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GIT-EQUIVALENCE AND SEMI-STABLE SUBCATEGORIES OF QUIVER REPRESENTATIONS

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ABSTRACT. In this paper, we answer the question of when the subcategory of semi-stable representations is the same for two rational vectors for an acyclic quiver. This question has been previously answered by Ingalls, Paquette, and Thomas in the tame case in [14]. Here we take a more invariant theoretic approach, to answer this question in general. We recover the known result in the tame case.

CONTENTS

1. INTRODUCTION

Throughout, K denotes an algebraically closed field of characteristic zero. By a quiver, we mean a finite, connected, acyclic quiver. All representations and modules are assumed to be finite-dimensional. By a module, we always mean a left module.

A central question in Geometric Invariant Theory is to determine when two rational characters of a reductive group G give rise to the same semi-stable locus inside an ambient G-variety X (see for example [1], [3], [10], [12]). In the context of quiver invariant theory, one can replace the group of rational characters of G by the space of rational vectors of a quiver Q, and X by the category $\operatorname{rep}(Q)$ of representations of Q. Within this more categorical framework, it is natural to ask when two rational vectors of Q share the same subcategory of semi-stable representations. This question has been answered by Ingalls, Paquette, and Thomas in the tame case in [14]. Here we take a representation-type free approach, to answer this question in general. We recover the known result in the tame case. We also point out that the study of the possible interactions between semi-stable subcategories is important for applications to cluster algebras and Artin groups (see [4, 5], [15]).

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Let Q be a quiver with vertex set Q_0 and consider the isomorphism of vector spaces

$$\begin{aligned}\mathrm{wt}: \mathbb{Q}^{Q_0} &\to \mathbb{Q}^{Q_0}\\ \alpha &\to (\langle \alpha, \mathbf{e}_x \rangle)_{x \in Q_0},\end{aligned}$$

where the $\mathbf{e}_x$, $x \in Q_0$, are the standard unit vectors of $\mathbb{Q}^{Q_0}$ and $\langle \cdot, \cdot \rangle$ is the Euler form of Q. Two rational vectors α_1 and $\alpha_2 \in \mathbb{Q}^{Q_0}$ are said to be GIT-**equivalent** if they share the same subcategory of semi-stable representations, i.e., if

$$\mathrm{rep}(Q)^{ss}_{\mathrm{wt}(\alpha_1)} = \mathrm{rep}(Q)^{ss}_{\mathrm{wt}(\alpha_2)}.$$

The GIT-**cone** associated to a dimension vector $\beta \in \mathbb{Z}^{Q_0}_{\geq 0}$ and a rational vector $\alpha \in \mathbb{Q}^{Q_0}$ is

$$\mathcal{C}(\beta)_\alpha = \{\alpha' \in \mathcal{D}(\beta) | \, \mathrm{rep}(Q, \beta)^{ss}_{\mathrm{wt}(\alpha)} \subseteq \mathrm{rep}(Q, \beta)^{ss}_{\mathrm{wt}(\alpha')}\},$$

where $\mathcal{D}(\beta)$ is the domain of semi-invariants associated to Q and β (we refer to Section 2.1 for the details behind our notation).

For each Schur root β of Q, define $\mathcal{C}(\beta)$ to be the collection of maximal (with respect to dimension) GIT-cones of $\mathcal{D}(\beta)$. Let

$$\mathcal{I} = \bigcup \mathcal{C}(\beta),$$

where the union is over all Schur roots β of Q. For any rational vector $\alpha \in \mathbb{Q}^{Q_0}$, define

$$\mathcal{I}_\alpha = \{\mathcal{C} \in \mathcal{I} \mid \alpha \in \mathcal{C}\}.$$

Now we are ready to state our first result.

Theorem 1. *Let Q be a quiver with vertex set Q_0 and let α_1 and α_2 be two rational vectors in $\mathbb{Q}^{Q_0}$. Then α_1 and α_2 are GIT-equivalent if and only if $\mathcal{I}_{\alpha_1} = \mathcal{I}_{\alpha_2}$.*

In Theorem 2 below we show that $\mathcal{I}$ can be explicitly described in the tame case. Let us assume now that Q is a Euclidean quiver and let N be the number of non-homogeneous regular tubes in the Auslander-Reiten quiver of Q. Let r_i be the rank of the i^{th} tube and let R be the set of all multi-indices of the form $I = (a_i)_{i=1}^N$ with $1 \leq a_i \leq r_i$, $\forall 1 \leq i \leq N$. Let $\{\beta_{i,j}\}_{i=1,\ldots,N, j=1,\ldots,r_i}$ be the regular simple roots in the non-homogeneous tubes. For every multi-index $I = (a_i)_{i=1}^N \in R$, set

$$\alpha_I := \delta + \sum_{i=1}^{N} \sum_{j \neq a_i} \beta_{i,j},$$

i.e. α_I is the sum of δ, and all other regular simples except β_{i,a_i} for $i = 1, \ldots, N$.

Now, we are ready to state our second result, which recovers Theorem 7.4 in [14].

Theorem 2. *(1) Assume that Q is a Dynkin quiver. Then $\mathcal{I} = \bigcup_\beta \mathcal{D}(\beta)$, where the union is over all positive roots of Q.*

(2) Assume that Q is a Euclidean quiver. Then

$$\mathcal{I} = \left(\bigcup_{I \in R} \mathcal{C}(\delta)_{\alpha_I} \right) \cup \left(\bigcup_{\beta} \mathcal{D}(\beta) \right),$$

where δ is the unique isotropic Schur root and the β's run through all real Schur roots of Q. Furthermore, for every $I = (a_i)_{i=1}^N \in R$, $\mathcal{C}(\delta)_{\alpha_I}$ is the cone given by all non-negative rational linear combinations of δ, and the quasi-simple roots of Q except $\beta_{1,a_1}, \ldots, \beta_{N,a_N}$..

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2. Background

Let $Q = (Q_0, Q_1, t, h)$ be a quiver with vertex set Q_0 and arrow set Q_1. The two functions $t, h : Q_1 \to Q_0$ assign to each arrow $a \in Q_1$ its tail *ta* and head *ha*, respectively.

A **representation** V of Q over K is a collection $(V(x), V(a))_{x \in Q_0, a \in Q_1}$ of finite-dimensional K-vector spaces $V(x)$, $x \in Q_0$, and K-linear maps $V(a) : V(ta) \to V(ha)$, $a \in Q_1$. The dimension vector of a representation V of Q is the function $\underline{\dim}\, V : Q_0 \to \mathbb{Z}$ defined by $(\underline{\dim}\, V)(x) = \dim_K V(x)$ for $x \in Q_0$. The one-dimensional representation of Q supported at vertex $x \in Q_0$ is denoted by S_x and its dimension vector is denoted by e_x. By a dimension vector of Q, we simply mean a vector $\beta \in \mathbb{Z}_{\geq 0}^{Q_0}$.

Let V and W be two representations of Q. A **morphism** $\varphi : V \to W$ is defined to be a collection $(\varphi(x))_{x \in Q_0}$ of K-linear maps with $\varphi(x) \in \mathrm{Hom}_K(V(x), W(x))$ for each $x \in Q_0$, such that $\varphi(ha)V(a) = W(a)\varphi(ta)$ for each $a \in Q_1$. We denote by $\mathrm{Hom}_Q(V, W)$ the K-vector space of all morphisms from V to W. We say that V is a subrepresentation of W if $V(x)$ is a subspace of $W(x)$ for each $x \in Q_0$ and $(i_x : V(x) \hookrightarrow W(x))_{x \in Q_0}$ is a morphism of representations, i.e. $V(a)$ is the restriction of $W(a)$ to $V(ta)$ for each $a \in Q_1$. The category of all representations of Q is denoted by $\mathrm{rep}(Q)$. It turns out that $\mathrm{rep}(Q)$ is an abelian hereditary category. A representation $V \in \mathrm{rep}(Q)$ is called a **Schur representation** if $\mathrm{End}_Q(V) \cong K$. A Schur representation $V \in \mathrm{rep}(Q)$ is called **exceptional** if $\mathrm{Ext}^1_Q(V, V) = 0$.

The path algebra, KQ, of Q has a K-basis consisting of all paths (including the trivial ones), and multiplication in KQ is given by concatenation of paths. It is easy to see that any KQ-module defines a representation of Q, and vice-versa. Furthermore, the category $\mathrm{mod}(KQ)$ of KQ-modules is equivalent to the category $\mathrm{rep}(Q)$. In what follows, we identify $\mathrm{mod}(KQ)$ and $\mathrm{rep}(Q)$, and use the same notation for a module and the corresponding representation.

The Euler form of Q is the bilinear form $\langle -, - \rangle : \mathbb{Z}^{Q_0} \times \mathbb{Z}^{Q_0} \to \mathbb{Z}$ defined by

$$\langle \alpha, \beta \rangle = \sum_{x \in Q_0} \alpha(x)\beta(x) - \sum_{a \in Q_1} \alpha(ta)\beta(ha), \forall \alpha, \beta \in \mathbb{Z}^{Q_0}.$$

This extends to a bilinear form on $\mathbb{Q}^{Q_0} \times \mathbb{Q}^{Q_0}$. The corresponding Tits quadratic form is $q : \mathbb{Z}^{Q_0} \to \mathbb{Z}$, $q(\alpha) = \langle \alpha, \alpha \rangle, \forall \alpha \in \mathbb{Z}^{Q_0}$.

For any two representations $V, W \in \mathrm{rep}(Q)$, we have (see [22])

$$\langle \underline{\dim}\, V, \underline{\dim}\, W \rangle = \dim_K \mathrm{Hom}_Q(V, W) - \dim_K \mathrm{Ext}^1_Q(V, W).$$

Since Q is assumed to be acyclic, the linear map

$$\mathbb{Q}^{Q_0} \to (\mathbb{Q}^{Q_0})^*$$
$$\alpha \mapsto \langle \alpha, \cdot \rangle$$

is an isomorphism. In fact, composing this map on the right with the canonical isomorphism $(\mathbb{Q}^{Q_0})^* \to \mathbb{Q}^{Q_0}$, we get the isomorphism $\mathrm{wt} : \mathbb{Q}^{Q_0} \to \mathbb{Q}^{Q_0}$. Given $\theta, \beta \in \mathbb{Q}^{Q_0}$, we denote by $\theta(\beta) := \sum_{x \in Q_0} \theta(x)\beta(x)$. Then for any $\alpha, \beta \in \mathbb{Q}^{Q_0}$, we have that

$$\mathrm{wt}(\alpha)(\beta) = \langle \alpha, \beta \rangle.$$

We call a dimension vector β a (Schur, exceptional) **root** of Q if there exists a β-dimensional (Schur, exceptional) indecomposable representation. If β is a root of Q, then $q(\beta) \leq 1$ (see for example [16, Section 1.1]). More specifically, a root β is said to be **real** if $q(\beta) = 1$. We call β an (imaginary) **isotropic root** if $q(\beta) = 0$. If $q(\beta) < 0$ we say that β is **strictly imaginary**.

One of the fundamental results in the representation theory of finite-dimensional algebras is the **Auslander-Reiten formula**, which we state for quiver representations (see for example [2]): For two indecomposable representations V and W of Q, we have

$$\mathrm{Hom}_Q(V, \tau W) \cong \mathrm{Hom}_Q(\tau^- V, W) \cong \mathrm{D}\,\mathrm{Ext}^1_Q(W, V),$$

where τ is the Auslander-Reiten translation of Q, and $\mathrm{D} = \mathrm{Hom}_K(-, K)$ is the duality functor over K.

2.1. **Domains of semi-invariants of quivers.** Let $\theta \in \mathbb{Q}^{Q_0}$. A representation $V \in \mathrm{rep}(Q)$ is said to θ**-semi-stable** if

$$\theta(\underline{\dim}\, V) = 0 \text{ and } \theta(\underline{\dim}\, V') \leq 0, \forall\, V' \leq V.$$

We say that V is θ-stable if V is not the zero representation, $\theta(\underline{\dim}\, V) = 0$, and $\theta(\underline{\dim}\, V') < 0$ for all proper subrepresentations $0 \neq V' < V$. We define $\mathrm{rep}(Q)^{ss}_\theta$ to be the full subcategory of $\mathrm{rep}(Q)$ whose objects are the θ-semi-stable representations of Q. Note that the simple objects of $\mathrm{rep}(Q)^{ss}_\theta$ are precisely the θ-stable representations. Furthermore, $\mathrm{rep}(Q)^{ss}_\theta$ is an abelian subcategory of $\mathrm{rep}(Q)$, closed under extensions. It is also Artinian and Noetherian and hence any $V \in \mathrm{rep}(Q)^{ss}_\theta$ has a filtration whose factors are θ-stable.

Let β be a dimension vector of Q. We say that a dimension vector β is θ-(semi)-stable if there exists a β-dimensional θ-(semi)-stable representation. We define the **domain of semi-invariants** associated to Q and β to be

$$\mathcal{D}(\beta) = \{\alpha \in \mathbb{Q}^{Q_0} | \beta \text{ is } \mathrm{wt}(\alpha)\text{-semi-stable}\}.$$

In what follows, we write $\beta' \hookrightarrow \beta$ to mean that any β-dimensional representation has a subrepresentation of dimension vector β'. Then, according to King's criterion for semi-stability of quiver representations (see [18]), we have

$$\mathcal{D}(\beta) = \{\alpha \in \mathbb{Q}^{Q_0} \mid \langle \alpha, \beta \rangle = 0, \langle \alpha, \beta' \rangle \leq 0, \forall\, \beta' \hookrightarrow \beta\}.$$

This makes it clear that $\mathcal{D}(\beta)$ is a rational convex polyhedral cone. Furthermore, it follows from [7, Theorem 2.4] and [23, Theorem 5.4] that

$$\mathcal{D}(\beta) \cap \mathbb{Z}^{Q_0}_{\geq 0} = \{\alpha \in \mathbb{Z}^{Q_0}_{\geq 0} \mid \alpha \text{ is } -\langle \cdot, \beta \rangle - \text{semi-stable}\}.$$

Next, we give a description of $\mathcal{D}(\beta)$ in terms of generators. We say that a dimension vector α is β**-simple** if α is $-\langle\cdot,\beta\rangle$-stable. Recall that the projective cover of the simple representation S_x is denoted P_x, and its dimension vector is denoted by γ_x, for $x \in Q_0$. For any $\rho \in \mathbb{Z}_{\geq 0}^{Q_0}$, we can construct a projective representation $P_\rho = \bigoplus_{x\in Q_0} P_x^{\rho(x)}$.

Lemma 3. *Let β be a dimension vector. Then $\mathcal{D}(\beta)$ is generated by the β-simple roots, together with $-\gamma_x$ for $x \in Q_0$ such that $\beta(x) = 0$.*

Proof. Let $\alpha \in \mathcal{D}(\beta)$. If α is not integral, multiply by a large enough integer to obtain an integral vector.

Define the following projective representation: P_ρ, where $\rho(x) = 1$ if $\beta(x) = 0$, and $\rho(x) = 0$ otherwise. From [13, Lemma 6.5.7], we know that for each $y \in Q_0$ with $\alpha(y) < 0$, there exists an $x \in Q_0$ with $\beta(x) = 0$ and $\dim_K P_x(y) > 0$. Thus, there exists a positive integer m_y such that $\alpha(y) + m_y \dim_K P_x(y) \geq 0$. Taking a sufficiently large $m \gg 0$, then, we have

$$\alpha + \underline{\dim}\, P_\rho^m \in \mathbb{Z}_{\geq 0}^{Q_0}.$$

Now, $\underline{\dim}\, P_\rho \in \mathcal{D}(\beta)$, because $\langle \underline{\dim}\, P_x, \beta\rangle = \beta(x) = 0$ and, if $\beta' \hookrightarrow \beta$, then $\langle \underline{\dim}\, P_x, \beta'\rangle = \beta'(x) \leq \beta(x) = 0$ for each x with $\rho(x) = 1$. That is, β is $\langle \underline{\dim}\, P_x, \cdot\rangle$-semi-stable for each x with $\beta(x) = 0$.

So, $\alpha + \underline{\dim}\, P_\rho^m \in \mathcal{D}(\beta) \cap \mathbb{Z}_{\geq 0}^{Q_0}$, which means there exists a V with $\underline{\dim}\, V = \alpha + \underline{\dim}\, P_\rho^m$, such that V is $-\langle\cdot,\beta\rangle$-semi-stable. Next, consider a Jordan-Hölder filtration of V in the category $\mathrm{rep}(Q)^{ss}_{-\langle\cdot,\beta\rangle}$

$$0 = V_0 < V_1 < \cdots < V_{r-1} < V_r = V,$$

where the composition factors V_i/V_{i-1} are $-\langle\cdot,\beta\rangle$-stable; in particular, $\underline{\dim}\, V_i/V_{i-1}$ is β-simple, for all $1 \leq i \leq r$. So,

$$\underline{\dim}\, V = \alpha + \underline{\dim}\, P_\rho^m = \sum_{i=1}^{r} \underline{\dim}\, V_i/V_{i-1}$$

is a sum of β-simples. Finally, we see that $\alpha = \underline{\dim}\, V - \underline{\dim}\, P_\rho^m$ is a sum of β-simples, plus $-\gamma_x$'s for x such that $\beta(x) = 0$. ☐

A description of the faces of $\mathcal{D}(\beta)$ was found by Derksen and Weyman in [9]. Specifically, let $\mathcal{F}$ be a (non-trivial) face of $\mathcal{D}(\beta)$ and choose a lattice point α in the relative interior of $\mathcal{F}$. It turns out that there exist unique $\mathrm{wt}(\alpha)$-stable dimension vectors $\beta_1, \ldots, \beta_l$ such that a generic representation in $\mathrm{rep}(Q,\beta)$ has a filtration (in $\mathrm{rep}(Q)^{ss}_{\mathrm{wt}(\alpha)}$) whose factors are $\mathrm{wt}(\alpha)$-stable representations of dimension $\beta_1, \ldots, \beta_l$ (in some order). We also write

$$\beta = \beta_1 \dotplus \ldots \dotplus \beta_l$$

and call this the $\mathrm{wt}(\alpha)$-stable decomposition of β. It is proved in [9] that

$$\mathcal{F} = \mathcal{D}(\beta_1) \cap \ldots \cap \mathcal{D}(\beta_l).$$

For any $\alpha \in \mathcal{D}(\beta)$, the GIT-class of α relative to β is the set $\{\alpha' \in \mathbb{Q}^{Q_0} | \mathrm{rep}(Q,\beta)^{ss}_{\mathrm{wt}(\alpha)} = \mathrm{rep}(Q,\beta)^{ss}_{\mathrm{wt}(\alpha')}\}$. The following lemma will be needed in the proof of Proposition 6.

Lemma 4. *Let $\mathcal{F}$ be a face of $\mathcal{D}(\beta)$ and $\alpha \in \mathbb{Q}^{Q_0}$ a rational vector lying in the relative interior of $\mathcal{F}$. Then the GIT-class of α relative to β is included in $\mathcal{F}$.*

Proof. From the discussion above we can write $\mathcal{F} = \mathcal{D}(\beta_1) \cap \ldots \cap \mathcal{D}(\beta_l)$, where $\beta = \beta_1 \dot{+} \ldots \dot{+} \beta_l$ is the $\mathrm{wt}(\alpha)$-stable decomposition of β. For each $1 \leq i \leq l$, choose $V_i \in \mathrm{rep}(Q, \beta_i)^s_{\mathrm{wt}(\alpha)}$ and set $V := \bigoplus_{i=1}^{l} V_i \in \mathrm{rep}(Q, \beta)^{ss}_{\mathrm{wt}(\alpha)}$.

Now, let α' be a rational vector such that $\mathrm{rep}(Q, \beta)^{ss}_{\mathrm{wt}(\alpha)} = \mathrm{rep}(Q, \beta)^{ss}_{\mathrm{wt}(\alpha')}$. Then $V \in \mathrm{rep}(Q, \beta)^{ss}_{\mathrm{wt}(\alpha')}$ and so each V_i becomes $\mathrm{wt}(\alpha')$-semi-stable, i.e. $\alpha' \in \mathcal{D}(\beta_i)$ for all $1 \leq i \leq l$. This shows that $\alpha' \in \mathcal{F}$. □

3. GIT-equivalence for rational vectors

Let Q be a quiver. Given a rational vector $\alpha \in \mathbb{Q}^{Q_0}$, recall that $\mathrm{rep}(Q)^{ss}_{\mathrm{wt}(\alpha)}$ is a full subcategory of $\mathrm{rep}(Q)$ whose class of objects consists of all $\mathrm{wt}(\alpha)$-semi-stable representations of Q. Moreover, two vectors $\alpha_1, \alpha_2 \in \mathbb{Q}^{Q_0}$ are GIT-equivalent, written $\alpha_1 \sim_{\mathrm{GIT}} \alpha_2$, if $\mathrm{rep}(Q)^{ss}_{\mathrm{wt}(\alpha_1)} = \mathrm{rep}(Q)^{ss}_{\mathrm{wt}(\alpha_2)}$.

In what follows, we give a characterization of this equivalence relation in terms of certain rational convex polyhedral cones of codimension one in $\mathbb{Q}^{Q_0}$. These cones arise most naturally in the context of variation of GIT-quotients. As we will see in Section 4, in case Q is tame, we can explicitly describe these cones and show they coincide with those found by Ingalls-Paquette-Thomas in [14].

We know from [23, Theorem 6.1] that a dimension vector β is a Schur root if and only if $\mathcal{D}^0(\beta) := \{\alpha \in \mathbb{Q}^{Q_0} \mid \langle \alpha, \beta \rangle = 0, \langle \alpha, \beta' \rangle < 0, \forall \beta' \hookrightarrow \beta, \beta' \neq \mathbf{0}, \beta\}$ is non-empty. In this case, $\mathcal{D}^0(\beta)$ is precisely the relative interior of $\mathcal{D}(\beta)$ and $\dim_K \mathcal{D}(\beta) = |Q_0| - 1$.

Recall that for a vector $\alpha \in \mathbb{Q}^{Q_0}$ and dimension vector $\beta \in \mathbb{Z}^{Q_0}_{\geq 0}$, the GIT-cone of α relative to β is the rational convex polyhedral cone $\mathcal{C}(\beta)_\alpha = \{\alpha' \in \mathcal{D}(\beta) | \mathrm{rep}(Q, \beta)^{ss}_{\mathrm{wt}(\alpha)} \subseteq \mathrm{rep}(Q, \beta)^{ss}_{\mathrm{wt}(\alpha')}\}$; it contains α in its relative interior (for more details, see [6, Lemmas 2.2, 3.3, and 4.1]). Finally, we define the GIT**-fan associated to** (Q, β) to be

$$\mathcal{F}(\beta) := \{\mathcal{C}(\beta)_\alpha \mid \alpha \in \mathcal{D}(\beta)\} \cup \{\mathbf{0}\}.$$

Theorem 5. *Keeping the same notation as above, $\mathcal{F}(\beta)$ is a finite fan with support $\mathcal{D}(\beta)$.*

Remark. A proof of this result within the context of quiver invariant theory, which is based on [21, Theorem 5.2], can be found in [6, Theorem 1.1] (see also [3, Theorem 2.11]). We point out that in [6] it is actually proved that the image of $\mathcal{F}(\beta)$ through wt forms a fan covering of $\mathrm{wt}(\mathcal{D}(\beta))$.

Proposition 6. *Assume that β is a Schur root. Then $\mathcal{F}(\beta)$ is a pure fan of dimension $|Q_0| - 1$.*

Proof. Since β is a Schur root, we know that $\dim_K \mathcal{D}(\beta) = |Q_0| - 1$. Now, let $\mathcal{C}(\beta)_{\alpha_0} \in \mathcal{F}(\beta)$ be a non-zero GIT-cone. From the semi-continuity property of semi-stable loci (see for example [20, Lemma 3.10] or [11, Proposition 4.2.2]), we know that there exists an open (with respect to the Euclidean topology) neighborhood $\mathcal{U}_0$ of α_0 in $\mathbb{Q}^{Q_0}$ such that

$$\mathrm{rep}(Q, \beta)^s_{\mathrm{wt}(\alpha_0)} \subseteq \mathrm{rep}(Q, \beta)^s_{\mathrm{wt}(\alpha)} \subseteq \mathrm{rep}(Q, \beta)^{ss}_{\mathrm{wt}(\alpha)} \subseteq \mathrm{rep}(Q, \beta)^{ss}_{\mathrm{wt}(\alpha_0)}, \tag{1}$$

for all $\alpha \in \mathcal{D}(\beta) \cap \mathcal{U}_0$. In particular, we get that $\mathcal{C}(\beta)_{\alpha_0} \subseteq \mathcal{C}(\beta)_\alpha$ and hence $\mathcal{C}(\beta)_{\alpha_0}$ is a face of $\mathcal{C}(\beta)_\alpha$ for all $\alpha \in \mathcal{D}(\beta) \cap \mathcal{U}_0$ by Theorem 5.

Now, let us assume that $\mathcal{C}(\beta)_{\alpha_0}$ is maximal with respect to either dimension or inclusion. Then $\mathcal{C}(\beta)_\alpha = \mathcal{C}(\beta)_{\alpha_0}$ for all $\alpha \in \mathcal{D}(\beta) \cap \mathcal{U}_0$, i.e. α_0 and α belong to the same GIT-class relative to β for all $\alpha \in \mathcal{D}(\beta) \cap \mathcal{U}_0$ which, in particular, implies that $\mathcal{C}(\beta)_{\alpha_0}$ contains $\mathcal{D}(\beta) \cap \mathcal{U}_0$.

Next, we claim that $\alpha_0 \in \mathcal{D}^0(\beta)$. Assume this is not the case. Then there exists a proper face $\mathcal{F}$ of $\mathcal{D}(\beta)$ with $\alpha_0 \in \operatorname{relint} \mathcal{F}$. Since $\mathcal{U}_0$ is an open neighborhood of $\alpha_0 \in \mathcal{D}(\beta)$ and $\mathcal{D}^0(\beta)$ is the relative interior of $\mathcal{D}(\beta)$, we can always choose an $\alpha \in \mathcal{D}^0(\beta) \cap \mathcal{U}_0$. Then α and α_0 belong to the same GIT-class relative to β by the paragraph above, and hence $\alpha \in \mathcal{F}$ by Lemma 4. But this is a contradiction since α is in the relative interior of $\mathcal{D}(\beta)$. This proves our claim above.

Finally note $\mathcal{D}^0(\beta) \cap \mathcal{U}_0$ is a non-empty open subset of the hyperplane $\mathbb{H}(\beta) := \{\theta \in \mathbb{Q}^{Q_0} \mid \theta(\beta) = 0\}$ as it contains α_0. Since $\mathcal{D}^0(\beta) \cap \mathcal{U}_0 \subseteq \mathcal{C}(\beta)_{\alpha_0}$, we get that $\dim \mathcal{C}(\beta)_{\alpha_0} = |Q_0| - 1$. Hence, $\mathcal{F}(\beta)$ is a pure fan of dimension $|Q_0| - 1$. □

Remark 1. The proof above shows that if β is a Schur root then a GIT-cone $\mathcal{C}(\beta)_{\alpha_0}$ is maximal with respect to inclusion if and only if it is maximal with respect to dimension. Moreover, if $\mathcal{C}(\beta)_{\alpha_0}$ is maximal then $\dim \mathcal{C}(\beta)_{\alpha_0} = |Q_0| - 1$ and $\alpha_0 \in \mathcal{D}^0(\beta)$. □

Now, we are ready to prove Theorem 1.

Proof of Theorem 1. ($\Longrightarrow$) Assume that $\alpha_1 \sim_{\mathrm{GIT}} \alpha_2$. Let $\mathcal{C}(\beta)_\alpha \in \mathcal{I}_{\alpha_1}$ where β is a Schur root and $\alpha \in \mathcal{D}(\beta)$. Then $\operatorname{rep}(Q,\beta)^{ss}_{\operatorname{wt}(\alpha)} \subseteq \operatorname{rep}(Q,\beta)^{ss}_{\operatorname{wt}(\alpha_1)}$ and, since $\operatorname{rep}(Q,\beta)^{ss}_{\operatorname{wt}(\alpha_1)} = \operatorname{rep}(Q,\beta)^{ss}_{\operatorname{wt}(\alpha_2)}$ as $\alpha_1 \sim_{\mathrm{GIT}} \alpha_2$, we get that $\alpha_2 \in \mathcal{C}(\beta)_\alpha$; hence, $\mathcal{C}(\beta)_\alpha \in \mathcal{I}_{\alpha_2}$. This proves that $\mathcal{I}_{\alpha_1} \subseteq \mathcal{I}_{\alpha_2}$. The other inclusion is proved similarly and so $\mathcal{I}_{\alpha_1} = \mathcal{I}_{\alpha_2}$.
($\Longleftarrow$) Let us assume now that $\mathcal{I}_{\alpha_1} = \mathcal{I}_{\alpha_2}$. For this, we will first check that

$$\operatorname{rep}(Q,\beta)^{ss}_{\operatorname{wt}(\alpha_1)} \subseteq \operatorname{rep}(Q,\beta)^{ss}_{\operatorname{wt}(\alpha_2)}, \tag{2}$$

for $\operatorname{wt}(\alpha_1)$-stable dimension vectors β. So, let β be a $\operatorname{wt}(\alpha_1)$-stable dimension vector. Then $\mathcal{C}(\beta)_{\alpha_1} \cap \mathcal{D}^0(\beta) \neq \emptyset$ since α_1 belongs to this intersection. Furthermore, since β is a Schur root, we know that the fan $\mathcal{F}(\beta)$ is pure of dimension $|Q_0| - 1$ by Lemma 6. It now follows from [17, Lemma 2.5] that

$$\mathcal{C}(\beta)_{\alpha_1} = \bigcap_{\mathcal{C}} \mathcal{C},$$

where the union is over all maximal GIT-cones $\mathcal{C}$ of $\mathcal{F}(\beta)$ with $\alpha_1 \in \mathcal{C}$. Since $\mathcal{I}_{\alpha_1} = \mathcal{I}_{\alpha_2}$, any such maximal GIT-cone $\mathcal{C}$ contains α_2 as well; in particular, $\mathcal{C}$ contains $\mathcal{C}(\beta)_{\alpha_2}$. So, we must have that $\mathcal{C}(\beta)_{\alpha_2} \subseteq \mathcal{C}(\beta)_{\alpha_1}$, i.e. $\operatorname{rep}(Q,\beta)^{ss}_{\operatorname{wt}(\alpha_1)} \subseteq \operatorname{rep}(Q,\beta)^{ss}_{\operatorname{wt}(\alpha_2)}$. Similarly one checks that the opposite inclusion holds if β is $\operatorname{wt}(\alpha_2)$-stable.

Next, let us check that $\operatorname{rep}(Q)^{ss}_{\operatorname{wt}(\alpha_1)} \subseteq \operatorname{rep}(Q)^{ss}_{\operatorname{wt}(\alpha_2)}$. Let $M \in \operatorname{rep}(Q)^{ss}_{\operatorname{wt}(\alpha_1)}$ and consider a Jordan-Hölder filtration of M in $\operatorname{rep}(Q)^{ss}_{\operatorname{wt}(\alpha_1)}$

$$0 = M_0 < M_1 < \ldots < M_n = M,$$

where M_i/M_{i-1} is $\operatorname{wt}(\alpha_1)$-stable for every $1 \leq i \leq n$. Set $\beta_i := \underline{\dim}\, M_i/M_{i-1}$ for all $1 \leq i \leq n$. Then we have that $\operatorname{rep}(Q,\beta_i)^{ss}_{\operatorname{wt}(\alpha_1)} \subseteq \operatorname{rep}(Q,\beta_i)^{ss}_{\operatorname{wt}(\alpha_2)}$ by (2). This implies that M is $\operatorname{wt}(\alpha_2)$-semi-stable since $\operatorname{rep}(Q)^{ss}_{\operatorname{wt}(\alpha_2)}$ is closed under extensions. We have just showed that $\operatorname{rep}(Q)^{ss}_{\operatorname{wt}(\alpha_1)} \subseteq \operatorname{rep}(Q)^{ss}_{\operatorname{wt}(\alpha_2)}$. Similarly one proves that $\operatorname{rep}(Q)^{ss}_{\operatorname{wt}(\alpha_2)} \subseteq \operatorname{rep}(Q)^{ss}_{\operatorname{wt}(\alpha_1)}$, and this completes the proof. □

4. The tame case

Throughout this section, we assume that Q is a Euclidean quiver unless otherwise specified. We denote by δ the unique isotropic Schur root of Q. Let τ be the Auslander-Reiten translate of Q. As a general reference for the representation theory of tame quivers, we refer the reader to [2, 24].

4.1. **Regular representations of tame quivers.** For V indecomposable, we say V is

- **preprojective** if $\tau^m(V) = 0$ for some $m \gg 0$
- **preinjective** if $\tau^{-m}(V) = 0$ for some $m \gg 0$
- **regular** otherwise, that is, if $\tau^m(V) \neq 0$ for all integers m.

For arbitrary V, we say that V is preprojective, preinjective, or regular (respectively) if all of its indecomposable direct summands are.

Proposition 7. [8] *If V is indecomposable, then V is preprojective, regular, or preinjective if $\langle \delta, \underline{\dim} V \rangle$ is negative, 0, or positive (respectively).*

Proposition 8. [8] *Let V and W be indecomposable.*

(1) If W is preprojective and V is not, then $\mathrm{Hom}_Q(V, W) = 0$ and $\mathrm{Ext}^1_Q(W, V) = 0$.
(2) If W is preinjective and V is not, then $\mathrm{Hom}_Q(W, V) = 0$ and $\mathrm{Ext}^1_Q(V, W) = 0$.

Proposition 9. [8] *The set of all regular representations of Q, denoted by $\mathcal{R}eg$, is an extension closed, abelian subcategory of $\mathrm{rep}(Q)$, and is equal to $\mathrm{rep}(Q)^{ss}_{\mathrm{wt}(\delta)}$.*

A representation $V \in \mathcal{R}eg$ which is simple in this subcategory (that is, a $\mathrm{wt}(\delta)$-stable representation) will be called **regular simple**. We will also call $\underline{\dim} V$ regular simple where no ambiguity will result.

Proposition 10. [8] *Suppose V is regular simple.*

(1) τV is also regular simple.
(2) V is Schur, thus $\underline{\dim} V$ is a Schur root.
(3) $\tau V \cong V$ if and only if $\underline{\dim} V = \delta$.
(4) There exists $p \in \mathbb{N}$ such that $\tau^p V \cong V$. p is called the period of V.
(5) If V has period p, then $\underline{\dim} V + \underline{\dim} \tau V + \cdots + \underline{\dim} \tau^{p-1} V = \delta$.

Proposition 11. [8] *Every indecomposable regular representation V has a unique filtration*

$$0 = V_0 < V_1 < \cdots < V_{r-1} < V_r = V$$

such that V_i/V_{i-1} is regular simple. This is called a **uniserial filtration**. *Furthermore, given a regular simple V_1 and a length r, there is a unique indecomposable regular representation V with regular socle V_1 and regular length r. It's composition factors, from the socle, are V_1, $\tau^- V_1, \tau^{-2} V_1, \ldots, \tau^{1-r} V_1$.*

Now, we can organize all indecomposable regular representations into groups, called **tubes**, in the following way: Begin with a regular simple representation V and set $\mathcal{V} = \{V, \tau V, \ldots, \tau^{p-1} V\}$, the τ-orbit of V. Then, include in the tube any indecomposable regular representation whose composition factors are taken from $\mathcal{V}$. We say that p is the period of the tube.

Proposition 12. [8] *Every regular indecomposable representation belongs to a unique tube.*

As a consequence of Proposition 10, we see that the dimension vectors of the regular simples must be either real Schur roots, or must be equal to δ. There are infinitely many (occurring in families) δ-dimensional indecomposable representations. Each one generates a tube of period 1, that is, a homogeneous tube. Real Schur roots correspond to exceptional representations, so we have finitely many non-homogeneous tubes generated by exceptional representations which are also regular simple. For the purpose of distinguishing from δ, call the dimension vectors of exceptional regular simple representations **quasi-simple**.

If β is the dimension vector of a regular representation of Q, we define $\tau\beta$ as $\beta\Phi$ and $\tau^-\beta$ as $\beta\Phi^{-1}$, where Φ is the Coxeter matrix of Q. If V is regular, then $\underline{\dim}\,(\tau V) = \tau\underline{\dim}\,V$, and $\underline{\dim}\,(\tau^- V) = \tau^-\underline{\dim}\,V$ (see [2]).

Lemma 13. *If β_1 and β_2 are quasi-simple, then*

$$\langle \beta_1, \beta_2 \rangle = \begin{cases} 1 \text{ if } \beta_1 = \beta_2 \\ -1 \text{ if } \beta_2 = \tau\beta_1 \\ 0 \text{ otherwise} \end{cases}$$

Proof. Let V_i be the unique indecomposable representation of dimension β_i for $i = 1, 2$. If $\beta_1 = \beta_2$, then $\langle \beta_1, \beta_2 \rangle = q(\beta_1) = 1$ since β_1 is a real Schur root.

If $\beta_2 = \tau\beta_1$, then since $V_1 \not\cong V_2$, and V_1 and V_2 are simple in $\mathrm{rep}(Q)^{ss}_{\langle\delta,\cdot\rangle}$, we have that $\mathrm{Hom}_Q(V_1, V_2) = 0$. So $\langle \beta_1, \beta_2 \rangle = -\dim_K \mathrm{Ext}^1_Q(V_1, \tau V_1) = -\dim_K \mathrm{Hom}_Q(\tau V_1, \tau V_1) = -1$.

If $\beta_2 \neq \beta_1, \tau\beta_1$, we get $\langle \beta_1, \beta_2 \rangle = -\dim_K \mathrm{Ext}^1_Q(V_1, V_2) = -\dim_K \mathrm{Hom}_Q(V_2, \tau V_1) = 0$. □

4.2. **A Summary of Work by Ingalls, Paquette, and Thomas.** We start by defining a rational convex polyhedral cone H^{ss}_δ to be the cone of non-negative rational linear combinations of regular simple dimension vectors. The generators of this cone are δ, and the quasi-simple roots, which we recall are divided into finitely many (say N) tubes. For convenience, label the quasi-simple roots $\beta_{i,j}$, so that $\beta_{i,j+1} = \tau\beta_{i,j}$, and $i = 1, \ldots, N$ indicates which tube the root is in. Call the rank, or period, of the i^{th} tube r_i.

Remark. It turns out that $H^{ss}_\delta = \mathcal{D}(\delta)$. Indeed, $\mathcal{D}(\delta)$ is generated by the dimension vectors that are stable with respect to $-\langle \cdot, \delta \rangle$ by Lemma 3. But, since $\langle \delta, \cdot \rangle = -\langle \cdot, \delta \rangle$, these dimension vectors are precisely the regular simple dimension vectors of Q. Hence, $\mathcal{D}(\delta)$ is precisely H^{ss}_δ.

Next, define the following cover of H^{ss}_δ. Recall that for any multi-index $I = (a_1, \ldots, a_N)$, with $1 \leq a_i \leq r_i$, C_I is the cone given by non-negative rational linear combinations of δ, and all quasi-simple roots *except* $\beta_{1,a_1}, \ldots, \beta_{N,a_N}$. Let R be the set of all such multi-indices and let

$$\mathcal{J} = \{C_I\}_{I \in R} \cup \{\mathcal{D}(\beta)\}_\beta,$$

where β runs through the set of real Schur roots of Q. Furthermore, for any $\alpha \in \mathbb{Z}^{Q_0}$ define $\mathcal{J}_\alpha = \{J \in \mathcal{J} | \alpha \in J\}$.

The theorem we will recover in the tame case is as follows.

Theorem. [14, Theorem 7.4] *Let Q be a Euclidean quiver and $\alpha_1, \alpha_2 \in \mathbb{Z}^{Q_0}$. Then $\alpha_1 \sim \alpha_2$ if and only if $\mathcal{J}_{\alpha_1} = \mathcal{J}_{\alpha_2}$.*

4.3. **Proof of Theorem 2.** Let I be a multi-index as above, and let $\mathrm{wt}(\alpha_I)$ be the weight given by $\alpha_I = \delta + \sum_{j \neq a_i} \sum_{i=1}^{N} \beta_{i,j}$. We want to prove the following two substantial lemmas, which together will prove Theorem 2.

Lemma 14. *$C_I = \mathcal{C}(\delta)_{\alpha_I}$, and $\mathcal{C}(\delta)_{\alpha_I}$ is a maximal GIT-cone.*

Lemma 15. *Let β be a Schur root of Q. Then any maximal GIT-cone $\mathcal{C}(\beta)_\alpha$ is either $\mathcal{D}(\beta)$, if β is real, or C_I for some multi-index $I \in R$, if β is isotropic.*

The proof of these two lemmas will be delayed, as we need some auxiliary results first. We start with a technical lemma:

Lemma 16. *Given an indecomposable regular representation V, and a quasi-simple root $\beta_{i,j}$, to check that V is $\mathrm{wt}(\beta_{i,j})$-semi-stable, it is enough to check that $\langle \beta_{i,j}, \underline{\dim}\, V\rangle = 0$, and $\langle \beta_{i,j}, \underline{\dim}\, V'\rangle \leq 0$ for all regular subrepresentations V' of V. Furthermore, for the weight $\alpha_I = \delta + \sum_{j \neq a_i} \sum_{i=1}^{N} \beta_{i,j}$, to check that V is $\mathrm{wt}(\alpha_I)$-stable, it is enough to check that $\langle \alpha_I, \underline{\dim}\, V\rangle = 0$ and $\langle \alpha_I, \underline{\dim}\, V'\rangle < 0$ for all proper, non-zero regular subrepresentations V' of V.*

Proof. Suppose we have indeed checked the criteria indicated, and let $Y \leq V$ be a non-regular subrepresentation. Then the indecomposable direct summands of Y are either preprojective or regular. Without loss of generality, we can assume Y is preprojective and indecomposable. Let $B_{i,j}$ be the unique exceptional representation of dimension vector $\beta_{i,j}$. Then:

$$\langle \beta_{i,j}, \underline{\dim}\, Y\rangle = \dim_K \mathrm{Hom}_Q(B_{i,j}, Y) - \dim_K \mathrm{Ext}^1_Q(B_{i,j}, Y). \tag{3}$$

Now since $B_{i,j}$ is not preprojective (it is regular), and Y is preprojective, by Proposition 8, we have $\mathrm{Hom}_Q(B_{i,j}, Y) = 0$. So, $\langle \beta_{i,j}, \underline{\dim}\, Y\rangle = -\dim_K \mathrm{Ext}^1_Q(B_{i,j}, Y) \leq 0$. Hence, V is $\mathrm{wt}(\beta_{i,j})$-semi-stable. Furthermore,

$$\langle \alpha_I, \underline{\dim}\, Y\rangle = \langle \delta, \underline{\dim}\, Y\rangle + \sum_{j \neq a_i} \sum_{i=1}^{N} \langle \beta_{i,j}, \underline{\dim}\, Y\rangle < 0,$$

since $\langle \delta, \underline{\dim}\, Y\rangle < 0$, for Y preprojective, and $\langle \beta_{i,j}, \underline{\dim}\, Y\rangle \leq 0$ by equation (3). So V is $\mathrm{wt}(\alpha_I)$-stable. □

Recall that the orbit cone of any representation V is defined as follows (see [3, Definition 2.1] or [6, Definition 3.1])

$$\Omega(V) = \{\alpha \in \mathbb{Q}^{Q_0} | V \in \mathrm{rep}(Q)^{ss}_{\mathrm{wt}(\alpha)}\}.$$

We have the following straightforward but useful lemma:

Lemma 17. *If $V = \bigoplus_{i=1}^{m} V_i$ then $\Omega(V) = \bigcap_{i=1}^{m} \Omega(V_i)$.*

Proof. Suppose $\alpha \in \Omega(V)$. Then we have $\langle \alpha, \underline{\dim}\, V_i\rangle \leq 0$ since $V_i \leq V$ and V is $\mathrm{wt}(\alpha)$-semi-stable. On the other hand,

$$\langle \alpha, \underline{\dim}\, V\rangle = \sum_{i=1}^{m} \langle \alpha, \underline{\dim}\, V_i\rangle = 0,$$

so $\langle \alpha, \underline{\dim}\, V_i\rangle = 0$ for all $i = 1, \ldots, m$. Next, for each i, a subrepresentation V_i' of V_i is a subrepresentation of V and, therefore, $\langle \alpha, \underline{\dim}\, V_i'\rangle \leq 0$. So, $\alpha \in \Omega(V_i)$ for each $1 \leq i \leq n$.

Conversely, if $\alpha \in \Omega(V_i)$ for all $i = 1, \ldots, m$, we immediately have $\alpha \in \Omega(V)$ since $\mathrm{rep}(Q)^{ss}_{\mathrm{wt}(\alpha)}$ is closed under extensions. □

Recall now the multi-index $I = (a_1, \ldots, a_N)$, and the corresponding list $\{\beta_{i,a_i}\}_{i=1}^N$ of quasi-simple roots, one from each tube of the Auslander-Reiten quiver of Q. By Proposition 10, we know that there exists a unique δ-dimensional indecomposable representation Z_i of length r_i and whose regular socle is of dimension β_{i,a_i}. Following the notation from Ingalls-Paquette-Thomas in [14], define $Z_I = \bigoplus_{i=1}^N Z_i$. We have the following useful result.

Lemma 18. ([14, Lemma 7.6]) *Keeping the notation above, $C_I = \Omega(Z_I)$.*

Corollary 19. *The Z_i as defined above is $\mathrm{wt}(\alpha_I)$-stable for all $1 \leq i \leq N$.*

Proof. We already know they are $\mathrm{wt}(\alpha_I)$-semi-stable. Let

$$0 = Z_{i,0} < Z_{i,1} < \cdots < Z_{i,r_i} = Z_i$$

be the uniserial filtration for Z_i. By Lemma 16, it is enough to check that $\langle \alpha_I, \underline{\dim}\, Z_{i,r} \rangle < 0$ for all $1 \leq r < r_i$.

Recall that $\underline{\dim}\,(Z_{i,r}/Z_{i,r-1}) = \tau^{-(r-1)}(\beta_{i,a_i})$. Also note that the weight α_I is

$$\alpha_I = \delta + \tau^{-}\beta_{i,a_i} + \cdot + \tau^{-(r_i-1)}(\beta_{i,a_i}) + \text{quasi-simples from other tubes} \tag{4}$$

We will proceed by induction on r to show that $\langle \alpha_I, Z_{i,r} \rangle = -1 < 0$ for all $1 \leq r < r_i$.

When $r = 1$:

$$\langle \alpha_I, \underline{\dim}\, Z_{i,1} \rangle = \langle \alpha_I, \beta_{i,a_i} \rangle = -1,$$

from (4), since $\tau^{-}\beta_{i,a_i}$ appears in the sum for α_I, but β_{i,a_i} does not.
Assume now that $\langle \alpha_I, \underline{\dim}\, Z_{i,s} \rangle = -1$ for $s = 1, \ldots, r-1$. Then

$$\langle \alpha_I, \underline{\dim}\, Z_{i,r}/Z_{i,r-1} \rangle = \langle \alpha_I, \underline{\dim}\, Z_{i,r} \rangle - \langle \alpha_I, \underline{\dim}\, Z_{i,r-1} \rangle$$

and so

$$\langle \alpha_I, \tau^{-(r-1)}(\beta_{i,a_i}) \rangle = \langle \alpha_I, \underline{\dim}\, Z_{i,r} \rangle + 1.$$

Then using (4) again, since $r < r_i$, we have

$$0 = \langle \alpha_I, \underline{\dim}\, Z_{i,r} \rangle + 1.$$

That is, $\langle \alpha_I, \underline{\dim}\, Z_{i,r} \rangle = -1$. So, Z_i is $\mathrm{wt}(\alpha_I)$-stable. □

Lemma 20. *If V is an indecomposable representation of Q with $\underline{\dim}\, V < \delta$, $\langle \alpha_I, \underline{\dim}\, V \rangle = 0$, then V is regular.*

Proof. The weight in question is

$$\alpha_I = \delta + \sum \beta_{i,j}.$$

For this proof it is not particularly important which $\beta_{i,j}$'s are included in the sum. For our specific weight α_I, the sum is over all i, and all $j \neq a_i$.

Suppose V was preinjective. Then we have $\langle \delta, \underline{\dim}\, V \rangle > 0$, so $\langle \sum \beta_{i,j}, \underline{\dim}\, V \rangle < 0$, since $\langle \alpha_I, \underline{\dim}\, V \rangle = 0$. Now $\beta_{i,j}$ is a quasi-simple root (in particular it is a real Schur root), so there is a unique indecomposable $B_{i,j}$ with $\underline{\dim}\, B_{i,j} = \beta_{i,j}$. We have

$$\sum \left(\dim_K \operatorname{Hom}_Q(B_{i,j}, V) - \dim_K \operatorname{Ext}^1_Q(B_{i,j}, V)\right) < 0.$$

But $B_{i,j}$ is regular, while V is preinjective, so $\dim_K \operatorname{Ext}^1_Q(B_{i,j}, V) = 0$ for all i, j by Proposition 8. Thus we conclude $\sum \dim_K \operatorname{Hom}_Q(B_{i,j}, V) < 0$, which is a contradiction. The case that V is preprojective is similar. So, V must be regular. □

Lemma 21. *If $V = \bigoplus_{l=1}^m V_l$ is the decomposition of V into indecomposables, V is δ-dimensional, and $\langle \alpha_I, \underline{\dim}\, V_l \rangle = 0$ for each l, then the V_l's all appear in the same non-homogeneous tube of the Auslander-Reiten quiver of Q.*

Proof. We know each V_l is regular by Lemma 20. The dimension of each V_l can be written as a sum of (τ-consecutive) quasi-simples $\beta_{i,j}$, using the composition factors of the uniserial filtration for V_l. That is, we have $\sum_{j\in J_i} \beta_{i,j} = \underline{\dim}\, V_l$ if V_l is in the ith tube. Since $\sum_{l=1}^m \underline{\dim}\, V_l = \delta$, we have:

$$\sum_{j\in J_1} c_{1,j}\beta_{1,j} + \cdots + \sum_{j\in J_N} c_{N,j}\beta_{N,j} = \delta, \tag{5}$$

where J_i is some subset of $\{1, \ldots, r_i\}$.

Note that we are not assuming there is one V_l in each tube, we are merely combining quasi-simples from the same tube into a single sum. There may be constants, thus the $c_{i,j}$'s, if V has repeated direct summands, or if different V_l's share composition factors. None of the constants are 0, by construction, since we are not including any trivial summands. Note that at least one of the constants must be 1, since $\delta(0) = 1$ where 0 is a vertex of Q such that $Q \setminus \{0\}$ is a Dynkin diagram. Exactly one quasi-simple from each tube has $\beta_{i,j}(0) = 1$, and exactly one of these must appear in the sum (5). Without loss, assume this occurs for the first tube. That is, there is a j_0 such that $\beta_{1,j_0}(0) = 1$, and $c_{1,j_0} = 1$ in the sum (5).

Set $J_1' = J_1 \setminus \{j_0\}$, so we have

$$\delta = \beta_{1,j_0} + \sum_{j\in J_1'} c_{1j}\beta_{1,j} = \sum_{j\in J_2} c_{2j}\beta_{2,j} + \cdots + \sum_{j\in J_N} c_{N,j}\beta_{N,j},$$

$$\delta - \beta_{1,j_0} = \sum_{j\in J_1'} c_{1,j}\beta_{1,j} = \sum_{j\in J_2} c_{2,j}\beta_{2,j} + \cdots + \sum_{j\in J_N} c_{N,j}\beta_{N,j}.$$

Using the relations among quasi-simples,

$$\sum_{j\neq j_0} \beta_{1,j} = \sum_{j\in J_1'} c_{1,j}\beta_{1,j} + \sum_{j\in J_2} c_{2,j}\beta_{2,j} + \cdots + \sum_{j\in J_N} c_{N,j}\beta_{N,j}. \tag{6}$$

Now, taking the Euler inner product of each side with β_{1,j_0}, we see

$$\langle \beta_{1,j_0}, \sum_{j\neq j_0} \beta_{1,j} \rangle = -1.$$

On the other hand,

$$\langle \beta_{1,j_0}, \sum_{j\in J_1'} c_{1,j}\beta_{1,j} + \sum_{j\in J_2} c_{2,j}\beta_{2,j} + \cdots + \sum_{j\in J_N} c_{N,j}\beta_{N,j}\rangle = \langle \beta_{1,j_0}, \sum_{j\in J_1'} c_{1,j}\beta_{1,j}\rangle.$$

If $\tau\beta_{1,j_0} = \beta_{1,j_1}$ does not appear in the right hand sum, that is, if $j_1 \notin J_1'$, this inner product is 0, so we have a contradiction, and the proof is complete. If it does appear, the inner product is $-c_{1,j_1}$, so we have $c_{1,j_1} = 1$. In that case, we return to equation (6), add the new information, and we have

$$\sum_{j\neq j_0,j_1} \beta_{1,j} = \sum_{j\in J_1''} c_{1,j}\beta_{1,j} + \sum_{j\in J_2} c_{2,j}\beta_{2,j} + \cdots + \sum_{j\in J_N} c_{N,j}\beta_{N,j} \tag{7}$$

where $J_1'' = J_1' \setminus \{j_1\}$. If $\tau\beta_{1,j_1} = \beta_{1,j_2}$ appears in the sum (that is, if $j_2 \in J_1''$), then $c_{1,j_2} = 1$ as well. Continuing in this fashion, either we arrive at a contradiction that completes the proof, or $c_{1,j} = 1$ for all $j \in J_1$. So, we have reduced to the following:

$$\sum_{j\in J_1} \beta_{1,j} + \sum_{j\in J_2} c_{2,j}\beta_{2,j} + \cdots + \sum_{j\in J_N} c_{N,j}\beta_{N,j} = \delta. \tag{8}$$

We want to prove all but one of these sums are trivial. Since we have already labeled the tubes in a manner to make the first sum non-trivial, we want to show the others are indeed trivial. Assume the V_l's are not all in the same tube. That is, assume at least two of the sums in equation (8) are non-trivial. We have:

$$\delta - \sum_{j\in J_1} \beta_{1,j} = \sum_{j\in J_2} c_{2j}\beta_{2,j} + \cdots + \sum_{j\in J_N} c_{N,j}\beta_{N,j} \iff$$
$$\sum_{j\in J_1^c} \beta_{1,j} = \sum_{j\in J_2} c_{2,j}\beta_{2,j} + \cdots + \sum_{j\in J_N} c_{N,j}\beta_{N,j}.$$

Note that both J_1 and J_1^c are non-empty. We claim that we can find a β_{1,j_0} with $j_0 \in J_1^c$ such that $\tau\beta_{1,j_0} = \beta_{1,j_0'}$ where $j_0' \in J_1$. Indeed, if it were not, that is if β_{1,j_0} with $j_0 \in J_1^c$ implied $j_0' \in J_1^c$, where $\beta_{1,j_0'} = \tau\beta_{1,j_0}$, then by applying τ successively, we would get $J_1^c = \{1, \ldots, r_1\}$, and J_1 empty.

Now, we have $\langle \beta_{1,j_0}, \sum_{j\in J_1^c} \beta_{1,j}\rangle = -1$ since $\langle \beta_{1,j_0}, \tau\beta_{1,j_0}\rangle = -1$, while

$$\langle \beta_{1,j_0}, \sum_{j\in J_2} c_{2,j}\beta_{2,j} + \cdots + \sum_{j\in J_N} c_{N,j}\beta_{N,j}\rangle = 0$$

which is a contradiction. So, it must be that all V_l's are in the same tube. □

Given a rational weight $\theta \in \mathbb{Q}^{Q_0}$, a representation $W \in \mathrm{rep}(Q)$ is said to be **θ-polystable** if and only if the indecomposable direct summands of W are θ-stable. A key result we will use says that if $\alpha \in \mathcal{D}(\beta)$ then

$$\mathcal{C}(\beta)_\alpha = \bigcap_W \Omega(W), \tag{9}$$

where the intersection is over all $\mathrm{wt}(\alpha)$-polystable representations $W \in \mathrm{rep}(Q, \beta)$ (see [6, Lemma 4.1] for a proof).

Lemma 22. *If V is δ-dimensional and $\mathrm{wt}(\alpha_I)$-polystable, then V is actually indecomposable and thus $\mathrm{wt}(\alpha_I)$-stable.*

Proof. Suppose by way of contradiction that a decomposable such V exists. Set $V = \bigoplus_{l=1}^{m} V_l$ to be the decomposition into indecomposables. Since V is $\mathrm{wt}(\alpha_I)$-polystable, each V_l is $\mathrm{wt}(\alpha_I)$-stable. Applying Lemma 20, each V_l is regular, and applying Lemma 21, they all lie in the same tube.

Each V_l has a uniserial filtration, so that $\underline{\dim}\, V_l$ is a sum of quasi-simple roots. Since $\sum_{l=1}^{m} \underline{\dim}\, V_l = \delta$, at most one of the V_l's may have regular socle of dimension β_{i,a_i}. Since we are assuming V is decomposable, there are at least 2 direct summands, and we can choose V_{l_o} so that the regular socle of V_{l_o} is of dimension $\beta_{i,j}$ with $j \neq a_i$. But then $\langle \alpha_I, \beta_{i,j} \rangle = 0$ or 1, a contradiction to V_l being $\mathrm{wt}(\alpha_I)$-stable. ☐

Lemma 23. *If V is δ-dimensional and homogeneous (i.e., regular simple), then $\Omega(V) = \mathcal{D}(\delta)$.*

Proof. Certainly $\Omega(V) \subseteq \mathcal{D}(\delta)$. Suppose $\alpha \in \mathcal{D}(\delta)$, and let $0 \neq V' < V$. Since V is regular simple, it has no proper regular subrepresentations. So V' must be preprojective. Without loss, assume V' is indecomposable.

If $\underline{\dim}\, V' \hookrightarrow \delta$, then $\langle \alpha, \underline{\dim}\, V' \rangle \leq 0$ since $\alpha \in \mathcal{D}(\beta)$, so $\alpha \in \Omega(V)$. If not, then by [9, Theorem 2.7] we know $\mathrm{Ext}^1_Q(V', V/V') \neq 0$. Let $V/V' = \bigoplus_{i=1}^{m} V_i$ be the decomposition of V/V' into indecomposable representations. Since V is regular, by Proposition 8 V/V' has no preprojective direct summand. So, by our assumption we have $\bigoplus_{i=1}^{m} \mathrm{Ext}^1_Q(V', V_i) \neq 0$. But, V' is preprojective and V_i is not, so $\mathrm{Ext}^1_Q(V', V_i) = 0$ by Proposition 8, which is a contradiction. ☐

Recall that Z_i is the unique δ-dimensional indecomposable representation of Q with regular socle of dimension β_{i,a_i}.

Lemma 24. *The list of δ-dimensional, $\mathrm{wt}(\alpha_I)$-polystable representations is exactly: Z_i for each $i = 1, \ldots, N$, together with the homogeneous Schur representations of Q.*

Proof. First of all, we know from Lemma 22 that any δ-dimensional, $\mathrm{wt}(\alpha_I)$-polystable representation is in fact regular indecomposable, thus $\mathrm{wt}(\alpha_I)$-stable.

Next, the Z_i's are $\mathrm{wt}(\alpha_I)$-stable by Corollary 19. On the other hand, if V is any other δ-dimensional indecomposable from a non-homogeneous tube, thus not isomorphic to one of the Z_i's, then the regular socle of V is of dimension $\beta_{i,j}$ for some $j \neq a_i$ and $1 \leq i \leq N$. So V has a subrepresentation V' with $\langle \alpha_I, \underline{\dim}\, V' \rangle = \langle \alpha_I, \beta_{i,j} \rangle = 0$ or 1, so V is not $\mathrm{wt}(\alpha_I)$-stable.

Finally, let Z be an indecomposable representation of dimension δ lying in a homogeneous tube. Then Z has no regular subrepresentations since it is regular simple, and hence Z is $\mathrm{wt}(\alpha_I)$-stable by Lemma 16. ☐

We need one more (general) lemma before we can prove our results from the beginning of this section. Let us recall that for any quiver Q, dimension vector $\beta \in \mathbb{Z}^{Q_0}$, and weight $\sigma \in \mathbb{Z}^{Q_0}$, the moduli space of σ-semi-stable β-dimensional representations is $\mathcal{M}(Q,\beta)^{ss}_{\sigma} = \mathrm{Proj}(\bigoplus_{n\geq 0} \mathrm{SI}(Q,\beta)_{n\sigma})$ where $\mathrm{SI}(Q,\beta)_{n\sigma} = \{f \in K[\mathrm{rep}(Q,\beta)] \mid g \cdot f = (\prod_{x \in Q_0} \det(g(x))^{n\sigma(x)}) f, \forall g = (g(x))_{x \in Q_0} \in \mathrm{GL}(\beta)\}$ is the space of semi-invariants on $\mathrm{rep}(Q,\beta)$ of weight $n\sigma$ for all $n \geq 0$. It follows from King's results in [19, Proposition 3.2] that the (closed) points of $\mathcal{M}(Q,\beta)^{ss}_{\sigma}$ correspond bijcetively to the isomorphism classes of the β-dimensional σ-polystable representations of Q.

Lemma 25. *Let Q be a quiver (not necessarily Euclidean) and β a real Schur root of Q. Then $\mathcal{C}(\beta) = \{\mathcal{D}(\beta)\}$, i.e. the only maximal GIT-cone of $\mathcal{D}(\beta)$ is $\mathcal{D}(\beta)$ itself.*

Proof. Let $V_0 \in \operatorname{rep}(Q,\beta)$ be a Schur representation. Then, as β is assumed to be real, we get

$$1 = q(\beta) = \dim_K \operatorname{End}_Q(V_0) - \dim_K \operatorname{Ext}^1_Q(V_0,V_0) = 1 - \dim_K \operatorname{Ext}^1_Q(V_0,V_0)$$

which implies that $\operatorname{Ext}^1_Q(V_0,V_0) = 0$. On the other hand, it well-known that the dimension of $\operatorname{Ext}^1_Q(V_0,V_0)$ equals the codimension in $\operatorname{rep}(Q,\beta)$ of $\operatorname{GL}(\beta)V_0$. (This formula holds in full generality, for any dimension vector β and representation V_0.) We conclude that $\operatorname{GL}(\beta)V_0$ is open in $\operatorname{rep}(Q,\beta)$. Consequently, for any $\alpha \in \mathcal{D}(\beta)$, $\operatorname{rep}(Q,\beta)^{ss}_{\operatorname{wt}(\alpha)}$ and $\operatorname{GL}(\beta)V_0$ are non-empty open subsets of $\operatorname{rep}(Q,\beta)$; therefore they have a non-empty intersection, implying that V_0 is $\operatorname{wt}(\alpha)$-semi-stable. This shows that $\Omega(V_0) = \mathcal{D}(\beta)$.

Now, let $\mathcal{C}(\beta)_{\alpha_0}$ be a maximal GIT-cone and ,without loss of generality, assume that $\alpha_0 \in \mathbb{Z}^{Q_0}$. By Remark 1, we know that $\alpha_0 \in \mathcal{D}^0(\beta)$ which means that the general β-dimensional representation is $\operatorname{wt}(\alpha_0)$-stable, and hence V_0 must $\operatorname{wt}(\alpha_0)$-stable; in particular, V_0 is $\operatorname{wt}(\alpha_0)$-polystable.

Next, we claim that V_0 is the only β-dimensional $\operatorname{wt}(\alpha_0)$-polystable representation, up to isomorphism. Indeed, since $\operatorname{GL}(\beta)$ acts on $\operatorname{rep}(Q,\beta)$ with a dense orbit, any weight space of semi-invariants is of dimension at most 1. Consequently, $\mathcal{M}(Q,\beta)_{\operatorname{wt}(\alpha_0)}$ is just a point, meaning that there is only one isomorphism class of $\operatorname{wt}(\alpha_0)$-polystable representations, i.e. V_0 is the only β-dimensional $\operatorname{wt}(\alpha_0)$-polystable representation.

Putting everything together and using (9), we finally get that $\mathcal{C}(\beta)_{\alpha_0} = \Omega(V_0) = \mathcal{D}(\beta)$. □

Proof of Lemma 14. We have shown that $C_I = \Omega(Z_I)$ in Lemma 18. Next, using Lemma 24 and equation (9), we get that:

$$\Omega(Z_I) = \bigcap_{i=1}^{N} \Omega(Z_i) = \left(\bigcap_{i=1}^{N} \Omega(Z_i)\right) \cap D(\delta) = \bigcap_{\substack{X \text{ is } \delta\text{-dimensional} \\ \operatorname{wt}(\alpha_I)\text{-polystable}}} \Omega(X) = \mathcal{C}(\delta)_{\alpha_I}.$$

Lastly, C_I is of maximal dimension, so $\mathcal{C}(\delta)_{\alpha_I}$ is of maximal dimension as well. □

Proof of Lemma 15. If β is real, then $\mathcal{C}(\beta)_\alpha$ maximal implies $\mathcal{C}(\beta)_\alpha = \mathcal{D}(\beta)$ by Lemma 25.

If β is not real, then it must be equal to δ, since it is Schur and Q is Euclidean. Since the C_I's cover $\mathcal{D}(\delta)$ as I varies, $\alpha \in C_I$ for some I. We know that $C_I = \mathcal{C}(\delta)_{\alpha_I}$ for the weight α_I by Lemma 14.

Finally, note that the intersection $\mathcal{C}(\delta)_\alpha \cap \mathcal{C}(\delta)_{\alpha_I}$ is a face of both cones, and it also contains α which lies in the relative interior of $\mathcal{C}(\delta)_\alpha$. Consequently, $\mathcal{C}(\delta)_\alpha \cap \mathcal{C}(\delta)_{\alpha_I} = \mathcal{C}(\delta)_\alpha$, making $\mathcal{C}(\delta)_\alpha$ a face of $\mathcal{C}(\delta)_{\alpha_I}$. Since these two cones are maximal, we see that $\mathcal{C}(\delta)_\alpha$ and $\mathcal{C}(\delta)_{\alpha_I}$ must be equal. □

We are now ready to prove Theorem 2 which, in the Euclidean case, simply says that $\mathcal{J} = \mathcal{I}$.

Proof of Theorem 2. (1) For a Dunkin quiver Q, the positive roots are precisely the real Schur roots of Q. So, the proof of this part of the theorem follows Lemma 25.

(2) It follows from Lemmas 14 and 15. □

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