



# Generators of Koszul homology with coefficients in a $J$ -closed module



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## ABSTRACT

We introduce a class of modules, called  $J$ -closed modules, inspired by the weak complete intersection ideals studied by Rahmati, Striuli, and Yang. We present explicit formulas for the generators of Koszul homology with coefficients in such modules. This generalizes work of Herzog and of Corso, Goto, Huneke, Polini, and Ulrich. We use these formulas to study connections between  $J$ -closed ideals and weak complete intersection ideals.

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## 1. Introduction

Let  $Q$  be a commutative Noetherian ring and let  $J$  be a complete intersection ideal in  $Q$ . In this paper, we study a class of ideals called  $J$ -closed ideals, inspired by the weak complete intersection ideals studied by Rahmati, Striuli, and Yang in their recent paper [10]. In a local ring  $Q$ , they define weak complete intersection ideals to be the ideals  $I$  such that the differentials in the minimal free resolution  $F$  of the quotient  $Q/I$  land in  $IF$ . In any commutative Noetherian ring  $Q$ , we define  $J$ -closed ideals to be the ideals  $I$  such that there is a projective resolution  $P$  of  $Q/I$  whose differentials land in  $JP$ . We define the more general notion of a  $J$ -closed module in a similar way. Expanding the allowed range for the differentials allows additional flexibility, which as we show in this paper, becomes useful in studying weak complete intersection ideals. Of course, when  $Q$  is a regular local ring and  $\mathfrak{m}$  is its maximal ideal, then any ideal is an  $\mathfrak{m}$ -closed ideal, but more interesting examples are abundant. We show that a weak complete intersection ideal is a  $J$ -closed ideal if and only if it is a complete intersection ideal embedded in  $J$ . Although neither class of

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ideals is contained in the other, we find other connections between them. To accomplish this, we study the Koszul homology  $H(\underline{g}; R)$ , where  $R = Q/I$  is the quotient by a  $J$ -closed ideal and  $\underline{g} = g_1, \dots, g_s$  is a regular sequence that generates  $J$ .

One approach to understanding Koszul homology is to describe the generators. In his 1991 paper [6] (see also [8]), Herzog gives explicit formulas for generators of the homology  $H(\underline{x}; R)$  of the Koszul complex on the minimal generators of the irrelevant maximal ideal  $m = (x_1, \dots, x_n)$  of a finitely generated graded  $k$ -algebra  $R$ , where  $k$  is a field of characteristic zero. Recently, the authors of [2] provided explicit formulas for generators of Koszul homology in a more general setting. They studied the homology of the Koszul complex on a sequence  $\underline{g} = x_1^{a_1}, \dots, x_n^{a_n}$  with  $M$ -coefficients, where  $J = (\underline{g})$  and  $M$  is what we call a  $J$ -closed module throughout this paper. We further generalize to the setting where  $\underline{g}$  is any full regular sequence generating  $J$  and  $M$  is a  $J$ -closed module. The formulas we obtain have new terms, which vanish in the case considered by Herzog.

One of the main tools we use to obtain these formulas is the classical perturbation lemma; see for example [3]. We also utilize the theory developed in [4] on the formulation of a sort of partial derivative with respect to a regular sequence  $\underline{g}$  and the de Rham contraction built from it. The formulas we provide are given in terms of these partials.

The formulas for generators of Koszul homology given in this paper become useful in studying the connections between  $J$ -closed ideals and weak complete intersection ideals. In particular, we study the ideal  $J/I$  in the quotient  $Q/I$ , where  $I$  is a  $J$ -closed ideal, and we give a general condition under which  $J/I$  is a weak complete intersection. The condition we give involves the partials with respect to a regular sequence that generates  $J$ , mentioned above. This provides a new family of examples of weak complete intersections.

## 2. Preliminaries

In this section we introduce the notion of  $J$ -closed ideals and modules and we discuss the main tools used throughout the paper, including the perturbation lemma and a version of the de Rham contraction developed in [4].

### 2.1. The perturbation lemma

In this section we discuss the classical perturbation lemma. We use it in Section 3 as the main tool for providing explicit formulas for the generators of Koszul homology. We begin with the definitions, which can be found in [3].

**Definition 2.1.** A *deformation retract datum*

$$\left( (F, \partial_F) \xrightleftharpoons[i]{p} (G, \partial_G), H \right)$$

consists of the following:

- (i) complexes  $(F, \partial_F)$  and  $(G, \partial_G)$
- (ii) quasi-isomorphisms  $p$  and  $i$  such that  $pi = \text{Id}_F$ .
- (iii) a homotopy  $H$  between  $ip$  and  $\text{Id}_G$  (i.e.  $\partial_G H + H \partial_G = ip - \text{Id}_G$ )

We call  $F$  a *deformation retract* of  $G$ . A *special deformation retract datum* is a deformation retract datum that also satisfies the equalities

$$Hi = 0, \quad pH = 0, \quad H^2 = 0.$$

Given a deformation retract datum, one can define a (small) perturbation of the datum as follows.

**Definition 2.2.** A *perturbation* of a deformation retract datum is a map

$$G \xrightarrow{\epsilon} G$$

of degree  $-1$ , such that  $(\partial_G + \epsilon)^2 = 0$ . The perturbation is *small* if the map  $\text{Id}_G - \epsilon H$  is invertible.

Now we state the perturbation lemma; see for example [3, 2.4].

**Theorem 2.3 (Perturbation Lemma).** If  $\epsilon$  is a small perturbation of the deformation retract datum

$$\left( (F, \partial_F) \xrightleftharpoons[i]{p} (G, \partial_G), H \right)$$

then the perturbed datum

$$\left( (F, \widetilde{\partial}_F) \xrightleftharpoons[\tilde{i}]{\tilde{p}} (G, \partial_G + \epsilon), \tilde{H} \right)$$

is a deformation retract datum, where

$$\widetilde{\partial}_F = \partial_F + pAi, \quad \tilde{p} = p + pAH, \quad \tilde{i} = i + HAi, \quad \tilde{H} = H + HAH$$

and  $A = (\text{Id}_G - \epsilon H)^{-1}\epsilon$ . In particular,  $\tilde{i}$  is a homotopy equivalence.

In Section 3, we modify a deformation retract datum constructed in [4], and apply the perturbation lemma to provide formulas for generators of Koszul homology.

## 2.2. $J$ -closed ideals and modules

Let  $Q$  be a commutative Noetherian ring and let  $J$  be a complete intersection ideal. In this section we introduce a class of ideals called  $J$ -closed ideals, inspired by the weak complete intersection ideals defined in [10].

**Definition 2.4.** A finitely generated  $Q$ -module  $M$  is a  *$J$ -closed module* if there is a projective resolution  $(P, \partial)$  of  $M$  which satisfies the property

$$\text{Im } \partial_i \subseteq JP_{i-1} \tag{1}$$

for every  $i$ . We call an ideal  $I \subseteq Q$  a  *$J$ -closed ideal* if  $Q/I$  is a  $J$ -closed module.

If  $Q$  is local, we note that it suffices to consider the minimal free resolution. We also have the following characterization of  $J$ -closed modules in the local case.

**Remark 2.5.** If  $Q$  is local, then a module  $M$  is  $J$ -closed if and only if  $\text{Tor}_i^Q(Q/J, M)$  is a free  $Q/J$ -module for every  $i$ ; see for example [10, Lemma 2.2].

We now give some examples of  $J$ -closed ideals.

**Example 2.6.** The complete intersection ideal  $J \subseteq Q$  is a  $J$ -closed ideal. Indeed,  $Q/J$  is resolved by the Koszul complex on a minimal set of generators of  $J$ , whose differentials certainly land in the ideal  $J$ . Similarly, any embedded complete intersection ideal  $I \subseteq J$  is also a  $J$ -closed ideal because  $Q/I$  is resolved by the Koszul complex on a minimal set of generators of  $I$ . In a regular local ring, every module is an  $\mathfrak{m}$ -closed module, where  $\mathfrak{m}$  is the maximal ideal.

The next example gives a large class of  $J$ -closed ideals that are not complete intersection ideals.

**Example 2.7.** Let  $J$  be a complete intersection ideal in a local ring  $Q$  and take  $\underline{g} = g_1, \dots, g_s$  to be a regular sequence generating  $J$ . Then any ideal  $I$  generated by monomials in  $\underline{g}$  is a  $J$ -closed ideal. Indeed,  $Q/I$  is resolved (possibly non-minimally) by the Taylor resolution for monomials in a regular sequence. The entries in the differentials are either monomials in  $\underline{g}$  or units. After change of bases, the minimal free resolution splits off. The entries in the differentials of the minimal resolution are still contained in  $J$  as the appropriate row and column operations do not disturb this property.

The next example shows that there are non-monomial ideals that are  $J$ -closed.

**Example 2.8.** Let  $Q = k[x, y, z]$  be a polynomial ring and let  $I = (x^2y^4 + y^3z^7, y^6, x^4y^2)$ . According to Macaulay2, a free resolution of  $Q/I$  over  $Q$  is given by

$$0 \rightarrow Q \xrightarrow{\partial_3} Q^3 \xrightarrow{\partial_2} Q^3 \xrightarrow{\partial_1} Q \rightarrow Q/I \rightarrow 0$$

where the differentials are given by the following matrices:

$$\partial_3 = \begin{bmatrix} -z^7 - x^2y \\ -x^4 \\ y^3 \end{bmatrix}, \quad \partial_2 = \begin{bmatrix} -y^4 & 0 & -yz^7 - x^2y^2 \\ x^4 & -z^7 - x^2y & 0 \\ 0 & y^3 & x^4 \end{bmatrix}, \quad \partial_1 = \begin{bmatrix} x^4y^2 & y^6 & x^2y^4 + y^3z^7 \end{bmatrix}.$$

It is easy to see that  $I$  is a  $J$ -closed ideal, where  $J$  is any complete intersection ideal containing  $(x^2, y^3, z^7)$ .

Although the definition of  $J$ -closed ideals was inspired by the definition of weak complete intersection ideals, for a fixed complete intersection ideal  $J$ , the two classes of ideals are distinct and neither one is contained in the other. In fact, the intersection of the two classes is contained in the class of complete intersection ideals, as shown in the following proposition.

**Proposition 2.9.** *Let  $Q$  be a local (or graded) ring and let  $J$  be a (homogeneous) complete intersection ideal in  $Q$ .*

- (1) *Every finitely generated  $J$ -closed module has finite projective dimension via the resolution in Definition 2.4.*
- (2) *A weak complete intersection ideal is a  $J$ -closed ideal if and only if it is a complete intersection ideal embedded in  $J$ .*

**Proof.** We give a proof of the local case; the one in the graded case is similar.

(1) Let  $M$  be a finitely generated  $J$ -closed module and let  $F$  be a free resolution of  $M$  with  $\text{Im } \partial_i \subseteq JF_{i-1}$  for all  $i$ . Pick a regular sequence  $\underline{g} = g_1, \dots, g_s$  that generates  $J$  and let  $H_\ell(\underline{g}; M) = H_\ell(K(\underline{g}; Q) \otimes_Q M)$ , where  $K(\underline{g}; Q)$  is the Koszul complex on  $\underline{g}$  over  $Q$ . Then we have isomorphisms

$$H_\ell(\underline{g}; M) = H_\ell(K(\underline{g}; Q) \otimes_Q M) \cong \text{Tor}_\ell^Q(Q/J, M) \cong H_\ell(Q/J \otimes_Q F) \cong Q/J \otimes_Q F_\ell$$

where the first isomorphism follows from the fact that  $\underline{g}$  is a regular sequence and the last isomorphism follows directly from Definition 2.4. Note that  $H_\ell(\underline{g}; M) = 0$  for all  $\ell > s$ . Then  $Q/J \otimes_Q F_\ell = 0$  by above, and hence  $F_\ell = 0$  for all  $\ell > s$  by Nakayama's Lemma. Thus,  $\text{pd}_Q M < \infty$ .

(2) Let  $I$  be a weak complete intersection ideal and suppose it is also  $J$ -closed. Note that the minimal generators of  $I$  are contained in the ideal  $J$  since they are the entries in the first differential of the minimal free resolution  $F$  of  $Q/I$  over  $Q$ . And by (1),  $\text{pd}_Q Q/I < \infty$ . Thus the result follows from [11, Cor 1].  $\square$

In Section 4, we investigate further the connection between  $J$ -closed ideals and weak complete intersection ideals.

### 2.3. The de Rham contraction

In this section we use the theory in [4] on connections to formulate a sort of partial derivative with respect to the elements of a regular sequence  $\underline{g}$ . We use these partial derivatives in the formulas given in Section 3.

Throughout this section we assume that  $k$  is a field of characteristic zero and  $Q$  is a Noetherian  $k$ -algebra. We let  $\underline{g}$  be a regular sequence in  $Q$  such that  $Q/(\underline{g})$  is a finite dimensional  $k$ -vector space and  $Q$  is complete<sup>2</sup> in the  $(\underline{g})$ -adic topology.

We let  $\Omega_{k[\underline{g}]/k}^1$  be the module of Kähler differentials, that is,

$$\Omega_{k[\underline{g}]/k}^1 = \bigoplus_{i=1}^s k[\underline{g}] dg_i,$$

and we denote by  $\Omega_{k[\underline{g}]/k}^\bullet$  the exterior algebra over  $\Omega_{k[\underline{g}]/k}^1$  with differential induced by the map  $\Omega_{k[\underline{g}]/k}^1 \rightarrow k[\underline{g}]$  sending  $dg_i$  to  $g_i$ , where we note that  $Q$  is an algebra over the subring  $k[\underline{g}]$ , which is a polynomial ring since  $\underline{g}$  is a regular sequence. The Koszul complex  $K(\underline{g}; Q)$  on  $\underline{g}$  over  $Q$  is given by  $Q \otimes_{k[\underline{g}]} \Omega_{k[\underline{g}]/k}^\bullet$ . Indeed,

$$\begin{aligned} Q \otimes_{k[\underline{g}]} \Omega_{k[\underline{g}]/k}^\bullet &= Q \otimes_{k[\underline{g}]} \bigwedge \Omega_{k[\underline{g}]/k}^1 \\ &= Q \otimes_{k[\underline{g}]} k[\underline{g}] \langle dg_1, \dots, dg_s \mid \delta(dg_i) = g_i \rangle \\ &= Q \langle dg_1, \dots, dg_s \mid \delta(dg_i) = g_i \rangle. \end{aligned}$$

Let  $\pi : Q \rightarrow Q/(\underline{g})$  be the quotient map and fix a  $k$ -linear splitting

$$\sigma : Q/(\underline{g}) \rightarrow Q$$

of  $\pi$ . The following lemma is a well known; see for example [4, Appendix B] or [9, Lemma 3.1.1].

**Lemma 2.10.** *For every element  $q \in Q$ , there exist unique residue classes  $\overline{q_N} \in Q/(\underline{g})$  such that*

$$q = \sum_{N \in \mathbb{N}^s} \sigma(\overline{q_N}) g^N,$$

where  $N = (n_1, \dots, n_s)$  and  $g^N = g_1^{n_1} \dots g_s^{n_s}$ .

<sup>2</sup> We use the term “complete” to mean “complete and separated”.

By writing elements of  $Q$  in this way, Dyckerhoff and Murfet in [4] give an explicit  $k$ -linear connection on  $Q$ ; that is, a map

$$\nabla^0: Q \rightarrow Q \otimes_{k[g]} \Omega_{k[g]/k}^1$$

which satisfies the Leibniz rule. In this context, the Leibniz rule is

$$\nabla^0(aq) = a\nabla^0(q) + q \otimes d^0(a),$$

for  $a \in k[g]$  and  $q \in Q$ , where  $d^0: k[g] \rightarrow \Omega_{k[g]/k}^1$  is the Kähler differential; see for example [4, Def 2.7]. Explicitly, they define

$$\nabla^0(q) = \sum_{i=1}^s \sum_N N_i \sigma(\overline{q_N}) g^{N-e_i} \otimes dg_i$$

where  $e_i$  are the standard basis vectors of  $\mathbb{Z}^s$  and  $g^N$  is defined to be zero whenever some component of  $N$  is negative. By means of this connection, one can define  $\frac{\partial}{\partial g_i}$  to be the  $k$ -linear map given by the composition

$$\frac{\partial}{\partial g_i}: Q \xrightarrow{\nabla^0} Q \otimes_{k[g]} \Omega_{k[g]/k}^1 \xrightarrow{(dg_i)^*} Q,$$

where

$$(dg_i)^*(q \otimes dg_j) = \begin{cases} q & i = j \\ 0 & i \neq j \end{cases}.$$

**Remark 2.11.** We note that in order for  $\frac{\partial}{\partial g_j}$  to be well-defined, it is important to fix a splitting  $\sigma$ .

We will need the following lemmas in the proof of Theorem 3.5.

**Lemma 2.12.**  $\frac{\partial^2}{\partial g_j \partial g_i} = \frac{\partial^2}{\partial g_i \partial g_j}.$

**Proof.** Writing  $q = \sum_N \sigma(\overline{q_N}) g^N$  as in Lemma 2.10, we have the equalities

$$\frac{\partial}{\partial g_i}(q) = \sum_N N_i \sigma(\overline{q_N}) g^{N-e_i} = \sum_M \sigma(\overline{r_M}) g^M$$

where  $\overline{r_M} = N_i \overline{q_N}$  and  $M = N - e_i$ , and where the second equality follows from the fact that  $\sigma$  is  $k$ -linear. Thus we get that

$$\frac{\partial^2}{\partial g_j \partial g_i}(q) = \sum_M M_j \sigma(\overline{r_M}) g^{M-e_j} = \sum_N N_i N_j \sigma(\overline{q_N}) g^{N-e_i-e_j}$$

again by  $k$ -linearity. A similar argument for  $\frac{\partial^2}{\partial g_i \partial g_j}(q)$ , gives the desired equality.  $\square$

The next lemma is a version of the Leibniz rule for  $\frac{\partial}{\partial g_j}$ . In the Lemma, we write elements  $q, r \in Q$  as  $q = \sum_N \sigma(\overline{q_N}) g^N$  and  $r = \sum_M \sigma(\overline{r_M}) g^M$ , respectively.

**Lemma 2.13.** *The map  $\frac{\partial}{\partial g_j}$  satisfies the rule*

$$\frac{\partial}{\partial g_j}(qr) = \frac{\partial}{\partial g_j}(q)r + \frac{\partial}{\partial g_j}(r)q + \sum_{M,N} \left( \frac{\partial}{\partial g_j}(\sigma(\overline{q_N})\sigma(\overline{r_M})) \right) g^{M+N}.$$

**Proof.** We begin by noting that  $qr = \sum_{M,N} \sigma(\overline{q_N})\sigma(\overline{r_M})g^{M+N}$  and writing  $\sigma(\overline{q_N})\sigma(\overline{r_M}) = \sum_P \sigma(\overline{s_P})g^P$ . Now we compute

$$\begin{aligned} \frac{\partial}{\partial g_j}(qr) &= \frac{\partial}{\partial g_j} \left( \sum_{M,N,P} \sigma(\overline{s_P})g^{M+N+P} \right) = \sum_{M,N,P} (M_j + N_j + P_j) \sigma(\overline{s_P})g^{M+N+P-e_j} \\ &= \sum_{M,N,P} (M_j + N_j) \sigma(\overline{s_P})g^{M+N+P-e_j} + \sum_{M,N,P} P_j \sigma(\overline{s_P})g^{M+N+P-e_j} \\ &= \sum_{M,N} (M_j + N_j) \left( \sum_P \sigma(\overline{s_P})g^P \right) g^{M+N-e_j} + \sum_{M,N} \left( \sum_P P_j \sigma(\overline{s_P})g^{P-e_j} \right) g^{M+N} \\ &= \sum_{M,N} (M_j + N_j) \sigma(\overline{q_N})\sigma(\overline{r_M})g^{M+N-e_j} + \sum_{M,N} \left( \frac{\partial}{\partial g_j}(\sigma(\overline{q_N})\sigma(\overline{r_M})) \right) g^{M+N} \end{aligned}$$

and we see that

$$\begin{aligned} \sum_{M,N} (M_j + N_j) \sigma(\overline{q_N})\sigma(\overline{r_M})g^{M+N-e_j} &= \sum_{M,N} M_j \sigma(\overline{q_N})\sigma(\overline{r_M})g^{M+N-e_j} + \sum_{M,N} N_j \sigma(\overline{q_N})\sigma(\overline{r_M})g^{M+N-e_j} \\ &= \left( \sum_M M_j \sigma(\overline{r_M})g^{M-e_j} \right) \left( \sum_N \sigma(\overline{q_N})g^N \right) + \left( \sum_N N_j \sigma(\overline{q_N})g^{N-e_j} \right) \left( \sum_M \sigma(\overline{r_M})g^M \right) \\ &= \frac{\partial}{\partial g_j}(q)r + \frac{\partial}{\partial g_j}(r)q \end{aligned}$$

which completes the proof.  $\square$

**Remark 2.14.** We see from Lemma 2.13 that  $\frac{\partial}{\partial g_j}$  satisfies the usual Leibniz rule,

$$\frac{\partial}{\partial g_j}(qr) = \frac{\partial}{\partial g_j}(q)r + q \frac{\partial}{\partial g_j}(r),$$

whenever for each pair  $(M, N)$ , either  $\sigma(\overline{q_N}) \in k$  or  $\sigma(\overline{r_M}) \in k$ . Indeed, in this case  $\frac{\partial}{\partial g_j}(\sigma(\overline{q_N})\sigma(\overline{r_M})) = 0$ . In particular, we have that

$$\frac{\partial}{\partial g_j}(qg_k) = \begin{cases} \frac{\partial}{\partial g_j}(q)g_k + q & j = k \\ \frac{\partial}{\partial g_j}(q)g_k & j \neq k \end{cases}.$$

However, examples which do not satisfy the usual product rule are plentiful. Consider the regular sequence  $g_1 = x^2$ ,  $g_2 = y^3$ ,  $g_3 = z^5$  in  $k[x, y, z]$ . We have that

$$\frac{\partial}{\partial g_1}(xy^3 \cdot xz^5) = \frac{\partial}{\partial g_1}(x^2y^3z^5) = y^3z^5,$$

but

$$\frac{\partial}{\partial g_1}(xy^3)xz^5 + \frac{\partial}{\partial g_1}(xz^5)xy^3 = 0.$$

Note however that

$$\frac{\partial}{\partial g_1}(x \cdot x)y^3z^5 = y^3z^5$$

which illustrates Lemma 2.13.

Using the map  $\frac{\partial}{\partial g_j}$ , one defines  $\nabla$  to be the  $k$ -linear map

$$\nabla: Q \otimes_{k[\underline{g}]} \Omega_{k[\underline{g}]/k}^\bullet \rightarrow Q \otimes_{k[\underline{g}]} \Omega_{k[\underline{g}]/k}^\bullet \quad (2)$$

given by

$$\nabla(q \otimes \omega) = \sum_{i=1}^s \frac{\partial}{\partial g_i}(q) \otimes dg_i \wedge \omega + q \otimes d\omega.$$

Recall that  $\delta$  is the differential on  $K(\underline{g}; Q)$ . Since  $\text{char } k = 0$ , we have that  $\delta\nabla + \nabla\delta$  is invertible in nonzero degrees by [4, 8.1], so one can make the following definition; see [4, Definition 8.8].

**Definition 2.15.** Let  $H_\nabla$  be the  $k$ -linear map

$$\begin{aligned} H_\nabla: Q \otimes_{k[\underline{g}]} \Omega_{k[\underline{g}]/k}^\bullet &\rightarrow Q \otimes_{k[\underline{g}]} \Omega_{k[\underline{g}]/k}^\bullet \\ H_\nabla &= (\delta\nabla + \nabla\delta)^{-1}\nabla. \end{aligned}$$

This map is called the *de Rham contraction*.

The de Rham contraction is a homotopy on the Koszul complex, as we see in the following result of Dyckerhoff and Murfet.

**Theorem 2.16.** [4, 8.10] *Let  $Q$  be a Noetherian  $k$ -algebra where  $k$  a field of characteristic zero and let  $\underline{g}$  be a regular sequence in  $Q$  such that  $Q/(\underline{g})$  is a finite dimensional  $k$ -vector space and  $Q$  is complete in the  $(\underline{g})$ -adic topology. The following is a special deformation retract datum*

$$\left( (Q/(\underline{g}), 0) \xrightleftharpoons[\sigma]{\pi} (Q \otimes_{k[\underline{g}]} \Omega_{k[\underline{g}]/k}^\bullet, \delta), H_\nabla \right).$$

We modify this datum in the next section and apply the perturbation lemma to provide the formulas in Theorem 3.5.

### 3. Generators of Koszul homology

Let  $Q$  be a Noetherian  $k$ -algebra with  $k$  a field of characteristic zero. We fix a regular sequence  $\underline{g}$  in  $Q$  and the ideal  $J$  generated by it such that  $Q/J$  is a finite dimensional  $k$ -vector space. Let  $M$  be a finitely generated  $J$ -closed  $Q$ -module that has a free resolution satisfying (1) in Definition 2.4. In this section we study the homology of the Koszul complex on  $\underline{g}$  with coefficients in  $M$ .



Now we fix some notation to be used throughout the section. Let  $F$  be a free resolution of  $M$  over  $Q$  such that  $\text{Im } \partial_F \subseteq JF$ . Let  $\pi : Q \rightarrow Q/J$  be the quotient map and fix a  $k$ -linear splitting  $\sigma : Q/J \rightarrow Q$ . Set  $K = K(\underline{g}; Q) = Q \otimes_{k[\underline{g}]} \Omega_{k[\underline{g}]/k}^\bullet$ .

Consider the isomorphisms

$$H_\ell(\underline{g}; M) = H_\ell(M \otimes_Q K) \cong \text{Tor}_\ell^Q(M, Q/J) \cong H_\ell(F \otimes_Q Q/J) \cong F_\ell \otimes_Q Q/J \quad (3)$$

of  $Q/J$ -modules, which appeared in the proof of Proposition 2.9. Thus, the Koszul homology we are interested in is isomorphic to the homology of the complex  $F \otimes_Q K$ . We begin by giving a modification, involving a related complex, of the special deformation retract datum of Theorem 2.16. We apply the perturbation lemma to this datum to yield the desired formulas in Theorem 3.5. Before stating it, we make the following remark about the maps that appear in Lemma 3.2.

**Remark 3.1.** We note that  $\sigma$  and  $H_\nabla$  are only  $k$ -linear maps, so to define maps  $1 \otimes \sigma$  and  $1 \otimes H_\nabla$ , we first fix a basis  $h_1^\ell, \dots, h_{b_\ell}^\ell$  for each module  $F_\ell$  in the resolution  $F$ . Now we have the isomorphisms

$$F_\ell \otimes_Q K_i \cong Q^{b_\ell} \otimes_Q K_i \cong K_i^{b_\ell} \quad (4)$$

and

$$F_\ell \otimes_Q Q/J \cong Q^{b_\ell} \otimes_Q Q/J \cong (Q/J)^{b_\ell},$$

for  $i \geq 0$ . We define  $1 \otimes \sigma : F_\ell \otimes_Q Q/J \rightarrow F_\ell \otimes_Q K_0$  by applying  $\sigma$  to each of the  $b_\ell$  summands. We consider the composition

$$F \otimes_Q Q/J \xrightarrow{1 \otimes \sigma} F \otimes_Q K_0 \hookrightarrow F \otimes_Q K$$

where the second map is the natural inclusion; abusing notation slightly, we call this map  $1 \otimes \sigma$ . Again using the isomorphisms in (4), we define  $1 \otimes H_\nabla$  by applying  $H_\nabla$  to each of the  $b_\ell$  summands. Throughout the remainder of this section, we use the notation  $1 \otimes \sigma$  and  $1 \otimes H_\nabla$  with the understanding that the maps are defined with respect to the fixed bases above. Also, we note that  $(1 \otimes \pi)(F \otimes_Q K_i) = 0$  for all  $i \geq 1$ .

We will see that, in order to give a basis for the Koszul homology, it is enough to find a  $k$ -linear map which agrees with the  $Q$ -linear isomorphism in (3), and apply this map to our fixed bases above. We use the  $k$ -linear maps  $1 \otimes \sigma$  and  $1 \otimes H_\nabla$  to produce such a map.

Now we state the lemma.

**Lemma 3.2.** *Let  $Q$  be complete in the  $J$ -adic topology. The following is a special deformation retract datum*

$$\left( (F \otimes_Q Q/J, 0) \xrightleftharpoons[1 \otimes \sigma]{1 \otimes \pi} (F \otimes_Q K, (0, \delta)), 1 \otimes H_\nabla \right).$$

**Proof.** We note that we have the equalities

$$\begin{aligned} (1 \otimes H_\nabla)(1 \otimes \sigma) &= 0 \\ (1 \otimes \pi)(1 \otimes H_\nabla) &= 0 \\ (1 \otimes H_\nabla)^2 &= 0 \end{aligned}$$

since the deformation retract datum from Theorem 2.16 is special. It is also clear that  $(1 \otimes \pi) \circ (1 \otimes \sigma) = \text{Id}_{F \otimes Q/J}$ . Thus we need only check that  $1 \otimes H_\nabla$  is a homotopy between  $(1 \otimes \sigma) \circ (1 \otimes \pi)$  and  $\text{Id}_{F \otimes K}$ , and the fact that  $1 \otimes \sigma$  and  $1 \otimes \pi$  are quasi-isomorphisms will follow. But the fact that  $1 \otimes H_\nabla$  is a homotopy follows directly from the definitions given in Remark 3.1 and from Theorem 2.16.  $\square$

Recall that  $\delta$  is the differential on the Koszul complex  $K = Q \otimes_{k[\underline{g}]} \Omega_{k[\underline{g}]/k}^\bullet$  and let  $\nabla$  be the  $k$ -linear map (2) defined in Section 2.3. The next lemma gives an explicit description of the map  $(\delta\nabla + \nabla\delta)^{-1}$ . This fact is known to experts, but to our knowledge, is not directly stated in the literature, so we state and prove it here.

**Lemma 3.3.** *The map  $(\delta\nabla + \nabla\delta)^{-1} : K \rightarrow K$  is given by*

$$(\delta\nabla + \nabla\delta)^{-1}(q \otimes dg_{i_1} \dots dg_{i_r}) = \sum_N \frac{1}{|N|+r} (\sigma(\overline{q_N}) g^N \otimes dg_{i_1} \dots dg_{i_r}),$$

where  $q = \sum_N \sigma(\overline{q_N}) g^N$  with  $N = (n_1, \dots, n_s)$  and  $|N| = n_1 + \dots + n_s$ .

**Proof.** We begin by computing

$$\begin{aligned} & (\delta\nabla + \nabla\delta)(q \otimes dg_{i_1} \dots dg_{i_r}) \\ &= \delta \left( \sum_{j=1}^s \frac{\partial}{\partial g_j} (q) \otimes dg_j dg_{i_1} \dots dg_{i_r} \right) + \nabla \left( q \otimes \left( \sum_{\ell=1}^k (-1)^{\ell+1} g_{i_\ell} dg_{i_1} \dots \widehat{dg_{i_\ell}} \dots dg_{i_r} \right) \right) \\ &= \sum_{j=1}^s \delta \left( \frac{\partial}{\partial g_j} (q) \otimes dg_j dg_{i_1} \dots dg_{i_r} \right) + \sum_{\ell=1}^k (-1)^{\ell+1} \nabla \left( q g_{i_\ell} \otimes (dg_{i_1} \dots \widehat{dg_{i_\ell}} \dots dg_{i_r}) \right) \\ &= \sum_{j=1}^s \left( \frac{\partial}{\partial g_j} (q) \otimes \left( dg_j dg_{i_1} \dots dg_{i_r} + \sum_{m=1}^k (-1)^m g_{i_m} dg_j dg_{i_1} \dots \widehat{dg_{i_m}} \dots dg_{i_r} \right) \right. \\ &\quad \left. + \sum_{\ell=1}^k (-1)^{\ell+1} \left( \sum_{p=1}^s \frac{\partial}{\partial g_p} (q g_{i_\ell}) \otimes (dg_p dg_{i_1} \dots \widehat{dg_{i_\ell}} \dots dg_{i_r}) \right) \right) \\ &= \sum_{j=1}^s \frac{\partial}{\partial g_j} (q) g_j \otimes dg_{i_1} \dots dg_{i_r} + \sum_{j=1}^s \sum_{m=1}^k (-1)^m \frac{\partial}{\partial g_j} (q) g_{i_m} \otimes dg_j dg_{i_1} \dots \widehat{dg_{i_m}} \dots dg_{i_r} \\ &\quad + \sum_{\ell=1}^k \sum_{p=1}^s (-1)^{\ell+1} \frac{\partial}{\partial g_p} (q) g_{i_\ell} \otimes dg_p dg_{i_1} \dots \widehat{dg_{i_\ell}} \dots dg_{i_r} + \sum_{\ell=1}^k (-1)^{\ell+1} q \otimes dg_{i_\ell} dg_{i_1} \dots \widehat{dg_{i_\ell}} \dots dg_{i_r} \end{aligned}$$

where the last equality follows from Remark 2.14. We note that the middle two sums cancel with each other and we are left with the equality

$$(\delta\nabla + \nabla\delta)(q \otimes dg_{i_1} \dots dg_{i_r}) = \sum_{j=1}^s \frac{\partial}{\partial g_j} (q) g_j \otimes dg_{i_1} \dots dg_{i_r} + r q \otimes dg_{i_1} \dots dg_{i_r}. \quad (5)$$

But we have that

$$\sum_{j=1}^s \frac{\partial}{\partial g_j} (q) g_j = \sum_{j=1}^s \sum_N N_j \sigma(\overline{q_N}) g^{N-e_j} g_j$$

$$\begin{aligned}
&= \sum_N \sum_{j=1}^s N_j \sigma(\overline{q_N}) g^N \\
&= \sum_N |N| \sigma(\overline{q_N}) g^N,
\end{aligned}$$

and thus, writing  $q = \sum_N \sigma(\overline{q_N}) g^N$  in (5), we get

$$(\delta \nabla + \nabla \delta) \left( \sum_N \sigma(\overline{q_N}) g^N \otimes dg_{i_1} \dots dg_{i_r} \right) = \sum_N (|N| + r) \left( \sigma(\overline{q_N}) g^N \otimes dg_{i_1} \dots dg_{i_r} \right). \quad (6)$$

Therefore, since  $\sigma$  is  $k$ -linear, we have

$$\begin{aligned}
(\delta \nabla + \nabla \delta)^{-1} \left( \sum_N \sigma(\overline{q_N}) g^N \otimes dg_{i_1} \dots dg_{i_r} \right) &= (\delta \nabla + \nabla \delta)^{-1} \left( \sum_N \frac{|N| + r}{|N| + r} \left( \sigma(\overline{q_N}) g^N \otimes dg_{i_1} \dots dg_{i_r} \right) \right) \\
&= (\delta \nabla + \nabla \delta)^{-1} \left( \sum_N (|N| + r) \left( \sigma \left( \frac{\overline{q_N}}{|N| + r} \right) g^N \otimes dg_{i_1} \dots dg_{i_r} \right) \right) \\
&= \sum_N \sigma \left( \frac{\overline{q_N}}{|N| + r} \right) g^N \otimes dg_{i_1} \dots dg_{i_r} \\
&= \sum_N \frac{1}{|N| + r} \left( \sigma(\overline{q_N}) g^N \otimes dg_{i_1} \dots dg_{i_r} \right),
\end{aligned}$$

where the third equality follows from (6), and this completes the proof.  $\square$

We establish some notation which we use throughout the rest of the section. We denote by  $\widehat{Q^J}$  the  $J$ -adic completion of  $Q$ .

**Definition 3.4.** Let  $q = \sum_N \sigma(\overline{q_N}) g^N$  be an element of  $\widehat{Q^J}$ . We define  $\frac{\partial^*}{\partial g_j}$  by

$$\frac{\partial^*}{\partial g_j}(q) = \sum_N \frac{N_j}{|N|} \sigma(\overline{q_N}) g^{N - e_j}.$$

For  $f \in Q$ , we denote by  $\widehat{f}$  the image of  $f$  in  $\widehat{Q^J}$ .

**Theorem 3.5.** If  $M$  is a finitely generated  $J$ -closed module over  $Q$  with free resolution  $(F, \partial_F)$  satisfying (1) in Definition 2.4, then a  $Q/J$ -basis of  $H_\ell(\underline{g}; M)$  is given by the homology classes of the elements

$$z_{j_1} = \sum_{1 \leq k_1, \dots, k_\ell \leq s} \sum_{j_2=1}^{b_{\ell-1}} \dots \sum_{j_{\ell+1}=1}^{b_0} m_{j_{\ell+1}} \frac{\partial^*}{\partial g_{k_\ell}} \left( \widehat{f_{j_{\ell+1}, j_\ell}^1} \frac{\partial^*}{\partial g_{k_{\ell-1}}} \left( \widehat{f_{j_\ell, j_{\ell-1}}^2} \dots \frac{\partial^*}{\partial g_{k_1}} (\widehat{f_{j_2, j_1}^\ell}) \dots \right) \right) dg_{k_1} \dots dg_{k_\ell}$$

for  $j_1 = 1, \dots, b_\ell$ , where  $h_1^i, \dots, h_{b_i}^i$  is a basis for  $F_i$ , where  $\partial_F(h_p^i) = \sum_{m=1}^{b_{i-1}} f_{m,p}^i h_m^{i-1}$ , where  $m_j$  is the image of  $h_j^0$  under the augmentation map  $F \rightarrow M$ , and where we identify  $z_{j_1} \in H(\underline{g}; \widehat{M^J})$  with its image under the isomorphism  $H(\underline{g}; \widehat{M^J}) \cong H(\underline{g}; M)$ .

**Proof.** We first reduce to the case where  $Q$  is complete with respect to the  $J$ -adic topology. Suppose that we can find such a basis for  $H(\underline{g}; \widehat{M^J})$  over the completion  $\widehat{Q/J^J}$ . Thus we have such a basis for

$$H_\ell(\widehat{\underline{g}; M})^J \cong H_\ell(\underline{g}; M) \otimes \widehat{Q}^J \cong H_\ell(\underline{g}; M \otimes \widehat{Q}^J) \cong H_\ell(\underline{g}; \widehat{M}^J)$$

by flatness. But

$$H_\ell(\widehat{\underline{g}; M})^J = \varprojlim_t H_\ell(\underline{g}; M)/J^t H_\ell(\underline{g}; M) = H_\ell(\underline{g}; M)$$

where the last equality follows from the fact that  $J \subseteq \text{ann}_Q H_\ell(\underline{g}; M)$ . Thus, it suffices to find such a basis in the complete case.

Applying Lemma 3.2, we get a special deformation retract datum

$$\left( (F \otimes_Q Q/J, 0) \xrightleftharpoons[1 \otimes \sigma]{1 \otimes \pi} (F \otimes_Q K, \delta), 1 \otimes H_\nabla \right).$$

We note that  $\partial_F \otimes 1: F \otimes K \rightarrow F \otimes K$ , which we write as  $\partial_F$  for ease of notation, is a perturbation of the special deformation retract datum above. Indeed, the degree of  $\partial_F$  is  $-1$  and  $(\partial_F + \delta)^2 = 0$  since it is the differential on the total complex. By Proposition 2.9 (i),  $F \otimes Q/J$  is a finite complex, say  $F_i \otimes Q/J = 0$  for all  $i > r$ , so we have the following equalities

$$\begin{aligned} & \left( 1 - \partial_F(1 \otimes H_\nabla) \right) \left( 1 + \partial_F(1 \otimes H_\nabla) + (\partial_F(1 \otimes H_\nabla))^2 + \dots + (\partial_F(1 \otimes H_\nabla))^r \right) \\ &= 1 + \partial_F(1 \otimes H_\nabla) + (\partial_F(1 \otimes H_\nabla))^2 + \dots + (\partial_F(1 \otimes H_\nabla))^r \\ &\quad - \left( \partial_F(1 \otimes H_\nabla) + (\partial_F(1 \otimes H_\nabla))^2 + \dots + (\partial_F(1 \otimes H_\nabla))^{r+1} \right) \\ &= 1 - (\partial_F(1 \otimes H_\nabla))^{r+1} = 1. \end{aligned}$$

Hence, we have that

$$(1 - \partial_F(1 \otimes H_\nabla))^{-1} = 1 + \partial_F(1 \otimes H_\nabla) + (\partial_F(1 \otimes H_\nabla))^2 + \dots + (\partial_F(1 \otimes H_\nabla))^r$$

is invertible and thus  $\partial_F$  is small. By the perturbation lemma, the perturbed datum is a deformation retract datum. In particular, we have the homotopy equivalence

$$(F \otimes_Q Q/J, \tilde{0}) \xrightleftharpoons[1 \otimes \sigma]{1 \otimes \pi} (F \otimes_Q K, \delta + \partial_F).$$

We note that  $\delta + \partial_F$  is the usual differential on the total complex and that the perturbed map

$$\begin{aligned} \tilde{0} &= 0 + (1 \otimes \pi)A(1 \otimes \sigma) \\ &= (1 \otimes \pi)(1 + \partial_F(1 \otimes H_\nabla) + (\partial_F(1 \otimes H_\nabla))^2 + \dots + (\partial_F(1 \otimes H_\nabla))^r) \partial_F(1 \otimes \sigma) \\ &= (1 \otimes \pi) \partial_F(1 \otimes \sigma) \end{aligned}$$

where the last equality follows from the fact that  $1 \otimes \pi$  composed with  $(\partial_F(1 \otimes H_\nabla))^i$  for  $i > 0$  is zero. Now since  $\text{Im } \partial_F \subseteq JF$ , we get that  $\pi$  composed with  $\partial_F$  is zero, thus  $\tilde{0} = 0$ , which gives the homotopy equivalence

$$F \otimes_Q Q/J \xrightarrow{1 \otimes \sigma} F \otimes_Q K$$

where

$$\begin{aligned}\widetilde{1 \otimes \sigma} &= (1 \otimes \sigma) + (1 \otimes H_{\nabla})A(1 \otimes \sigma) \\ &= (1 \otimes \sigma) + (1 \otimes H_{\nabla})\left(1 + \partial_F(1 \otimes H_{\nabla}) + (\partial_F(1 \otimes H_{\nabla}))^2 + \dots + (\partial_F(1 \otimes H_{\nabla}))^r\right)\partial_F(1 \otimes \sigma) \\ &= \left(1 + (1 \otimes H_{\nabla})\partial_F + ((1 \otimes H_{\nabla})\partial_F)^2 + \dots + ((1 \otimes H_{\nabla})\partial_F)^r\right)(1 \otimes \sigma).\end{aligned}$$

We also note that  $\widetilde{1 \otimes \pi}$  is just the  $Q$ -linear map  $1 \otimes \pi$  since it sends all elements to zero except ones lying in the first row of the double complex,  $F \otimes K$ . Thus, the induced map on homology

$$F_{\ell} \otimes_Q Q/J \xrightarrow{\cong} \text{Tor}_{\ell}^Q(M, Q/J) \cong H_{\ell}(M \otimes_Q K(\underline{g}; Q)) \cong H_{\ell}(\underline{g}; M)$$

is an isomorphism whose inverse is induced by the  $Q$ -linear map  $1 \otimes \pi$ , and hence agrees with the  $Q$ -linear isomorphism (3). As a result, a basis for  $H_{\ell}(\underline{g}; M)$  is given by first applying  $\widetilde{1 \otimes \sigma}$  to the basis elements of  $F_{\ell} \otimes Q/(\underline{g})$  and then applying the augmentation map  $F_0 \rightarrow M$  and taking homology classes of the results.

To this end, we compute

$$\begin{aligned}((1 \otimes H_{\nabla})\partial_F)^{\ell}(h_{j_1}^{\ell} \otimes 1 \otimes 1) &= ((1 \otimes H_{\nabla})\partial_F)^{\ell-1}(1 \otimes H_{\nabla})\left(\sum_{j_2=1}^{b_{\ell-1}} f_{j_2, j_1}^{\ell} h_{j_2}^{\ell-1} \otimes 1 \otimes 1\right) \\ &= \sum_{j_2=1}^{b_{\ell-1}} ((1 \otimes H_{\nabla})\partial_F)^{\ell-1}(1 \otimes H_{\nabla})(h_{j_2}^{\ell-1} \otimes f_{j_2, j_1}^{\ell} \otimes 1) \\ &= \sum_{j_2=1}^{b_{\ell-1}} ((1 \otimes H_{\nabla})\partial_F)^{\ell-1}(h_{j_2}^{\ell-1} \otimes \sum_{k_1=1}^s \frac{\partial^*}{\partial g_{k_1}}(f_{j_2, j_1}^{\ell}) \otimes dg_{k_1}) \\ &= \sum_{k_1=1}^s \sum_{j_2=1}^{b_{\ell-1}} ((1 \otimes H_{\nabla})\partial_F)^{\ell-1}(h_{j_2}^{\ell-1} \otimes \frac{\partial^*}{\partial g_{k_1}}(f_{j_2, j_1}^{\ell}) \otimes dg_{k_1})\end{aligned}$$

Applying the procedure above  $\ell - 1$  more times, we get that

$$\begin{aligned}((1 \otimes H_{\nabla})\partial_F)^{\ell}(h_{j_1}^{\ell} \otimes 1 \otimes 1) &= \\ \sum_{1 \leq k_1, \dots, k_{\ell} \leq s} \sum_{j_2=1}^{b_{\ell-1}} \dots \sum_{j_{\ell+1}=1}^{b_0} \left( h_{j_{\ell+1}}^0 \otimes \frac{\partial^*}{\partial g_{k_{\ell}}} \left( f_{j_{\ell+1}, j_{\ell}}^1 \frac{\partial^*}{\partial g_{k_{\ell-1}}} \left( f_{j_{\ell}, j_{\ell-1}}^2 \dots \frac{\partial^*}{\partial g_{k_1}}(f_{j_2, j_1}^{\ell}) \dots \right) \right) \otimes dg_{k_1} \dots dg_{k_{\ell}} \right).\end{aligned}$$

Applying the augmentation map  $F_0 \otimes_Q K_{\ell} \rightarrow M \otimes_Q K_{\ell}$ , we obtain the desired formula.  $\square$

We make the following remark regarding Theorem 3.5.

**Remark 3.6.** (1) We see that in the case that  $M = Q/I$  and  $I$  is a  $J$ -closed ideal, the formulas in Theorem 3.5 are given by

$$z_{j_1} = \sum_{1 \leq k_1, \dots, k_{\ell} \leq s} \sum_{j_2=1}^{b_{\ell-1}} \dots \sum_{j_{\ell}=1}^{b_1} \overline{\frac{\partial^*}{\partial g_{k_{\ell}}} \left( \widehat{f_{j_1, j_{\ell}}^1} \frac{\partial^*}{\partial g_{k_{\ell-1}}} \left( \widehat{f_{j_{\ell}, j_{\ell-1}}^2} \dots \frac{\partial^*}{\partial g_{k_1}}(\widehat{f_{j_2, j_1}^{\ell}}) \dots \right) \right)} dg_{k_1} \dots dg_{k_{\ell}}$$

where the bar denotes the image in the quotient.

(2) The proof of Theorem 3.5 actually yields the following more general result. Let  $F$  be any complex such that  $\text{Im } \partial_i \subseteq JF_{i-1}$  for all  $i$ . Define  $K(\underline{g}; F) = K(\underline{g}; Q) \otimes_Q F$  and let  $H(\underline{g}; F)$  be its homology. Then the homology classes of the elements given in Theorem 3.5 are a basis for  $H(\underline{g}; F)$ .

We now give useful versions of these formulas for some special cases of interest. In the following corollaries, we consider the case where  $M$  is a cyclic module, but similar formulas can be given in the non-cyclic case as in Theorem 3.5. First we establish some notation. We denote

$$\frac{\partial(f_1, \dots, f_i)}{\partial(g_{k_1}, \dots, g_{k_i})} = \det \left( \frac{\partial}{\partial g_{k_j}}(f_\ell) \right)_{j,\ell}.$$

**Definition 3.7.** We call  $f$  *homogeneous in  $\underline{g}$  of degree  $n$*  if there is an integer  $n$  such that

$$f = \sum_N \sigma(f_N) g^N$$

for  $N = (n_1, \dots, n_s) \in \mathbb{N}^s$  satisfying  $\sum_{\ell=1}^s n_\ell = n$ .

In the following corollary, we provide more explicit versions of the formulas from Theorem 3.5. These formulas have new terms which vanish in the classical case, where  $\underline{g} = x_1, \dots, x_n$  are minimal generators of the maximal ideal.

**Corollary 3.8.** *If  $I$  is a  $J$ -closed ideal in  $Q$  with  $M = Q/I$  and  $(F, \partial_F)$  a  $Q$ -free resolution of  $M$  such that the entries in the matrices  $\partial_F$  are homogeneous in  $\underline{g}$  with coefficients in  $Q$ , then a  $Q/J$ -basis of  $H_\ell(\underline{g}; M)$  is given by the homology classes of the elements*

$$z_{\ell_{j_1}} = \sum_{j_2=1}^{b_{\ell-1}} \dots \sum_{j_\ell=1}^{b_1} D_{j_1, \dots, j_\ell} \left( \sum_{1 \leq k_1 < \dots < k_\ell \leq s} \frac{\overline{\partial(f_{1,j_\ell}^1, f_{j_\ell, j_{\ell-1}}^2, \dots, f_{j_2, j_1}^\ell)}}{\partial(g_{k_1}, \dots, g_{k_\ell})} + \varepsilon \right) dg_{k_1} \dots dg_{k_\ell}$$

for  $j_1 = 1, \dots, b_\ell$ , where

$$\varepsilon = \sum_{1 \leq k_1, \dots, k_\ell \leq s} \sum_{m=2}^\ell \prod_{n=m+1}^\ell \frac{\partial}{\partial g_{k_n}}(f_{j_{n+1}, j_n}^{\ell-n+1}) \sum_{M, N} \frac{\partial}{\partial g_{k_m}} \left( \sigma(f_{j_{m+1}, j_m}^{\ell-m+1}) \sigma(y_{(m-1)j_1}^{(m-1)N}) \right) g^{M+N}$$

and the elements  $y_{(m-1)j_1}$  are defined inductively by

$$z_{(m-1)j_1} = \sum_{1 \leq k_1, \dots, k_{m-1} \leq s} \sum_{j_2=1}^{b_{m-2}} \dots \sum_{j_{m-1}=1}^{b_1} D_{j_1, \dots, j_{m-1}} y_{(m-1)j_1} dg_{k_1} \dots dg_{k_{m-1}}$$

and where

$$D_{j_1, \dots, j_\ell} = \frac{1}{d_{j_2, j_1}^\ell} \cdot \frac{1}{d_{j_3, j_2}^{\ell-1} + d_{j_2, j_1}^\ell - 1} \cdot \dots \cdot \frac{1}{d_{1, j_\ell}^1 + d_{j_\ell, j_{\ell-1}}^2 + \dots + d_{j_2, j_1}^\ell - \ell + 1}$$

with  $d_{i,j}^k$  the degree of  $f_{i,j}^k$ .

**Proof.** We first note that for a  $\underline{g}$ -homogeneous polynomial,  $f_{i,j}^k$  we have that

$$\frac{\partial^*}{\partial g_j}(f_{i,j}^k) = \frac{1}{d_{i,j}^k} \frac{\partial}{\partial g_j}(f_{i,j}^k).$$

Thus, since  $\frac{\partial}{\partial g_j}$  reduces the  $\underline{g}$ -degree of its argument by one whenever it is nonzero, the formulas from Theorem 3.5 become

$$z_{\ell_{j_1}} = \sum_{1 \leq k_1, \dots, k_\ell \leq s} \sum_{j_2=1}^{b_{\ell-1}} \dots \sum_{j_\ell=1}^{b_1} D_{j_1, \dots, j_\ell} \frac{\partial}{\partial g_{k_\ell}} \left( f_{1,j_\ell}^1 \frac{\partial}{\partial g_{k_{\ell-1}}} \left( f_{j_\ell, j_{\ell-1}}^2 \dots \frac{\partial}{\partial g_{k_1}} (f_{j_2, j_1}^\ell) \dots \right) \right) dg_{k_1} \dots dg_{k_\ell}, \quad (7)$$

where we omit the bar for ease of exposition. It is now clear that we have the desired formula for  $z_{1_{j_1}}$ . To obtain the desired formula for  $z_{\ell_{j_1}}$ , we apply the Leibniz rule from Lemma 2.13 to (7), to get

$$\begin{aligned} z_{\ell_{j_1}} &= \sum_{1 \leq k_1, \dots, k_\ell \leq s} \sum_{j_2=1}^{b_{\ell-1}} \dots \sum_{j_\ell=1}^{b_1} D_{j_1, \dots, j_\ell} \left( \frac{\partial}{\partial g_{k_\ell}} (f_{1,j_\ell}^1) \frac{\partial}{\partial g_{k_{\ell-1}}} \left( f_{j_\ell, j_{\ell-1}}^2 \dots \frac{\partial}{\partial g_{k_1}} (f_{j_2, j_1}^\ell) \dots \right) \right. \\ &\quad \left. + f_{1,j_\ell}^1 \frac{\partial}{\partial g_{k_\ell}} \left( \frac{\partial}{\partial g_{k_{\ell-1}}} \left( f_{j_\ell, j_{\ell-1}}^2 \dots \frac{\partial}{\partial g_{k_1}} (f_{j_2, j_1}^\ell) \dots \right) \right) \right. \\ &\quad \left. + \sum_{M,N} \frac{\partial}{\partial g_{k_\ell}} \left( \sigma(\overline{f_{1,j_\ell}^1}) \sigma \left( \frac{\partial}{\partial g_{k_{\ell-1}}} \left( f_{j_\ell, j_{\ell-1}}^2 \dots \frac{\partial}{\partial g_{k_1}} (f_{j_2, j_1}^\ell) \dots \right) \right)_N \right) g^{M+N} \right) dg_{k_1} \dots dg_{k_\ell}. \end{aligned}$$

But by Lemma 2.12, we have that

$$\sum_{1 \leq k_1, \dots, k_\ell \leq s} \frac{\partial}{\partial g_{k_\ell}} \left( \frac{\partial}{\partial g_{k_{\ell-1}}} \left( f_{j_\ell, j_{\ell-1}}^2 \dots \frac{\partial}{\partial g_{k_1}} (f_{j_2, j_1}^\ell) \dots \right) \right) dg_{k_1} \dots dg_{k_\ell} = 0,$$

which gives the following formula

$$\begin{aligned} z_{\ell_{j_1}} &= \sum_{1 \leq k_1, \dots, k_\ell \leq s} \sum_{j_2=1}^{b_{\ell-1}} \dots \sum_{j_\ell=1}^{b_1} D_{j_1, \dots, j_\ell} \left( \frac{\partial}{\partial g_{k_\ell}} (f_{1,j_\ell}^1) \frac{\partial}{\partial g_{k_{\ell-1}}} \left( f_{j_\ell, j_{\ell-1}}^2 \dots \frac{\partial}{\partial g_{k_1}} (f_{j_2, j_1}^\ell) \dots \right) \right. \\ &\quad \left. + \sum_{M,N} \frac{\partial}{\partial g_{k_\ell}} \left( \sigma(\overline{f_{1,j_\ell}^1}) \sigma \left( \frac{\partial}{\partial g_{k_{\ell-1}}} \left( f_{j_\ell, j_{\ell-1}}^2 \dots \frac{\partial}{\partial g_{k_1}} (f_{j_2, j_1}^\ell) \dots \right) \right)_N \right) g^{M+N} \right) dg_{k_1} \dots dg_{k_\ell}. \end{aligned}$$

Now we apply the Leibniz rule from Lemma 2.13 repeatedly, simplifying at each step as above to obtain

$$\begin{aligned} z_{\ell_{j_1}} &= \sum_{1 \leq k_1, \dots, k_\ell \leq s} \sum_{j_2=1}^{b_{\ell-1}} \dots \sum_{j_\ell=1}^{b_1} D_{j_1, \dots, j_\ell} \left( \frac{\partial}{\partial g_{k_\ell}} (f_{1,j_\ell}^1) \frac{\partial}{\partial g_{k_{\ell-1}}} (f_{j_\ell, j_{\ell-1}}^1) \dots \frac{\partial}{\partial g_{k_1}} (f_{j_2, j_1}^\ell) \right. \\ &\quad \left. + \sum_{m=2}^\ell \prod_{n=m+1}^\ell \frac{\partial}{\partial g_{k_n}} (f_{j_{n+1}, j_n}^{\ell-n+1}) \sum_{M,N} \frac{\partial}{\partial g_{k_m}} \left( \sigma(\overline{f_{j_{m+1}, j_m}^{\ell-m+1}}) \sigma(\overline{y_{(m-1)j_1}})_N \right) g^{M+N} \right) dg_{k_1} \dots dg_{k_\ell}, \end{aligned}$$

where

$$y_{(m-1)j_1} = \frac{\partial}{\partial g_{k_{m-1}}} \left( f_{j_m, j_{m-1}}^{\ell-m+2} \dots \frac{\partial}{\partial g_{k_1}} (f_{j_2, j_1}^\ell) \dots \right).$$

We observe that the first part of the above formula can be written as a determinant, thus giving the desired formulas.  $\square$

We now give formulas in the case that the entries in the matrices given by  $\partial_F$  are homogeneous in  $\underline{g}$  with coefficients in  $k$  rather than  $Q$ . This happens for example when  $M = Q \otimes_{k[\underline{g}]} \widetilde{M}$  where  $\widetilde{M}$  is a  $k[\underline{g}]$ -module.

**Corollary 3.9.** *If  $I$  is a  $J$ -closed ideal in  $Q$  with  $M = Q/I$  and  $(F, \partial_F)$  a  $Q$ -free resolution of  $M$  such that the entries in the matrices  $\partial_F$  are homogeneous polynomials in  $\underline{g}$  with coefficients in  $k$ , then a  $Q/J$ -basis of  $H_\ell(\underline{g}; M)$  is given by the homology classes of the elements*

$$z_{j_1} = \sum_{j_2=1}^{b_{\ell-1}} \cdots \sum_{j_\ell=1}^{b_1} D_{j_1, \dots, j_\ell} \sum_{1 \leq k_1 < \dots < k_\ell \leq s} \frac{\partial(f_{1,j_\ell}^1, f_{j_\ell, j_{\ell-1}}^2, \dots, f_{j_2, j_1}^\ell)}{\partial(g_{k_1}, \dots, g_{k_\ell})} dg_{k_1} \dots dg_{k_\ell}$$

for  $j_1 = 1, \dots, b_\ell$ .

**Proof.** This follows immediately from Corollary 3.8 and the fact that  $\frac{\partial}{\partial g_j}$  satisfies the usual Leibniz rule in the case that the coefficients involved in the product are elements of  $k$ , as discussed in Remark 2.14.  $\square$

**Remark 3.10.** Taking  $\underline{g} = x_1, \dots, x_n$  to be the minimal generators of the maximal ideal, the corollary recovers the formulas given by Herzog in [6].

We finish this section by giving an example which illustrates our results. We note that for this example, the new terms in the formulas given in Corollary 3.8, which vanish in the case where  $\underline{g} = x_1, \dots, x_n$  are minimal generators of the maximal ideal, do not vanish.

**Example 3.11.** Let  $Q = k[x, y, z]$  be a polynomial ring with  $k$  a field of characteristic zero and let  $J$  be a complete intersection ideal generated by the regular sequence  $g_1 = x^2 + yz$ ,  $g_2 = y^3$ ,  $g_3 = z^5$ . Consider the ideal  $I = (x^2y^4 + y^5z + xz^{10}, y^6, x^4y^2 + x^2y^3z)$ . Let us compute the generators of  $H_\ell(\underline{g}; R)$ , where  $R = Q/I$ . We begin by fixing a  $k$ -linear splitting of  $\pi : Q \rightarrow Q/J$ . One can find the following basis for  $Q/J$  as a  $k$ -vector space either by hand or using Macaulay2

$$\begin{aligned} &\bar{1}, \bar{x}, \bar{xy}, \overline{xy^2}, \overline{xy^2z}, \overline{xy^2z^2}, \overline{xy^2z^3}, \overline{xy^2z^4}, \overline{xyz}, \overline{xyz^2}, \overline{xyz^3}, \overline{xyz^4}, \bar{xz}, \overline{xz^2}, \\ &\overline{xz^3}, \overline{xz^4}, \bar{y}, \bar{y^2}, \overline{y^2z}, \overline{y^2z^2}, \overline{y^2z^3}, \overline{y^2z^4}, \bar{yz}, \overline{yz^2}, \overline{yz^3}, \overline{yz^4}, \bar{z}, \bar{z^2}, \bar{z^3}, \bar{z^4}. \end{aligned}$$

We choose the splitting  $\sigma(\bar{a}) = a$  for every basis element  $\bar{a}$ . According to Macaulay2, a free resolution of  $R$  over  $Q$  is given by

$$0 \rightarrow Q \xrightarrow{\partial_3} Q^3 \xrightarrow{\partial_2} Q^3 \xrightarrow{\partial_1} Q \rightarrow R \rightarrow 0$$

where the differentials are given by the following matrices:

$$\begin{aligned} \partial_3 &= \begin{bmatrix} z^{10} + xy^4 \\ -y^4 \\ x^3 + xyz \end{bmatrix}, \quad \partial_2 = \begin{bmatrix} -y^4 & -z^{10} - xy^4 & 0 \\ x^4 + x^2yz & -x^3yz - xy^2z^2 & -xz^{10} - x^2y^4 - y^5z \\ 0 & x^3y^2 + xy^3z & y^6 \end{bmatrix}, \\ \partial_1 &= \begin{bmatrix} x^4y^2 + x^2y^3z & y^6 & x^2y^4 + y^5z + xz^{10} \end{bmatrix}. \end{aligned}$$

We now write the entries in the differentials as in Lemma 2.10. Note for example that the first entry in  $\partial_1$  can be written as follows:

$$x^4y^2 + x^2y^3z = x^2y^2g_1$$

but note that  $x^2y^2$  is not in the image of our splitting map  $\sigma$ , so we write



$$\begin{aligned}
 x^4y^2 + x^2y^3z &= x^2y^2g_1 \\
 &= (g_1 - yz)y^2g_1 \\
 &= y^2g_1^2 - y^3zg_1 \\
 &= y^2g_1^2 - zg_1g_2
 \end{aligned}$$

and we can see that the coefficients are now in the image of  $\sigma$ . Using a similar procedure on the other entries, we obtain the following matrices

$$\begin{aligned}
 \partial_3 &= \begin{bmatrix} xyg_2 + g_3^2 \\ -yg_2 \\ xg_1 \end{bmatrix}, \quad \partial_2 = \begin{bmatrix} -yg_2 & -xyg_2 - g_3^2 & 0 \\ g_1^2 - yzg_1 & -xyzg_1 & -yg_1g_2 - xg_3^2 \\ 0 & xy^2g_1 & g_2^2 \end{bmatrix}, \\
 \partial_1 &= \begin{bmatrix} y^2g_1^2 - zg_1g_2 & g_2^2 & yg_1g_2 + xg_3^2 \end{bmatrix}.
 \end{aligned}$$

Applying Theorem 3.5, we get the following set of elements whose homology classes generate  $H_1(\underline{g}; R)$ ,

$$\begin{aligned}
 \widetilde{h}_1^1 &= \overline{y^2g_1 - zg_2}dg_1 - \frac{1}{2}\overline{zg_1}dg_2 = \overline{x^2y}dg_1 - \frac{1}{2}\overline{x^2z + yz^2}dg_2 \\
 \widetilde{h}_2^1 &= \overline{g_2}dg_2 = \overline{y^3}dg_2 \\
 \widetilde{h}_3^1 &= \frac{1}{2}\overline{yg_2}dg_1 + \frac{1}{2}\overline{yg_1}dg_2 + \overline{xg_3}dg_3 = \frac{1}{2}\overline{y^4}dg_1 - \frac{1}{2}\overline{x^2y + y^2z}dg_2 + \overline{xz^5}dg_3
 \end{aligned}$$

where  $\widetilde{h}_i^j$  is the generator which corresponds to the basis element  $h_i^j$  of  $F_j$ . We get the following generators for  $H_2(\underline{g}; R)$

$$\begin{aligned}
 \widetilde{h}_1^2 &= -\frac{1}{2}\overline{y^4z}dg_1dg_2 \\
 \widetilde{h}_2^2 &= \left(-\frac{1}{2}\overline{xy^4z} - \frac{1}{3}\overline{z^{11}}\right)dg_1dg_2 - \frac{1}{3}\overline{y^3z^6}dg_1dg_3 + \frac{1}{3}\overline{x^2z^6 + yz^7}dg_2dg_3 \\
 \widetilde{h}_3^2 &= -\frac{1}{3}\overline{y^7}dg_1dg_2
 \end{aligned}$$

and for  $H_3(\underline{g}; R)$

$$\widetilde{h}_1^3 = \left(\frac{1}{6}\overline{x^2y^3z^5} + \frac{2}{3}\overline{y^4z^6} + \frac{1}{9}\overline{x^2z^5} - \frac{1}{18}\overline{yz^6}\right)dg_1dg_2dg_3.$$

#### 4. Applications to weak complete intersection ideals

In this section we use the formulas for generators of Koszul homology given in the previous section to study the ideal  $\bar{J} = J/I$  of the quotient  $R = Q/I$ , where  $Q$  is local and  $I$  is a  $J$ -closed ideal of  $Q$ .

In Proposition 4.4, we provide a general condition under which  $\bar{J}$  is a weak complete intersection ideal in  $R$ , which expands the class of known examples of weak complete intersection ideals given in [10]. However, there are examples of ideals  $\bar{J}$  which are not weak complete intersection ideals in quotients by  $J$ -closed ideals. We discuss one example in the following remark.

**Remark 4.1.** Taking  $\underline{g} = x^2, y^3, z^5$  in Example 2.8, the ideal  $\bar{J}$  of  $R = Q/I$  is not a weak complete intersection ideal. One can verify this by looking at the beginning of the minimal free resolution of  $R/\bar{J}$  over  $R$  on Macaulay2. There are entries in the differentials which are not elements of  $\bar{J}$ , for example,  $y^2$  is one such entry.

We will need the following definition in the proof of Proposition 4.4; see for example [10, Definition 2.7] or [1, Remark 5.2.1].

**Definition 4.2.** Let  $R$  be a local ring and assume  $H_\ell(\underline{g}; R)$  is a free  $R/(\underline{g})$ -module for all  $\ell$ . We say that  $K(\underline{g}; R)$  admits a *trivial Massey operation* if for some basis  $\mathcal{B} = \{h_\lambda\}_{\lambda \in \Lambda}$  of  $H_{\geq 1}(\underline{g}; R)$ , there is a function

$$\mu : \bigsqcup_{n=1}^{\infty} \mathcal{B}^n \rightarrow K(\underline{g}; R)$$

such that  $\mu(h_\lambda) = z_\lambda$  is a cycle with  $\text{cls}(z_\lambda) = h_\lambda$  and

$$\partial^K \mu(h_{\lambda_1}, \dots, h_{\lambda_p}) = \sum_{j=1}^{p-1} \overline{\mu(h_{\lambda_1}, \dots, h_{\lambda_j})} \mu(h_{\lambda_{j+1}}, \dots, h_{\lambda_p}) \quad (8)$$

where  $\bar{a} = (-1)^{|a|+1}a$ .

Now we establish some notation.

**Definition 4.3.** Let  $I$  be an ideal in  $Q$ . We define  $\frac{\partial}{\partial \underline{g}}(I)$  to be the ideal of  $\widehat{Q^J}$  generated by the elements  $\{\frac{\partial}{\partial g_j}(\hat{f}) | f \in I, j = 1, \dots, s\}$ .

The following result gives a condition under which  $\bar{J}$  is a weak complete intersection ideal in  $R$ .

**Proposition 4.4.** If  $I$  is a  $J$ -closed ideal of  $Q$  and  $\left(\frac{\partial}{\partial \underline{g}}(I)\right)^2 \subseteq \widehat{I^J}$ , then  $\bar{J} = J/I$  is a weak complete intersection ideal in  $R = Q/I$ .

**Proof.** First we note that the weak complete intersection property descends along the  $J$ -adic completion, so we may assume that  $Q$  is complete in the  $J$ -adic topology.

Since  $J/J^2$  is a free  $Q/J$ -module and  $H_\ell(\underline{g}; R)$  is a free  $Q/J$ -module for every  $\ell$ , it suffices to show that  $K(\underline{g}; R)$  admits a trivial Massey operation by [10, Theorem 2.9].

We take the  $z_\lambda$  to be the elements given in Theorem 3.5 and lift them to form a basis  $\mathcal{B}$  of  $H(\underline{g}; R)$ . We define  $\mu(h_{\lambda_1}, \dots, h_{\lambda_p}) = 0$  for all  $p \geq 2$  and  $h_{\lambda_i} \in \mathcal{B}$ . By Theorem 3.5, we see that every  $z_\lambda$  has coefficients in  $\frac{\partial}{\partial \underline{g}}(I)$ , since the elements  $f_{1,j_\ell}^1$  are the entries in the first differential in the minimal free resolution of  $R$ , and hence are elements of  $I$ . Thus, the coefficients of  $\mu(h_{\lambda_i})\mu(h_{\lambda_j})$  are elements of  $\frac{\partial}{\partial \underline{g}}(I)^2 \subseteq I$  for all  $i$  and  $j$ , so the products are zero in  $K(\underline{g}; R)$ . It is now easy to see that our definition of  $\mu$  satisfies

$$\partial^K \mu(h_{\lambda_1}, \dots, h_{\lambda_p}) = \sum_{j=1}^{p-1} \overline{\mu(h_{\lambda_1}, \dots, h_{\lambda_j})} \mu(h_{\lambda_{j+1}}, \dots, h_{\lambda_p}).$$

Thus,  $\mu$  is a trivial Massey operation on  $K(\underline{g}; R)$ , as desired.  $\square$

We note that taking  $\underline{g} = x_1, \dots, x_n$  to be minimal generators of the maximal ideal in the proof above, we get that  $R$  is Golod, and this is precisely the result [7, Theorem 1.1] of Herzog and Huneke; see also [5, Theorem 3.5] for a similar result regarding Golod modules. The proof of Proposition 4.4 shows that the condition on  $\frac{\partial}{\partial \underline{g}}(I)$  cannot hold in the case that  $I \subseteq J$  is an embedded complete intersection ideal, as  $H(\underline{g}; R)$  cannot have trivial products in this case.

The following example illustrates Proposition 4.4 and shows that it produces new examples of weak complete intersection ideals.

**Example 4.5.** Let  $Q = k[x, y]$  with  $\text{char } k = 0$  and let  $J$  be a complete intersection ideal in  $Q$  generated by the regular sequence  $g_1 = x^2 + y^2$ ,  $g_2 = y^3$ . We consider the ideal  $I = (g_1^2 g_2, g_1^4, g_2^3) \subseteq Q$ . A free resolution of  $Q/I$  is

$$0 \rightarrow Q^2 \xrightarrow{\partial_2} Q^3 \xrightarrow{\partial_1} Q \rightarrow R \rightarrow 0$$

with differentials

$$\partial_2 = \begin{bmatrix} -g_1^2 & -g_2^2 \\ g_2 & 0 \\ 0 & g_1^2 \end{bmatrix}, \quad \partial_1 = \begin{bmatrix} g_1^2 g_2 & g_1^4 & g_2^3 \end{bmatrix},$$

so  $I$  is a  $J$ -closed ideal. The ideal  $\frac{\partial}{\partial g}(I)$  is generated by the elements  $g_1 g_2, g_1^2, g_2^2$ . Indeed, elements of  $I$  are of the form  $f = a g_1^2 g_2 + b g_1^4 + c g_2^3$  for  $a, b, c \in Q$ , and

$$\begin{aligned} \frac{\partial}{\partial g_1}(f) &= 2a g_1 g_2 + 4b g_1^3 + \frac{\partial}{\partial g_1}(c) g_2^3 \\ \frac{\partial}{\partial g_2}(f) &= a g_1^2 + \frac{\partial}{\partial g_2}(b) g_1^4 + 3c g_2^2 \end{aligned}$$

which are both elements of the ideal  $(g_1 g_2, g_1^2, g_2^2)$ . It is easy to check that  $\frac{\partial}{\partial g}(I)^2 \subseteq I$ . Thus, by Proposition 4.4,  $\bar{J}$  is a weak complete intersection ideal in  $R = Q/I$ .

The converse of Proposition 4.4 is not true in general, as shown by the following example.

**Example 4.6.** Let  $Q = k[x, y, z]$  with  $\text{char } k = 0$  and let  $J$  be a complete intersection ideal in  $Q$  generated by the regular sequence  $g_1 = x^2$ ,  $g_2 = y^3$ ,  $g_3 = z^5$ . The ideal  $I = (x^2 y^8, y^8 z^9, x^3 z^{14} + x^5 y^5) \subseteq Q$  is a  $J$ -closed ideal; a free resolution of  $R = Q/I$  being

$$0 \rightarrow Q^2 \xrightarrow{\partial_2} Q^3 \xrightarrow{\partial_1} Q \rightarrow R \rightarrow 0$$

with differentials

$$\partial_2 = \begin{bmatrix} -z^9 & -x^3 y^5 \\ x^2 & -x^3 z^5 \\ 0 & y^8 \end{bmatrix}, \quad \partial_1 = \begin{bmatrix} x^2 y^8 & y^8 z^9 & x^3 z^{14} + x^5 y^5 \end{bmatrix}.$$

By Theorem 3.5, the homology classes of the elements  $\{\tilde{h}_1^1, \tilde{h}_2^1, \tilde{h}_3^1, \tilde{h}_1^2, \tilde{h}_2^2\}$ , where

$$\begin{aligned} \tilde{h}_1^1 &= \frac{1}{3} \overline{y^8} dg_1 + \frac{2}{3} \overline{x^2 y^5} dg_2 \\ \tilde{h}_2^1 &= \frac{2}{3} \overline{y^5 z^9} dg_2 + \frac{1}{3} \overline{y^8 z^4} dg_3 \\ \tilde{h}_3^1 &= \frac{1}{3} \overline{(x z^{14} + 2 x^3 y^5)} dg_1 + \frac{1}{3} \overline{x^5 y^2} dg_2 + \frac{2}{3} \overline{x^3 z^9} dg_3 \\ \tilde{h}_1^2 &= \frac{2}{3} \overline{y^5 z^9} dg_1 dg_2 - \frac{2}{3} \overline{x^2 y^5 z^4} dg_2 dg_3 \\ \tilde{h}_2^2 &= \frac{1}{6} \overline{x^3 y^{10}} dg_1 dg_2 - \frac{1}{6} \overline{x^2 y^8 z^4} dg_1 dg_3 + \frac{1}{3} \overline{x^3 y^5 z^9} dg_2 dg_3 \end{aligned}$$

is a basis for  $H_{\geq 1}(\underline{g}; R)$ . To obtain a trivial Massey operation on  $K(\underline{g}; R)$ , we first multiply the cycles

$$\begin{aligned}\tilde{h}_1^1 \cdot \tilde{h}_2^1 &= \frac{1}{9} \overline{y^{16} z^4} dg_1 dg_3 \\ \tilde{h}_1^1 \cdot \tilde{h}_3^1 &= \frac{4}{9} \overline{x^5 y^5 z^9} dg_2 dg_3 \\ \tilde{h}_2^1 \cdot \tilde{h}_3^1 &= \frac{2}{9} \overline{xy^5 z^{23}} dg_1 dg_2\end{aligned}$$

and we observe that all multiplications involving  $\tilde{h}_1^2$  and  $\tilde{h}_2^2$  are zero. So we define  $\mu$  as follows

$$\begin{aligned}\mu([\tilde{h}_1^1], [\tilde{h}_2^1]) &= -\frac{1}{9} y^{13} z^4 dg_1 dg_2 dg_3 \\ \mu([\tilde{h}_1^1], [\tilde{h}_3^1]) &= \frac{4}{9} x^3 y^5 z^9 dg_1 dg_2 dg_3 \\ \mu([\tilde{h}_2^1], [\tilde{h}_3^1]) &= \frac{2}{9} xy^5 z^{18} dg_1 dg_2 dg_3\end{aligned}$$

where  $[\cdot]$  denotes the homology class, and otherwise we define  $\mu$  to be zero. It is straightforward to check that  $\mu$  satisfies (8), thus it is a trivial Massey operation. Therefore, by [10, Theorem 2.9],  $\bar{J}$  is a weak complete intersection ideal in  $R$ . However,  $y^8 \in \frac{\partial}{\partial g}(I)$ , thus  $y^{16}$  is in  $\frac{\partial}{\partial g}(I)^2$ , but not in  $I$ . This shows that  $\frac{\partial}{\partial g}(I)^2 \subseteq I$  is not a necessary condition for  $\bar{J}$  to be a weak complete intersection ideal in  $R$ .

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## References

- [1] L.L. Avramov, Infinite free resolutions, in: Six Lectures on Commutative Algebra, in: Mod. Birkhäuser Class., Birkhäuser Verlag, Basel, 2010, pp. 1–118.
- [2] A. Corso, S. Goto, C. Huneke, C. Polini, B. Ulrich, Iterated socles and integral dependence in regular rings, Trans. Am. Math. Soc. 370 (2018) 53–72.
- [3] M. Crainic, On the perturbation lemma, and deformations, arXiv Mathematics e-prints, arXiv:math/0403266, 2004.
- [4] T. Dyckerhoff, D. Murfet, Pushing forward matrix factorizations, Duke Math. J. 162 (2013) 1249–1311.
- [5] A. Gupta, Ascent and descent of the Golod property along algebra retracts, J. Algebra 480 (2017) 124–143.
- [6] J. Herzog, Canonical Koszul cycles, in: International Seminar on Algebra and Its Applications (Spanish), México City, 1991, in: Aportaciones Mat. Notas Investigación, vol. 6, Soc. Mat. Mexicana, México, 1992, pp. 33–41.
- [7] J. Herzog, C. Huneke, Ordinary and symbolic powers are Golod, Adv. Math. 246 (2013) 89–99.
- [8] J. Herzog, R.A. Maleki, Koszul cycles and Golod rings, Manuscr. Math. 157 (2018) 483–495.
- [9] J. Lipman, Residues and Traces of Differential Forms via Hochschild Homology, Contemporary Mathematics, vol. 61, American Mathematical Society, Providence, RI, 1987.
- [10] H. Rahmati, J. Striuli, Z. Yang, Poincaré series of fiber products and weak complete intersection ideals, J. Algebra 498 (2018) 129–152.
- [11] W.V. Vasconcelos, Ideals generated by R-sequences, J. Algebra 6 (1967) 309–316.