}}^{-1}(\lambda) f \otimes p_{\mathfrak{g}/\mathfrak{k}}(0) (p_{\mathfrak{g}/\mathfrak{k}}^{-1}(\lambda) g \otimes 1_Z) \right).$$

Applying  $\Phi$  yields  $f \star g$  in the left-hand side.  $\square$

The new operation makes  $\mathcal{A}^{\mathfrak{k}}$  a right  $U_q(\mathfrak{g})$ -module algebra with respect to the action  $a \triangleleft x = (a^{(1)}, x) a^{(2)}$  for all  $a \in \mathcal{A}$  and  $x \in U_q(\mathfrak{g})$ . This action commutes with the left action  $\triangleright$  participating in  $\star$ , and the initial multiplication on  $\mathcal{A}$  is  $\triangleleft$ -equivariant. Therefore  $\star$  is equivariant too.

## 5. Application to vector bundles on quantum spheres

We conclude this presentation by illustrating Theorems 2.2 and 3.2 with an example relevant to quantum spheres of even dimension, [21]. Here  $Z$  is fixed to a base module that supports the quantization of  $\mathbb{C}[\mathbb{S}^{2n}]$  as a subalgebra in  $\text{End}_{\mathbb{C}}(Z)$ , [22]. The module  $V$  varies over all equivalence classes of finite dimensional quasi-classical<sup>1</sup> irreducible representations of  $U_q(\mathfrak{so}(2n+1))$ . Unlike in Section 4, the subspaces  $V_Z^+ \subset V$  are hard to evaluate while their reciprocals  $Z_V^+ \subset Z$  are readily known from [21]. That enables us to compute  $\theta_{Z,V}$  using (3.7) and (3.8). Thus Theorem 2.2 benefits from alternative parameterizations of singular vectors that prove to be the most convenient for particular calculations.

In this section, we fix  $\mathfrak{g} = \mathfrak{so}(2n+1)$  and  $\mathfrak{k} = \mathfrak{so}(2n) \subset \mathfrak{g}$ . Note that there is no natural quantization of  $U(\mathfrak{k})$  as a subalgebra in  $U_q(\mathfrak{g})$ , contrary to the case of Levi  $\mathfrak{k}$  considered in the previous section. Let  $\{\varepsilon_i\}_{i=1}^n$  denote the orthonormal basis of short roots in  $\mathbb{R}^+$ . We enumerate the basis of simple positive roots as  $\alpha_n = \varepsilon_n - \varepsilon_{n-1}, \dots, \alpha_2 = \varepsilon_2 - \varepsilon_1, \alpha_1 = \varepsilon_1$ . We choose  $\lambda \in \mathfrak{h}^*$  such that  $q^{2(\lambda, \varepsilon_i)} = -q^{-1}$  for all  $i = 1, \dots, n$  and define  $Z$  as the irreducible module of highest weight  $\lambda$  whose canonical generator  $1_Z$  is annihilated by  $f_{\alpha_i}$  with  $i > 1$  and by  $[[f_{\alpha_2}, f_{\alpha_1}]_q, f_{\alpha_1}]_{\bar{q}}$ , where we set  $\bar{q} = q^{-1}$ . Put  $e_{\varepsilon_1} = e_{\alpha_1}$  and  $f_{\varepsilon_1} = f_{\alpha_1}$  and furthermore

$$e_{\varepsilon_{i+1}} = [e_{\alpha_{i+1}}, e_{\varepsilon_i}]_q, \quad f_{\varepsilon_{i+1}} = [f_{\varepsilon_i}, f_{\alpha_{i+1}}]_{\bar{q}}$$

for  $i > 1$ . Weight vectors  $f_{\varepsilon_1}^{m_1} \dots f_{\varepsilon_n}^{m_n} 1_Z$  with  $m_i$  taking all possible values in  $\mathbb{Z}_+$  deliver an orthogonal basis in  $Z$ , [22].

<sup>1</sup> A representation of  $U_q(\mathfrak{g})$  is called quasi-classical if it is a deformation of a representation of  $U(\mathfrak{g})$ . That imposes conditions on the set of weights, see e.g. [12].



The module  $Z$  is a quotient of a parabolic Verma module relative to the Levi subalgebra  $\mathfrak{l} \subset \mathfrak{g}$  with the basis of simple roots  $\Pi_{\mathfrak{l}} = \{\alpha_i\}_{i=2}^n$ . Therefore it is locally finite over  $U_q(\mathfrak{l})$ . All weights in  $\Lambda(Z)$  are multiplicity free. Next we describe the structure of  $Z$  as a module over the quantum  $\mathfrak{sl}(2)$ -subalgebra corresponding to a positive root from  $R_{\mathfrak{g}/\mathfrak{l}}^+$ ; we will need it for calculation of the extremal twist. The set  $R_{\mathfrak{g}/\mathfrak{l}}^+$  consists of short roots  $\{\varepsilon_i\}_{i=1}^n$  and long roots  $\{\varepsilon_i + \varepsilon_j\}_{i < j}$ .

**Lemma 5.1.** *The module  $Z$  is completely reducible over  $U_q(\mathfrak{g}^\alpha + \mathfrak{h})$  for each  $\alpha \in R_{\mathfrak{g}/\mathfrak{l}}^+$ . It is a direct sum of irreducible Verma modules whose highest vectors carry weights  $\lambda - \sum_{l=1}^n m_l \varepsilon_l$  with  $m_i = 0$  for  $\alpha = \varepsilon_i$  and  $m_i m_j = 0$  for  $\alpha = \varepsilon_i + \varepsilon_j$ ,  $i \neq j$ .*

**Proof.** We will sketch the proof. There is a normal ordering of positive roots such that the root vectors  $f_{\varepsilon_i}$  and  $f_{\varepsilon_i + \varepsilon_j} \propto [f_{\varepsilon_i}, f_{\varepsilon_j}]$  enter the PBW basis. It follows from [22], Corollary 3.2, that the short root vectors, as operators on  $Z$ , satisfy commutation relations  $f_{\varepsilon_i} f_{\varepsilon_j} = q f_{\varepsilon_j} f_{\varepsilon_i}$ ,  $i < j$ . This implies that  $Z$  is freely generated over  $U_q(\mathfrak{g}^-)$  by  $U_q(\mathfrak{g}_+^\alpha)$ -invariants whose weights are as indicated in the statement: the Verma modules they generate do not intersect. Their irreducibility can be seen from examining their highest weights against a criterion of [5].  $\square$

Fix a finite dimensional  $U_q(\mathfrak{g})$ -module  $V$  of highest weight  $\nu$  and put  $\ell_i = (\nu, \alpha_i^\vee) \in \mathbb{Z}_+$ ,  $i = 1, \dots, n$ . These integers are coordinates of  $\nu$  in the basis of fundamental weights of  $\mathfrak{g}$ . Its expression through the basis  $\{\varepsilon_i\}_{i=1}^n$  (with reverse enumeration order) can be found e.g. in [24]. The ideal  $I_V^+$  determining  $Z_V^+ = \ker I_V^+ \subset Z$ , is generated by  $\{e_{\alpha_i}^{\ell_i+1}\}_{i=1}^n$ . There is an orthogonal decomposition  $Z = Z_V^+ \oplus \omega(I_V^+)Z$  with

$$Z_V^+ = \text{Span}\{f_{\varepsilon_1}^{m_1} \dots f_{\varepsilon_n}^{m_n} 1_Z\}_{m_1 \leq \ell_1, \dots, m_n \leq \ell_n}, \quad \omega(I_V^+)Z = \text{Span}\{f_{\varepsilon_1}^{k_1} \dots f_{\varepsilon_n}^{k_n} 1_Z\}_{k_1, \dots, k_n},$$

where  $k_i > \ell_i$  for some  $i = 1, \dots, n$ .

The dominant weight  $\nu$  is expanded over the orthonormal basis of short roots as

$$\nu = \frac{\ell_1}{2} \sum_{i=1}^n \varepsilon_i + \sum_{i=2}^n \ell_i \sum_{j=i}^n \varepsilon_j, \quad (\nu, \varepsilon_k) = \frac{\ell_1}{2} + \sum_{i=2}^n \ell_i \sum_{j=i}^n \delta_{j,k} = \frac{\ell_1}{2} + \sum_{i=2}^k \ell_i, \quad k = 1, \dots, n.$$

Using this explicit expression, we can check regularity of the extremal projector.

**Proposition 5.2.** *For any quasi-classical finite dimensional module  $V$ , the extremal projector  $p_{\mathfrak{g}} = p_{\mathfrak{g}/\mathfrak{l}} p_{\mathfrak{l}}$  is well defined on  $1_V \otimes Z_V^+$ .*

**Proof.** Denote by  $W = \sum_{\xi \in \Lambda(Z_V^+)} (V \otimes Z)[\nu + \xi]$  the sum of weight spaces in  $V \otimes Z$  of all weights of singular vectors. It contains  $1_V \otimes Z_V^+$  as a vector subspace. We will show that factors  $p_\alpha(t)$  for all  $\alpha \in R^+$  are well defined on  $W$  at  $t = (\rho, \alpha^\vee) = \rho_\alpha$ . That is true for  $\alpha \in R_{\mathfrak{l}}^+$  since  $Z$  is locally finite over  $U_q(\mathfrak{l})$ . Moreover,  $p_{\mathfrak{l}}(0)W$  is in  $U_q(\mathfrak{l}_+)$ -invariants since all weights in  $\Lambda(W)$  are  $\mathfrak{l}$ -dominant (by virtue of Proposition 3.6 for  $\mathfrak{g} = \mathfrak{l}$ ). So we can further assume that  $\alpha \in R_{\mathfrak{g}/\mathfrak{l}}^+$ .

Present  $\xi \in \Lambda(Z_V^+)$  as  $\xi = \lambda - \sum_{i=1}^n m_i \varepsilon_i$  with  $m_i \leq \ell_i$ . For  $\alpha = \varepsilon_i + \varepsilon_j$  with  $i < j$  and  $k \in \mathbb{N}$  we find

$$[(\nu + \xi + \rho, \alpha^\vee) + k]_{q_\alpha} = [\ell_1 + \sum_{l=2}^i \ell_l + \sum_{l=2}^j \ell_l - m_i - m_j + i + j - 2 + k]_{q_\alpha}.$$

The integer in the square brackets in the right-hand side is positive, and  $q$  is not a root of unity, hence  $p_\alpha(\rho_\alpha)$  is well defined on  $W[\nu + \xi]$  as no denominator in (3.7) turns zero.

For short roots  $\alpha = \varepsilon_i$ ,  $i = 1, \dots, n$ , the expression  $[(\nu + \xi + \rho, \alpha^\vee) + k]_{q_\alpha}$  does not turn zero at all  $k \in \mathbb{Z}$  as it is proportional to  $q^{\frac{1}{2}+k'} + q^{-\frac{1}{2}-k'}$  for some integer  $k'$  ( $q$  is not a root of unity). So the series (3.6) for

$p_\alpha(t)$  is regular at  $t = (\rho, \alpha^\vee)$ . This also proves that the extremal projector  $p_{\alpha_1}(1)$  is well defined on  $W$ . Finally,  $p_{\mathfrak{g}}(0)W$  is annihilated by each  $e_\alpha$  with  $\alpha \in \Pi$  since one can choose a normal order with  $\alpha$  on the left.  $\square$

As all weights of  $Z$  are multiplicity free, we can write, up to a non-zero factor:

$$\theta_{Z,V} w \propto \prod_{\alpha \in R_{\mathfrak{g}/I}^+} \prod_{k=1}^{l_{\xi,\alpha}} \frac{[(\nu + \rho + \xi, \alpha^\vee) + k]_{q_\alpha}}{[(\nu + \rho, \alpha^\vee) - k]_{q_\alpha}} w, \quad w \in Z_V^+[\xi],$$

where  $l_{\xi,\alpha} = \max\{l \in \mathbb{Z} : e_\alpha^l w \neq 0\}$ . This is a corollary of Lemma 5.1 and formula (3.7). In particular, for  $\xi = \lambda - \sum_{i=1}^n m_i \varepsilon_i$  we have  $l_{\xi,\varepsilon_i} = m_i$  and  $l_{\xi,\varepsilon_i + \varepsilon_j} = \min(m_i, m_j)$ , where  $i \neq j$ . With the use of the shortcuts  $\phi_{\xi,\alpha,k}$  for  $\frac{[(\nu + \rho + \xi, \alpha^\vee) + k]_{q_\alpha}}{[(\nu + \rho, \alpha^\vee) - k]_{q_\alpha}}$  we can write

$$\det(\theta_{Z,V}) \propto \prod_{\xi} \prod_{\alpha \in R_{\mathfrak{g}/I}^+} \prod_{k=1}^{l_{\xi,\alpha}} \phi_{\xi,\alpha,k}, \quad \text{where } \xi \in \{\lambda - \sum_{i=1}^n m_i \varepsilon_i\}_{m_i \leq \ell_i}.$$

Note that factors corresponding to roots  $\alpha \in R_I^+$  are absent in the product because the operator  $p_I(\nu)$  is invertible on  $Z_V^+$  due to local finiteness of  $Z$  with respect to  $U_q(\mathfrak{l})$ .

**Proposition 5.3.** *The operator  $\theta_{Z,V}$  is invertible.*

**Proof.** We should prove that  $\phi_{\xi,\alpha,k} \neq 0$  for all  $\alpha \in R_{\mathfrak{g}/I}^+$ . For short  $\alpha$ , neither the denominator nor enumerator in  $\phi_{\xi,\alpha,k}$  turn zero since they are of the form  $[(\lambda, \alpha^\vee) + k]_{q^{\frac{1}{2}}}$  with  $k \in \mathbb{Z}$ , cf. the proof of Proposition 5.2. So we have to check it only for  $\alpha = \varepsilon_i + \varepsilon_j \in R_{\mathfrak{l}/I}^+$ ,  $i < j$ . Then

$$\phi_{\xi,\alpha,k} = \frac{[\ell_1 + \sum_{l=2}^i \ell_l + \sum_{l=2}^j \ell_l - m_i - m_j + i + j - 2 + k]_q}{[\ell_1 + \sum_{l=2}^i \ell_l + \sum_{l=2}^j \ell_l + i + j - 2 - k]_q}$$

does not vanish because all  $m_i \leq \ell_i$ . Since  $k \leq l_{\xi,\alpha} = \min\{m_i, m_j\} \leq \min\{\ell_i, \ell_j\}$ , the denominator is not zero too.  $\square$

**Corollary 5.4.** *For any quasi-classical finite dimensional  $U_q(\mathfrak{g})$ -module  $V$ , the tensor product  $V \otimes Z$  is completely reducible.*

The irreducible components of  $V \otimes Z$  are pseudo-parabolic Verma modules described in [21].

## References

- [1] A. Alekseev, A. Lachowska, Invariant  $*$ -product on coadjoint orbits and the Shapovalov pairing, *Comment. Math. Helv.* 80 (2005) 795–810.
- [2] R.M. Asherova, Yu.F. Smirnov, V.N. Tolstoy, Projection operators for the simple Lie groups, *Theor. Math. Phys.* 8 (1971) 813–825.
- [3] V. Chari, A. Pressley, *A Guide to Quantum Groups*, Cambridge University Press, Cambridge, 1994.
- [4] V. Drinfeld, Quantum groups, in: A.V. Gleason (Ed.), *Proc. Int. Congress of Mathematicians*, Berkeley, 1986, AMS, Providence, 1987, pp. 798–820.
- [5] C. De Concini, V.G. Kac, Representations of quantum groups at roots of 1, in: *Operator Algebras, Unitary Representations, Enveloping Algebras, and Invariant Theory*, Paris, 1989, in: *Progress in Mathematics*, vol. 92, Birkhäuser, 1990, pp. 471–506.
- [6] J. Donin, A. Mudrov, Dynamical Yang–Baxter equation and quantum vector bundles, *Commun. Math. Phys.* 254 (2005) 719–760.

- [7] B. Enriquez, P. Etingof, Quantization of classical dynamical  $r$ -matrices with nonabelian base, *Commun. Math. Phys.* 254 (2005) 603–650.
- [8] P. Etingof, B. Enriquez, I. Marshall, Quantization of some Poisson-Lie dynamical  $r$ -matrices and Poisson homogeneous spaces, *Contemp. Math.* 433 (2007) 135–176.
- [9] P. Etingof, A. Varchenko, Dynamical Weyl groups and applications, *Adv. Math.* 167 (2002) 74–127.
- [10] L. Faddeev, N. Reshetikhin, L. Takhtajan, Quantization of Lie groups and Lie algebras, *Leningr. Math. J.* 1 (1990) 193–226.
- [11] J.E. Humphreys, *Introduction to Lie Algebras and Representation Theory*, Grad. Texts in Math., vol. 9, Springer-Verlag, New York, 1973.
- [12] J.C. Jantzen, *Lectures on Quantum Groups*, Grad. Stud. in Math., vol. 6, AMS, Providence, RI, 1996.
- [13] J.C. Jantzen, Kontravariante Formen und Induzierten Darstellungen halbeinfacher Lie-Algebren, *Math. Ann.* 226 (1977) 53–65.
- [14] S. Khoroshkin, Extremal projector and dynamical twist, *Theor. Math. Phys.* 139 (1) (2004) 582–597.
- [15] P. Kulish, A. Mudrov, Dynamical reflection equation, *Contemp. Math.* 433 (2007) 281–310.
- [16] E. Karolinsky, K. Muzykin, A. Stolin, V. Tarasov, Dynamical Yang-Baxter equations, quasi-Poisson homogeneous spaces, and quantization, *Lett. Math. Phys.* 71 (2005) 179–197.
- [17] E. Karolinsky, A. Stolin, V. Tarasov, Irreducible highest weight modules and equivariant quantization, *Adv. Math.* 211 (2007) 266–283.
- [18] S.M. Khoroshkin, V.N. Tolstoy, Extremal projector and universal  $R$ -matrix for quantized contragredient Lie (super)algebras, in: *Quantum Groups and Related Topics*, Wrocław, 1991, in: *Math. Phys. Stud.*, vol. 13, Kluwer Acad. Publ., Dordrecht, 1992, pp. 23–32.
- [19] S. Khoroshkin, M. Nazarov, Mickelsson algebras and representations of Yangians, *Trans. Am. Math. Soc.* 364 (2012) 1293–1367.
- [20] A. Mudrov, Contravariant form on tensor product of highest weight modules, *SIGMA* 15 (2019) 026.
- [21] A. Mudrov, Equivariant vector bundles over quantum spheres, *J. Noncommut. Geom.* 15 (1) (2021) 79–111.
- [22] A. Mudrov, Star-product on complex sphere  $\mathbb{S}^{2n}$ , *Lett. Math. Phys.* 108 (6) (2018) 1443–1454.
- [23] M. Semenov-Tian-Shansky, Poisson-Lie groups, quantum duality principle, and the quantum double, *Contemp. Math.* 175 (1994) 219–248.
- [24] E. Vinberg, A. Onishchik, *Seminar po Gruppam Li i Algebraicheskim Gruppam (A Seminar on Lie Groups and Algebraic Groups)* (in Russian), Moscow 1988.
- [25] D.P. Zhelobenko, *Representations of Reductive Lie Algebras*, Nauka, Moscow, 1994.
- [26] D.P. Zhelobenko, *Compact Lie Groups and Their Representations*, AMS, 1973.