



## Corrigendum

## Corrigendum to “Intrinsic algebraic entropy” [J. Pure Appl. Algebra 219 (7) (2015) 2933–2961]

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## ABSTRACT

We correct an argument in the proof of the Logarithmic Law for the intrinsic algebraic entropy, published as Lemma 3.12 in the article: D. Dikranjan, A. Giordano Bruno, L. Salce, S. Virili. Intrinsic algebraic entropy, J. Pure Appl. Algebra 219 (2015) 2933–2961.

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The Logarithmic Law for the intrinsic algebraic entropy, Lemma 3.12 of [1], was proved using an incorrect identity in [1, Lem. 3.11(b)]. More precisely, using the notation as in [1], the identity

$$\widetilde{\text{ent}}(\varphi^k, H) = \widetilde{\text{ent}}(\varphi^k, H')$$

does not hold true in general, as the following counterexample shows.

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**Example 1.** Let  $\mathbb{Z}(2) = \{0, 1\}$  be the group of integers modulo 2, and

$$G := \bigoplus_{i \in \mathbb{N}} \mathbb{Z}(2) = \{(x_i)_{i \in \mathbb{N}} \in \prod_{i \in \mathbb{N}} \mathbb{Z}(2) \mid x_i \neq 0 \text{ for finitely many } i \in \mathbb{N}\}.$$

Consider the right Bernoulli shift  $\beta: G \rightarrow G$ , mapping  $(x_0, x_1, x_2, \dots) \mapsto (0, x_0, x_1, x_2, \dots)$ . Let  $H := \{(x_i)_{i \in \mathbb{N}} \in G \mid x_i = 0 \text{ for } i \neq 0\}$ , and  $H' := T_2(\beta, H)$ , so that

$$H' = \{(x_i)_{i \in \mathbb{N}} \in G \mid x_i = 0 \text{ for } i \neq 0, 1\}.$$

Both  $H$  and  $H'$  are finite, hence  $\beta$ -inert and  $\beta^2$ -inert subgroups of  $G$ . Now, for a positive integer  $n$ ,

$$T_n(\beta^2, H') = \{(x_i)_{i \in \mathbb{N}} \in G \mid x_i = 0 \text{ for } i \geq 2n\},$$

so  $|T_n(\beta^2, H')/H'| = 2^{2n-2}$  and  $\widetilde{\text{ent}}(\beta^2, H') = 2 \cdot \log(2)$ . On the other hand,

$$T_n(\beta^2, H) = \{(x_i)_{i \in \mathbb{N}} \in G \mid x_i = 0 \text{ for either } i > 2n - 2, \text{ or } i \text{ even}\},$$

so  $|T_n(\beta^2, H)/H| = 2^{n-1}$  and  $\widetilde{\text{ent}}(\beta^2, H) = \log(2)$ . In particular,  $\widetilde{\text{ent}}(\beta^2, H) \neq \widetilde{\text{ent}}(\beta^2, H')$ .

We now give a correct proof of the Logarithmic Law, [1, Lem. 3.12]. For this we will use some deep properties of the intrinsic entropy proved in [1] (more precisely, we will use [1, Prop. 5.6, Lem. 2.7, Prop. 3.16(b), Lem. 3.14]), whose proofs do not rely on the wrong [1, Lem. 3.11(b)].

**Logarithmic Law** ([1, Lem. 3.12]). *If  $\phi: G \rightarrow G$  is an endomorphism of an Abelian group  $G$ , and  $k \in \mathbb{N}_+$ , then  $\widetilde{\text{ent}}(\phi^k) = k \cdot \widetilde{\text{ent}}(\phi)$ .*

**Proof.** We first prove the Logarithmic Law under the additional assumption that

$$G = T(\phi, F),$$

for a finitely generated subgroup  $F \leq G$ , that is, when  $G_\phi$  is finitely generated as a  $\mathbb{Z}[X]$ -module. Then,  $G = T(\phi^k, T_k(\phi, F))$ , so also the  $\mathbb{Z}[X]$ -module  $G_{\phi^k}$  is finitely generated. Now, by [1, Prop. 5.6] (which applies to finitely generated  $\mathbb{Z}[X]$ -modules),  $\widetilde{\text{ent}}(\phi) = \infty$  if and only if  $\text{rk}(G) = \infty$  if and only if  $\widetilde{\text{ent}}(\phi^k) = \infty$ , so let us suppose that  $\widetilde{\text{ent}}(\phi) < \infty$ . In that case, again by [1, Prop. 5.6], there exists  $m \in \mathbb{N}_+$  such that  $T_m(\phi, F)$  is  $\phi$ -inert and, by [1, Lem. 3.11(a)], also the subgroup  $H := T_{m+k-1}(\phi, F) = T_k(\phi, T_m(\phi, F))$  is  $\phi$ -inert. Moreover, by [1, Lem. 2.7(b)],  $H$  is also  $\phi^k$ -inert. Furthermore,  $H$  is a finitely generated subgroup of  $G$  such that  $G = T(\phi, H) = T(\phi^k, H)$ . Then, [1, Prop. 3.16(b)] applies respectively to  $\phi$  and  $\phi^k$  to give the following equalities:

$$\widetilde{\text{ent}}(\phi) = \widetilde{\text{ent}}(\phi, H), \text{ and } \widetilde{\text{ent}}(\phi^k) = \widetilde{\text{ent}}(\phi^k, H).$$

To conclude this part of the proof, we now verify that  $\widetilde{\text{ent}}(\phi^k, H) = k \cdot \widetilde{\text{ent}}(\phi, H)$ . Indeed, for every  $n \in \mathbb{N}_+$ ,  $T_n(\phi^k, H) = T_{kn-k+1}(\phi, H)$ , and so

$$\begin{aligned} \widetilde{\text{ent}}(\phi^k, H) &= \lim_{n \rightarrow \infty} \frac{\log |T_n(\phi^k, H)/H|}{n} \\ &= \lim_{n \rightarrow \infty} \frac{\log |T_{kn-k+1}(\phi, H)/H|}{kn-k+1} \cdot \frac{kn-k+1}{n} = k \cdot \widetilde{\text{ent}}(\phi, H). \end{aligned}$$

In the general case, we use the fact that  $G$  is the direct limit of the family of  $\phi$ -invariant subgroups  $\{T(\phi, F) \mid F \leq G, F \text{ finitely generated}\}$ . Then, [1, Lem. 3.14] implies that

$$\begin{aligned}\widetilde{\text{ent}}(\phi) &= \sup\{\widetilde{\text{ent}}(\phi \upharpoonright_{T(\phi,F)}) \mid F \leq G, F \text{ finitely generated}\}, \text{ and} \\ \widetilde{\text{ent}}(\phi^k) &= \sup\{\widetilde{\text{ent}}(\phi^k \upharpoonright_{T(\phi,F)}) \mid F \leq G, F \text{ finitely generated}\}.\end{aligned}\tag{1}$$

For any finitely generated subgroup  $F \leq G$ , the first part of the proof gives that  $\widetilde{\text{ent}}(\phi^k \upharpoonright_{T(\phi,F)}) = k \cdot \widetilde{\text{ent}}(\phi \upharpoonright_{T(\phi,F)})$ , so (1) yields  $\widetilde{\text{ent}}(\phi^k) = k \cdot \widetilde{\text{ent}}(\phi)$ .  $\square$

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## References

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