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A birational embedding with two Galois points for quotient curves

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ABSTRACT

A criterion for the existence of a birational embedding with two Galois points for quotient curves is presented. We apply our criterion to several curves, for example, some cyclic subcovers of the Giulietti–Korchmáros curve or of the curves constructed by Skabelund. New examples of plane curves with two Galois points are described, as plane models of such quotient curves.

1. Introduction

The notion of a *Galois point* was introduced by Hisao Yoshihara in 1996 ([1,14,17]): for a plane curve $\mathcal{C} \subset \mathbb{P}^2$, if the field extension $k(\mathcal{C})/\pi_P^* k(\mathbb{P}^1)$ of function fields induced by the projection π_P from a point $P \in \mathbb{P}^2$ is Galois, then the point P is called a Galois point. If a Galois point P is contained in $\mathcal{C} \setminus \mathrm{Sing}(\mathcal{C})$ (resp. in $\mathbb{P}^2 \setminus \mathcal{C}$), where $\mathrm{Sing}(\mathcal{C})$ is the set of all singular points of $\mathcal{C}$, then we call it an inner Galois point (resp. an outer Galois point). The associated Galois group at P is denoted by G_P. It is interesting that many important families of algebraic curves (in positive characteristic), such as the Hermitian, Suzuki, Ree and Giulietti–Korchmáros curves, admit a plane model with two or more Galois points ([4–6,12]).

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Let $\mathcal{X}$ be a (reduced, irreducible) smooth projective curve over an algebraically closed field k of characteristic $p \geq 0$ and let $k(\mathcal{X})$ be its function field. Recently, a criterion for the existence of a birational embedding with two Galois points was presented by the first author ([3]), and by this criterion, several new examples of plane curves with two Galois points were described. We recall this criterion.

Fact 1. *Let G_1, G_2 be finite subgroups of $\mathrm{Aut}(\mathcal{X})$ and let P_1, P_2 be different points of $\mathcal{X}$. Then the three conditions*

(a) $\mathcal{X}/G_1 \cong \mathbb{P}^1$, $\mathcal{X}/G_2 \cong \mathbb{P}^1$,
(b) $G_1 \cap G_2 = \{1\}$, *and*
(c) $P_1 + \sum_{\sigma \in G_1} \sigma(P_2) = P_2 + \sum_{\tau \in G_2} \tau(P_1)$

are satisfied, if and only if there exists a birational embedding $\varphi : \mathcal{X} \to \mathbb{P}^2$ of degree $|G_1|+1$ such that $\varphi(P_1)$ and $\varphi(P_2)$ are different inner Galois points for $\varphi(\mathcal{X})$ and $G_{\varphi(P_i)} = G_i$ for $i = 1, 2$.

Fact 2. *Let G_1, G_2 be finite subgroups of $\mathrm{Aut}(\mathcal{X})$ and let $Q \in \mathcal{X}$. Then the three conditions*

(a) $\mathcal{X}/G_1 \cong \mathbb{P}^1$, $\mathcal{X}/G_2 \cong \mathbb{P}^1$,
(b) $G_1 \cap G_2 = \{1\}$, *and*
(c) $\sum_{\sigma \in G_1} \sigma(Q) = \sum_{\tau \in G_2} \tau(Q)$

are satisfied, if and only if there exists a birational embedding $\varphi : \mathcal{X} \to \mathbb{P}^2$ of degree $|G_1|$ and different outer Galois points $P_1, P_2 \in \mathbb{P}^2 \setminus \varphi(\mathcal{X})$ exist for $\varphi(\mathcal{X})$ such that $G_{P_i} = G_i$ for $i = 1, 2$ and points $\varphi(Q)$, P_1 and P_2 are collinear.

Some known examples of plane curves with two Galois points are regarded as quotient curves $\mathcal{X}/H$ of curves $\mathcal{X}$ with a subgroup $H \subset \mathrm{Aut}(\mathcal{X})$ such that $\mathcal{X}$ admits a birational embedding with two Galois points. Typical examples are quotient curves of the Hermitian curve ([5,12]), and the Hermitian curve as a Galois subcover of the Giulietti–Korchmáros curve ([4,6]). Quotient curves are important in the study of maximal curves with respect to the Hasse–Weil bound (see, for example, [7–10]).

Motivated by this observation, the aim of this article is to present a criterion for the existence of a plane model with two Galois points for quotient curves. For a finite subgroup H of $\mathrm{Aut}(\mathcal{X})$ and a point $Q \in \mathcal{X}$, the quotient map is denoted by $f_H : \mathcal{X} \to \mathcal{X}/H$ and the image $f_H(Q)$ is denoted by $\overline{Q}$. Assume that H is a normal subgroup of a subgroup $G \subset \mathrm{Aut}(\mathcal{X})$. Then it follows that for each $\sigma \in G$, the pullback $\sigma^* : k(\mathcal{X}) \to k(\mathcal{X})$ satisfies $\sigma^*(k(\mathcal{X})^H) = k(\mathcal{X})^H$. Therefore, there exists a natural homomorphism $G \to \mathrm{Aut}(\mathcal{X}/H)$; $\sigma \mapsto \overline{\sigma}$, where $\overline{\sigma}$ corresponds to the restriction $\sigma^*|_{k(\mathcal{X})^H}$. The image is denoted by $\overline{G}$, which is isomorphic to G/H. The following two theorems are our main results.

Theorem 1. *Let $H, G_1, G_2 \subset \mathrm{Aut}(\mathcal{X})$ be finite subgroups with $H \lhd G_i$ for $i = 1, 2$, and let $P_1, P_2 \in \mathcal{X}$. Then the four conditions*

(a') $\mathcal{X}/G_1 \cong \mathbb{P}^1$, $\mathcal{X}/G_2 \cong \mathbb{P}^1$,
(b') $G_1 \cap G_2 = H$,
(c') $\sum_{h \in H} h(P_1) + \sum_{\sigma \in G_1} \sigma(P_2) = \sum_{h \in H} h(P_2) + \sum_{\tau \in G_2} \tau(P_1)$, *and*
(d') $HP_1 \neq HP_2$, *where HP_i is the orbit of P_i, i.e., $HP_i = \{h(P_i) \mid h \in H\}$ for $i = 1, 2$,*

are satisfied, if and only if there exists a birational embedding $\varphi : \mathcal{X}/H \to \mathbb{P}^2$ of degree $|G_1/H| + 1$ such that $\varphi(\overline{P_1})$ and $\varphi(\overline{P_2})$ are different inner Galois points for $\varphi(\mathcal{X}/H)$ and $G_{\varphi(\overline{P_i})} = \overline{G_i}$ for $i = 1, 2$.

Theorem 2. *Let $H, G_1, G_2 \subset \mathrm{Aut}(\mathcal{X})$ be finite subgroups with $H \lhd G_i$ for $i = 1, 2$, and let $Q \in \mathcal{X}$. Then the three conditions*

(a') $\mathcal{X}/G_1 \cong \mathbb{P}^1$, $\mathcal{X}/G_2 \cong \mathbb{P}^1$,
(b') $G_1 \cap G_2 = H$, *and*
(c') $\sum_{\sigma \in G_1} \sigma(Q) = \sum_{\tau \in G_2} \tau(Q)$

are satisfied, if and only if there exists a birational embedding $\varphi : \mathcal{X}/H \to \mathbb{P}^2$ of degree $|G_1/H|$ and different outer Galois points $P_1, P_2 \in \mathbb{P}^2 \setminus \varphi(\mathcal{X}/H)$ exist for $\varphi(\mathcal{X}/H)$ such that $G_{P_i} = \overline{G_i}$ for $i = 1, 2$ and points $\varphi(\overline{Q})$, P_1 and P_2 are collinear.

As an application, for the case where $\mathcal{X}$ admits a birational embedding with two Galois points, the following two results hold.

Corollary 1. *Let G_1, G_2, H be finite subgroups of $\mathrm{Aut}(\mathcal{X})$, and let P_1, P_2 be different points of $\mathcal{X}$. Assume that the three conditions*

(a) $\mathcal{X}/G_1 \cong \mathbb{P}^1$, $\mathcal{X}/G_2 \cong \mathbb{P}^1$,
(b) $G_1 \cap G_2 = \{1\}$, *and*
(c) $P_1 + \sum_{\sigma \in G_1} \sigma(P_2) = P_2 + \sum_{\tau \in G_2} \tau(P_1)$

are satisfied. If the three conditions

(d) $H \cap G_1G_2 = \{1\}$,
(e) $HG_1 = H \rtimes G_1$, $HG_2 = H \rtimes G_2$, *and*
(f) $HP_1 \neq HP_2$

are satisfied, then there exists a birational embedding $\psi : \mathcal{X}/H \to \mathbb{P}^2$ of degree $|G_1| + 1$ such that $\psi(\overline{P_1})$ and $\psi(\overline{P_2})$ are different inner Galois points for $\psi(\mathcal{X}/H)$ and $G_{\psi(\overline{P_i})} \cong G_i$ for $i = 1, 2$.

Corollary 2. *Let G_1, G_2, H be finite subgroups of $\mathrm{Aut}(\mathcal{X})$, and let $Q \in \mathcal{X}$. Assume that the three conditions*

(a) $\mathcal{X}/G_1 \cong \mathbb{P}^1$, $\mathcal{X}/G_2 \cong \mathbb{P}^1$,
(b) $G_1 \cap G_2 = \{1\}$, *and*
(c) $\sum_{\sigma \in G_1} \sigma(Q) = \sum_{\tau \in G_2} \tau(Q)$

are satisfied. If the two conditions

(d) $H \cap G_1G_2 = \{1\}$, *and*
(e) $HG_1 = H \rtimes G_1$, $HG_2 = H \rtimes G_2$

are satisfied, then there exists a birational embedding $\psi : \mathcal{X}/H \to \mathbb{P}^2$ of degree $|G_1|$ and different outer Galois points $P_1, P_2 \in \mathbb{P}^2 \setminus \psi(\mathcal{X}/H)$ exist for $\psi(\mathcal{X}/H)$ such that $G_{P_i} \cong G_i$ for $i = 1, 2$ and points $\psi(\overline{Q})$, P_1 and P_2 are collinear.

In Sections 3 and 4, we will apply Corollary 1 to the Giulietti–Korchmáros curve, and the curves constructed by Skabelund. Theorems 3, 4, 5 and 6 provide new examples of plane curves with two Galois points (see the Table in [18]). In Section 5, we discuss the relations between Corollaries 1, 2 and the previous works.

2. Proof of the main theorems

Before proving our main theorems, we prove Fact 2 for outer Galois points, since this fact is just stated in a remark [3, Remark 1] and is needed for the proof of Theorem 2.

Proof of Fact 2. We consider the only-if part. Assume that conditions (a), (b) and (c) are satisfied. Let D be the divisor

$$D = \sum_{\sigma \in G_1} \sigma(Q) = \sum_{\tau \in G_2} \tau(Q),$$

by (c). Let $f, g \in k(\mathcal{X})$ be generators of $k(\mathcal{X})^{G_1}$, $k(\mathcal{X})^{G_2}$ such that $(f)_\infty = D$ and $(g)_\infty = D$, by (a), where $(f)_\infty$ is the pole divisor of f. Then $f, g \in \mathcal{L}(D)$. Let $\varphi : \mathcal{X} \to \mathbb{P}^2$ be given by $(f : g : 1)$. Similarly to [3, Proposition 1], by (b), φ is birational onto its image. The sublinear system of $|D|$ corresponding to $\langle f, g, 1\rangle$ is base-point-free, since $\mathrm{supp}(D) \cap \mathrm{supp}((f) + D) = \emptyset$. Therefore, $\deg \varphi(\mathcal{X}) = \deg D$, and the morphism $(f : 1)$ (resp. $(g : 1)$) coincides with the projection from the point $P_1 = (0 : 1 : 0) \in \mathbb{P}^2 \setminus \varphi(\mathcal{X})$ (resp. $P_2 = (1 : 0 : 0) \in \mathbb{P}^2 \setminus \varphi(\mathcal{X})$).

We consider the if part. Assume that P_1 and P_2 are outer Galois points for $\varphi(\mathcal{X})$ such that $G_{P_i} = G_i$ for $i = 1, 2$, and $\varphi(Q) \in \overline{P_1P_2}$, where $\overline{P_1P_2}$ is the line passing through P_1 and P_2. Since $k(\mathcal{X})^{G_i} = k(\varphi(\mathcal{X}))^{G_{P_i}}$ for $i = 1, 2$, condition (a) is satisfied. According to [2, Lemma 7], condition (b) is satisfied. Let D be the divisor induced by the intersection of $\varphi(\mathcal{X})$ and the hyperplane $\overline{P_1P_2}$, that is, $D = \varphi^*\overline{P_1P_2} = \sum_{P \in \mathcal{X}} (\mathrm{ord}_P \varphi^* \overline{P_1P_2}) \cdot P$. We can consider the line $\overline{P_1P_2}$ as a point in the images of $\pi_{P_1} \circ \varphi$ and $\pi_{P_2} \circ \varphi$. Since $\pi_{P_1} \circ \varphi$ (resp. $\pi_2 \circ \varphi$) is a Galois covering and $Q \in \varphi^{-1}(\varphi(\mathcal{X}) \cap \overline{P_1P_2})$,

$$(\pi_{P_1} \circ \varphi)^* \overline{P_1P_2} = \sum_{\sigma \in G_1} \sigma(Q) \quad \left(\text{resp. } (\pi_{P_2} \circ \varphi)^* \overline{P_1P_2} = \sum_{\tau \in G_2} \tau(Q)\right)$$

as divisors (see, for example, [16, III.7.1, III.7.2, III.8.2]), where $(\pi_{P_1} \circ \varphi)^*$ denotes the pullback (see, for example, [11, p. 137]). On the other hand, it follows that $(\pi_{P_1} \circ \varphi)^* \overline{P_1P_2} = D$ (resp. $(\pi_{P_2} \circ \varphi)^* \overline{P_1P_2} = D$). Therefore,

$$D = \sum_{\sigma \in G_1} \sigma(Q) = \sum_{\tau \in G_2} \tau(Q),$$

which is nothing but assertion (c). □

Proof of Theorem 1. We consider the only-if part. Assume that conditions (a'), (b'), (c') and (d') of Theorem 1 are satisfied. By condition (d'), $\overline{P_1} \neq \overline{P_2}$. We would like to prove that conditions (a), (b) and (c) of Fact 1 are satisfied for the 4-tuple $(\overline{G_1}, \overline{G_2}, \overline{P_1}, \overline{P_2})$. Since $k(\mathcal{X}/H)^{\overline{G_i}} = k(\mathcal{X})^{G_i}$, by condition (a'), the fixed field $k(\mathcal{X}/H)^{\overline{G_i}}$ is rational. It follows from condition (b') that $\overline{G_1} \cap \overline{G_2} = \{1\}$. Therefore, conditions (a) and (b) for the 4-tuple $(\overline{G_1}, \overline{G_2}, \overline{P_1}, \overline{P_2})$ are satisfied. Since

$$\sum_{\sigma \in G_1} \sigma(P_2) = \sum_{H\sigma \in G_1/H} \sum_{h \in H} h\sigma(P_2),$$

it follows that

$$(f_H)_* \left(\sum_{\sigma \in G_1} \sigma(P_2) \right) = \sum_{H\sigma \in G_1/H} |H| \cdot \overline{\sigma(P_2)} = |H| \sum_{\overline{\sigma} \in \overline{G_1}} \overline{\sigma}(\overline{P_2}),$$

where $(f_H)_* : \text{Div } \mathcal{X} \to \text{Div } \mathcal{X}/H$ is a homomorphism such that $(f_H)_*(\sum n_i P_i) = \sum n_i f_H(P_i)$ for any divisor $\sum n_i P_i$ on $\mathcal{X}$ ([11, IV, Exercise 2.6]). On the other hand, $(f_H)_*(\sum_{h \in H} h(P_1)) = |H|\overline{P_1}$. It follows from condition (c') that

$$|H|\left(\overline{P_1} + \sum_{\overline{\sigma} \in \overline{G_1}} \overline{\sigma}(\overline{P_2})\right) = |H|\left(\overline{P_2} + \sum_{\overline{\tau} \in \overline{G_2}} \overline{\tau}(\overline{P_1})\right).$$

Since $|H| \cdot D = 0$ implies $D = 0$ for any divisor D, we are able to cut the multiplier $|H|$. Condition (c) for the 4-tuple $(\overline{G_1}, \overline{G_2}, \overline{P_1}, \overline{P_2})$ is satisfied.

We consider the if part. By Fact 1, we have that conditions (a), (b) and (c) of Fact 1 are satisfied for the 4-tuple $(\overline{G_1}, \overline{G_2}, \overline{P_1}, \overline{P_2})$. Since $k(\mathcal{X})^{G_i} = k(\mathcal{X}/H)^{\overline{G_i}}$, by condition (a), the fixed field $k(\mathcal{X})^{G_i}$ is rational. Condition (a') is satisfied. Since $\overline{G_1} \cap \overline{G_2} = \{1\}$, condition (b') is satisfied. Since $\varphi(\overline{P_1}) \neq \varphi(\overline{P_2})$, condition (d') is satisfied. By condition (c),

$$\overline{P_1} + \sum_{\overline{\sigma} \in \overline{G_1}} \overline{\sigma}(\overline{P_2}) = \overline{P_2} + \sum_{\overline{\tau} \in \overline{G_2}} \overline{\tau}(\overline{P_1}).$$

Since $f_H^*(\overline{Q}) = \sum_{h \in H} h(Q)$ for each $Q \in \mathcal{X}$, where f_H^* denotes the pullback (see, for example, [16, III.7.1, III.7.2, III.8.2]),

$$\begin{aligned} f_H^*\left(\overline{P_1} + \sum_{\overline{\sigma} \in \overline{G_1}} \overline{\sigma}(\overline{P_2})\right) &= f_H^*(\overline{P_1}) + \sum_{\overline{\sigma} \in \overline{G_1}} f_H^*(\overline{\sigma}(\overline{P_2})) \\ &= \sum_{h \in H} h(P_1) + \sum_{H\sigma \in G_1/H} \sum_{h \in H} h\sigma(P_2) \\ &= \sum_{h \in H} h(P_1) + \sum_{\sigma \in G_1} \sigma(P_2). \end{aligned}$$

Similarly,

$$f_H^*\left(\overline{P_2} + \sum_{\overline{\tau} \in \overline{G_2}} \overline{\tau}(\overline{P_1})\right) = \sum_{h \in H} h(P_2) + \sum_{\tau \in G_2} \tau(P_1).$$

Condition (c') is satisfied. $\square$

Proof of Theorem 2. We consider the only-if part. Assume that conditions (a'), (b') and (c') of Theorem 2 are satisfied. We would like to prove that conditions (a), (b) and (c) of Fact 2 are satisfied for the triple $(\overline{G_1}, \overline{G_2}, \overline{Q})$. Since $k(\mathcal{X}/H)^{\overline{G_i}} = k(\mathcal{X})^{G_i}$, by condition (a'), the fixed field $k(\mathcal{X}/H)^{\overline{G_i}}$ is rational. It follows from condition (b') that $\overline{G_1} \cap \overline{G_2} = \{1\}$. Therefore, conditions (a) and (b) for the triple $(\overline{G_1}, \overline{G_2}, \overline{Q})$ are satisfied. Since

$$\sum_{\sigma \in G_1} \sigma(Q) = \sum_{H\sigma \in G_1/H} \sum_{h \in H} h\sigma(Q),$$

it follows that

$$(f_H)_*\left(\sum_{\sigma \in G_1} \sigma(Q)\right) = \sum_{H\sigma \in G_1/H} |H| \cdot \overline{\sigma(Q)} = |H| \sum_{\overline{\sigma} \in \overline{G_1}} \overline{\sigma}(\overline{Q}).$$

It follows from condition (c') that

$$|H|\left(\sum_{\overline{\sigma}\in\overline{G_1}}\overline{\sigma}(\overline{Q})\right)=|H|\left(\sum_{\overline{\tau}\in\overline{G_2}}\overline{\tau}(\overline{Q})\right).$$

Since $|H|\cdot D=0$ implies $D=0$ for any divisor D, we are able to cut the multiplier $|H|$. Condition (c) for the triple $(\overline{G_1},\overline{G_2},\overline{Q})$ is satisfied.

We consider the if part. By Fact 2, we have that conditions (a), (b) and (c) of Fact 2 are satisfied for the triple $(\overline{G_1},\overline{G_2},\overline{Q})$. Since $k(\mathcal{X})^{G_i}=k(\mathcal{X}/H)^{\overline{G_i}}$, by condition (a), the fixed field $k(\mathcal{X})^{G_i}$ is rational. Condition (a') is satisfied. Since $\overline{G_1}\cap\overline{G_2}=\{1\}$, condition (b') is satisfied. By condition (c),

$$\sum_{\overline{\sigma}\in\overline{G_1}}\overline{\sigma}(\overline{Q})=\sum_{\overline{\tau}\in\overline{G_2}}\overline{\tau}(\overline{Q}).$$

Since $f_H^*(\overline{Q})=\sum_{h\in H}h(Q)$ for each $Q\in\mathcal{X}$ (see, for example, [16, III.7.1, III.7.2, III.8.2]),

$$\begin{aligned}f_H^*\left(\sum_{\overline{\sigma}\in\overline{G_1}}\overline{\sigma}(\overline{Q})\right)&=\sum_{\overline{\sigma}\in\overline{G_1}}f_H^*(\overline{\sigma}(\overline{Q}))=\sum_{H\sigma\in G_1/H}\sum_{h\in H}h\sigma(Q)\\&=\sum_{\sigma\in G_1}\sigma(Q).\end{aligned}$$

Similarly,

$$f_H^*\left(\sum_{\overline{\tau}\in\overline{G_2}}\overline{\tau}(\overline{Q})\right)=\sum_{\tau\in G_2}\tau(Q).$$

Condition (c') is satisfied. $\square$

Proof of Corollary 1. By condition (d), $H\cap G_i=\{1\}$ for $i=1,2$. By condition (e), $HG_i=H\rtimes G_i$. Let $\hat{G}_i=H\rtimes G_i$ for $i=1,2$. Note that $H\triangleleft\hat{G}_i$ for $i=1,2$. We would like to prove that conditions (a'), (b'), (c') and (d') of Theorem 1 are satisfied for the 5-tuple $(\hat{G}_1,\hat{G}_2,H,P_1,P_2)$. Condition (f) is the same as condition (d'). Since $k(\mathcal{X})^{\hat{G}_i}\subset k(\mathcal{X})^{G_i}$, by condition (a) and Lüroth's theorem, it follows that $\mathcal{X}/\hat{G}_i\cong\mathbb{P}^1$. Condition (a') is satisfied.

Let $\eta\in\hat{G}_1\cap\hat{G}_2$. Then there exist $h_1,h_2\in H$, $\sigma\in G_1$ and $\tau\in G_2$ such that $\eta=h_1\sigma=h_2\tau$. Then $\sigma\tau^{-1}=h_1^{-1}h_2\in H$. By condition (d), $\sigma\tau^{-1}=1$ and hence, $\sigma=\tau\in G_1\cap G_2$. By condition (b), $\sigma=\tau=1$. This implies that $\eta\in H$. It follows that $\hat{G}_1\cap\hat{G}_2=H$. Condition (b') is satisfied.

By condition (c), it follows that

$$P_1+\sum_{\sigma\in G_1}\sigma(P_2)=P_2+\sum_{\tau\in G_2}\tau(P_1).$$

For each $h\in H$,

$$h(P_1)+\sum_{\sigma\in G_1}h\sigma(P_2)=h(P_2)+\sum_{\tau\in G_2}h\tau(P_1).$$

Therefore,

$$\sum_{h\in H} h(P_1) + \sum_{h\in H}\sum_{\sigma\in G_1} h\sigma(P_2) = \sum_{h\in H} h(P_2) + \sum_{h\in H}\sum_{\tau\in G_2} h\tau(P_1).$$

Condition (c') is satisfied, since each element of $\hat{G}_1$ (resp. of $\hat{G}_2$) is represented uniquely as $h\sigma$ (resp. $h\tau$) for some $h \in H$ and $\sigma \in G_1$ (resp. $\tau \in G_2$). $\square$

Proof of Corollary 2. Similarly to the proof of Corollary 1, we prove that conditions (a'), (b') and (c') of Theorem 2 are satisfied for the 4-tuple $(\hat{G}_1, \hat{G}_2, H, Q)$, where $\hat{G}_i = H \rtimes G_i$ for $i = 1, 2$. The proof for conditions (a') and (b') is the same as the proof of Corollary 1. By condition (c), it follows that $\sum_{\sigma\in G_1} \sigma(Q) = \sum_{\tau\in G_2} \tau(Q)$. For each $h \in H$, $\sum_{\sigma\in G_1} h\sigma(Q) = \sum_{\tau\in G_2} h\tau(Q)$. Therefore,

$$\sum_{h\in H}\sum_{\sigma\in G_1} h\sigma(Q) = \sum_{h\in H}\sum_{\tau\in G_2} h\tau(Q).$$

Condition (c') is satisfied, since each element of $\hat{G}_1$ (resp. of $\hat{G}_2$) is represented uniquely as $h\sigma$ (resp. $h\tau$) for some $h \in H$ and $\sigma \in G_1$ (resp. $\tau \in G_2$). $\square$

3. An application to cyclic subcovers of the Giulietti–Korchmáros curve

Let $p > 0$ and let q be a power of p. We consider the Giulietti–Korchmáros curve $\mathcal{X} \subset \mathbb{P}^3$, which is defined by

$$x^q + x - y^{q+1} = 0 \quad \text{and} \quad y((x^q + x)^{q-1} - 1) - z^{q^2-q+1} = 0$$

(see [8]). The group

$$G_1 := \left\{ \begin{pmatrix} 1 & b^q & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mid a, b \in \mathbb{F}_{q^2},\ a^q + a - b^{q+1} = 0 \right\} \subset \mathrm{PGL}(4, k)$$

of order q^3 acts on $\mathcal{X}$ (see [8, Lemma 7]). This group acts on the set $\mathcal{X} \cap \{Z = 0\} = \mathcal{X}(\mathbb{F}_{q^2})$ of all $\mathbb{F}_{q^2}$-rational points of $\mathcal{X}$, and fixes a point $P_1 := (1 : 0 : 0 : 0) \in \mathcal{X}$. Let

$$\xi(x, y, z) = \left(\frac{1}{x}, -\frac{y}{x}, \frac{z}{x}\right).$$

Then ξ acts on $\mathcal{X}$ ([8, Lemma 7]). This automorphism acts on $\mathcal{X}(\mathbb{F}_{q^2})$, and $P_2 := \xi(P_1) = (0 : 0 : 0 : 1)$. Let $G_2 := \xi G_1 \xi^{-1}$, which fixes P_2. According to [6, Theorem 2], conditions (a), (b) and (c) of Fact 1 are satisfied for the 4-tuple (G_1, G_2, P_1, P_2).

It follows from [8, Equation (9)] that the cyclic group

$$C_{q^2-q+1} := \left\{ (x, y, z) \mapsto (x, y, \zeta z) \mid \zeta^{q^2-q+1} = 1 \right\}$$

acts on $\mathcal{X}$. We prove the following.

Theorem 3. *Let H be a subgroup of C_{q^2-q+1}. Then there exists a birational embedding $\psi : \mathcal{X}/H \to \mathbb{P}^2$ of degree $q^3 + 1$ with two inner Galois points.*

Proof. Note that H fixes all points of $\mathcal{X}(\mathbb{F}_{q^2})$ $(= \mathcal{X} \cap \{Z = 0\})$. Therefore, $HP_1 = \{P_1\} \neq \{P_2\} = HP_2$. Since $\sigma|_{\mathcal{X}(\mathbb{F}_{q^2})} \neq \tau|_{\mathcal{X}(\mathbb{F}_{q^2})}$ for any $\sigma \in G_1 \setminus \{1\}$ and $\tau \in G_2 \setminus \{1\}$, $H \cap G_1G_2 = \{1\}$ follows. It is easily verified that $HG_1 = H \times G_1$. Since $\xi h = h\xi$ for each element $h \in H$, $HG_2 = H \times G_2$ follows. Therefore, conditions (d), (e) and (f) of Corollary 1 are satisfied for the 5-tuple (G_1, G_2, P_1, P_2, H). By Corollary 1, the assertion follows. $\square$

4. The curves constructed by Skabelund and their quotient curves

We consider the cyclic cover $\tilde{\mathcal{S}}$ of the Suzuki curve $\mathcal{S}$, constructed by Skabelund ([15]). Let $p = 2$, let q_0 be a power of 2, and let $q = 2q_0^2$. The curve $\tilde{\mathcal{S}}$ is the smooth model of the curve defined by

$$y^q + y = x^{q_0}(x^q + x) \quad \text{and} \quad x^q + x = z^{q-2q_0+1}$$

in $\mathbb{P}^3$. Let $P_1 \in \tilde{\mathcal{S}}$ be the pole of x. It is known that the group

$$G_1 := \left\{ \begin{pmatrix} 1 & 0 & 0 & a \\ a^{q_0} & 1 & 0 & b \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mid a, b \in \mathbb{F}_q \right\} \subset \mathrm{PGL}(4, k)$$

of order q^2 acts on $\tilde{\mathcal{S}}$ (see [15, Lemma 3.3], [9, Section 2]). This group acts on the set $\tilde{\mathcal{S}}(\mathbb{F}_q)$ of all $\mathbb{F}_q$-rational points of $\tilde{\mathcal{S}}$, and fixes P_1. Let $\alpha := y^{2q_0} + x^{2q_0+1}$, $\beta := xy^{2q_0} + \alpha^{2q_0}$ and let

$$\xi(x, y, z) = \left(\frac{\alpha}{\beta}, \frac{y}{\beta}, \frac{z}{\beta}\right).$$

Then ξ acts on $\tilde{\mathcal{S}}$ (see [15, Proofs of Lemmas 3.3 and 3.4], [9, Section 2]). This automorphism acts on $\tilde{\mathcal{S}}(\mathbb{F}_q)$, and $P_2 := \xi(P_1) = (0 : 0 : 0 : 1)$ (see [15, Proofs of Lemmas 3.3 and 3.4], [9, Section 2]). Let $G_2 := \xi G_1 \xi^{-1}$, which fixes P_2. Then we have the following.

Theorem 4. *The curve $\tilde{\mathcal{S}}$ admits a plane model of degree $q^2 + 1$ with two inner Galois points.*

Proof. We prove that conditions (a), (b) and (c) of Fact 1 are satisfied for the 4-tuple (G_1, G_2, P_1, P_2). It is not difficult to check that $k(\tilde{\mathcal{S}})^{G_1} = k(z)$ and $k(\tilde{\mathcal{S}})^{G_2} = k(z/\beta)$. Since no nontrivial element of G_1 fixes P_2, $G_1 \cap G_2 = \{1\}$. Conditions (a) and (b) are satisfied. Condition (c) is satisfied, since

$$P_1 + \sum_{\sigma \in G_1} \sigma(P_2) = \sum_{Q \in \tilde{\mathcal{S}}(\mathbb{F}_q)} Q = P_2 + \sum_{\tau \in G_2} \tau(P_1). \quad \square$$

It follows from the shape of the second equation that the cyclic group

$$C_{q-2q_0+1} := \left\{(x, y, z) \mapsto (x, y, \zeta z) \mid \zeta^{q-2q_0+1} = 1\right\}$$

acts on $\tilde{\mathcal{S}}$. Similarly to the proof of Theorem 3, the following holds.

Theorem 5. *Let H be a subgroup of C_{q-2q_0+1}. Then there exists a birational embedding $\psi : \tilde{\mathcal{S}}/H \to \mathbb{P}^2$ of degree $q^2 + 1$ with two inner Galois points.*

We consider the cyclic cover $\tilde{\mathcal{R}}$ of the Ree curve $\mathcal{R}$, constructed by Skabelund. Let $p = 3$, let q_0 be a power of 3 and let $q = 3q_0^2$. The curve $\tilde{\mathcal{R}}$ is the smooth model of the curve defined by

$$y^q - y = x^{q_0}(x^q - x),\ z^q - z = x^{2q_0}(x^q - x) \text{ and } x^q - x = t^{q-3q_0+1}.$$

Let $P_1 \in \tilde{\mathcal{R}}$ be the pole of x. It is known that the group

$$G_1 := \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 & a \\ a^{q_0} & 1 & 0 & 0 & b \\ a^{2q_0} & -a^{q_0} & 1 & 0 & c \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \mid a, b, c \in \mathbb{F}_q \right\} \subset \mathrm{PGL}(5, k)$$

of order q^3 acts on $\tilde{\mathcal{R}}$ (see [15, Lemma 4.2], [9, Section 2]). This group acts on the set $\tilde{\mathcal{R}}(\mathbb{F}_q)$ of all $\mathbb{F}_q$-rational points of $\tilde{\mathcal{R}}$, and fixes P_1. There exists an involution ξ of $\tilde{\mathcal{R}}$ such that ξ acts on $\tilde{\mathcal{R}}(\mathbb{F}_q)$ and $P_2 := \xi(P_1) = (0 : 0 : 0 : 0 : 1)$ (see [15, Proofs of Lemmas 4.2 and 4.3], [9, Section 2]). Let $G_2 := \xi G_1 \xi^{-1}$, which fixes P_2.

It follows from the shape of the third equation that the cyclic group

$$C_{q-3q_0+1} := \left\{(x, y, z, t) \mapsto (x, y, z, \zeta t) \mid \zeta^{q-3q_0+1} = 1\right\}$$

acts on $\tilde{\mathcal{R}}$. Similarly to Theorems 4 and 5, the following result holds.

Theorem 6. *Let H be a subgroup of C_{q-3q_0+1}. Then the curves $\tilde{\mathcal{R}}$ and $\tilde{\mathcal{R}}/H$ admit plane models of degree q^3+1 with two inner Galois points.*

5. Relations with the previous works

We can provide another proof of Theorems 1 and 2 in [5], by Corollaries 1, 2 and the analysis of the Hermitian curve $\mathcal{H} \subset \mathbb{P}^2$: $x^q + x = y^{q+1}$. We recover Theorem 1(1) in [5] here. Precisely:

Fact 3. *Let a positive integer m divide $q+1$. Then the smooth model of the curve $y^m = x^q + x$ possesses a birational embedding into $\mathbb{P}^2$ of degree $q+1$ with two inner Galois points.*

Proof. Let $P_1 = (1 : 0 : 0)$ and $P_2 = (0 : 0 : 1) \in \mathbb{P}^2$. Then P_1 and P_2 are inner Galois points for the Hermitian curve $\mathcal{H} \subset \mathbb{P}^2$ ([12]). The associated Galois groups at P_1, P_2 are represented by

$$G_1 := \left\{ \begin{pmatrix} 1 & 0 & \alpha \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid \alpha^q + \alpha = 0 \right\},\ G_2 := \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \alpha & 0 & 1 \end{pmatrix} \mid \alpha^q + \alpha = 0 \right\}$$

respectively. Then conditions (a), (b) and (c) of Fact 1 are satisfied for the 4-tuple (G_1, G_2, P_1, P_2). Let $sm = q+1$ and let C_s be a cyclic group of order s generated by the automorphism group $(x, y) \mapsto (x, \zeta y)$, where ζ is a primitive s-th root of unity. Note that C_s fixes all points in the line $Y = 0$. Therefore, $C_s P_1 = \{P_1\} \neq \{P_2\} = C_s P_2$. It is easily verified that $C_s \cap G_1 G_2 = \{1\}$ and $C_s G_i = C_s \times G_i$. Conditions (d), (e) and (f) of Corollary 1 are satisfied. By Corollary 1, the quotient curve $\mathcal{H}/C_s$ has a birational embedding of degree $q+1$ with two inner Galois points. On the other hand, the quotient curve $\mathcal{H}/C_s$ has a plane model defined by $y^m = x^q + x$. $\square$

A similar argument is applicable to the curve $\mathcal{C} \subset \mathbb{P}^2$ defined by $x^3 + y^4 + 1 = 0$, which has two inner Galois points $P_1 = (1 : 0 : 0)$ and $P_2 = (-1 : 0 : 1)$ on the line $Y = 0$ (under the assumption $p \neq 2, 3$), by taking $H = \langle \eta \rangle$ with $\eta(x, y) = (x, -y)$ (see [13,14,17]). Here, the associated Galois groups G_1, G_2 at P_1, P_2 are generated by matrices

$$\begin{pmatrix} \omega & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} \frac{-\omega}{-\omega+1} & 0 & \frac{2}{-\omega+1} \\ 0 & 1 & 0 \\ \frac{1}{-\omega+1} & 0 & \frac{\omega^2}{-\omega+1} \end{pmatrix}$$

respectively, where $\omega^2 + \omega + 1 = 0$ (see [13, Lemma 1] for the explicit description of the generators). Then the quotient curve $\mathcal{C}/H$ is the elliptic curve $y^2 + x^3 + 1 = 0$. It is well known that this curve is isomorphic to the Fermat curve. (An elliptic curve E admitting a triple Galois covering $E \to \mathbb{P}^1$ is uniquely determined [11, IV, Corollary 4.7]. One proof is given in [3, p. 100].) Since the Galois group G_i at P_i fixes P_i, the group $\overline{G_i}$ fixes $\overline{P_i}$ for $i = 1, 2$. Then the point $\overline{P_i}$ is a ramification point of index $e_{\overline{P_i}} = |\overline{G_i}| = 3$ for the covering $\mathcal{C}/H \to (\mathcal{C}/H)/\overline{G_i}$ (see [16, III.8.2]). Let ψ be the induced birational embedding, according to Corollary 1. Then $\psi(\overline{P_i})$ is an inner Galois point for $\psi(\mathcal{C}/H) \subset \mathbb{P}^2$. Since $e_{\overline{P_i}} + 1 = I_{\psi(\overline{P_i})}(\psi(\mathcal{C}/H), T_{\psi(\overline{P_i})}\psi(\mathcal{C}/H))$ for the projection from $\psi(\overline{P_i})$, where $I_{\psi(\overline{P_i})}(\psi(\mathcal{C}/H), T_{\psi(\overline{P_i})}\psi(\mathcal{C}/H))$ is the intersection multiplicity of $\psi(\mathcal{C}/H)$ and the tangent line $T_{\psi(\overline{P_i})}\psi(\mathcal{C}/H)$ of $\psi(\mathcal{C}/H)$ at $\psi(\overline{P_i})$, it follows that $\psi(\overline{P_i})$ is a total inflection point. The following result is similar to [3, Theorem 3], but the proofs are different.

Theorem 7. *Let $p \neq 2, 3$. For the cubic Fermat curve, there exists a plane model of degree four with two inner Galois points such that they are total inflection points.*

In [3, Theorem 3], we were not able to assert that two Galois points are inflection points, for the embedding provided in the proof.

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References

[1] S. Fukasawa, Galois points for a plane curve in arbitrary characteristic, in: Proceedings of the IV Iberoamerican Conference on Complex Geometry, Geom. Dedic. 139 (2009) 211–218.
[2] S. Fukasawa, Classification of plane curves with infinitely many Galois points, J. Math. Soc. Jpn. 63 (2011) 195–209.
[3] S. Fukasawa, A birational embedding of an algebraic curve into a projective plane with two Galois points, J. Algebra 511 (2018) 95–101.
[4] S. Fukasawa, Birational embeddings of the Hermitian, Suzuki and Ree curves with two Galois points, Finite Fields Appl. 57 (2019) 60–67.
[5] S. Fukasawa, K. Higashine, A birational embedding with two Galois points for certain Artin–Schreier curves, Finite Fields Appl. 52 (2018) 281–288.
[6] S. Fukasawa, K. Higashine, Galois lines for the Giulietti–Korchmáros curve, Finite Fields Appl. 57 (2019) 268–275.
[7] A. Garcia, H. Stichtenoth, A maximal curve which is not a Galois subcover of the Hermitian curve, Bull. Braz. Math. Soc. 37 (2006) 139–152.
[8] M. Giulietti, G. Korchmáros, A new family of maximal curves over a finite field, Math. Ann. 343 (2009) 229–245.
[9] M. Giulietti, M. Montanucci, L. Quoos, G. Zini, On some Galois covers of the Suzuki and Ree curves, J. Number Theory 189 (2018) 220–254.
[10] M. Giulietti, M. Montanucci, G. Zini, On maximal curves that are not quotients of the Hermitian curve, Finite Fields Appl. 41 (2016) 72–88.
[11] R. Hartshorne, Algebraic Geometry, Graduate Texts in Mathematics, vol. 52, Springer-Verlag, New York, 1977.
[12] M. Homma, Galois points for a Hermitian curve, Commun. Algebra 34 (2006) 4503–4511.
[13] M. Kanazawa, T. Takahashi, H. Yoshihara, The group generated by automorphisms belonging to Galois points of the quartic surface, Nihonkai Math. J. 12 (2001) 89–99.
[14] K. Miura, H. Yoshihara, Field theory for function fields of plane quartic curves, J. Algebra 226 (2000) 283–294.
[15] D.C. Skabelund, New maximal curves as ray class fields over Deligne–Lusztig curves, Proc. Am. Math. Soc. 146 (2018) 525–540.
[16] H. Stichtenoth, Algebraic Function Fields and Codes, Universitext, Springer-Verlag, Berlin, 1993.
[17] H. Yoshihara, Function field theory of plane curves by dual curves, J. Algebra 239 (2001) 340–355.
[18] H. Yoshihara, S. Fukasawa, List of problems, available at: http://hyoshihara.web.fc2.com/openquestion.html.