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Bricks over preprojective algebras and join-irreducible elements in Coxeter groups

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A B S T R A C T

A (semi)brick over an algebra A is a module S such that its endomorphism ring $\mathsf{End}_A(S)$ is a (product of) division algebra. For each Dynkin diagram Δ, there is a bijection from the Coxeter group W of type Δ to the set of semibricks over the preprojective algebra Π of type Δ, which is restricted to a bijection from the set of join-irreducible elements of W to the set of bricks over Π. This paper is devoted to giving an explicit description of these bijections in the case $\Delta = \mathbb{A}_n$ or $\mathbb{D}_n$. First, for each join-irreducible element $w \in W$, we describe the corresponding brick $S(w)$ in terms of "Young diagram-like" notation. Next, we determine the canonical join representation $w = \bigvee_{i=1}^m w_i$ of an arbitrary element $w \in W$ based on Reading's work, and prove that $\bigoplus_{i=1}^m S(w_i)$ is the semibrick corresponding to w.

Contents

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0. Introduction

The representation theory of *preprojective algebras* Π of Dynkin type Δ has been developed by investigating its relationship with the *Coxeter groups* $W = W(\Delta)$ associated to Δ. In particular, the ideal $I(w)$ of Π associated to each element $w \in W$ introduced by [16,10] plays an important role. For example, see [2,3,7,14,17,21,22,24].

The Coxeter group W has a partial order $\leq$ called *the right weak order*. The partially ordered set $(W, \leq)$ is a *lattice* [8] in the sense that W admits the two binary operations called the *join* $x \vee y$ and the *meet* $x \wedge y$ for any $x, y \in W$. In our study, we shall use *join-irreducible* elements in a lattice L. We write $\mathsf{j\text{-}irr}\, L$ for the set of join-irreducible elements in L.

Reading [23] introduced the important notion of canonical join representations. For a given element $x \in L$, a subset $U = \{u_1, u_2, \ldots, u_m\} \subset L$ is called the *canonical join representation* of x if U satisfies $x = \bigvee_{i=1}^{m} u_i$ and some additional minimal conditions. In this case, $U \subset \mathsf{j\text{-}irr}\, L$ holds.

Any element in a Coxeter group of Dynkin type has a unique canonical join representation, since the Coxeter group is a *semidistributive* lattice, see [15] for the detail. One of the aims of this paper is to show that the canonical join representations of the elements in the Coxeter group W are strongly related to the representation theory of Π. We will explain the details later in this section.

We will show some of our results in a more general setting. Let A be a finite-dimensional algebra over a field K. We write $\mathsf{torf}\, A$ for the set of *torsion-free classes* in the category $\mathsf{mod}\, A$ of finite-dimensional A-modules. The set $\mathsf{torf}\, A$ has a natural partial order $\subset$ defined by inclusion relations, and then, the partially ordered set $(\mathsf{torf}\, A, \subset)$ is also a lattice.

In the rest of this paper, we assume that A is τ-tilting finite, that is, $\mathsf{torf}\, A$ is a finite set. There are many bijections between $\mathsf{torf}\, A$ and many important objects in $\mathsf{mod}\, A$ or in its bounded derived category $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, A)$ [1,4,9,18,19]. In particular, we have a bijection F from the set $\mathsf{sbrick}\, A$ of *semibricks* in $\mathsf{mod}\, A$ to the set $\mathsf{torf}\, A$, where $\mathsf{F}(S)$ is defined as the minimum torsion-free class containing a semibrick S. Here, a semibrick S is defined as a module in $\mathsf{mod}\, A$ which admits a decomposition $S = \bigoplus_{i=1}^{m} S_i$ with $\mathsf{End}_A(S_i)$ a division K-algebra (that is, S_i is a *brick*) and with $\mathsf{Hom}_A(S_i, S_j) = 0$ for $i \neq j$. The sets $\mathsf{torf}\, A$ and $\mathsf{sbrick}\, A$ have bijections from the set $\mathsf{s}\tau^{-1}\text{-}\mathsf{tilt}\, A$ of *support τ^{-1}-tilting A-modules* satisfying the following commutative diagram [1,4]:

$$\begin{tikzcd}
\mathsf{s}\tau^{-1}\text{-}\mathsf{tilt}\, A \arrow[r, "\mathsf{Sub}"] \arrow[rr, bend right, "M \mapsto \mathsf{soc}_{\mathsf{End}_A(M)} M"'] & \mathsf{torf}\, A & \mathsf{sbrick}\, A \arrow[l, "\mathsf{F}"']
\end{tikzcd} .$$

Moreover, the bijection F is restricted to a bijection from the set $\mathsf{brick}\, A$ of bricks in $\mathsf{mod}\, A$ to the set $\mathsf{j\text{-}irr}(\mathsf{torf}\, A)$, and we have the following commutative diagram of bijections:

$$\begin{tikzcd}
\mathsf{i}\tau^{-1}\text{-}\mathsf{rigid}\,A \arrow[r, "\mathsf{Sub}"] \arrow[rr, bend right, "M \mapsto \mathsf{soc}_{\mathsf{End}_A(M)} M"'] & \mathsf{j}\text{-}\mathsf{irr}(\mathsf{torf}\,A) & \mathsf{brick}\,A \arrow[l, "\mathsf{F}"']
\end{tikzcd}.$$

Here, $\mathsf{i}\tau^{-1}\text{-}\mathsf{rigid}\,A$ denotes the set of indecomposable τ^{-1}-rigid modules in $\mathsf{mod}\,A$.

As the first step, we will show that the canonical join representation of a torsion-free class is given by the decomposition of the corresponding semibrick as a direct sum of bricks. This fact was independently obtained also in [6].

Theorem 0.1 *(Theorem 1.8). Let $\mathcal{F} \in \mathsf{torf}\,A$, take the unique semibrick $S \in \mathsf{sbrick}\,A$ satisfying $\mathcal{F} = \mathsf{F}(S)$, and decompose S as $\bigoplus_{i=1}^m S_i$ with $S_i \in \mathsf{brick}\,A$. Then, the representation $\mathcal{F} = \bigvee_{i=1}^m \mathsf{F}(S_i)$ is the canonical join representation.*

For the preprojective algebra Π, Mizuno [20] proved that the two lattices $(W, \le)$ and $(\mathsf{torf}\,\Pi, \subset)$ are isomorphic by the correspondence $w \mapsto \mathsf{Sub}(\Pi/I(w))$ and that $\Pi/I(w)$ is a support τ^{-1}-tilting Π-module. Therefore, we obtain a bijection $S(?)\colon W \to \mathsf{sbrick}\,\Pi$ given by $S(w) := \mathsf{soc}_{\mathsf{End}_\Pi(\Pi/I(w))}(\Pi/I(w))$. The main aim of this paper is to describe the semibrick $S(w)$ for each element $w \in W$ as a quiver representation in the case $\Delta = \mathbb{A}_n$ or $\mathbb{D}_n$:

$$\mathbb{A}_n\colon\ 1 \text{ --- } 2 \text{ --- } 3 \text{ --- } \cdots \text{ --- } n\ ,$$

$$\mathbb{D}_n\colon\ \begin{tikzcd}[column sep=small, row sep=tiny] 1 \arrow[dr, dash] & & & & \\ & 2 \arrow[r, dash] & 3 \arrow[r, dash] & \cdots \arrow[r, dash] & n-1 \\ -1 \arrow[ur, dash] & & & & \end{tikzcd}\ .$$

If $\Delta = \mathbb{A}_n$, then W is the symmetric group $\mathfrak{S}_{n+1}$, and if $\Delta = \mathbb{D}_n$, then W is the subgroup of the automorphism group of the set $\{\pm 1, \pm 2, \dots, \pm n\}$ consisting of all elements w such that $w(-i) = -w(i)$ holds for each i and that the number $\#\{i > 0 \mid w(i) < 0\}$ is even. In either case, we can express every $w \in W$ in the form $(w(1), w(2), \dots, w(m))$, and our description of the semibrick $S(w)$ is constructed from this expression.

Mizuno's isomorphism $W \to \mathsf{torf}\,\Pi$ of lattices is restricted to a bijection $\mathsf{j}\text{-}\mathsf{irr}\,W \to \mathsf{j}\text{-}\mathsf{irr}(\mathsf{torf}\,\Pi)$ between the join-irreducible elements, so we also obtain a restricted bijection $S(?)\colon \mathsf{j}\text{-}\mathsf{irr}\,W \to \mathsf{brick}\,\Pi$. By [15] (types $\mathbb{A}_n$ and $\mathbb{D}_n$) and [11] (type $\mathbb{E}_n$, with computer-assisted calculation), the cardinality of each set is

$$\begin{cases} 2^{n+1} - n - 2 & (\Delta = \mathbb{A}_n) \\ 3^n - n \cdot 2^{n-1} - n - 1 & (\Delta = \mathbb{D}_n) \\ 1272 & (\Delta = \mathbb{E}_6) \\ 17635 & (\Delta = \mathbb{E}_7) \\ 881752 & (\Delta = \mathbb{E}_8) \end{cases}.$$

Moreover, we obtain the following property immediately from Theorem 0.1.

Corollary 0.2 *(Corollary 2.3). Let $w \in W$, and take $w_1, w_2, \dots, w_m \in \mathsf{j}\text{-}\mathsf{irr}\,W$ such that $S(w) = \bigoplus_{i=1}^m S(w_i)$. Then, $w = \bigvee_{i=1}^m w_i$ holds, and it is the canonical join representation of w in W.*

In this paper, we will give a description of the semibrick $S(w)$ by the following two steps:

(a) we find the canonical join representation $\bigvee_{i=1}^m w_i$ of w; and
(b) we explicitly describe the brick $S(w_i)$ for each $w_i \in \mathsf{j}\text{-}\mathsf{irr}\,W$.

There is a combinatorial “Young diagram-like” description by Iyama–Reading–Reiten–Thomas [15] of the module $J(w) := (\Pi/I(w))e_l$ for $w \in \mathsf{j\text{-}irr}\, W$ in the case Δ is $\mathbb{A}_n$ or $\mathbb{D}_n$, where l is the unique *descent* of $w \in \mathsf{j\text{-}irr}\, W$ and e_l is the primitive idempotent of Π corresponding to l. The module $J(w)$ is an indecomposable direct summand of $\Pi/I(w) \in \mathsf{s}\tau^{-1}\text{-}\mathsf{tilt}\, A$ satisfying $\mathsf{Sub}\, J(w) = \mathsf{Sub}(\Pi/I(w))$, so $S(w) = \mathsf{soc}_{\mathsf{End}_\Pi(J(w))} J(w)$ follows.

For example, we consider the two elements $w = (2, 5, 8, 1, 3, 4, 6, 7, 9) \in W(\mathbb{A}_8)$ and $w' = (6, 9, -7, -4, 1, 2, 3, 5, 8) \in W(\mathbb{D}_9)$. Then, the modules $J(w)$, $J(w')$, $S(w)$, and $S(w')$ are expressed by the following figures:

$$J(w) = \begin{array}{ccc} 3 & 2 & 1 \\ 4 & 3 & \\ 5 & 4 & \\ 6 & & \\ 7 & & \end{array}, \qquad S(w) = \begin{array}{ccc} & 2 & 1 \\ & 3 & \\ 5 & 4 & \\ 6 & & \\ 7 & & \end{array},$$

$$J(w') = \begin{array}{ccccccc} 2 & {}^{-1}_{\ 1} & -2 & -3 & \color{blue}{-4} & \color{blue}{-5} & -6 \\ 3 & 2 & {}^{\ 1}_{-1} & \color{blue}{-2} & \color{blue}{-3} & & \\ 4 & 3 & \color{blue}{2} & {}_{1} & & & \\ 5 & \color{blue}{4} & \color{blue}{3} & 2 & & & \\ 6 & \color{blue}{5} & 4 & 3 & & & \\ 7 & 6 & 5 & & & & \\ 8 & & & & & & \end{array}, \qquad S(w') = \begin{array}{ccccccc} & & & & \color{blue}{(4)} & \color{blue}{(5)} & -6 \\ & & {}_{-1} & \color{blue}{(2)} & \color{blue}{(3)} & & \\ & & \color{blue}{(2)} & {}_{1} & & & \\ & \color{blue}{(4)} & \color{blue}{(3)} & 2 & & & \\ & \color{blue}{(5)} & 4 & 3 & & & \\ 7 & 6 & 5 & & & & \\ 8 & & & & & & \end{array}$$

Here, for each module M above, each square $\boxed{i}$ in the figure for M denotes a one-dimensional subspace of $e_i M$ if $i \geq -1$; and of $e_{|i|} M$ if $i \leq -2$. As a K-vector space, M is the direct sum of these one-dimensional subspaces. In the figure for $S(w')$, for each $i = 2, 3, 4, 5$, the two squares $\boxed{(i)}$ together denote a certain one-dimensional subspace of the two-dimensional vector space corresponding to the two squares $\boxed{i}$ and $\boxed{-i}$ in the figure for $J(w')$.

We can check that $w \in W(\mathbb{A}_n)$ consists of two strictly increasing sequences, and that the right-most entry of each row in the figure for $J(w)$ appears in the latter increasing sequence. Similarly, $w' \in W(\mathbb{D}_n)$ also consists of two strictly increasing sequences. If i is the right-most entry of some row in the figure for $J(w')$, then i appears in the latter increasing sequence if $i \geq -1$, and $i - 1$ appears there if $i \leq -2$.

The bricks $S(w)$ and $S(w')$ can be expressed more simply by using quiver representations as follows, where the symbol $-i$ ($i = 2, 3, 4, 5$) in the quiver representation of $S(w')$ below corresponds to the one-dimensional vector subspace denoted by the two squares $\boxed{(i)}$ in the figure for $S(w')$ above:

$$S(w) = 1 \leftarrow 2 \rightarrow 3 \rightarrow 4 \leftarrow 5 \rightarrow 6 \rightarrow 7,$$

$$S(w') = \begin{tikzcd} -1 \arrow[r] & \color{blue}{-2} \arrow[r] \arrow[dl] & \color{blue}{-3} \arrow[dl] \arrow[dr] & \color{blue}{-4} \arrow[l] \arrow[r] & \color{blue}{-5} \arrow[r] \arrow[dl] \arrow[dr] & -6 & & \\ 1 \arrow[r] & 2 \arrow[r] & 3 & 4 \arrow[l] \arrow[r] & 5 & 6 \arrow[l] & 7 \arrow[l] \arrow[r] & 8 \end{tikzcd}.$$

In this paper, we give a combinatorial algorithm to obtain a quiver representation of the brick $S(w)$ for each $w \in \mathsf{j\text{-}irr}\, W$ in the case $\Delta = \mathbb{A}_n$ or $\mathbb{D}_n$; then, the step (b) is done.

If $\Delta = \mathbb{A}_n$, then we have obtained the following result. Here, $[x, y]$ denotes $\{i \in \mathbb{Z} \mid x \leq i \leq y\}$ for $x, y \in \mathbb{Z}$.

Theorem 0.3 *(Theorem 3.1, Corollary 3.3). Let $w \in \mathsf{j\text{-}irr}\, W(\mathbb{A}_n)$ with its unique descent l. Then, the brick $S(w)$ is given as follows.*

- *Set $R := w([l+1, n+1])$, $a := w(l)$, $b := w(l+1)$, and $V := [b, a-1]$.*
- *The brick $S(w)$ has a K-basis $(\langle i \rangle)_{i \in V}$, where $\langle i \rangle$ belongs to $e_i S(w)$.*
- *For each $i \in V$, place a symbol i denoting the K-vector subspace $K\langle i \rangle$.*
- *For each $i \in V \setminus \{\max V\}$, we write exactly one arrow between i and $i+1$, where its orientation is $i \to i+1$ if $i+1 \in R$ and $i \leftarrow i+1$ if $i+1 \notin R$.*

If $\Delta = \mathbb{D}_n$, then the bricks are obtained from the following procedure.

Theorem 0.4 *(Theorem 3.7, Corollary 3.10). Let $w \in \mathsf{j\text{-}irr}\, W(\mathbb{D}_n)$ with its unique descent l. Then, the brick $S(w)$ is given as follows.*

- *Set $R := w([|l|+1, n])$, $a := w(l)$, $b := w(|l|+1)$, and*

$$r := \max\{k \geq 0 \mid [1,k] \subset \pm R\}, \quad c := \begin{cases} w(|w^{-1}(1)|) & (r \geq 1) \\ 1 & (r = 0) \end{cases},$$

$$(V_-, V_+) := \begin{cases} (\emptyset, [b, a-1]) & (b \geq 2) \\ (\emptyset, \{c\} \cup [2, a-1]) & (b = \pm 1) \\ ([b+1, -2] \cup \{-c\}, \{c\} \cup [2, a-1]) & (b \leq -2) \end{cases},$$

$$V := V_+ \amalg V_-.$$

- *The brick $S(w)$ has a K-basis $(\langle i \rangle)_{i \in V}$, where $\langle i \rangle$ belongs to $e_i S(w)$ if $i \geq -1$, and $e_{|i|} S(w)$ if $i \leq -2$.*
- *For each $i \in V$, place a symbol i denoting the K-vector subspace $K\langle i \rangle$.*
- *We write the following arrows.*
  - (i) *For each $i \in V_+ \setminus \{\max V_+\}$, draw an arrow $i \to |i|+1$ if $|i|+1 \in R$; and $i \leftarrow |i|+1$ otherwise.*
  - (ii) *For each $i \in V_- \setminus \{\min V_-\}$, draw an arrow $i \leftarrow -(|i|+1)$ if $-(|i|+1) \in R$; and $i \to -(|i|+1)$ otherwise.*
  - (iii) *If $r \geq 1$, for each $i \in V_-$ with $|i| \leq r$, draw an arrow $-i \leftarrow -(|i|+1)$ if $|i|+1 \in R$; and $i \to |i|+1$ otherwise.*
  - (iv) *If $r = 0$, draw an arrow $-c \leftarrow 2$ if $c \leftarrow 2$ exists in* (i)*, and draw an arrow $c \to -2$ if $-c \to -2$ exists in* (ii)*.*

Consequently, we obtain that any brick over the preprojective algebra of type $\mathbb{A}_n$ is a module over some path algebra of type $\mathbb{A}_n$. On the other hand, the preprojective algebra of type $\mathbb{D}_n$ does not have the corresponding property.

Finally, we consider an arbitrary element $w \in W$. In Propositions 4.4 and 4.8, we will explicitly determine the canonical join representation $\bigvee_{i=1}^m w_i$ of $w \in W$ by using the characterization of canonical join representations in the Coxeter groups of Dynkin type given by Reading [23]. Then, the step (a) is done, and in Theorems 4.6 and 4.10, we explicitly write down the semibrick $S(w) = \bigoplus_{i=1}^m S(w_i)$ by using the description of bricks.

For example, let $\Delta := \mathbb{A}_8$ and $w = (4, 9, 3, 6, 2, 8, 5, 1, 7)$. Then, its canonical join representation is $w_2 \vee w_4 \vee w_6 \vee w_7$, where

$$w_2 := (1, 2, 4, 9, 3, 5, 6, 7, 8), \qquad w_4 := (1, 3, 4, 6, 2, 5, 7, 8, 9),$$

$$w_6 := (1,2,3,4,6,8,5,7,9), \qquad w_7 := (2,3,4,5,1,6,7,8,9).$$

Thus, the semibrick $S(w)$ is the direct sum of the following bricks:

$$\begin{array}{rl} S(w_2) = & \phantom{1 \leftarrow 2 \leftarrow {}} 3 \leftarrow 4 \to 5 \to 6 \to 7 \to 8, \\ S(w_4) = & \phantom{1 \leftarrow {}} 2 \leftarrow 3 \leftarrow 4 \to 5 \phantom{{} \to 6 \to 7 \to 8}, \\ S(w_6) = & \phantom{1 \leftarrow 2 \leftarrow 3 \leftarrow 4 \to {}} 5 \leftarrow 6 \to 7 \phantom{{} \to 8}, \\ S(w_7) = & 1 \leftarrow 2 \leftarrow 3 \leftarrow 4 \phantom{{} \to 5 \to 6 \to 7 \to 8}. \end{array}$$

0.1. Notation

The composition of two maps $f\colon X \to Y$ and $g\colon Y \to Z$ is denoted by gf.

We define the multiplication on the automorphism group of a finite set X by $(\sigma\tau)(i) := \sigma(\tau(i))$ for $i \in X$. For $a, b \in X$, the notation $(a \quad b)$ means the transposition which exchanges a and b.

For integers $a, b \in \mathbb{Z}$, we define $[a, b] := \{i \in \mathbb{Z} \mid a \le i \le b\}$. For a set $X \subset \mathbb{Z}$, we set $-X := \{-i \mid i \in X\}$ and $\pm X := X \cup (-X)$.

Throughout this paper, K is a field and A is a finite-dimensional K-algebra. Unless otherwise stated, A-modules are finite-dimensional left A-modules, and we write $\mathsf{mod}\, A$ for the category of finite-dimensional left A-modules. Let $M \in \mathsf{mod}\, A$, and decompose M as $M \cong \bigoplus_{i=1}^{m} M_i^{\oplus l_i}$ with $M_i \not\cong M_j$ for $i \ne j$ and with $l_i \ge 1$ for each i. Then, we define the number $|M| := m$, and we say that M is *basic* if $l_i = 1$ for any i. We set the multiplication on the endomorphism algebra $\mathsf{End}_A(M)$ as $g \cdot f := gf$. Thus, M is also a left $\mathsf{End}_A(M)$-module by $fx := f(x)$ for $f \in \mathsf{End}_A(M)$ and $x \in M$.

For a quiver Q, the composition of the two arrows $\alpha\colon i \to j$ and $\beta\colon j \to k$ in Q is denoted by $\alpha\beta$, which is a path from i to k.

1. General observations of τ-tilting finite algebras

In this section, we observe some general properties holding for τ-tilting finite algebras A over a field K.

1.1. Lattices

First, we recall the notion of lattices.

Definition 1.1. Let $(L, \le)$ be a partially ordered set.

(1) For $x, y, z \in L$, the element z is called the *meet* of x and y if z is the maximum element satisfying $z \le x$ and $z \le y$. In this case, z is denoted by $x \wedge y$.
(2) For $x, y, z \in L$, the element z is called the *join* of x and y if z is the minimum element satisfying $z \ge x$ and $z \ge y$. In this case, z is denoted by $x \vee y$.
(3) The set L is called a *lattice* if L admits the meet $x \wedge y$ and the join $x \vee y$ for any $x, y \in L$.
(4) The set L is called a *finite lattice* if L is a finite set and a lattice.

The operations join and meet clearly satisfy the associative relations, so we may use the expressions $x \wedge y \wedge z$ and $x \vee y \vee z$. If $L \ne \emptyset$ is a finite lattice, there exist the maximum element $\mathsf{max}\, L$ and the minimum element $\mathsf{min}\, L$. In this case, we define $\bigwedge_{x \in \emptyset} x := \mathsf{max}\, L$ and $\bigvee_{x \in \emptyset} x := \mathsf{min}\, L$.

Later in this paper, we will consider the decomposition of an element in a lattice with respect to the operation join, so we recall the notion of join-irreducible elements.

Definition 1.2. Let L be a lattice. An element $x \in L$ is called a *join-irreducible* element if the following conditions hold:

- x is not the minimum element $\min L$; and
- for any $y, z \in L$, if $x = y \vee z$, then $y = x$ or $z = x$.

We write $\mathsf{j\text{-}irr}\, L$ for the set of join-irreducible elements in W.

We remark that $x \in \mathsf{j\text{-}irr}\, L$ is equivalent to that there exists a unique maximal element of the set $\{y \in L \mid y < x\}$ if L is a finite lattice. This fails if we drop the assumption that L is finite [6, Remark 3.1.2].

1.2. Torsion-free classes

Let A be a finite-dimensional algebra.

A full subcategory $\mathcal{F}$ of $\mathsf{mod}\, A$ is called a *torsion-free class* in $\mathsf{mod}\, A$ if $\mathcal{F}$ is closed under submodules and extensions, and we write $\mathsf{torf}\, A$ for the set of torsion-free classes in $\mathsf{mod}\, A$. For a full subcategory $\mathcal{C} \subset \mathsf{mod}\, A$, we define

- $\mathsf{add}\, \mathcal{C}$ as the subcategory of $\mathsf{mod}\, A$ consisting of $M \in \mathsf{mod}\, A$ which is a direct summand of $\bigoplus_{i=1}^{s} C_i$ for some $C_1, C_2, \dots, C_s \in \mathcal{C}$;
- $\mathsf{Filt}\, \mathcal{C}$ as the subcategory of $\mathsf{mod}\, A$ consisting of $M \in \mathsf{mod}\, A$ which has a sequence $0 = M_0 \subset M_1 \subset \cdots \subset M_l = M$ with $M_i/M_{i-1} \in \mathsf{add}\, \mathcal{C}$;
- $\mathsf{Sub}\, \mathcal{C}$ as the subcategory of $\mathsf{mod}\, A$ consisting of $M \in \mathsf{mod}\, A$ which is a submodule of some object in $\mathsf{add}\, \mathcal{C}$; and
- $\mathsf{F}(\mathcal{C})$ as $\mathsf{Filt}(\mathsf{Sub}\, \mathcal{C})$.

Then, $\mathsf{F}(\mathcal{C})$ is the smallest torsion-free class containing $\mathcal{C}$, see [19, Lemma 3.1].

The set $\mathsf{torf}\, A$ has a natural partial order defined by inclusions, and then, the partially ordered set $(\mathsf{torf}\, A, \subset)$ is a lattice with $\mathcal{F}_1 \wedge \mathcal{F}_2 = \mathcal{F}_1 \cap \mathcal{F}_2$ and $\mathcal{F}_1 \vee \mathcal{F}_2 = \mathsf{F}(\mathcal{F}_1 \cup \mathcal{F}_2)$. The notion of *torsion classes* is dually defined.

In general, a torsion-free class in $\mathsf{mod}\, A$ is not functorially finite in $\mathsf{mod}\, A$. Demonet–Iyama–Jasso [12] introduced the notion of *τ-tilting finiteness*, which is equivalent to that $\mathsf{torf}\, A$ is a finite set. In their paper, they proved that A is τ-tilting finite if and only if every torsion-free class is functorially finite. In the rest, A is assumed to be τ-tilting finite.

Functorially finite torsion-free classes are strongly connected with support τ^{-1}-tilting A-modules, which were introduced by Adachi–Iyama–Reiten [1].

Let $M \in \mathsf{mod}\, A$ and I be an injective A-module in $\mathsf{mod}\, A$. Then, M is called a *τ^{-1}-rigid module* if $\mathsf{Hom}_A(\tau^{-1}M, M) = 0$, and the pair (M, I) is called a *τ^{-1}-rigid pair* if M is τ^{-1}-rigid and $\mathsf{Hom}_A(M, I) = 0$. If a τ^{-1}-rigid pair (M, I) satisfies $|M| + |I| = |A|$, the pair (M, I) is called a *support τ^{-1}-tilting pair*, and an A-module M is called a *support τ^{-1}-tilting module* if there exists some injective module I such that (M, I) is a support τ-tilting pair. We write $\mathsf{s}\tau^{-1}\text{-}\mathsf{tilt}\, A$ for the set of basic support τ^{-1}-tilting modules in $\mathsf{mod}\, A$. The notion of *support τ-tilting modules* is dually defined.

If M is τ^{-1}-rigid, then the full subcategory $\mathsf{Sub}\, M$ is a torsion-free class [5, Theorem 5.10]. Based on this, Adachi–Iyama–Reiten obtained the following bijection.

Proposition 1.3. *[1, Theorem 2.7] The correspondence* $\mathsf{s}\tau^{-1}\text{-}\mathsf{tilt}\, A \ni M \mapsto \mathsf{Sub}\, M \in \mathsf{torf}\, A$ *is a bijection.*

Thus, we can induce a partial order $\leq$ on the set $\mathsf{s}\tau^{-1}\text{-}\mathsf{tilt}\,A$ from inclusion relations on $\mathsf{torf}\,A$; namely, $M \leq N$ holds if and only if $\mathsf{Sub}\,M \subset \mathsf{Sub}\,N$. Then, $(\mathsf{s}\tau^{-1}\text{-}\mathsf{tilt}\,A, \leq)$ is clearly a lattice.

1.3. Semibricks

We assume that A is τ-tilting finite as in the previous subsection. We define the notions of bricks and semibricks as follows.

Definition 1.4. Let S be an A-module.

(1) The module S is called a *brick* if the endomorphism ring $\mathsf{End}_A(S)$ is a division ring. We write $\mathsf{brick}\,A$ for the set of bricks.
(2) The module S is called a *semibrick* if S is decomposed as the direct sum $\bigoplus_{i=1}^m S_i$ of bricks $S_1, S_2, \dots, S_m \in \mathsf{brick}\,A$ satisfying $\mathsf{Hom}_A(S_i, S_j) = 0$ if $i \neq j$. We write $\mathsf{sbrick}\,A$ for the set of semibricks in $\mathsf{mod}\,A$.

The notion of semibricks is originally defined as sets of Hom-orthogonal bricks in [4], but it does not matter here, since A is assumed to be τ-tilting finite [4, Corollary 2.10]. Then, [4, Proposition 2.9] tells us that there is a bijection $\mathsf{F}\colon \mathsf{sbrick}\,A \to \mathsf{torf}\,A$ taking the minimum torsion-free class $\mathsf{F}(S)$ containing each semibrick S. Moreover, it satisfies the property below.

Proposition 1.5. *[4, Proposition 2.9] We have the following commutative diagram of bijections:*

$$\begin{tikzcd} \mathsf{s}\tau^{-1}\text{-}\mathsf{tilt}\,A \arrow[r, "\mathsf{Sub}"] \arrow[rr, bend right, "M \mapsto \mathsf{soc}_{\mathsf{End}_A(M)} M"'] & \mathsf{torf}\,A & \mathsf{sbrick}\,A \arrow[l, "\mathsf{F}"'] \end{tikzcd}.$$

Now, we set $\mathsf{i}\tau^{-1}\text{-}\mathsf{rigid}\,A$ as the set of indecomposable τ^{-1}-rigid A-modules in $\mathsf{mod}\,A$. Then, we also have another commutative diagram.

Proposition 1.6. *We have the following commutative diagram of bijections:*

$$\begin{tikzcd} \mathsf{i}\tau^{-1}\text{-}\mathsf{rigid}\,A \arrow[r, "\mathsf{Sub}"] \arrow[rr, bend right, "M \mapsto \mathsf{soc}_{\mathsf{End}_A(M)} M"'] & \mathsf{j}\text{-}\mathsf{irr}(\mathsf{torf}\,A) & \mathsf{brick}\,A \arrow[l, "\mathsf{F}"'] \end{tikzcd}.$$

Proof. The map $\mathsf{Sub}\colon \mathsf{i}\tau^{-1}\text{-}\mathsf{rigid}\,A \to \mathsf{j}\text{-}\mathsf{irr}(\mathsf{torf}\,A)$ is bijective by [15, Theorem 2.7]. On the other hand, it follows from [12, Theorem 4.2, Lemma 4.3] that the map $\mathsf{i}\tau^{-1}\text{-}\mathsf{rigid}\,A \ni M \mapsto \mathsf{soc}_{\mathsf{End}_A(M)} M \in \mathsf{brick}\,A$ is a bijection satisfying $\mathsf{F}(\mathsf{soc}_{\mathsf{End}_A(M)} M) = \mathsf{Sub}\,M$. Thus, we have the desired commutative diagram of bijections. $\square$

1.4. Canonical join representations

Now that the bijection $\mathsf{F}\colon \mathsf{sbrick}\,A \to \mathsf{torf}\,A$ is restricted to a bijection $\mathsf{brick}\,A \to \mathsf{j}\text{-}\mathsf{irr}(\mathsf{torf}\,A)$, the following natural question occurs:

Let $\mathcal{F} \in \mathsf{torf}\, A$, take the unique semibrick $S \in \mathsf{sbrick}\, A$ satisfying $\mathcal{F} = \mathsf{F}(S)$, and decompose S as $\bigoplus_{i=1}^m S_i$ with $S_i \in \mathsf{brick}\, A$. Then, what is the relationship between $\mathsf{F}(S) \in \mathsf{torf}\, A$ and $\mathsf{F}(S_1), \mathsf{F}(S_2), \dots, \mathsf{F}(S_m) \in \mathsf{j\text{-}irr}(\mathsf{torf}\, A)$?

Clearly, $\mathsf{F}(S) = \bigvee_{i=1}^m \mathsf{F}(S_i)$ holds, since $\mathsf{F}(S)$ is the minimum torsion-free class containing all $\mathsf{F}(S_i)$. Actually, this will turn out to be a canonical join representation. Here, the notion of canonical join representations was introduced by Reading [23], and defined as follows.

Definition 1.7. Let L be a finite lattice, $x \in L$, and $U \subset L$. Then, we say that U is a *canonical join representation* if

(a) $x = \bigvee_{u \in U} u$ holds;
(b) for any proper subset $U' \subsetneq U$, the join $\bigvee_{u \in U'} u$ never coincides with x; and
(c) if $V \subset L$ satisfies the properties (a) and (b), then, for every $u \in U$, there exists $v \in V$ such that $u \leq v$.

In this case, we also say $x = \bigvee_{u \in U} u$ is a canonical join representation.

If $x \in L$ has a canonical join representation U, then we can easily check that it is the unique canonical join representation for $x \in L$. Namely, let V be another canonical join representation of x. Then, for any $u \in U$, we can take $v \in V$ satisfying $u \leq v$ by the property (c) of U, and there exists $u' \in U$ satisfying $v \leq u'$ by the property (c) of V. Thus $u \leq v \leq u'$, which implies $u = u'$ by the property (b) of U; hence $u = v \in V$. Therefore, $U \subset V$ holds, so $U = V$ by the property (b) of V. It is also easy to see that U is a subset of $\mathsf{j\text{-}irr}\, L$.

The existence of a canonical join representation of each element is not guaranteed for a general finite lattice. In the case that $L = \mathsf{torf}\, A$, every $\mathcal{F} \in \mathsf{torf}\, A$ has a canonical join representation given by the indecomposable decomposition of semibricks.

Theorem 1.8. *Let $\mathcal{F} \in \mathsf{torf}\, A$, take the unique semibrick $S \in \mathsf{sbrick}\, A$ satisfying $\mathcal{F} = \mathsf{F}(S)$, and decompose S as $\bigoplus_{i=1}^m S_i$ with $S_i \in \mathsf{brick}\, A$. Then, the representation $\mathcal{F} = \bigvee_{i=1}^m \mathsf{F}(S_i)$ is the canonical join representation.*

Proof. We have seen the property (a): $\mathcal{F} = \mathsf{F}(S) = \bigvee_{i=1}^m \mathsf{F}(S_i)$.

We show the property (b). Let I be a proper subset of $[1, m]$. Take $j \in [1, m] \setminus I$. Then, the brick S_j cannot belong to $\mathsf{F}(\{S_i\}_{i \in I}) = \mathsf{F}(\bigcup_{i \in I} \mathsf{F}(S_i)) = \bigvee_{i \in I} \mathsf{F}(S_i)$, since $\mathsf{Hom}_A(S_j, S_i) = 0$ holds for each $i \in I$. This implies that $\mathsf{F}(S) \neq \bigvee_{i \in I} \mathsf{F}(S_i)$.

It remains to show the property (c). Let $\mathcal{F}_1, \dots, \mathcal{F}_{m'} \in \mathsf{torf}\, A$ satisfy the properties (a) and (b). For each $i \in [1, m]$, the brick S_i belongs to $\mathsf{F}(S) = \mathcal{F}$, which coincides with $\bigvee_{j=1}^{m'} \mathcal{F}_j = \mathsf{F}(\bigcup_{j=1}^{m'} \mathcal{F}_j)$. Thus, there must exist some $j \in [1, m']$ such that $\mathsf{Hom}_A(S_i, \mathcal{F}_j) \neq 0$. We take a semibrick S' such that $\mathcal{F}_j = \mathsf{F}(S')$ by Proposition 1.5, then there exists a nonzero homomorphism $f \colon S_i \to S'$. By [4, Lemma 2.7], f is injective, since $S_i, S' \in \mathcal{F} = \mathsf{F}(S)$ and S_i is a direct summand of S. This implies that $\mathsf{F}(S_i) \subset \mathsf{F}(S') = \mathcal{F}_j$. $\square$

In particular, the partially ordered set $\mathsf{torf}\, A$ admits a canonical join representation for any $\mathcal{F} \in \mathsf{torf}\, A$.

The notion of canonical join representations is defined in a fully combinatorial way, but decomposing semibricks into direct sums of bricks is a purely representation-theoretic problem. These two are related by Theorem 1.8.

The relationship between semibricks and torsion classes is independently discussed by Barnard–Carroll–Zhu [6] and Demonet–Iyama–Reiten–Reading–Thomas [13] in the setting that the algebra A is not necessarily τ-tilting finite. In particular, our Theorem 1.8 is generalized in [6, Proposition 3.2.5].

2. Preliminaries for preprojective algebras

In this section, we recall some properties on Coxeter groups and preprojective algebras of Dynkin type.

2.1. Coxeter groups

Coxeter groups of Dynkin type are strongly related to the corresponding preprojective algebras. In this subsection, we state the definition of Coxeter groups of Dynkin type, and prepare some basic terms on the combinatorics of Coxeter groups. For more information, see [8].

Let Δ be a Dynkin diagram whose vertices set is Δ_0. Then, the *Coxeter group* W for Δ is the group defined by the generators $\{s_i \mid i \in \Delta_0\}$ and the relations

- $s_i^2 = 1$ for each i;
- $s_i s_j = s_j s_i$ if there is no edge between i and j in Δ; and
- $s_i s_j s_i = s_j s_i s_j$ if there is exactly one edge between i and j in Δ.

It is well-known that the Coxeter group W associated to a Dynkin diagram Δ is a finite group.

Each element $w \in W$ has the minimum number l such that w can be written as a product $s_{i_1} s_{i_2} \cdots s_{i_l}$ of l generators. Such number is called the *length* of w, and is denoted by $l(w)$. If $l = l(w)$ and $w = s_{i_1} s_{i_2} \cdots s_{i_l}$, then $s_{i_1} s_{i_2} \cdots s_{i_l}$ is called a *reduced expression* of w, which is not necessarily unique.

If an element $w \in W$ has the maximum length among the elements of W, then w is called a *longest element* of W. Actually, such an element uniquely exists, and it is often denoted by w_0.

We can consider several partial orders on the Coxeter group W, but in this paper, we only use the *right weak order*: for $w, w' \in W$, the inequality $w \le w'$ holds if and only if $l(w') = l(w) + l(w^{-1}w')$. Then, the poset $(W, \le)$ is a lattice.

We write $\mathsf{j\text{-}irr}\, W$ for the set of join-irreducible elements of the partially ordered set $(W, \le)$. For $w \in W$, the maximal elements of the set $\{w' \in W \mid w' < w\}$ are ws_i for all $i \in \Delta_0$ satisfying $l(w) > l(ws_i)$. Therefore, $w \in W$ is join-irreducible if and only if there uniquely exists $i \in \Delta_0$ such that $l(w) > l(ws_i)$. In this case, we say that w is a join-irreducible element of *type* i.

When we consider the right weak order of the Coxeter group, the notion of inversions is useful. We call an element $t \in W$ a *reflection* of W if there exist some $w \in W$ and $i \in \Delta_0$ satisfying $t = ws_iw^{-1}$. Fix $w \in W$, then a reflection t of W is called an *inversion* of w if $l(tw) < l(w)$, and the set of inversions of w is denoted by $\mathsf{inv}(w)$. It is well-known that, for two elements $w, w' \in W$, the inequality $w \le w'$ holds if and only if $\mathsf{inv}(w) \subset \mathsf{inv}(w')$.

2.2. Bijections

Now that the preparation on Coxeter groups of Dynkin type is done, let us see how they are related to the corresponding preprojective algebras.

We quickly recall the definition of preprojective algebras of Dynkin type. Let Δ be a Dynkin diagram. We define the *double quiver* Q for Δ, that is, the set Q_0 of vertices of Q is Δ_0, and the set Q_1 of arrows of Q consists of $i \to j$ and $j \to i$ for each edge between i and j of Δ. For each arrow $\alpha \colon i \to j$ in Q_1, we write α^* for the reversed arrow $j \to i$. There is a subset $Q'_1 \subset Q_1$ such that, for each $\alpha \in Q_1$, the condition $\alpha \in Q'_1$ holds if and only if $\alpha^* \notin Q'_1$. Then, the *preprojective algebra* Π corresponding to Δ is given by $KQ/\langle \sum_{\alpha \in Q'_1} (\alpha\alpha^* - \alpha^*\alpha) \rangle$. Here, the choice of the subset Q'_1 is not unique in general, but Π is uniquely defined up to isomorphisms, since Δ is Dynkin. It is well-known that Π is self-injective. For each vertex $i \in Q_0$, we write e_i for the idempotent of Π corresponding to the vertex i.

Let Π be the preprojective algebra of Dynkin type Δ, and set $I_i := \Pi(1-e_i)\Pi$, which is a maximal ideal of Π. We write $\langle I_i \mid i \in \Delta_0\rangle$ for the set of ideals of the form $I_{i_1}I_{i_2}\cdots I_{i_k}$.

There is an important ideal $I(w)$ of Π associated to each element w of the Coxeter group W for Δ. The ideal $I(w)$ is defined as follows: take a reduced expression of $w = s_{i_1}s_{i_2}\cdots s_{i_k}$ and set $I(w) := I_{i_1}I_{i_2}\cdots I_{i_k}$. Clearly, $I(w)$ belongs to the set $\langle I_i \mid i \in \Delta_0\rangle$.

By [20, Theorem 2.14], $I(w)$ does not depend on the choice of a reduced expression of w, and the well-defined correspondence $w \mapsto I(w)$ gives a bijection $W \to \langle I_i \mid i \in \Delta_0\rangle$. We remark that a similar bijection exists for a preprojective algebra of non-Dynkin type, see [10, Theorem III.1.9].

Moreover, Mizuno proved the set $\langle I_i \mid i \in \Delta_0\rangle$ coincides with the set $\mathsf{s\tau\text{-}tilt}\,\Pi$ of support τ-tilting Π-modules. He also proved that the bijection $W \ni w \mapsto I(w) \in \mathsf{s\tau\text{-}tilt}\,\Pi$ is an isomorphism $(W,\le) \to (\mathsf{s\tau\text{-}tilt}\,\Pi,\ge)$ of lattices [20, Theorem 2.30].

In our convention, we need the dual version of this isomorphism. The torsion-free class corresponding to the torsion class $\mathsf{Fac}\,I(w)$ is $\mathsf{Sub}(\Pi/I(w))$, and it follows from Mizuno's isomorphism and [22, Proposition 6.4] that the module $\Pi/I(w)$ is a support τ^{-1}-tilting module. Thus, we obtain the following isomorphism of lattices.

Proposition 2.1. *There exists an isomorphism $(W,\le) \to (\mathsf{s\tau^{-1}\text{-}tilt}\,\Pi,\le)$ of lattices given by $w \mapsto \Pi/I(w)$.*

In this map, the longest element $w_0 \in W$ corresponds to the injective cogenerator Π, and the identity element id_W corresponds to 0.

Since the Coxeter group W for the Dynkin diagram Δ is a finite group, Π is τ-tilting finite. Therefore, we obtain the following bijections from Propositions 1.5, 1.6, and 2.1.

Proposition 2.2. *There exists a bijection $S(?)\colon W \to \mathsf{sbrick}\,\Pi$ defined by the formula $S(w) := \mathsf{soc}_{\mathsf{End}(\Pi/I(w))}(\Pi/I(w))$. As a restriction, we have another bijection $S(?)\colon \mathsf{j\text{-}irr}\,W \to \mathsf{brick}\,\Pi$.*

The aim of this paper is to describe the semibrick $S(w)$ for each $w \in W$ explicitly.

Since the partially ordered sets $(W,\le)$ and $(\mathsf{torf}\,A,\subset)$ are isomorphic, we obtain the following property immediately from Theorem 1.8.

Corollary 2.3. *Let $w \in W$, and take $w_1, w_2, \dots, w_m \in \mathsf{j\text{-}irr}\,W$ such that $S(w) = \bigoplus_{i=1}^m S(w_i)$. Then, $w = \bigvee_{i=1}^m w_i$ holds, and it is the canonical join representation of w in W.*

We will explicitly determine the canonical join representation for each $w \in W$ in Section 4. It is a purely combinatorial problem.

Then, the remaining task is to describe the brick $S(w)$ for each join-irreducible element $w \in \mathsf{j\text{-}irr}\,W$. For this purpose, we use the following bijection by Iyama–Reading–Reiten–Thomas [15].

Proposition 2.4. *[15, Theorem 4.1] For each $w \in \mathsf{j\text{-}irr}\,W$ of type l, we set a module $J(w) := (\Pi/I(w))e_l$, which is a direct summand of $\Pi/I(w)$. Then, $\mathsf{Sub}\,J(w) = \mathsf{Sub}(\Pi/I(w))$ holds, and this induces a bijection $J(?)\colon \mathsf{j\text{-}irr}\,W \to \mathsf{i\tau^{-1}\text{-}rigid}\,\Pi$.*

Thus, by Proposition 1.6, we obtain the following formula.

Proposition 2.5. *Let $w \in \mathsf{j\text{-}irr}\,W$ be of type l, and set $J(w) := (\Pi/I(w))e_l$. Then, the brick $S(w)$ is equal to $\mathsf{soc}_{\mathsf{End}_\Pi(J(w))}\,J(w)$.*

Moreover, they have already given a combinatorial "Young diagram–like" description of $J(w)$ for $\Delta = \mathbb{A}_n, \mathbb{D}_n$. This will be cited in the following subsections. By using this and Proposition 1.6, we will write down the explicit structure of the brick $S(w)$ for each $w \in \mathsf{j\text{-}irr}\,W$ in Section 3.

Now, we have recalled some properties holding for any preprojective algebra of Dynkin type. In the next two subsections, we will observe the preprojective algebras of type $\mathbb{A}_n$ and $\mathbb{D}_n$ in detail.

2.3. Type $\mathbb{A}_n$

Let $\Delta = \mathbb{A}_n$ in this subsection. The preprojective algebra Π of type $\mathbb{A}_n$ is given by the following quiver and relations:

$$\begin{tikzcd} 1 \arrow[r, shift left, "\alpha_1"] & 2 \arrow[l, shift left, "\beta_2"] \arrow[r, shift left, "\alpha_2"] & 3 \arrow[l, shift left, "\beta_3"] \arrow[r, shift left, "\alpha_3"] & \cdots \arrow[l, shift left, "\beta_4"] \arrow[r, shift left, "\alpha_{n-1}"] & n \arrow[l, shift left, "\beta_n"] \end{tikzcd};$$

$$\alpha_1\beta_2 = 0, \quad \alpha_i\beta_{i+1} = \beta_i\alpha_{i-1} \ (2 \leq i \leq n-1), \quad \beta_n\alpha_{n-1} = 0.$$

The Coxeter group W of type $\mathbb{A}_n$ is isomorphic to the symmetric group $\mathfrak{S}_{n+1}$ by sending each s_i to the transposition $(i \quad i+1)$. We identify the Coxeter group with $\mathfrak{S}_{n+1}$ by this isomorphism, and we express $w \in W$ as $(w(1), w(2), \dots, w(n+1))$.

The reflections of W are precisely the transpositions $(a \quad b)$ with $a, b \in [1, n+1]$ and $a > b$, and the set $\mathsf{inv}(w)$ of inversions of $w \in W$ is

$$\{(a \quad b) \mid a, b \in [1, n+1], \ a > b, \ w^{-1}(a) < w^{-1}(b)\}.$$

An element $w \in W$ is a join-irreducible element of type l if and only if l is the unique element in $[1, n]$ satisfying $w(l) > w(l+1)$. In this case, we have $w(l) \geq 2$.

We set a basis of each indecomposable projective module Πe_l as follows. Let $i, j, l \in Q_0 = [1, n]$ with $i \leq j \geq l$. We define a path $p(i, j, l)$ in Q as

$$p(i, j, l) := (\alpha_i\alpha_{i+1} \cdots \alpha_{j-1}) \cdot (\beta_j\beta_{j-1} \cdots \beta_{l+1}).$$

This is the shortest path starting from i, going through j, and ending at l. As an element in Π, the path $p(i, j, l)$ is not zero in Π if and only if $i \geq j - l + 1$, so set

$$\Gamma[l] := \{(i, j) \in Q_0 \times Q_0 \mid j - l + 1 \leq i \leq j \geq l\}.$$

We obtain the following assertion from straightforward calculation.

Lemma 2.6. *The set* $\{p(i, j, l) \mid (i, j) \in \Gamma[l]\}$ *forms a* K*-basis of* Πe_l*.*

This basis allows us to express Πe_l as

$$\begin{tikzcd} l \arrow[r] \arrow[d] & l-1 \arrow[r] \arrow[d] & \cdots \arrow[r] & 1 \arrow[d] \\ l+1 \arrow[r] \arrow[d] & l \arrow[r] \arrow[d] & \cdots \arrow[r] & 2 \arrow[d] \\ \vdots \arrow[d] & \vdots \arrow[d] & & \vdots \arrow[d] \\ n \arrow[r] & n-1 \arrow[r] & \cdots \arrow[r] & n-l+1 \end{tikzcd}. \tag{2.1}$$

Here, each number i in the row starting at j denotes a one-dimensional vector space $Kp(i, j, l)$ with a basis $p(i, j, l)$, and each arrow stands for the identity map $K \to K$ with respect to these bases.

In examples later, we sometimes write Πe_l like a Young diagram by enclosing each entry with a square and omitting arrows: for example, if $n = 8$ and $l = 3$, then Πe_l is denoted by

$$\begin{array}{|c|c|c|}\hline 3 & 2 & 1 \\ \hline 4 & 3 & 2 \\ \hline 5 & 4 & 3 \\ \hline 6 & 5 & 4 \\ \hline 7 & 6 & 5 \\ \hline 8 & 7 & 6 \\ \hline \end{array}. \tag{2.2}$$

We use similar notation for subfactor modules of Πe_l.

Under this preparation, we recall the result of [15] for type $\mathbb{A}_n$.

Proposition 2.7. *[15, Theorem 6.1] Let $w \in \mathsf{j\text{-}irr}\, W$ be a join-irreducible element of type l. Then, the module $J(w) \in \mathsf{i}\tau^{-1}\text{-}\mathsf{rigid}\, \Pi$ is expressed as follows.*

- *Consider the diagram (2.1).*
- *For each $j \in [l, n]$, in the row starting at j, keep the entries i satisfying $i \geq w(j + 1)$ and delete the others.*

Example 2.8. Let $n = 8$ and $w = (2, 5, 8, 1, 3, 4, 6, 7, 9)$. Then, w is a join-irreducible element of type $l = 3$, and Proposition 2.7 gives

$$J(w) = \begin{array}{|c|c|c|}\hline 3 & 2 & 1 \\ \hline 4 & 3 & \multicolumn{1}{c}{} \\ \cline{1-2} 5 & 4 & \multicolumn{1}{c}{} \\ \cline{1-2} 6 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-1} 7 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}.$$

2.4. Type $\mathbb{D}_n$

Let $\Delta = \mathbb{D}_n$ in this subsection. The preprojective algebra Π of type $\mathbb{D}_n$ is given by the following quiver and relations:

$$\begin{tikzcd} 1 \arrow[dr, shift left, "\alpha_1^+"] & & & & \\ & 2 \arrow[ul, shift left, "\beta_2^+"] \arrow[dl, shift left, "\beta_2^-"] \arrow[r, shift left, "\alpha_2"] & 3 \arrow[l, shift left, "\beta_3"] \arrow[r, shift left, "\alpha_3"] & \cdots \arrow[l, shift left, "\beta_4"] \arrow[r, shift left, "\alpha_{n-2}"] & n-1 \arrow[l, shift left, "\beta_{n-1}"] \\ -1 \arrow[ur, shift left, "\alpha_1^-"] & & & & \end{tikzcd};$$

$$\alpha_1^+\beta_2^+ = 0, \quad \alpha_1^-\beta_2^- = 0, \quad \alpha_2\beta_3 = \beta_2^+\alpha_1^+ + \beta_2^-\alpha_1^-,$$
$$\alpha_i\beta_{i+1} = \beta_i\alpha_{i-1}\ (3 \leq i \leq n-2), \quad \beta_{n-1}\alpha_{n-2} = 0.$$

To avoid complicated notation, we set $\alpha_1 := \alpha_1^+ + \alpha_1^-$ and $\beta_2 := \beta_2^+ + \beta_2^-$.

The Coxeter group W of type $\mathbb{D}_n$ is isomorphic to the group consisting of all automorphisms w on the set $\pm[1, n]$ satisfying the following conditions:

- $w(-i) = -w(i)$ holds for each $i \in [1, n]$; and
- the number of elements in $\{i \in [1, n] \mid w(i) < 0\}$ is even.

Here, $s_i \in W$ is sent to $(-1 \quad 2)(-2 \quad 1)$ if $i = -1$; and $(-i \quad -(i+1))(i \quad i+1)$ if $i \neq -1$. We identify W with the group above by this isomorphism. Since $w(-i) = -w(i)$ holds, we express $w \in W$ as $(w(1), w(2), \dots, w(n))$.

The reflections of W are precisely the elements of the form $(-a \quad -b)(a \quad b)$ with $a, b \in \pm[1, n]$ and $a > |b|$, and the set $\mathsf{inv}(w)$ of inversions of $w \in W$ is

$$\{(-a \quad -b)(a \quad b) \mid a, b \in \pm[1, n],\ a > |b|,\ w^{-1}(a) < w^{-1}(b)\}.$$

An element $w \in W$ is a join-irreducible element of type l if and only if l is the unique element in $\{-1\} \cup [1, n-1] = Q_0$ such that $w(l) > w(|l| + 1)$ holds.

We fix one or two bases of each indecomposable projective module Πe_l as follows. We divide the argument by whether $l = \pm 1$ or not.

We consider the case $l = \pm 1$ first. Let $i, j \in Q_0 = \{-1\} \cup [1, n-1]$ with $i \leq j \neq -l$. We define a path $p(i, j, l)$ by

$$p(i, j, \pm 1) := \begin{cases} (\alpha_i \alpha_{i+1} \cdots \alpha_{j-1}) \cdot (\beta_j \beta_{j-1} \cdots \beta_3) \beta_2^{\pm} & (i \geq 2) \\ \alpha_1^+ p(2, j, \pm 1) & (i = 1) \\ \alpha_1^- p(2, j, \pm 1) & (i = -1) \end{cases}.$$

This is a shortest path starting from i, going through j, and ending at l. As an element in Π, the path $p(i, j, l)$ is not zero in Π if and only if $i \neq (-1)^j l$, so set

$$\Gamma[l] := \{(i, j) \in Q_0 \times Q_0 \mid (-1)^j l \neq i \leq j \neq -l\}.$$

We obtain the following assertion from straightforward calculation.

Lemma 2.9. *The set $\{p(i, j, l) \mid (i, j) \in \Gamma[l]\}$ forms a K-basis of Πe_l.*

This basis allows us to express Πe_l as

$$\begin{tikzcd}
l \arrow[d] & & & & \\
2 \arrow[r] \arrow[d] & -l \arrow[d] & & & \\
\vdots \arrow[d] & \vdots \arrow[d] & & & \\
n-2 \arrow[r] \arrow[d] & n-3 \arrow[r] \arrow[d] & \cdots \arrow[r] & (-1)^{n-3} l \arrow[d] & \\
n-1 \arrow[r] & n-2 \arrow[r] & \cdots \arrow[r] & 2 \arrow[r] & (-1)^{n-2} l
\end{tikzcd} . \qquad (2.3)$$

Here, each number i in the row starting at j denotes a one-dimensional vector space $Kp(i, j, l)$ with a basis $p(i, j, l)$, and each arrow stands for the identity map $K \to K$ with respect to these bases.

If we use the "Young diagram-like" notation as (2.2) for the case $n = 9$ and $l = 1$, then Πe_l is denoted by

$$\begin{array}{cccccccc}
1 \\
2 & -1 \\
3 & 2 & 1 \\
4 & 3 & 2 & -1 \\
5 & 4 & 3 & 2 & 1 \\
6 & 5 & 4 & 3 & 2 & -1 \\
7 & 6 & 5 & 4 & 3 & 2 & 1 \\
8 & 7 & 6 & 5 & 4 & 3 & 2 & -1
\end{array} \quad . \tag{2.4}$$

The indecomposable τ^{-1}-rigid module $J(w)$ for $w \in \text{j-irr}\, W$ of type $l = \pm 1$ is given as follows.

Proposition 2.10. *[15, Theorem 6.5] Let $w \in \text{j-irr}\, W$ be a join-irreducible element of type $l = \pm 1$. Then, the module $J(w) \in \text{i}\tau^{-1}\text{-rigid}\, \Pi$ is expressed as follows.*

- *Consider the diagram (2.3).*
- *For each $j \in \{l\} \cup [2, n-1]$, in the row starting at j, keep the entries i satisfying $i \geq w(|j|+1)$ and delete the others.*

Example 2.11. Let $n = 9$ and $w = (9, -7, -6, -4, -1, 2, 3, 5, 8) \in \text{j-irr}\, W$ of type $l = 1$. Then, by Proposition 2.10, the module $J(w)$ is

$$J(w) = \begin{array}{cccc}
1 \\
2 & -1 \\
3 & 2 & 1 \\
4 & 3 & 2 & -1 \\
5 & 4 & 3 & 2 \\
6 & 5 & 4 & 3 \\
7 & 6 & 5 \\
8
\end{array} .$$

Next, we consider the case $l \geq 2$. Let $i \in \pm Q_0 = \pm[1, n-1]$ and $j \in Q_0 = \{-1\} \cup [1, n-1]$ with $i \leq j \geq l$. Set $t := (-1)^{j-l+1}$. We define two paths $p_1(i, j, l)$ and $p_{-1}(i, j, l)$ in Q by

$$p_\varepsilon(i, j, l) := \begin{cases} (\alpha_i \alpha_{i-1} \cdots \alpha_{j-1}) \cdot (\beta_j \beta_{j-1} \cdots \beta_{l+1}) & (i \geq 2) \\ \alpha_1^+ p_\varepsilon(2, j, l) & (i = 1) \\ \alpha_1^- p_\varepsilon(2, j, l) & (i = -1) \\ \beta_2 p_\varepsilon(\varepsilon t, j, l) & (i = -2) \\ (\beta_{-i} \beta_{-i-1} \cdots \beta_3) p_\varepsilon(-2, j, l) & (i \leq -3) \end{cases} .$$

For $\varepsilon = \pm 1$, the path $p_\varepsilon(i, j, l)$ is a shortest path

- starting from i, going through j, and ending at l if $i \geq -1$; and
- starting from $|i|$, going through εt and then j, and ending at l if $i \leq -2$.

As an element in Π, the path $p_\varepsilon(i,j,l)$ is not zero in Π if and only if $i \geq j-(n-1)-l$, so set

$$\Gamma[l] := \{(i,j) \in \pm Q_0 \times Q_0 \mid j-(n-1)-l \leq i \leq j \geq l\}.$$

We obtain the following assertion from straightforward calculation.

Lemma 2.12. *Let $\varepsilon = \pm 1$. Then, the set $\{p_\varepsilon(i,j,l) \mid (i,j) \in \Gamma[l]\}$ forms a K-basis of Πe_l.*

Each basis above allows us to express Πe_l as

$$\begin{tikzcd}[row sep=small, column sep=small]
 & & & & -\varepsilon \arrow[dr] & & & & & & & \\
l \arrow[r] \arrow[dd] & l-1 \arrow[r] \arrow[dd] & \cdots \arrow[r] & 2 \arrow[ur] \arrow[dr] \arrow[dd] & & -2 \arrow[r] \arrow[d] & \cdots \arrow[r] & -m \arrow[r] \arrow[dd] & -m-1 \arrow[r] \arrow[dd] & \cdots \arrow[r] & -n+2 \arrow[r] \arrow[dd] & -n+1 \arrow[dd] \\
 & & & & \varepsilon \arrow[ur, "-1"'] \arrow[d] & \varepsilon \arrow[dr] & & & & & & \\
l+1 \arrow[r] \arrow[dd] & l \arrow[r] \arrow[dd] & \cdots \arrow[r] & 3 \arrow[r] \arrow[dd] & 2 \arrow[ur] \arrow[dr] \arrow[dd] & & \cdots \arrow[r] & -m+1 \arrow[r] \arrow[dd] & -m \arrow[r] \arrow[dd] & \cdots \arrow[r] & -n+3 \arrow[r] \arrow[dd] & -n+2 \arrow[dd] \\
 & & & & & -\varepsilon \arrow[ur, "-1"'] \arrow[d] & & & & & & \\
\vdots \arrow[dd] & \vdots \arrow[dd] & & \vdots \arrow[dd] & \vdots \arrow[dd] & \vdots \arrow[dd] & & \vdots \arrow[d] & \vdots \arrow[dd] & & \vdots \arrow[dd] & \vdots \arrow[dd] \\
 & & & & & & & -\varepsilon t \arrow[dr] & & & & \\
n-1 \arrow[r] & n-2 \arrow[r] & \cdots \arrow[r] & m+1 \arrow[r] & m \arrow[r] & m-1 \arrow[r] & \cdots \arrow[ur] \arrow[dr] & & -2 \arrow[r] & \cdots \arrow[r] & -l+1 \arrow[r] & -l \\
 & & & & & & & \varepsilon t \arrow[ur, "-1"'] & & & &
\end{tikzcd}, \tag{2.5}$$

where $m := n-l$, $t := (-1)^{m-1}$. Here, each number i in the row starting at j denotes a one-dimensional vector space $Kp_\varepsilon(i,j,l)$ with a basis $p_\varepsilon(i,j,l)$. Each arrow with the label "-1" stands for the map $K \ni x \mapsto -x \in K$, and each of the other arrows stands for the identity map $K \to K$, with respect to these bases.

If we use the "Young diagram-like" notation as (2.2) for the case $n=9$, $l=2$, and $\varepsilon = 1$, then Πe_l is denoted by

2	$\begin{smallmatrix}-1\\1\end{smallmatrix}$	−2	−3	−4	−5	−6	−7	−8
3	2	$\begin{smallmatrix}1\\-1\end{smallmatrix}$	−2	−3	−4	−5	−6	−7
4	3	2	$\begin{smallmatrix}-1\\1\end{smallmatrix}$	−2	−3	−4	−5	−6
5	4	3	2	$\begin{smallmatrix}1\\-1\end{smallmatrix}$	−2	−3	−4	−5
6	5	4	3	2	$\begin{smallmatrix}-1\\1\end{smallmatrix}$	−2	−3	−4
7	6	5	4	3	2	$\begin{smallmatrix}1\\-1\end{smallmatrix}$	−2	−3
8	7	6	5	4	3	2	$\begin{smallmatrix}-1\\1\end{smallmatrix}$	−2

. (2.6)

We use similar notation for subfactor modules of Πe_l.

The indecomposable τ^{-1}-rigid module $J(w)$ for $w \in \mathsf{j\text{-}irr}\, W$ of type $l \neq \pm 1$ is given as follows.

Proposition 2.13. *[15, Theorem 6.12] Let $w \in \mathsf{j\text{-}irr}\, W$ be a join-irreducible element of type $l \neq \pm 1$. If $w(l+1) \leq 1$, then set*

$$m := \mathsf{max}\{k \in [l+1, n] \mid w(k) \leq 1\}, \quad \varepsilon := \begin{cases} (-1)^{m-(l+1)} & (w(m) \leq -2) \\ (-1)^{m-(l+1)} w(m) & (w(m) = \pm 1) \end{cases};$$

otherwise, set $\varepsilon := 1$. Then, the module $J(w) \in \mathsf{i}\tau^{-1}\text{-}\mathsf{rigid}\, \Pi$ is expressed as follows.

- *Consider the diagram (2.5).*

- *For each $j \in [l, n-1]$, in the row starting at j, keep the entries i satisfying*

$$\begin{cases} i \geq w(j+1) & (w(j+1) \geq 2) \\ i \geq 2 \quad \text{or} \quad i = w(j+1) & (w(j+1) = \pm 1) \\ i \geq w(j+1)+1 & (w(j+1) \leq -2) \end{cases}$$

and delete the others.

Example 2.14. Let $n = 9$ and $w = (-6, 9, -7, -4, -1, 2, 3, 5, 8) \in \mathsf{j\text{-}irr}\, W$ of type $l = 2$. By Proposition 2.13, the module $J(w)$ is described as follows:

$$J(w) = \begin{array}{|c|c|c|c|c|c|c|} \hline 2 & \begin{smallmatrix} 1 \\ -1 \end{smallmatrix} & -2 & -3 & -4 & -5 & -6 \\ \hline 3 & 2 & \begin{smallmatrix} -1 \\ 1 \end{smallmatrix} & -2 & -3 \\ \cline{1-5} 4 & 3 & 2 & {}_{-1} \\ \cline{1-4} 5 & 4 & 3 & 2 \\ \cline{1-4} 6 & 5 & 4 & 3 \\ \cline{1-4} 7 & 6 & 5 \\ \cline{1-3} 8 \\ \cline{1-1} \end{array}.$$

3. Description of bricks

In this section, we describe the bricks over the preprojective algebras Π of Dynkin type $\Delta = \mathbb{A}_n, \mathbb{D}_n$. For $w \in \mathsf{j\text{-}irr}\, W$, we have obtained that the brick $S(w)$ is given as $\mathsf{soc}_{\mathsf{End}_\Pi(J(w))} J(w)$ in Proposition 2.5, and the module $J(w) \in \mathsf{i}\tau^{-1}\text{-}\mathsf{rigid}\, \Pi$ is combinatorially determined in Propositions 2.7, 2.10, and 2.13.

We remark that the bricks in $\mathsf{mod}\, \Pi$ coincide with the layers of Π [15, Theorem 1.2]. Thus, the dimension vector of each brick in $\mathsf{mod}\, \Pi$ is a positive root by [3, Theorem 2.7]. Here, a module L in $\mathsf{mod}\, \Pi$ is called a *layer* if there exist some $w \in W$ and some vertex i in Δ satisfying $w < ws_i$ and $L \cong I(w)/I(ws_i)$ [3, Section 2].

3.1. Type $\mathbb{A}_n$

We state the result and give an example first.

Theorem 3.1. *Let $w \in \mathsf{j\text{-}irr}\, W$ be a join-irreducible element of type l. Set*

$$R := w([l+1, n+1]), \quad a := w(l), \quad b := w(l+1), \quad V := [b, a-1].$$

Then, the brick $S(w)$ is isomorphic to the Π-module $S'(w)$ defined as follows.

(a) *The brick $S'(w)$ has a K-basis $(\langle i \rangle)_{i \in V}$, and if $j = i$, then $e_j \langle i \rangle := \langle i \rangle$; otherwise, $e_j \langle i \rangle := 0$.*
(b) *Let $i \in V$. If $j \neq i-1$, then $\alpha_j \langle i \rangle := 0$. If $j \neq i+1$, then $\beta_j \langle i \rangle := 0$.*
(c) *If $i \in V \setminus \{\max V\}$, then*

$$\alpha_i \langle i+1 \rangle := \begin{cases} \langle i \rangle & (i+1 \notin R) \\ 0 & (i+1 \in R) \end{cases}, \quad \beta_{i+1} \langle i \rangle := \begin{cases} 0 & (i+1 \notin R) \\ \langle i+1 \rangle & (i+1 \in R) \end{cases}.$$

Example 3.2. Let $n = 8$ and $w = (2, 5, 8, 1, 3, 4, 6, 7, 9)$. Then, we have $l = 3$, $R = \{1, 3, 4, 6, 7, 9\}$, $a = 8$, $b = 1$, and $V = [1, 7]$. The module $S(w)$ has a K-basis $\langle 1\rangle, \langle 2\rangle, \dots, \langle 7\rangle$ and its structure as a Π-module can be written as

$$\langle 1\rangle \xleftarrow{\alpha_1} \langle 2\rangle \xrightarrow{\beta_3} \langle 3\rangle \xrightarrow{\beta_4} \langle 4\rangle \xleftarrow{\alpha_4} \langle 5\rangle \xrightarrow{\beta_6} \langle 6\rangle \xrightarrow{\beta_7} \langle 7\rangle.$$

In an abbreviated form, the brick $S(w)$ is denoted by

$$1 \leftarrow 2 \rightarrow 3 \rightarrow 4 \leftarrow 5 \rightarrow 6 \rightarrow 7. \tag{3.1}$$

If we use the notation as (2.2), then by Example 2.8, the module $J(w)$ and the "position" of a submodule $S(w)$ in $J(w)$ are described as follows:

$$J(w) = \begin{array}{|c|c|c|} \hline 3 & 2 & 1 \\ \hline 4 & 3 & \\ \cline{1-2} 5 & 4 & \\ \cline{1-2} 6 & & \\ \cline{1-1} 7 & & \\ \cline{1-1} \end{array}, \quad S(w) = \begin{array}{c|c|c|} \cline{2-3} & 2 & 1 \\ \cline{2-3} & 3 & \\ \cline{1-2} \multicolumn{1}{|c|}{5} & 4 & \\ \cline{1-2} \multicolumn{1}{|c|}{6} & & \\ \cline{1-1} \multicolumn{1}{|c|}{7} & & \\ \cline{1-1} \end{array}.$$

Compare this expression of the brick $S(w)$ to (3.1). If we use such abbreviated expressions of bricks as (3.1), then the theorem can be restated as follows.

Corollary 3.3. *Let $w \in \mathsf{j\text{-}irr}\, W$ be a join-irreducible element of type l, and use the setting of Theorem 3.1. We express the brick $S(w)$ in the following abbreviation rules.*

- *For each $i \in V$, the K-vector subspace $K\langle i\rangle$ is denoted by the symbol i.*
- *If the action of some $\gamma \in Q_1$ on $S(w)$ induces a nonzero K-linear map $K\langle i\rangle \to K\langle j\rangle$, then we draw an arrow from the symbol i to the symbol j.*

Then, for each $i \in V \setminus \{\max V\}$, there exists exactly one arrow between i and $i + 1$, and its orientation is $i \to i + 1$ if $i + 1 \in R$ and $i \leftarrow i + 1$ if $i + 1 \notin R$.

It is easy to see that there exists some path algebra A of type $\mathbb{A}_n$ such that the brick $S(w)$ is an A-module, and that any 2-cycle in Q annihilates all the bricks in Π. Let I be the ideal of Π generated by all the 2-cycles in Q, then [13, Corollary 5.20] implies that $\mathsf{torf}\, \Pi \cong \mathsf{torf}(\Pi/I)$ as lattices. Thus, there is an isomorphism from W to $\mathsf{torf}(\Pi/I)$ as lattices by Propositions 1.5 and 2.1. The relationship between W and Π/I is investigated from another point of view in [6, Section 4].

Now we start the proof of Theorem 3.1. For this purpose, we restate Proposition 2.7 as follows.

Lemma 3.4. *Let $w \in \mathsf{j\text{-}irr}\, W$ be a join-irreducible element of type l.*

(1) *Assume $(i, j) \in \Gamma[l]$. Then, $p(i, j, l) \notin I(w)$ holds if and only if $i \geq w(j + 1)$.*
(2) *Define $\Gamma(w) \subset \Gamma[l]$ as the subset consisting of the elements $(i, j) \in \Gamma[l]$ with $p(i, j, l) \notin I(w)$. Then, the set $\{p(i, j, l) \mid (i, j) \in \Gamma(w)\}$ induces a K-basis of $J(w)$.*

To express $S(w)$, we define the following set for $k \geq 1$:

$$\Gamma_k(w) := \{(i, j) \in \Gamma(w) \mid \min\{x \geq 1 \mid (i, j + x) \notin \Gamma(w)\} = k\}.$$

It is easy to see that $\Gamma(w)$ is the disjoint union of the $\Gamma_k(w)$'s. Moreover, we extend the definition of the path $p(i, j, l)$ to $\tilde{\Gamma}[l] := \{(i, j) \in Q_0 \times \mathbb{Z} \mid i \leq j \geq l\}$ by setting $p(i, j, l) := 0$ if $j \geq n + 1$.

Lemma 3.5. *Let $w \in \mathsf{j\text{-}irr}\, W$ be a join-irreducible element of type l. Consider the endomorphism $f := (\cdot p(l, l + 1, l))\colon J(w) \to J(w)$.*

(1) *We have $S(w) = \mathsf{Ker}\, f$.*
(2) *Let $(i, j) \in \Gamma(w)$. Then, $p(i, j, l) \in \mathsf{Ker}\, f$ holds if and only if $(i, j) \in \Gamma_1(w)$.*
(3) *The set $\{p(i, j, l) \mid (i, j) \in \Gamma_1(w)\}$ induces a K-basis of $\mathsf{Ker}\, f$.*

Proof. (1) Since $J(w) = (\Pi/I(w))e_l$, we can regard $\mathsf{End}_\Pi(J(w))$ as a factor algebra of $\mathsf{End}_\Pi(\Pi e_l) \cong e_l \Pi e_l$. It is easy to check that $\tilde{f} := (\cdot p(l, l + 1, l))\colon \Pi e_l \to \Pi e_l$ satisfies $\mathsf{rad}\, \mathsf{End}_\Pi(\Pi e_l) = \mathsf{End}_\Pi(\Pi e_l)\tilde{f}$. Under the quotient map $\mathsf{End}_\Pi(\Pi e_l) \to \mathsf{End}_\Pi(J(w))$, the image of $\mathsf{rad}\, \mathsf{End}_\Pi(\Pi e_l)$ is $\mathsf{rad}\, \mathsf{End}_\Pi(J(w))$ and $\tilde{f}$ is sent to f, so $\mathsf{rad}\, \mathsf{End}_\Pi(J(w)) = \mathsf{End}_\Pi(J(w))f$. Thus, for any $g \in \mathsf{rad}\, \mathsf{End}_\Pi(J(w))$, we have $\mathsf{Ker}\, g \supset \mathsf{Ker}\, f$, so

$$S(w) = \mathsf{soc}_{\mathsf{End}_\Pi(J(w))} J(w) = \bigcap_{g \in \mathsf{rad}\, \mathsf{End}_\Pi(J(w))} \mathsf{Ker}\, g = \mathsf{Ker}\, f.$$

(2) As an element in Π, we have $f(p(i, j, l)) = p(i, j, l)p(l, l + 1, l) = p(i, j + 1, l)$. Then, Lemma 3.4 implies the assertion.

(3) From Lemma 3.4, recall that the set $\{p(i, j, l) \mid (i, j) \in \Gamma(w)\}$ induces a basis of $J(w)$, so this set is linearly independent in $J(w)$.

Thus, the subset $\{p(i, j, l) \mid (i, j) \in \Gamma_1(w)\}$ is linearly independent in $J(w)$, and is contained in $\mathsf{Ker}\, f$ by (2).

On the other hand, in the proof of (2), we got $f(p(i, j, l)) = p(i, j + 1, l)$. If $(i, j) \in \Gamma(w) \setminus \Gamma_1(w)$, then $(i, j + 1) \in \Gamma(w)$. The set $\{p(i, j + 1, l) \mid (i, j) \in \Gamma(w) \setminus \Gamma_1(w)\}$ is linearly independent in $J(w)$. This implies $\dim_K \mathsf{Im}\, f \geq \#(\Gamma(w) \setminus \Gamma_1(w))$; hence,

$$\dim_K \mathsf{Ker}\, f = \dim_K J(w) - \dim_K \mathsf{Im}\, f \leq \#\Gamma(w) - \#(\Gamma(w) \setminus \Gamma_1(w)) = \#\Gamma_1(w).$$

Since $\{p(i, j, l) \mid (i, j) \in \Gamma_1(w)\}$ is linearly independent in $J(w)$ and has $\#\Gamma_1(w)$ elements, the set $\{p(i, j, l) \mid (i, j) \in \Gamma_1(w)\}$ induces a K-basis of $\mathsf{Ker}\, f$. □

Lemma 3.6. *Let $w \in \mathsf{j\text{-}irr}\, W$ be a join-irreducible element of type l, and define V as in Theorem 3.1. Then, there exists a bijection $\Gamma_1(w) \to V$ given by $(i, j) \mapsto i$.*

Proof. In the proof, we fully use the notation in Theorem 3.1.

We first show the well-definedness of the map $\Gamma_1(w) \to V$.

We remark that, for $k \in [l + 1, n + 1]$, the condition $w(k) = k$ holds if and only if $k > a$, and that this condition is also equivalent to $w(k) > a$. Lemma 3.4 and $(i, j) \in \Gamma(w)$ give $j \geq i \geq w(j + 1)$. Thus, $w(j + 1) \leq j$ holds, so we get $j + 1 \leq a$, or equivalently, $j < a$. Therefore, we obtain $i \leq j < a$.

On the other hand, Lemma 3.4 and $(i, j) \in \Gamma(w)$ also imply $j \geq i \geq w(j + 1) \geq w(l + 1) = b$.

These imply that the map $\Gamma_1(w) \to V$ is well-defined. It is clearly injective by Lemma 3.4.

We next prove that the map $\Gamma_1(w) \to V$ is also surjective. Let $i \in V$. Then, $i < a$ holds, so there exists some $j \in [l, n]$ such that $(i, j) \in \Gamma(w)$ by Lemma 3.4. Take the maximum j, then it is easy to obtain $(i, j) \in \Gamma_1(w)$ from Lemma 3.4.

Hence, the map $\Gamma_1(w) \to V$ is also surjective, and thus, bijective. □

Now, we show Theorem 3.1.

Proof of Theorem 3.1. By Lemma 3.6, we can define a map $\rho\colon V \to Q_0$ as follows: $\rho(i)$ is the unique element $j \in Q_0$ such that $(i,j) \in \Gamma_1(w)$. Set $\langle i\rangle := p(i,\rho(i),l)$ for each $i \in V$. It suffices to show that $(\langle i\rangle)_{i\in V}$ satisfies the properties (a), (b), and (c), since the three properties are enough to define an isomorphic class of Π-modules.

First, $(\langle i\rangle)_{i\in V}$ is a K-basis of $S(w)$ by Lemmas 3.5 and 3.6, and $K\langle i\rangle$ is clearly a subspace of $e_iS(w)$. Thus, the property (a) holds, and (b) follows from (a).

We begin the proof of (c).

Let $i \in V \setminus \{\max V\}$ and set $j := \rho(i+1)$. Then,

$$\alpha_i\langle i+1\rangle = \alpha_i p(i+1,j,l) = p(i,j,l) = \begin{cases} \langle i\rangle & (\text{if } i+1 \notin R, \text{ since } (i,j) \in \Gamma_1(w)) \\ 0 & (\text{if } i+1 \in R, \text{ since } (i,j) \notin \Gamma(w)) \end{cases}.$$

Next, let $i \in V \setminus \{\max V\}$ and set $j := \rho(i)$. Then,

$$\begin{aligned} \beta_{i+1}\langle i\rangle &= \beta_{i+1}p(i,j,l) = p(i+1,j+1,l) \\ &= \begin{cases} 0 & (\text{if } i+1 \notin R, \text{ since } (i+1,j+1) \notin \Gamma(w)) \\ \langle i+1\rangle & (\text{if } i+1 \in R, \text{ since } (i+1,j+1) \in \Gamma_1(w)) \end{cases}. \end{aligned}$$

From these, we have the property (c). $\square$

3.2. Type $\mathbb{D}_n$

We state the result and give some examples first. Recall $\alpha_1 = \alpha_1^+ + \alpha_1^-$ and $\beta_2 = \beta_2^+ + \beta_2^-$.

Theorem 3.7. *Let $w \in \mathsf{j\text{-}irr}\, W$ be a join-irreducible element of type l. Set*

$$\begin{aligned} &R := w([|l|+1,n]), \quad a := w(l), \quad b := w(|l|+1), \\ &r := \max\{k \geq 0 \mid [1,k] \subset \pm R\}, \quad c := \begin{cases} w(|w^{-1}(1)|) & (r \geq 1) \\ 1 & (r = 0) \end{cases}, \\ &(V_-,V_+) := \begin{cases} (\emptyset, [b,a-1]) & (b \geq 2) \\ (\emptyset, \{c\} \cup [2,a-1]) & (b = \pm 1) \\ ([b+1,-2] \cup \{-c\}, \{c\} \cup [2,a-1]) & (b \leq -2) \end{cases}, \quad V := V_+ \amalg V_-. \end{aligned}$$

Then, the brick $S(w)$ is isomorphic to the Π-module $S'(w)$ defined as follows.

(a) *The brick $S'(w)$ has a K-basis $(\langle i\rangle)_{i\in V}$, and if $j = |i| \geq 2$ or $j = i \in \{\pm 1\}$, then $e_j\langle i\rangle := \langle i\rangle$; otherwise $e_j\langle i\rangle := 0$.*

(b) *Let $i \in V$. If $j \neq |i|-1$, then $\alpha_j\langle i\rangle := 0$. If $j \neq |i|+1$, then $\beta_j\langle i\rangle := 0$.*

(c) *The remaining actions of arrows are given as follows, where we set $\langle j\rangle := 0$ if $j \notin V$ (in this case, the coefficient ξ_i^+, ξ_i^-, η_i^+, or η_i^- of $\langle j\rangle$ is set as zero below).*

   (i) *For $i \in V_+ \setminus \{\max V_+\}$, we have $\alpha_{|i|}\langle |i|+1\rangle := \xi_i^+\langle i\rangle + \xi_i^-\langle -i\rangle$, where*

$$\xi_i^+ := \begin{cases} 1 & (|i|+1 \notin R) \\ 0 & (|i|+1 \in R) \end{cases}, \quad \xi_i^- := \begin{cases} 1 & (|i| = 1,\ r = 0,\ 2 \notin R) \\ 0 & (\text{otherwise}) \end{cases}.$$

(ii) *For $i \in V_+ \setminus \{\max V_+\}$, we have* $\beta_{|i|+1}\langle i\rangle := \eta_i^+\langle |i|+1\rangle + \eta_i^-\langle -(|i|+1)\rangle$, *where*

$$\eta_i^+ := \begin{cases} 1 & (|i|+1 \in R) \\ 0 & (|i|+1 \notin R) \end{cases}, \quad \eta_i^- := \begin{cases} -1 & (|i|=1,\ r=0,\ -2 \notin R) \\ 0 & (\text{otherwise}) \end{cases}.$$

(iii) *For $i \in V_- \setminus \{\min V_-\}$, we have* $\alpha_{|i|}\langle -(|i|+1)\rangle := \xi_i^+\langle -i\rangle + \xi_i^-\langle i\rangle$, *where*

$$\xi_i^+ := \begin{cases} 1 & (|i| \le r,\ |i|+1 \in R) \\ 0 & (\text{otherwise}) \end{cases}, \quad \xi_i^- := \begin{cases} 1 & (-(|i|+1) \in R) \\ 0 & (-(|i|+1) \notin R) \end{cases}.$$

(iv) *For $i \in V_-$, we have* $\beta_{|i|+1}\langle i\rangle := \eta_i^+\langle |i|+1\rangle + \eta_i^-\langle -(|i|+1)\rangle$, *where*

$$\eta_i^+ := \begin{cases} 1 & (|i| \le r,\ |i|+1 \notin R) \\ 0 & (\text{otherwise}) \end{cases}, \quad \eta_i^- := \begin{cases} 1 & (|i| \ne r,\ -(|i|+1) \notin R) \\ -1 & (|i| = r) \\ 0 & (\text{otherwise}) \end{cases}.$$

The proof of the theorem given in later depends on whether the type l of the join-irreducible element $w \in \mathsf{j\text{-}irr}\, W$ is ± 1 or not, because the description of the indecomposable τ^{-1}-rigid module $J(w)$ does so. The following examples show the difference of the calculation of the brick $S(w)$ in these two cases.

Example 3.8. Let $n = 9$, $w = (9, -7, -6, -4, -1, 2, 3, 5, 8)$. Then, we have $l = 1$, $R = \{-7, -6, -4, -1, 2, 3, 5, 8\}$, $a = 9$, $b = -7$, $r = 8$, and $c = -1$. Thus, $(V_-, V_+) = ([-6, -2] \cup \{1\}, \{-1\} \cup [2, 8])$, and the desired brick $S(w)$ is written as

$$\begin{tikzcd}
\langle 1\rangle \arrow[r, "\beta_2^+"] & \langle -2\rangle \arrow[r, "\beta_3"] \arrow[dl, "\alpha_1^-"'] & \langle -3\rangle \arrow[dl, "\alpha_2"'] \arrow[dr, "\beta_4"] & \langle -4\rangle \arrow[l, "\alpha_3"'] \arrow[r, "\beta_5"] & \langle -5\rangle \arrow[dl, "\alpha_4"'] \arrow[dr, "\beta_6"] & \langle -6\rangle \arrow[l, "\alpha_5"'] \arrow[dr, "\beta_7"] & & \\
\langle -1\rangle \arrow[r, "\beta_2^-"] & \langle 2\rangle \arrow[r, "\beta_3"] & \langle 3\rangle & \langle 4\rangle \arrow[l, "\alpha_3"'] \arrow[r, "\beta_5"] & \langle 5\rangle & \langle 6\rangle \arrow[l, "\alpha_5"'] & \langle 7\rangle \arrow[l, "\alpha_6"'] \arrow[r, "\beta_8"] & \langle 8\rangle
\end{tikzcd}.$$

By omitting the labels of the arrows, the brick $S(w)$ can be written in the following abbreviated way, which is enough to determine $S(w)$ up to isomorphisms:

$$\begin{tikzcd}
1 \arrow[r] & -2 \arrow[r] \arrow[dl] & -3 \arrow[dl] \arrow[dr] & -4 \arrow[l] \arrow[r] & -5 \arrow[dl] \arrow[dr] & -6 \arrow[l] \arrow[dr] & & \\
-1 \arrow[r] & 2 \arrow[r] & 3 & 4 \arrow[l] \arrow[r] & 5 & 6 \arrow[l] & 7 \arrow[l] \arrow[r] & 8
\end{tikzcd}. \tag{3.2}$$

If we use the notation as (2.4), then by Example 2.11, the module $J(w)$ and the "position" of a submodule $S(w)$ in $J(w)$ are described as follows:

$$J(w) = \begin{array}{|c|c|c|c|}
\hline
1 \\
\cline{1-1}
2 & -1 \\
\cline{1-2}
3 & 2 & 1 \\
\cline{1-3}
4 & 3 & 2 & -1 \\
\hline
5 & 4 & 3 & 2 \\
\hline
6 & 5 & 4 & 3 \\
\hline
7 & 6 & 5 \\
\cline{1-3}
8 \\
\cline{1-1}
\end{array}, \quad S(w) = \begin{array}{cccc}
 & & \boxed{1} & \\
 & & \boxed{\color{red}2} & \boxed{-1} \\
 & \boxed{\color{red}4} & \boxed{\color{red}3} & \boxed{2} \\
\boxed{\color{red}6} & \boxed{\color{red}5} & \boxed{4} & \boxed{3} \\
\boxed{7} & \boxed{6} & \boxed{5} & \\
\boxed{8} & & &
\end{array}.$$

In the figure for $S(w)$, every square $\boxed{\color{red}i}$ with a red letter denotes $K\langle -i\rangle$, which is a subspace of $e_iS(w)$. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.) There are five such squares $\boxed{\color{red}2}, \boxed{\color{red}3}, \boxed{\color{red}4}, \boxed{\color{red}5}, \boxed{\color{red}6}$. Every other square $\boxed{i}$ denotes $K\langle i\rangle$, and it is a subspace of $e_iS(w)$. Compare this expression of the brick $S(w)$ to (3.2).

Example 3.9. Let $n = 9$, $w = (-6, 9, -7, -4, -1, 2, 3, 5, 8)$. Then, we have $l = 2$, $R = \{-7, -4, -1, 2, 3, 5, 8\}$, $a = 9$, $b = -7$, $r = 5$, and $c = -1$. Thus, $(V_-, V_+) = ([-6, -2] \cup \{1\}, \{-1\} \cup [2, 8])$, and the desired brick $S(w)$ is written as

$$\begin{tikzcd}
\langle 1\rangle \arrow[r, "\beta_2^+"] & \color{blue}\langle -2\rangle \arrow[r, "\beta_3"] \arrow[dl, "\alpha_1^-"'] & \color{blue}\langle -3\rangle \arrow[dl, "\alpha_2"'] \arrow[dr, "\beta_4"] & \color{blue}\langle -4\rangle \arrow[l, "\alpha_3"'] \arrow[r, "\beta_5"] & \color{blue}\langle -5\rangle \arrow[r, "-\beta_6"] \arrow[dl, "\alpha_4"'] \arrow[dr, "\beta_6"] & \langle -6\rangle & & & \\
\langle -1\rangle \arrow[r, "\beta_2^-"] & \langle 2\rangle \arrow[r, "\beta_3"] & \langle 3\rangle & \langle 4\rangle \arrow[l, "\alpha_3"'] \arrow[r, "\beta_5"] & \langle 5\rangle & \langle 6\rangle \arrow[l, "\alpha_5"'] & \langle 7\rangle \arrow[l, "\alpha_6"'] \arrow[r, "\beta_8"] & \langle 8\rangle
\end{tikzcd}.$$

The brick $S(w)$ can be written in the following abbreviated way:

$$\begin{tikzcd}
1 \arrow[r] & \color{blue}-2 \arrow[r] \arrow[dl] & \color{blue}-3 \arrow[dl] \arrow[dr] & \color{blue}-4 \arrow[l] \arrow[r] & \color{blue}-5 \arrow[r] \arrow[dl] \arrow[dr] & -6 & & \\
-1 \arrow[r] & 2 \arrow[r] & 3 & 4 \arrow[l] \arrow[r] & 5 & 6 \arrow[l] & 7 \arrow[l] \arrow[r] & 8
\end{tikzcd}. \tag{3.3}$$

Now we use the notation as (2.6), then by Example 2.14, the module $J(w)$ and the "position" of a submodule $S(w)$ in $J(w)$ are described as follows:

$$J(w) = \begin{array}{|c|c|c|c|c|c|c|}
\hline
2 & {}^{1}_{-1} & -2 & -3 & \color{blue}-4 & \color{blue}-5 & -6 \\
\hline
3 & 2 & {}^{-1}_{1} & \color{blue}-2 & \color{blue}-3 \\
\cline{1-5}
4 & 3 & \color{blue}2 & {}_{-1} \\
\cline{1-4}
5 & \color{blue}4 & \color{blue}3 & 2 \\
\cline{1-4}
6 & \color{blue}5 & 4 & 3 \\
\cline{1-4}
7 & 6 & 5 \\
\cline{1-3}
8 \\
\cline{1-1}
\end{array}, \quad S(w) = \begin{array}{ccccccc}
 & & & & \boxed{\color{blue}(4)} & \boxed{\color{blue}(5)} & \boxed{-6} \\
 & & \boxed{{}_{1}} & \boxed{\color{blue}(2)} & \boxed{\color{blue}(3)} & & \\
 & & \boxed{\color{blue}(2)} & \boxed{{}_{-1}} & & & \\
 & \boxed{\color{blue}(4)} & \boxed{\color{blue}(3)} & \boxed{2} & & & \\
 & \boxed{\color{blue}(5)} & \boxed{4} & \boxed{3} & & & \\
\boxed{7} & \boxed{6} & \boxed{5} & & & & \\
\boxed{8} & & & & & &
\end{array}.$$

In the figure for $S(w)$, for each $i = 2, 3, 4, 5$, the two squares $\boxed{\color{blue}(i)}$ together denote a certain one-dimensional subspace of the two-dimensional vector space corresponding to the two squares $\boxed{\color{blue}i}$ and $\boxed{\color{blue}-i}$ in the figure for $J(w)$. This one-dimensional vector space is actually $K\langle -i\rangle$, which is a subspace of $e_iS(w)$. Every other square $\boxed{i}$ in the figure for $S(w)$ denotes $K\langle i\rangle$, which is a subspace of $e_{|i|}S(w)$ if $i \leq -2$, and of $e_iS(w)$ if $i \geq -1$. We can check that this expression of the brick $S(w)$ can be rewritten as (3.3).

We mainly use such abbreviated expressions of bricks as (3.2) and (3.3) in the rest. Theorem 3.7 can be restated as follows by using the abbreviated expressions.

Corollary 3.10. *Let $w \in \mathsf{j\text{-}irr}\, W$ be a join-irreducible element of type l, and use the setting of Theorem 3.7. We express the brick $S(w)$ in the same abbreviation rules as Corollary 3.3. Then, there exist the following arrows, and no other arrows exist.*

(i) *For each $i \in V_+ \setminus \{\max V_+\}$, there exists an arrow $i \to |i|+1$ if $|i|+1 \in R$; and $i \leftarrow |i|+1$ otherwise.*
(ii) *For each $i \in V_- \setminus \{\min V_-\}$, there exists an arrow $i \leftarrow -(|i|+1)$ if $-(|i|+1) \in R$; and $i \to -(|i|+1)$ otherwise.*
(iii) *If $r \geq 1$, then for each $i \in V_-$ with $|i| \leq r$, there exists an arrow $-i \leftarrow -(|i|+1)$ if $|i|+1 \in R$; and $i \to |i|+1$ otherwise.*
(iv) *If $r = 0$, then there exists an arrow $-c \leftarrow 2$ if $c \leftarrow 2$ exists in* (i)*, and an arrow $c \to -2$ if $-c \to -2$ exists in* (ii)*.*

Proof. We remark that, for $i \in [1, r]$, the condition $-i \in R$ is equivalent to $i \notin R$. Then, Theorem 3.7 yields the assertion. $\square$

Unlike the case of type $\mathbb{A}_n$, for $w \in \mathsf{j\text{-}irr}\, W$, there may not exist a path algebra A of type $\mathbb{D}_n$ such that the brick $S(w)$ is an A-module. For example, the bricks obtained in Examples 3.8 and 3.9 cannot be modules over any path algebra of type $\mathbb{D}_n$, since the 2-cycle $\alpha_2\beta_3$ annihilates none of the two bricks. Our results imply that, if an element in Π is the product of some two 2-cycles, then it annihilates all the bricks in $\mathsf{brick}\, \Pi$.

We give more examples.

Example 3.11. In these examples, assume $n = 9$.

(1) Let $w = (3, 5, 8, -7, -4, 1, 2, 6, 9)$. Then, we have $l = 3$, $a = 8$, $b = -7$, $r = 2$, and $c = 1$. Thus, $(V_-, V_+) = ([-6, -1], [1, 7])$, and the desired brick $S(w)$ is written as

$$\begin{tikzcd} -1 \arrow[r] & -2 \arrow[r] \arrow[dl] \arrow[dr] & -3 & -4 \arrow[l] \arrow[r] & -5 \arrow[r] & -6 & \\ 1 \arrow[r] & 2 & 3 \arrow[l] & 4 \arrow[l] & 5 \arrow[l] \arrow[r] & 6 & 7 \arrow[l] \end{tikzcd}.$$

(2) Let $w = (1, 3, 5, 8, -7, -4, 2, 6, 9)$. Then, we have $l = 4$, $a = 8$, $b = -7$, $r = 0$, and $c = 1$. Thus, $(V_-, V_+) = ([-6, -1], [1, 7])$, and the desired brick $S(w)$ is written as

$$\begin{tikzcd} -1 \arrow[r] & -2 \arrow[r] & -3 & -4 \arrow[l] \arrow[r] & -5 \arrow[r] & -6 & \\ 1 \arrow[r] \arrow[ur] & 2 & 3 \arrow[l] & 4 \arrow[l] & 5 \arrow[l] \arrow[r] & 6 & 7 \arrow[l] \end{tikzcd}.$$

(3) Let $w = (1, 2, 3, 5, 8, -7, -4, 6, 9)$. Then, we have $l = 5$, $a = 8$, $b = -7$, $r = 0$, and $c = 1$. Thus, $(V_-, V_+) = ([-6, -1], [1, 7])$, and the desired brick $S(w)$ is written as

$$\begin{tikzcd}
-1 \arrow[r] & -2 \arrow[r] & -3 & -4 \arrow[l] \arrow[r] & -5 \arrow[r] & -6 & \\
1 \arrow[ur] & 2 \arrow[l] \arrow[ul] & 3 \arrow[l] & 4 \arrow[l] & 5 \arrow[l] \arrow[r] & 6 & 7 \arrow[l]
\end{tikzcd}.$$

We also have the list of bricks in the case $\Delta = \mathbb{D}_5$ in Appendix A.

Now, we start the proof of Theorem 3.7. We divide the argument by whether the type l of $w \in \mathsf{j\text{-}irr}\, W$ is ± 1 or not.

We first assume that $l = \pm 1$. We can restate Proposition 2.10 as follows.

Lemma 3.12. *Let $w \in \mathsf{j\text{-}irr}\, W$ be a join-irreducible element of type $l = \pm 1$.*

(1) *Assume $(i, j) \in \Gamma[l]$. Then, $p(i, j, l) \notin I(w)$ holds if and only if $i \geq w(|j| + 1)$.*
(2) *Consider the subset $\Gamma(w) \subset \Gamma[l]$ consisting of the elements $(i, j) \in \Gamma[l]$ with $p(i, j, l) \notin I(w)$. Then, the set $\{p(i, j, l) \mid (i, j) \in \Gamma(w)\}$ induces a K-basis of $J(w)$.*

In this lemma, we can replace the condition $i \geq w(|j|+1)$ by $|i| \geq w(|j|+1)$ in (1), since $w(m) = (-1)^{m-1} l$ holds for the number $m := |w^{-1}(1)|$.

To express $S(w)$, we define the following set for $k \geq 1$:

$$\Gamma_k(w) := \begin{cases} \{(i, j) \in \Gamma(w) \mid \mathsf{min}\{x \geq 1 \mid (i, |j| + x) \notin \Gamma(w)\} = k\} & (i \geq 2) \\ \{(i, j) \in \Gamma(w) \mid \mathsf{min}\{x \geq 1 \mid ((-1)^x i, |j| + x) \notin \Gamma(w)\} = k\} & (i = \pm 1) \end{cases}.$$

It is easy to see that $\Gamma(w)$ is the disjoint union of the $\Gamma_k(w)$'s. Moreover, we extend the definition of the path $p(i, j, l)$ to $\tilde{\Gamma}[l] := \{(i, j) \in Q_0 \times \mathbb{Z} \mid i \leq j \geq l\}$ by setting $p(i, j, l) := 0$ if $j \geq n$, and also define $w(k) := k$ if $k \geq n + 1$. In Example 3.8, the squares with black letters denote $\Gamma_1(w)$, and the squares with red letters denote $\Gamma_2(w)$.

Lemma 3.13. *Let $w \in \mathsf{j\text{-}irr}\, W$ be a join-irreducible element of type $l = \pm 1$. Consider the endomorphism $f := (\cdot p(l, 3, l)) \colon J(w) \to J(w)$.*

(1) *We have $S(w) = \mathsf{Ker}\, f$.*
(2) *Let $(i, j) \in \Gamma(w)$. Then, $p(i, j, l) \in \mathsf{Ker}\, f$ holds if and only if $(i, j) \in \Gamma_1(w) \amalg \Gamma_2(w)$.*
(3) *The set $\{p(i, j, l) \mid (i, j) \in \Gamma_1(w) \amalg \Gamma_2(w)\}$ induces a K-basis of $\mathsf{Ker}\, f$.*

Proof. Similar argument to Lemma 3.5 works. We remark that

$$f(p(i, j, l)) = p(i, j, l)p(l, 3, l) = p(i, |j| + 2, l)$$

hold in Π. □

Lemma 3.14. *Let $w \in \mathsf{j\text{-}irr}\, W$ be a join-irreducible element of type $l = \pm 1$. Define V_+ and V_- as in Theorem 3.1.*

(1) *There exists a bijection $\Gamma_1(w) \to V_+$ given by $(i, j) \mapsto i$.*
(2) *There exists a bijection $\Gamma_2(w) \to V_-$ given by $(i, j) \mapsto -i$ if $i \geq 2$; and $(i, j) \mapsto i$ if $i = \pm 1$.*

Proof. We use the notation in Theorem 3.7 in the proof.

(1) We see the well-definedness of the map $\Gamma_1(w) \to V_+$.

We first show that every $(i,j) \in \Gamma_1(w)$ satisfies that $i < a$. We remark that, for $k \in [2,n]$, the condition $w(k) = k$ holds if and only if $k > a$, and that this condition is also equivalent to $w(k) > a$. Lemma 3.12 and $(i,j) \in \Gamma(w)$ give $j \geq i \geq w(|j|+1)$. Thus, $w(|j|+1) \leq j$ holds, so we get $|j|+1 \leq a$, or equivalently, $|j| < a$. Therefore, we obtain $i \leq |j| < a$.

We also prove that, if $(i,j) \in \Gamma_1(w)$ and $|i| = 1$, then $i = c$ (*). Since $(i,j) \in \Gamma(w)$, we get $i = (-1)^{|j|-1}l$. In this case, Lemma 3.12 and $(i,j) \in \Gamma_1(w)$ yield $w(|j|+2) \geq 2$ and $i \geq w(|j|+1)$. If $w(|j|+1) \leq -2$, then $|w^{-1}(1)| = 1$, which contradicts $l = \pm 1$. We have $w(|j|+1) = \pm 1$. Thus, $|j|+1 = |w^{-1}(1)|$ holds; hence, we have $(-1)^{|j|-1}l = (-1)^{|w^{-1}(1)|}l = c$. Therefore, $i = c$.

Moreover, Lemma 3.12 and $(i,j) \in \Gamma(w)$ imply $j \geq i \geq w(|j|+1) \geq w(2) = b$.

These imply that the map $\Gamma_1(w) \to V_+$ is well-defined. By Lemma 3.12, it is clearly injective.

We next prove that the map $\Gamma_1(w) \to V_+$ is also surjective. Let $i \in V_+$, then $|i| < a$ holds. Thus, the first remark yields $w(|i|+1) < |i|+1$, so there exists some $j \in \{l\} \cup [2, n-1]$ such that $(i,j) \in \Gamma(w)$. Take the maximum j among such j's.

If $i \geq 2$, then (i,j) belongs to $\Gamma_1(w)$ by Lemma 3.12.

If $i = c$, then $(i,j) \in \Gamma(w)$ and $(i, |j|+2) \notin \Gamma(w)$ hold. On the other hand, we obtain $(-i, |j|+1) \notin \Gamma_1(w)$ from (*). From these, (i,j) must be in $\Gamma_1(w)$.

Therefore, $(i,j) \in \Gamma_1(w)$ holds, so the map $\Gamma_1(w) \to V_+$ is also surjective, and thus, bijective.

(2) First, for each $(i,j) \in \Gamma_2(w)$, we have $(i, |j|+1) \in \Gamma_1(w)$ if $i \geq 2$ and $(-i, |j|+1) \in \Gamma_1(w)$ if $i = \pm 1$. This correspondence gives a bijection

$$\Gamma_2(w) \to \{(i,j) \in \Gamma_1(w) \mid |j| > |i|\}.$$

Next, we show that, for any $(i,j) \in \Gamma_1(w)$, the conditions $|j| > |i|$ and $|i| < |b|$ are equivalent. Let $(i,j) \in \Gamma_1(w)$.

If $|j| > |i|$, then $|j| > w(|j|+1)$ must hold (otherwise, there exists no $(i,j) \in \Gamma(w)$ such that $|j| > |i|$) by Lemma 3.12, a contradiction), which implies that $|j| \leq |b|$ since $l = \pm 1$. Thus, $|i| < |b|$.

Conversely, if $|j| > |i|$ does not hold, then $|j| = |i|$. In this case, $w(|i|+2) = w(|j|+2) > |i|$, where the latter inequality comes from $(i,j) \in \Gamma_1(w)$. This means $w(|i|+2) \geq |i|+1$, which yields $|i|+1 > |b|$. Thus, $|i| \geq |b|$.

Therefore,

$$\{(i,j) \in \Gamma_1(w) \mid |j| > |i|\} = \{(i,j) \in \Gamma_1(w) \mid |i| < |b|\}.$$

By (1), we have a bijection

$$\{(i,j) \in \Gamma_1(w) \mid |i| < |b|\} \to \{i \in V_+ \mid |i| < |b|\},$$

and by definition, $i \to -i$ yields a bijection

$$\{i \in V_+ \mid |i| < |b|\} \to V_-.$$

The composite of these bijections is nothing but the map in the statement. $\square$

Now, we show Theorem 3.7 in the case $l = \pm 1$.

Proof of Theorem 3.7 in the case $l = \pm 1$. By Lemma 3.14, we can define a map $\rho \colon V \to Q_0$ as follows.

- If $i \in V_+$, then $\rho(i)$ is the unique element $j \in Q_0$ such that $(i,j) \in \Gamma_1(w)$.
- If $i \in V_-$ and $i = \pm 1$, then $\rho(i)$ is the unique element $j \in Q_0$ such that $(i,j) \in \Gamma_2(w)$.

- If $i \in V_-$ and $i \leq -2$, then $\rho(i)$ is the unique element $j \in Q_0$ such that $(|i|, j) \in \Gamma_2(w)$.

Set $\langle i \rangle := p(i, \rho(i), l)$ for each $i \in V$. It suffices to show that $(\langle i \rangle)_{i \in V}$ satisfies the properties (a), (b), and (c), since the three properties are enough to define an isomorphic class of Π-modules.

First, $(\langle i \rangle)_{i \in V}$ is a K-basis of $S(w)$ by Lemmas 3.13 and 3.14, and $K\langle i \rangle$ is clearly a subspace of $e_i S(w)$ if $i \geq -1$; and of $e_{|i|}S(w)$ if $i \leq -2$. Thus, the property (a) has been proved, and the property (b) follows from (a).

In the following observation, we fully use Lemma 3.12.

We begin the proof of (c)(i). First, we assume $2 \in V_+$, and set $j := \rho(2)$.

- If $2 \notin R$, then $w(j+1) = c$ and $w(j+2) \geq 3$ hold, so we have $(c, j) \in \Gamma_1(w)$ and $(-c, j) \notin \Gamma(w)$.
- If $2 \in R$, then $w(j+1) = 2$ holds, so we have $(c, j), (-c, j) \notin \Gamma(w)$.

These imply

$$\alpha_1 \langle 2 \rangle = \alpha_1 p(2, j, l) = p(c, j, l) + p(-c, j, l) = \begin{cases} \langle c \rangle & (2 \notin R) \\ 0 & (2 \in R) \end{cases}.$$

Second, let $i \in V_+ \setminus \{\max V_+\}$ and $i \geq 2$, and set $j := \rho(i+1)$. Then,

$$\alpha_i \langle i+1 \rangle = \alpha_i p(i+1, j, l) = p(i, j, l) = \begin{cases} \langle i \rangle & (\text{if } i+1 \notin R, \text{ since } (i, j) \in \Gamma_1(w)) \\ 0 & (\text{if } i+1 \in R, \text{ since } (i, j) \notin \Gamma(w)) \end{cases}.$$

Since $l = \pm 1$, we have $r \geq 1$. Thus, we have proved (c)(i).

Next, we begin the proof of (c)(ii). Let $i \in V_+ \setminus \{\max V_+\}$, and set $j := \rho(i)$. Then,

$$\begin{aligned} \beta_{|i|+1} \langle i \rangle &= \beta_{|i|+1} p(i, j, l) = p(|i|+1, j+1, l) \\ &= \begin{cases} 0 & (\text{if } i+1 \notin R, \text{ since } (|i|+1, j+1) \notin \Gamma(w)) \\ \langle |i|+1 \rangle & (\text{if } i+1 \in R, \text{ since } (|i|+1, j+1) \in \Gamma_1(w)) \end{cases}. \end{aligned}$$

Since $r \geq 1$, this implies (c)(ii).

Before continuing the proof, we remark the following: every $i \in V_-$ satisfies $|i| < r$, since $l = \pm 1$. Thus, if $i \in V_-$, then $|i| + 1 \notin R$ is equivalent to $-(|i|+1) \in R$.

We proceed to the proof of (c)(iii). First, assume $-c \in V_- \setminus \{\min V_-\}$, and set $j := \rho(-2)$.

- If $-2 \notin R$, then $w(j+1) = c$ and $w(j+2) = 2$ hold, so we have $(c, j) \in \Gamma_1(w)$ and $(-c, j) \notin \Gamma(w)$.
- If $-2 \in R$, then $w(j+1) = -2$ and $w(j+2) = c$ hold, so we have $(-c, j) \in \Gamma_1(w)$ and $(c, j) \notin \Gamma(w)$.

Thus,

$$\alpha_1 \langle -2 \rangle = \alpha_1 p(2, j, l) = p(c, j, l) + p(-c, j, l) = \begin{cases} \langle c \rangle & (-2 \notin R) \\ \langle -c \rangle & (-2 \in R) \end{cases}.$$

Second, let $i \in V_- \setminus \{\min V_-\}$ and $|i| \geq 2$, and set $j := \rho(-(|i|+1))$. Then,

$$\alpha_{|i|} \langle -(|i|+1) \rangle = \alpha_{|i|} p(|i|+1, j, l) = p(|i|, j, l)$$

$$= \begin{cases} \langle |i| \rangle = \langle -i \rangle & (\text{if } |i|+1 \in R, \text{ since } (|i|,j) \in \Gamma_1(w)) \\ \langle -|i| \rangle = \langle i \rangle & (\text{if } |i|+1 \notin R, \text{ since } (|i|,j) \in \Gamma_2(w)) \end{cases}.$$

These observations yield (c)(iii).

The remaining task is to check (c)(iv). Assume $i \in V_-$, and set $j := \rho(i)$. Then,

$$\beta_{|i|+1}\langle i \rangle = \beta_{|i|+1} p(i,j,l) = p(|i|+1, j+1, l)$$
$$= \begin{cases} \langle -(|i|+1) \rangle & (\text{if } |i|+1 \in R, \text{ since } (|i|+1,j+1) \in \Gamma_2(w)) \\ \langle |i|+1 \rangle & (\text{if } |i|+1 \notin R, \text{ since } (|i|+1,j+1) \in \Gamma_1(w)) \end{cases}.$$

Thus, we have obtained (c)(iv).

Now, all the desired properties have been proved. □

We next assume that the type l is not ± 1. We can restate Proposition 2.13 as follows.

Lemma 3.15. *Let $w \in \mathsf{j\text{-}irr}\, W$ be a join-irreducible element of type $l \neq \pm 1$, and set c as in Theorem 3.7. If $w(l+1) \leq 1$, then set $m := \max\{k \in [l+1,n] \mid w(k) \leq 1\}$ and $\varepsilon := (-1)^{m-(l+1)}c$; otherwise, set $\varepsilon := 1$.*

(1) *Assume $(i,j) \in \Gamma[l]$. Then, $p_\varepsilon(i,j,l) \notin I(w)$ holds if and only if*

$$\begin{cases} i \geq w(j+1) & (w(j+1) \geq 2) \\ i \geq 2 \quad \text{or} \quad i = w(j+1) & (w(j+1) = \pm 1) \\ i \geq w(j+1)+1 & (w(j+1) \leq -2) \end{cases}.$$

(2) *Consider the subset $\Gamma(w) \subset \Gamma[l]$ consisting of the elements $(i,j) \in \Gamma[l]$ with $p_\varepsilon(i,j,l) \notin I(w)$. Then, the set $\{p_\varepsilon(i,j,l) \mid (i,j) \in \Gamma(w)\}$ induces a K-basis of $J(w)$.*

To express $S(w)$, we define the following set for $k \geq 1$:

$$\Gamma_k(w) := \{(i,j) \in \Gamma(w) \mid \min\{x \geq 1 \mid (i,j+x) \notin \Gamma(w)\} = k\}.$$

Moreover, we extend the definition of the path $p_\varepsilon(i,j,l)$ to $\tilde{\Gamma}[l] := \{(i,j) \in \pm Q_0 \times \mathbb{Z} \mid i \leq j \geq l\}$ by setting $p_\varepsilon(i,j,l) := 0$ if $j \geq n$, and define $w(k) := k$ if $k \geq n+1$.

Then, straightforward calculation yields the relation

$$p_{-\varepsilon}(i,j,l) = -p_\varepsilon(i,j,l) + p_\varepsilon(|i|, j+|i|-1, l) \tag{3.4}$$

for $(i,j) \in \tilde{\Gamma}[l]$ with $i \leq -2$.

Lemma 3.16. *Let $w \in \mathsf{j\text{-}irr}\, W$ be a join-irreducible element of type $l \neq \pm 1$. Set R, a, b, r, c as in Theorem 3.7, and ε as in Lemma 3.15.*

(1) *Consider the endomorphisms*

$$f_1 := (\cdot p_\varepsilon(l, l+1, l)) \colon J(w) \to J(w) \quad \text{and} \quad f_2 := (\cdot p_\varepsilon(-l, l, l)) \colon J(w) \to J(w).$$

Then, $S(w) = \mathsf{Ker}\, f_1 \cap \mathsf{Ker}\, f_2$ holds.

(2) *Let $(i,j) \in \Gamma(w)$. Then, $p_{-\varepsilon}(i,j,l) \in \mathsf{Ker}\, f_1$ holds if and only if $(i,j) \in \Gamma_1(w)$.*

(3) *The set* $\{p_{-\varepsilon}(i,j,l) \mid (i,j) \in \Gamma_1(w)\}$ *induces a* K*-basis of* $\mathsf{Ker}\, f_1$.
(4) *Set* $\Lambda_1(w) := \{(i,j) \in \Gamma_1(w) \mid a-1 \geq i\}$. *Then, the set* $\{p_{-\varepsilon}(i,j,l) \mid (i,j) \in \Lambda_1(w)\}$ *induces a* K*-basis of* $S(w)$.
(5) *Assume* $b \leq -2$ *and* $r \geq 1$, *and let* $(i,j) \in \Gamma_1(w)$ *with* $-2 \geq i \geq b+1$. *Then,* $p_{-\varepsilon}(|i|, j+|i|-1, l) \notin \mathsf{Ker}\, f_1$ *holds if and only if* $|i| \leq r$. *In this case,* $(|i|, j+|i|-1)$ *belongs to* $\Gamma_2(w)$.
(6) *The submodule* $\mathsf{Ker}\, f_1 \cap \mathsf{Ker}\, f_2$ *has a basis formed by*
   - $p_\varepsilon(i,j,l)$ *for each* $(i,j) \in \Lambda_1(w)$ *with* $i \geq -1$ *or* $-r-1 \geq i$; *and*
   - $p_{-\varepsilon}(i,j,l)$ *for each* $(i,j) \in \Lambda_1(w)$ *with* $-2 \geq i \geq -r$.

Proof. The proofs of (1), (2) and (3) are similar to Lemma 3.5. We remark that $f_1(p_{-\varepsilon}(i,j,l)) = p_\varepsilon(i,j+1,l)$ holds in Π.

(4) Let $(i,j) \in \Gamma_1(w)$. We show that $p_{-\varepsilon}(i,j,l) \in \mathsf{Ker}\, f_2$ holds if and only if $i \leq w(l)-1$.

We first assume that $i \geq 2$. In this case, $f_2(p_{-\varepsilon}(i,j,l)) = p_{-\varepsilon}(i,j,l)p_\varepsilon(-l,l,l) = p_\varepsilon(-i, l+j-i, l)$ hold. Thus, $f_2(p_{-\varepsilon}(i,j,l)) = 0$ in $J(w)$ holds if and only if $p_\varepsilon(-i, l+j-i, l) \in \Pi$ belongs to $I(w)$. This is equivalent to $w(l+j-i+1)+1 > -i$ by Lemma 3.15, and also to $\#(R \cap [-n, -i-1]) < j-i+1$.

On the other hand, $(i,j) \in \Gamma_1(w)$ gives $w(j+2)-1 \geq i \geq w(j+1)$, because $i \geq 2$. This implies that $\#(R \cap [-n, i]) = j+1-l$.

Therefore, $f_2(p_{-\varepsilon}(i,j,l)) = 0$ in $J(w)$ holds if and only if $\#(R \cap [-i,i]) > i-l$. This condition is equivalent to that $\#(w([1,l]) \cap [-i,i]) < l$. This exactly means that there exists some $k \in [1,l]$ such that $|w(k)| > i$, and it is equivalent to $a > i$.

Now, the proof for $i \geq 2$ is complete.

Next, we assume that $i = \pm 1$. We must show $p_{-\varepsilon}(i,j,l) \in \mathsf{Ker}\, f_2$. In this case,

$$f_2(p_{-\varepsilon}(i,j,l)) = p_{-\varepsilon}(i,j,l)p_\varepsilon(-l,l,l) = \alpha_i p_\varepsilon(2,j,l)p_\varepsilon(-l,l,l) = \alpha_i p_\varepsilon(-2, l+j-2, l)$$
$$= \begin{cases} p_\varepsilon(i, l+j-1, l) & (i = \varepsilon(-1)^j) \\ 0 & (i = -\varepsilon(-1)^j) \end{cases},$$

since $p_\varepsilon(-2, l+j-2, l)$ factors through $\varepsilon(-1)^{j-1}$.

Thus, we may assume $i = \varepsilon(-1)^j$. First, $p_\varepsilon(i, l+j-1, l) \in I(w)$ is equivalent to that $(w(l+j) \geq 2$ or $w(l+j) = -i)$ by Lemma 3.15. On the other hand, $(i,j) \in \Gamma_1(w)$ implies that $w(j+2) \geq 2$ or $w(j+2) = -i$. Since $l \geq 2$, we have $(w(l+j) \geq 2$ or $w(l+j) = -i)$. Therefore, $p_\varepsilon(i, l+j-1, l) \in I(w)$.

Consequently, $i = \pm 1$ implies that $p_{-\varepsilon}(i,j,l) \in \mathsf{Ker}\, f_2$.

Finally, we assume that $i \leq -2$. Then, $p_{-\varepsilon}(i,j,l) \in \mathsf{Ker}\, f_2$ holds, because the path $p_{-\varepsilon}(i,j,l)$ has $p_{-\varepsilon}(1,j,l)$ or $p_{-\varepsilon}(-1,j,l)$ in its ending.

Now, we have proved that $p_{-\varepsilon}(i,j,l) \in \mathsf{Ker}\, f_2$ holds if and only if $i \leq a-1$, and obtained that $f_2(p_{-\varepsilon}(i,j,l)) = p_\varepsilon(-i, l+j-i, l) \neq 0$ in $J(w)$ if $(i,j) \in \Gamma_1(w)$ and $i \geq a$.

Thus, the set $\{f_2(p_{-\varepsilon}(i,j,l)) \mid (i,j) \in \Gamma_1(w) \setminus \Lambda_1(w)\}$ is linearly independent in $J(w)$, so $\{p_{-\varepsilon}(i,j,l) \mid (i,j) \in \Lambda_1(w)\}$ generates $\mathsf{Ker}\, f_1 \cap \mathsf{Ker}\, f_2$. This set is clearly linearly independent in $J(w)$. Therefore, we obtain the assertion from (1).

(5) Let $(i,j) \in \Gamma_1(w)$ with $-2 \geq i \geq b+1$.

For the first statement, it is easy to see that $f_1(p_{-\varepsilon}(|i|, j+|i|-1, l)) = p_\varepsilon(|i|, j+|i|, l)$ in Π, so $p_{-\varepsilon}(|i|, j+|i|-1, l) \notin \mathsf{Ker}\, f_1$ precisely means $p_\varepsilon(|i|, j+|i|, l) \notin I(w)$ in Π. Lemma 3.15 yields that this holds if and only if $w(j+|i|+1) \leq |i|$, because $|i| \geq 2$. It is equivalent to $\#(R \cap [-n, |i|]) \geq j+|i|+1-l$.

On the other hand, $(i,j) \in \Gamma_1(w)$ gives $w(j+2) \geq i = -|i| \geq w(j+1)+1$, because $i \leq -2$. This implies that $\#(R \cap [-n, -|i|-1]) = j+1-l$.

Therefore, $f_1(p_{-\varepsilon}(|i|, j+|i|-1, l)) \neq 0$ in $J(w)$ holds if and only if $\#(R \cap [-|i|, |i|]) \geq |i|$. This exactly means $[1, |i|] \subset \pm R$. By the definition of the number r, it is equivalent to $|i| \leq r$. In this case, $\#(R \cap [-|i|, |i|]) = |i|$.

The first statement has been proved.

Next, we show the second statement, so we assume $|i| \leq r$. It suffices to prove $(|i|, j+|i|) \in \Gamma(w)$ and $(|i|, j+|i|+1) \notin \Gamma(w)$. We already have $\#(R \cap [-|i|, |i|]) = |i|$, and by the argument above, this yields $\#(R \cap [-n, |i|]) = j+|i|+1-l$. Thus, we have $w(j+|i|+1) \leq |i|$ and $w(j+|i|+2) > |i|$. Since $|i| \geq 2$, Lemma 3.15 implies that $(|i|, j+|i|) \in \Gamma(w)$ and $(|i|, j+|i|+1) \notin \Gamma(w)$. Thus, $(|i|, j+|i|-1)$ belongs to $\Gamma_2(w)$.

(6) In (4), $p_{-\varepsilon}(i,j,l) = p_\varepsilon(i,j,l)$ holds for each $(i,j) \in \Lambda_1(w)$ with $i \geq -1$.

On the other hand, let $(i,j) \in \Lambda_1(w)$ with $i < -r$ and $i \leq -2$. By (3), $p_{-\varepsilon}(i,j,l) \in \mathsf{Ker}\, f_1$. By (5), we have $p_{-\varepsilon}(|i|, j+|i|-1, l) \in \mathsf{Ker}\, f_1$.

If $p_{-\varepsilon}(|i|, j+|i|-1, l) \neq 0$ in $J(w)$, then $\#(R \cap [-|i|, |i|]) \geq |i|-1$ follows from similar argument to the proof of the first statement of (5). This implies $|i| < a$, since $l \geq 2$. We have $(|i|, j+|i|-1, l) \in \Lambda_1(w)$. Thus, in the K-basis of $\mathsf{Ker}\, f_1 \cap \mathsf{Ker}\, f_2$ given in (4), we can replace $p_{-\varepsilon}(i,j,l)$ to $p_\varepsilon(i,j,l)$ to obtain another K-basis of $S(w)$ by (3.4).

If $p_{-\varepsilon}(|i|, j+|i|-1, l) = 0$ in $J(w)$, then $p_{-\varepsilon}(i,j,l) = p_\varepsilon(i,j,l)$ holds.

We apply this procedure to all $(i,j) \in \Lambda_1(w)$ with $i < -r$ and $i \leq -2$, and get that the elements in the statement form a K-basis of $\mathsf{Ker}\, f_1 \cap \mathsf{Ker}\, f_2$. □

The next assertion follows from the definition of $\Lambda_1(w)$.

Lemma 3.17. *Let $w \in \mathsf{j\text{-}irr}\, W$ be a join-irreducible element of type $l \neq \pm 1$. Then, there exists a bijection $\Lambda_1(w) \to V$ given by $(i,j) \mapsto i$.*

Proof. The well-definedness can be checked by Lemma 3.16.

We clearly have $\mathsf{max}\{k \in [l+1, n] \mid w(k) < k\} - 1 \geq a-1$. Then, Lemma 3.15 and the definition of V yield that, for any $i \in V$, there exists some j such that $(i,j) \in \Gamma(w)$. Thus, the definition of $\Lambda_1(w)$ and $i \leq a-1$ imply that there uniquely exists j such that $(i,j) \in \Lambda_1(w)$. This means that the map $\Lambda_1(w) \to V$ is bijective. □

Now, we show Theorem 3.7 in the case $l \neq \pm 1$.

Proof of Theorem 3.7 in the case $l \neq \pm 1$. By Lemma 3.17, we can define a map $\rho\colon V \to Q_0$ as follows: $\rho(i)$ is the unique element $j \in Q_0$ such that $(i,j) \in \Lambda_1(w)$. Set ε as in Lemma 3.15, and define $\langle i \rangle$ as

$$\begin{cases} p_\varepsilon(i, \rho(i), l) & (i \geq -1 \text{ or } -r-1 \geq i) \\ p_{-\varepsilon}(i, \rho(i), l) & (-2 \geq i \geq -r) \end{cases}$$

for each $i \in V$. It suffices to show that $(\langle i \rangle)_{i \in V}$ satisfies the properties (a), (b), and (c), since the three properties are enough to define an isomorphic class of Π-modules.

First, $(\langle i \rangle)_{i \in V}$ is a K-basis of $S(w)$ by Lemmas 3.16 and 3.17, and $K\langle i \rangle$ is clearly a subspace of $e_i S(w)$ if $i \geq -1$; and of $e_{|i|} S(w)$ if $i \leq -2$. Thus, the property (a) has been obtained, and the property (b) follows from (a).

In the rest, we fully use Lemma 3.15.

We begin the proof of (c)(i). First, we assume $2 \in V_+$, and set $j := \rho(2)$.

- If $2 \notin R$ and $r \geq 1$, then $w(j+1) = c$. Thus, $(c, j) \in \Lambda_1(w)$ and $(-c, j) \notin \Gamma(w)$ follow.
- If $2 \notin R$ and $r = 0$, then $w(j+1) \leq -2$. Thus, $(c,j), (-c,j) \in \Lambda_1(w)$ follows.
- If $2 \in R$, then $w(j+1) = 2$. Thus, $(c,j), (-c,j) \notin \Gamma(w)$ follows.

Therefore,

$$\alpha_1\langle 2\rangle = \alpha_1 p_\varepsilon(2,j,l) = p_\varepsilon(c,j,l) + p_\varepsilon(-c,j,l) = \begin{cases} \langle c\rangle & (2 \notin R,\ r \geq 1) \\ \langle c\rangle + \langle -c\rangle & (2 \notin R,\ r = 0) \\ 0 & (2 \in R) \end{cases}.$$

Second, we assume $i \in V_+ \setminus \{\max V_+\}$ and $i \geq 2$, and set $j := \rho(i+1)$. Then,

$$\alpha_i\langle i+1\rangle = \alpha_i p_\varepsilon(i+1,j,l) = p_\varepsilon(i,j,l) = \begin{cases} \langle i\rangle & (\text{if } i+1 \notin R, \text{ since } (i,j) \in \mathit{\Gamma}_1(w)) \\ 0 & (\text{if } i+1 \in R, \text{ since } (i,j) \notin \mathit{\Gamma}(w)) \end{cases}.$$

Thus, we have the property (c)(i).

We begin the proof of (c)(ii). First, let $c \in V_+ \setminus \{\max V_+\}$, and set $j := \rho(c)$. In this case, $w(j+1) \leq 1$, $w(j+2) \geq 2$, and $\varepsilon = (-1)^{j-l}c$ hold, so the path $p_\varepsilon(-2,j,l)$ factors through $-c$. We observe the following properties.

- If $-2 \notin R$ and $r = 0$, then $(-2,j) \in \mathit{\Lambda}_1(w)$; otherwise $(-2,j) \notin \mathit{\Gamma}(w)$.
- If $2 \in R$, then $(2,j+1) \in \mathit{\Lambda}_1(w)$; otherwise $(2,j+1) \notin \mathit{\Gamma}(w)$.

Thus, we have

$$\begin{aligned} \beta_2\langle c\rangle &= \beta_2 p_\varepsilon(c,j,l) = p_{-\varepsilon}(-2,j,l) \\ &= -p_\varepsilon(-2,j,l) + p_\varepsilon(2,j+1,l) = \eta_c^-\langle -2\rangle + \eta_c^+\langle 2\rangle. \end{aligned}$$

Second, let $i \in V_+ \setminus \{\max V_+\}$ and $i \geq 2$, and set $j := \rho(i)$. Then,

$$\begin{aligned} \beta_{|i|+1}\langle i\rangle &= \beta_{|i|+1} p_\varepsilon(i,j,l) = p_\varepsilon(|i|+1,j+1,l) \\ &= \begin{cases} 0 & (\text{if } i+1 \notin R, \text{ since } (|i|+1,j+1) \notin \mathit{\Gamma}(w)) \\ \langle |i|+1\rangle & (\text{if } i+1 \in R, \text{ since } (|i|+1,j+1) \in \mathit{\Gamma}_1(w)) \end{cases}. \end{aligned}$$

These observations imply the property (c)(ii).

We next consider the elements in V_-. In order to observe the actions of the arrows to $\langle -i\rangle$ ($i \in [2,r]$), we define sets $\mathit{\Omega}(w)$ and $\mathit{\Lambda}_2(w)$ as

$$\mathit{\Omega}(w) := \{(i,j) \in \mathit{\Lambda}_1(w) \mid i \in [-r,-2]\}, \quad \mathit{\Lambda}_2(w) := \{(i,j) \in \mathit{\Gamma}_2(w) \mid i \in [2,r]\}.$$

The element $\langle -i\rangle$ is equal to the path $p_{-\varepsilon}(i,j,l)$ with $(i,j) \in \mathit{\Omega}(w)$, but we want to deal with the paths of the form $p_\varepsilon(i',j',l)$. In the formula (3.4), $p_{-\varepsilon}(i,j,l)$ is a linear combination of $p_\varepsilon(i,j,l)$ and $p_\varepsilon(|i|,j+|i|-1,l)$. By Lemma 3.16 (5), $(|i|,j+|i|-1)$ belongs to $\mathit{\Lambda}_2(w)$. Moreover, $\phi\colon \mathit{\Omega}(w) \ni (i,j) \mapsto (|i|,j+|i|-1) \in \mathit{\Lambda}_2(w)$ is a bijection.

In the figure for $J(w)$ in Example 3.9, the squares with positive blue numbers are the elements of $\mathit{\Lambda}_2(w)$, and that the squares with negative blue numbers are the elements of $\mathit{\Omega}(w)$.

Now, we begin the proof of (c)(iii). We first assume $-c \in V_- \setminus \{\min V_-\}$, and set $j := \rho(-2)$.

- If $-2 \in R$ and $r \geq 2$, then $w(j+1) \leq -3$, $w(j+2) = -2$, $w(j+3) = c$, $w(j+4) \geq 3$, and $\varepsilon = (-1)^{j+2-l}c$ hold, so the path $p_\varepsilon(-2,j,l)$ factors through $-c$, and $(-c,j+1) \in \mathit{\Lambda}_1(w)$ follows. Thus,

$$\begin{aligned}\alpha_1\langle -2\rangle &= \alpha_1 p_{-\varepsilon}(-2,j,l) = -\alpha_1 p_\varepsilon(-2,j,l) + \alpha_1 p_\varepsilon(2,j+1,l)\\ &= -p_\varepsilon(c,j+1,l) + (p_\varepsilon(c,j+1,l) + p_\varepsilon(-c,j+1,l))\\ &= p_\varepsilon(-c,j+1,l) = \langle -c\rangle.\end{aligned}$$

- If $-2 \in R$ and $r = 0$, then $w(j+1) \le -3$, $w(j+2) = -2$, $w(j+3) \ge 3$, and $\varepsilon = (-1)^{j+1-l}c$ hold, so the path $p_\varepsilon(-2,j,l)$ factors through c, and $(-c,j+1) \in \Lambda_1(w)$ follows. Thus,

$$\alpha_1\langle -2\rangle = \alpha_1 p_\varepsilon(-2,j,l) = p_\varepsilon(-c,j+1,l) = \langle -c\rangle.$$

- If $-2 \notin R$ and $r \ge 2$, then $w(j+1) \le -3$, $w(j+2) = c$, $w(j+3) = 2$, and $\varepsilon = (-1)^{j+1-l}c$ hold, so the path $p_\varepsilon(-2,j,l)$ factors through c, and $(c,j+1) \in \Lambda_1(w)$ follows. Thus,

$$\begin{aligned}\alpha_1\langle -2\rangle &= \alpha_1 p_{-\varepsilon}(-2,j,l) = -\alpha_1 p_\varepsilon(-2,j,l) + \alpha_1 p_\varepsilon(2,j+1,l)\\ &= -p_\varepsilon(-c,j+1,l) + (p_\varepsilon(c,j+1,l) + p_\varepsilon(-c,j+1,l))\\ &= p_\varepsilon(c,j+1,l) = \langle c\rangle.\end{aligned}$$

- If $-2 \notin R$ and $r = 1$, then $w(j+1) \le -3$, $w(j+2) = c$, $w(j+3) \ge 3$, and $\varepsilon = (-1)^{j+1-l}c$ hold, so the path $p_\varepsilon(-2,j,l)$ factors through c, and $(-c,j+1) \notin \Gamma(w)$ follows. Thus,

$$\alpha_1\langle -2\rangle = \alpha_1 p_\varepsilon(-2,j,l) = p_\varepsilon(-c,j+1,l) = 0.$$

- If $-2 \notin R$ and $r = 0$, then $w(j+1) \le -3$, $w(j+2) \ge 2$, and $\varepsilon = (-1)^{j-l}c$ hold, so the path $p_\varepsilon(-2,j,l)$ factors through $-c$, and $(c,j+1) \notin \Gamma(w)$ follows. Thus,

$$\alpha_1\langle -2\rangle = \alpha_1 p_\varepsilon(-2,j,l) = p_\varepsilon(c,j+1,l) = 0.$$

Second, we assume $i \in V_- \setminus \{\min V_-\}$ and $|i| \ge 2$, and set $j := \rho(-(|i|+1))$.

- If $|i| < r$, then $(-(|i|+1),j) \in \Omega(w)$ and $\phi(-(|i|+1),j) = (|i|+1, j+|i|) \in \Lambda_2(w)$ hold, and

$$\begin{aligned}\alpha_{|i|}\langle -(|i|+1)\rangle &= \alpha_{|i|} p_{-\varepsilon}(-(|i|+1),j,l)\\ &= -\alpha_{|i|} p_\varepsilon(-(|i|+1),j,l) + \alpha_{|i|} p_\varepsilon(|i|+1,j+|i|,l)\\ &= -p_\varepsilon(-|i|,j+1,l) + p_\varepsilon(|i|,j+|i|,l)\\ &= \begin{cases}\langle -|i|\rangle & (\text{if } -(|i|+1) \in R, \text{ since } (-|i|,j+1) \in \Omega(w))\\ p_\varepsilon(|i|,j+|i|,l) & (\text{if } -(|i|+1) \notin R, \text{ since } (-|i|,j+1) \notin \Gamma(w))\end{cases}\\ &= \begin{cases}\langle i\rangle & (\text{if } -(|i|+1) \in R)\\ \langle -i\rangle & (\text{if } -(|i|+1) \notin R, \text{ since } |i|+1 \in R \text{ and } (|i|,j+|i|) \in \Lambda_1(w))\end{cases}.\end{aligned}$$

- If $|i| \ge r$, then

$$\begin{aligned}\alpha_i\langle -(|i|+1)\rangle &= \alpha_i p_\varepsilon(-(|i|+1),j,l) = p_\varepsilon(-|i|,j+1,l)\\ &= \begin{cases}\langle -|i|\rangle = \langle i\rangle & (\text{if } -(|i|+1) \in R, \text{ since } (-|i|,j+1) \in \Lambda_1(w))\\ 0 & (\text{if } -(|i|+1) \notin R, \text{ since } (-|i|,j+1) \notin \Gamma(w))\end{cases}.\end{aligned}$$

These observations and the definition of r tell us that (c)(iii) holds.

Finally, we would like to show the property (c)(iv). First, we assume $-c \in V_-$, and set $j := \rho(-c)$.

- If $r \geq 2$, then $w(j+1) \leq -2$, $w(j+2) = c$, $w(j+3) \geq 2$, and $\varepsilon = (-1)^{j+1-l}c$ hold, so the path $p_\varepsilon(-2,j,l)$ factors through c. Thus,

$$\begin{aligned}\beta_2\langle -c\rangle &= \beta_2 p_\varepsilon(-c,j,l) = -p_\varepsilon(-2,j,l) + p_\varepsilon(2,j+1,l) \\ &= \begin{cases} p_\varepsilon(2,j+1,l) & (\text{if } -2 \in R, \text{ since } (-2,j) \notin \Gamma(w)) \\ p_{-\varepsilon}(-2,j,l) & (\text{if } -2 \notin R) \end{cases} \\ &= \begin{cases} \langle 2\rangle & (\text{if } -2 \in R, \text{ since } 2 \notin R \text{ and } (2,j+1) \in \Lambda_1(w)) \\ \langle -2\rangle & (\text{if } -2 \notin R, \text{ since } (-2,j) \in \Omega(w)) \end{cases}.\end{aligned}$$

- If $r = 1$, the path $p_\varepsilon(-2,j,l)$ factors through c by the same reason as above. Since $r = 1$, we have $-2, 2 \notin R$, so $(-2,j), (2,j+1) \in \Lambda_1(w)$. Thus,

$$\beta_2\langle -c\rangle = \beta_2 p_\varepsilon(-c,j,l) = -p_\varepsilon(-2,j,l) + p_\varepsilon(2,j+1,l) = -\langle -2\rangle + \langle 2\rangle.$$

- If $r = 0$, then $w(j+1) \leq -2$, $w(j+2) \geq 2$, and $\varepsilon = (-1)^{j-l}c$ hold, so the path $p_\varepsilon(-2,j,l)$ factors through $-c$. Thus,

$$\begin{aligned}\beta_2\langle -c\rangle &= \beta_2 p_\varepsilon(-c,j,l) = p_\varepsilon(-2,j,l) \\ &= \begin{cases} 0 & (\text{if } -2 \in R, \text{ since } (-2,j) \notin \Gamma(w)) \\ \langle -2\rangle & (\text{if } -2 \notin R, \text{ since } (-2,j) \in \Lambda_1(w)) \end{cases}.\end{aligned}$$

Second, we assume $i \in V_-$, $|i| \geq 2$, and set $j := \rho(i)$.

- If $|i| < r$, then $(i,j) \in \Omega(w)$ and $\phi(i,j) = (|i|, j+|i|-1) \in \Lambda_2(w)$ hold, so

$$\begin{aligned}\beta_{|i|+1}\langle i\rangle &= \beta_{|i|+1} p_{-\varepsilon}(i,j,l) = -\beta_{|i|+1} p_\varepsilon(i,j,l) + \beta_{|i|+1} p_\varepsilon(|i|, j+|i|-1, l) \\ &= -p_\varepsilon(-(|i|+1), j, l) + p_\varepsilon(|i|+1, j+|i|, l) \\ &= \begin{cases} p_\varepsilon(|i|+1, j+|i|, l) & (\text{if } -(|i|+1) \in R, \text{ since } (-(|i|+1), j) \notin \Gamma(w)) \\ p_{-\varepsilon}(-(|i|+1), j, l) & (\text{if } -(|i|+1) \notin R) \end{cases} \\ &= \begin{cases} \langle |i|+1\rangle & \begin{pmatrix} \text{if } -(|i|+1) \in R, \\ \text{since } |i|+1 \notin R \text{ and } (|i|+1, j+|i|) \in \Lambda_1(w) \end{pmatrix} \\ \langle -(|i|+1)\rangle & (\text{if } -(|i|+1) \notin R, \text{ since } (-(|i|+1), j) \in \Omega(w)) \end{cases}.\end{aligned}$$

- If $|i| = r$, then $(i,j) \in \Omega(w)$ and $\phi(i,j) = (|i|, j+|i|-1) \in \Lambda_2(w)$ hold. Since $|i| = r$, we have $-(|i|+1), |i|+1 \notin R$, so $(|i|+1, j+|i|), (-(|i|+1), j) \in \Lambda_1(w)$ hold. Thus,

$$\begin{aligned}\beta_{|i|+1}\langle i\rangle &= \beta_{|i|+1} p_{-\varepsilon}(i,j,l) = -\beta_{|i|+1} p_\varepsilon(i,j,l) + \beta_{|i|+1} p_\varepsilon(|i|, j+|i|-1, l) \\ &= -p_\varepsilon(-(|i|+1), j, l) + p_\varepsilon(|i|+1, j+|i|, l) = -\langle -(|i|+1)\rangle + \langle |i|+1\rangle.\end{aligned}$$

- If $|i| > r$, then

$$\beta_{|i|+1}\langle i\rangle = \beta_{|i|+1}p_\varepsilon(i,j,l) = p_\varepsilon(-(|i|+1),j,l)$$
$$= \begin{cases} 0 & (\text{if } -(|i|+1) \in R, \text{ since } (-(|i|+1),j) \notin \varGamma(w)) \\ \langle -(|i|+1)\rangle & (\text{if } -(|i|+1) \notin R, \text{ since } (-(|i|+1),j) \in \varLambda_1(w)) \end{cases}.$$

The property (c)(iv) follows from these observations and the definition of r.

Now, all the proof is complete. □

4. Description of semibricks

4.1. Canonical join representations in Coxeter groups

Let $\varDelta$ be a Dynkin diagram $\mathbb{A}_n$ or $\mathbb{D}_n$, and $\varPi$ and W be the corresponding preprojective algebra and the Coxeter group, respectively. We obtained a canonical bijection $S(?)\colon W \to \mathsf{sbrick}\,\varPi$ in Proposition 2.2. The aim of this section is to give the explicit description of this map. In the previous section, this aim has been achieved for the restricted bijection $S(?)\colon \mathsf{j\text{-}irr}\,W \to \mathsf{brick}\,\varPi$. To extend this to all elements in W, it is enough to determine the canonical join representations in W for $\varDelta = \mathbb{A}_n, \mathbb{D}_n$ by Corollary 2.3.

It would be difficult to prove that a set of join-irreducible elements gives a canonical join representation of a given element in W by directly checking the conditions in Definition 1.7. Fortunately, Reading [23] has obtained a nice property characterizing canonical join representations in finite Coxeter groups. To explain this, we prepare some notation.

Let $\varDelta_0$ be the vertices set of $\varDelta$. Then, W has the canonical generators $\{s_i \mid i \in \varDelta_0\}$. For each $w \in W$, set $\mathsf{des}(w)$ and $\mathsf{cov}(w)$ as the set of *descents* and the set of *cover reflections* of w, respectively: that is,

$$\mathsf{des}(w) := \{i \in \varDelta_0 \mid ws_i < w\}, \quad \mathsf{cov}(w) := \{ws_iw^{-1} \mid i \in \mathsf{des}(w)\}.$$

There exists a natural bijection $\mathsf{des}(w) \to \mathsf{cov}(w)$ defined by $i \mapsto ws_iw^{-1}$. By using the set $\mathsf{cov}(w)$, we can write the canonical join representation of w as follows.

Proposition 4.1. *[23, Theorem 10-3.9] Let $w \in W$. For each $t \in \mathsf{cov}(w)$, the set $\{v \in W \mid v \le w,\ t \in \mathsf{inv}(v)\}$ has a unique minimal element w_t. Moreover, $\bigvee_{t\in\mathsf{cov}(w)} w_t$ is the canonical join representation of w.*

Hence, we have the following way to find canonical join representations.

Proposition 4.2. *Let $w \in W$. Assume that, for each $d \in \mathsf{des}(w)$, there exists a join-irreducible element $w_d \in \mathsf{j\text{-}irr}\,W$ satisfying $w_d \le w$ and $\mathsf{cov}(w_d) = \{ws_dw^{-1}\}$. Then, $\bigvee_{d\in\mathsf{des}(w)} w_d$ is the canonical join representation of w.*

Proof. Let $d \in \mathsf{des}(w)$ and set $t := ws_dw^{-1} \in \mathsf{cov}(w)$. By Proposition 4.1, it suffices to show that w_d is a minimal element of $V := \{v \in W \mid v \le w,\ t \in \mathsf{inv}(v)\}$. We assume that $v \in V$ satisfies $v < w_d$ and deduce a contradiction. Take the unique descent d' of $w_d \in \mathsf{j\text{-}irr}\,W$, then $t = w_ds_{d'}w_d^{-1}$ holds.

Since $w_ds_{d'} = tw_d$ and $d' \in \mathsf{des}(w_d)$, we get $l(t{\cdot}w_ds_{d'}) = l(t{\cdot}tw_d) = l(w_d) > l(w_ds_{d'})$. Thus, $t \notin \mathsf{inv}(w_ds_{d'})$.

On the other hand, the inequality $v \le w_ds_{d'}$ holds, since w_d is a join-irreducible element with its unique descent d'. Thus, we have $\mathsf{inv}(v) \subset \mathsf{inv}(w_ds_{d'})$. By assumption, t belongs to $\mathsf{inv}(v)$, so t must be in $\mathsf{inv}(w_ds_{d'})$.

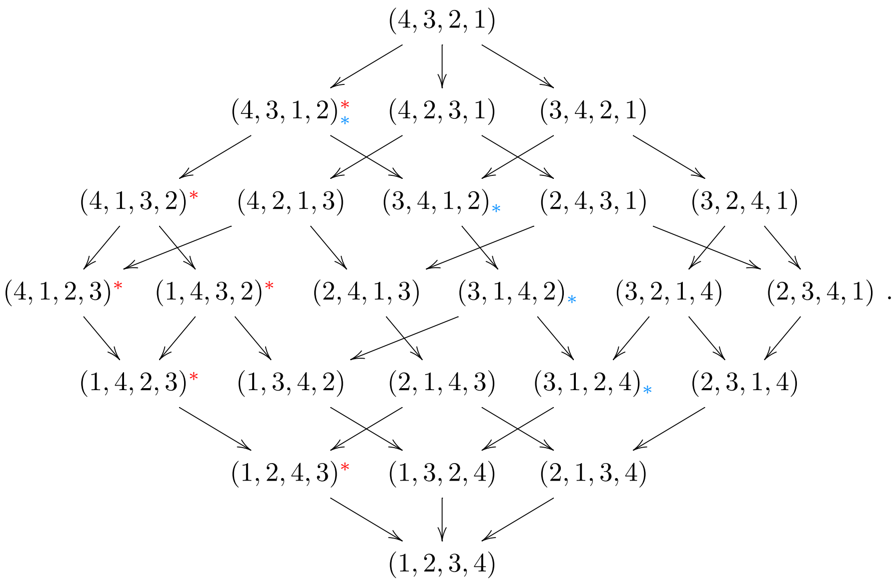
These two results contradict to each other. Thus, there exists no $v \in V$ such that $v < w_d$. This exactly means that w_d is a minimal element of V. □

We remark that $w_d \neq w'_d$ for any $d \neq d' \in \mathsf{des}(w)$, because $ws_dw^{-1} \neq ws_{d'}w^{-1}$. The proposition above also implies that the uniqueness of w_d, since the canonical join representation of w is unique.

Before proceeding to the next subsection, we give an example of canonical join representations. We recall that the *Hasse quiver* of W is defined as follows.

- The vertices are the elements of W.
- For any $w, w' \in W$, we write an arrow $w \to w'$ if and only if $w > w'$ holds and there exists no $v \in W$ such that $w > v > w'$.

Example 4.3. Let $\Delta = \mathbb{A}_3$. Then, the Hasse quiver of W is



We determine the canonical join representation of the element $w := (4, 3, 1, 2)$ from the Hasse quiver. In this case, we have $\mathsf{des}(w) = \{1, 2\}$ and $\mathsf{cov}(w) = \{(4\ \ 3), (3\ \ 1)\}$. Thus, we consider the following sets:

- $\{v \in W \mid v \leq w,\ (4\ \ 3) \in \mathsf{inv}(v)\}$, whose elements are indicated by *; and
- $\{v \in W \mid v \leq w,\ (3\ \ 1) \in \mathsf{inv}(v)\}$, whose elements are indicated by $_*$.

These sets have $(1, 2, 4, 3)$ and $(3, 1, 2, 4)$ as their unique minimal elements, respectively. By Proposition 4.1, the canonical join representation of w is $(1, 2, 4, 3) \vee (3, 1, 2, 4)$. We also remark that $\mathsf{cov}((1, 2, 4, 3)) = \{(4\ \ 3)\}$ and $\mathsf{cov}((3, 1, 2, 4)) = \{(3\ \ 1)\}$ hold.

4.2. Type $\mathbb{A}_n$

Let $\Delta = \mathbb{A}_n$. For each element w in $\mathsf{j\text{-}irr}\, W$ of type l, we set

$$L(w) := w([1, l]), \quad R(w) := w([l + 1, n + 1]).$$

It is easy to see that the correspondence $w \mapsto R(w)$ is injective.

The following procedure gives the canonical join representation of a given element of the Coxeter group W. This coincides with [23, Theorem 10-5.6].

Proposition 4.4. *Let $w \in W$, and set $a_d := w(d)$, $b_d := w(d+1)$ for each $d \in \mathsf{des}(w)$. Then, the canonical join representation of w is $\bigvee_{d \in \mathsf{des}(w)} w_d$, where $w_d \in \mathsf{j\text{-}irr}\, W$ is the unique join-irreducible element such that $R(w_d)$ coincides with R_d defined as follows:*

$$X_d := w([d+1, n+1]), \quad R_d := ([b_d, a_d - 1] \cap X_d) \cup [a_d + 1, n+1].$$

Proof. Let $d \in \mathsf{des}(w)$. It is easy to see that there uniquely exists $w_d \in \mathsf{j\text{-}irr}\, W$ with $R(w_d) = R_d$. Then, $L(w_d) = [1, b_d - 1] \cup ([b_d + 1, a_d] \setminus X_d)$. From this, we can straightforwardly check that $\mathsf{inv}(w_d) \subset \mathsf{inv}(w)$, which is equivalent to $w_d \leq w$. Moreover, the unique cover reflection of w_d is $(a_d \ \ b_d)$, and it is equal to $w s_d w^{-1}$. Therefore, the assertion follows from Proposition 4.2. $\square$

Example 4.5. Let $n = 8$ and $w = (4, 9, 3, 6, 2, 8, 5, 1, 7)$. Then, we have $\mathsf{des}(w) = \{2, 4, 6, 7\}$. The canonical join representation of w is $\bigvee_{d \in \mathsf{des}(w)} w_d$, where w_d is given as follows for each $d \in \mathsf{des}(w)$.

d	a_d	b_d	$R(w_d)$	w_d
2	9	3	$\{3, 5, 6, 7, 8\}$	$(1, 2, 4, 9, 3, 5, 6, 7, 8)$
4	6	2	$\{2, 5, 7, 8, 9\}$	$(1, 3, 4, 6, 2, 5, 7, 8, 9)$
6	8	5	$\{5, 7, 9\}$	$(1, 2, 3, 4, 6, 8, 5, 7, 9)$
7	5	1	$\{1, 6, 7, 8, 9\}$	$(2, 3, 4, 5, 1, 6, 7, 8, 9)$

Combining Corollary 3.3 and Proposition 4.4, we can obtain the semibrick $S(w)$ directly.

Theorem 4.6. *Let $w \in W$. Then, the semibrick $S(w)$ is $\bigoplus_{d \in \mathsf{des}(w)} S_d$, where S_d is the brick whose abbreviated description as in Corollary 3.3 is given as follows.*

- *Set R_d as in Proposition 4.4, and $a_d := w(d)$, $b_d := w(d+1)$, $V_d := [b_d, a_d - 1]$.*
- *The brick S_d has a K-basis $(\langle i \rangle_d)_{i \in V_d}$, where $\langle i \rangle_d$ belongs to $e_i S_d$.*
- *For each $i \in V_d$, place a symbol i denoting the K-vector subspace $K\langle i \rangle_d$.*
- *For each $i \in V_d \setminus \{\max V_d\}$, we write exactly one arrow between i and $i+1$, and its orientation is $i \to i+1$ if $i+1 \in R_d$ and $i \leftarrow i+1$ if $i+1 \notin R_d$.*

Proof. For each $d \in \mathsf{des}(w)$, let w_d be the join-irreducible element in the canonical join representation given in Proposition 4.4. Then, we can check that the abbreviated description of $S(w_d)$ in Corollary 3.3 coincides with the statement. $\square$

We remark that R_d in Theorem 4.6 can be replaced by $R_d \cap V_d = [b_d, a_d - 1] \cap X_d$.

Example 4.7. Let $n = 8$ and $w = (4, 9, 3, 6, 2, 8, 5, 1, 7)$ as in Example 4.5. Then, the semibrick $S(w)$ is the direct sum of the following bricks:

$$\begin{aligned}
S_2 &= \phantom{1 \leftarrow 2 \leftarrow{}} 3 \leftarrow 4 \to 5 \to 6 \to 7 \to 8, \\
S_4 &= \phantom{1 \leftarrow{}} 2 \leftarrow 3 \leftarrow 4 \to 5 \qquad\qquad\quad , \\
S_6 &= \phantom{1 \leftarrow 2 \leftarrow 3 \leftarrow 4 \to{}} 5 \leftarrow 6 \to 7 \qquad , \\
S_7 &= 1 \leftarrow 2 \leftarrow 3 \leftarrow 4 \qquad\qquad\qquad\quad .
\end{aligned}$$

4.3. Type $\mathbb{D}_n$

Let $\Delta = \mathbb{D}_n$. For each element w in $\mathsf{j\text{-}irr}\, W$ of type l, we set

$$L(w) := \{|w(k)| \mid k \in [1, |l|]\}, \quad R(w) := w([|l|+1, n]).$$

As in the case of type $\mathbb{A}_n$, it is easy to see that the correspondence $w \mapsto R(w)$ is injective.

The canonical join representations of the elements of the Coxeter group W are given by the following procedure.

Proposition 4.8. *Let $w \in W$, and set $a_d := w(d)$, $b_d := w(|d|+1)$, $X_d := w([|d|+1, n])$ for each $d \in \mathsf{des}(w)$. Then, the canonical join representation of w is $\bigvee_{d \in \mathsf{des}(w)} w_d$, where $w_d \in \mathsf{j\text{-}irr}\, W$ is the unique join-irreducible element such that $R(w_d)$ coincides with R_d defined as follows.*

(A) *If $a_d + b_d < 0$ and $w([1, |d|]) \subset \pm[a_d, n]$, then*

$$R_d := \begin{cases} \{-a_d\} \cup (\pm[1, a_d - 1] \cap X_d) \cup ([a_d + 1, -b_d - 1] \setminus (-X_d)) \cup [-b_d + 1, n] & (a_d > 0) \\ ([-a_d, -b_d - 1] \setminus (-X_d)) \cup [-b_d + 1, n] & (a_d < 0) \end{cases}.$$

(B) *Otherwise,*

$$R_d := \begin{cases} ([b_d, a_d - 1] \cap X_d) \cup [a_d + 1, n] & (a_d + b_d > 0) \\ ([b_d, a_d - 1] \cap X_d) \cup ([a_d + 1, -b_d - 1] \setminus (-X_d)) \cup [-b_d + 1, n] & (a_d + b_d < 0) \end{cases}.$$

Proof. The proof is similar to the one for type $\mathbb{A}_n$. In this case, the set $L(w_d)$ is given as follows.

(A) If $a_d + b_d < 0$ and $w([1, |d|]) \subset \pm[a_d, n]$, then

$$L(w_d) = \begin{cases} [a_d + 1, -b_d] \cap (-X_d) & (a_d > 0) \\ [1, -a_d - 1] \cup ([-a_d + 1, -b_d] \cap (-X_d)) & (a_d < 0) \end{cases}.$$

(B) Otherwise,

$$L(w_d) = \begin{cases} [1, b_d - 1] \cup ([b_d + 1, a_d] \setminus X_d) & (b_d > 0) \\ ([1, -b_d - 1] \setminus (\pm X_d)) \cup ([-b_d + 1, a_d] \setminus X_d) & (b_d < 0,\ a_d + b_d > 0) \\ [1, a_d] \setminus (\pm X_d) & (a_d + b_d < 0) \end{cases}.$$

By using these, we can check $w_d \leq w$ and $\mathsf{cov}(w_d) = \{w s_d w^{-1}\}$. $\square$

In the rest, the symbols (A) and (B) mean the conditions (A) and (B) in Proposition 4.8, respectively.

Example 4.9. Let $n = 9$ and $w = (5, 3, -7, 4, -6, -8, 9, -1, 2)$. Then, we have $\mathsf{des}(w) = \{1, 2, 4, 5, 7\}$. The canonical join representation of w is $\bigvee_{d \in \mathsf{des}(w)} w_d$, where w_d is given as follows for each $d \in \mathsf{des}(w)$.

d	a_d	b_d	(A) or (B)	$R(w_d)$	w_d
1	5	3	(B)	$\{3,4,6,7,8,9\}$	$(1, 2, 5, 3, 4, 6, 7, 8, 9)$
2	3	-7	(A)	$\{-3,-1,2,4,5,8,9\}$	$(6, 7, -3, -1, 2, 4, 5, 8, 9)$
4	4	-6	(B)	$\{-6,-1,2,5,7,8,9\}$	$(3, 4, -6, -1, 2, 5, 7, 8, 9)$
5	-6	-8	(A)	$\{6,7,9\}$	$(1, 2, 3, 4, 5, 8, 6, 7, 9)$
7	9	-1	(B)	$\{-1,2\}$	$(-3, 4, 5, 6, 7, 8, 9, -1, 2)$

Now, by combining Corollary 3.10 and Proposition 4.8, we can obtain the semibrick $S(w)$ from $w \in W$ directly. We need to define a few notations: for integers $a > b$ and $c = \pm 1$, we set

$$(V_-(a,b,c), V_+(a,b,c)) := \begin{cases} (\emptyset, [b, a-1]) & (b \geq 2) \\ (\emptyset, \{c\} \cup [2, a-1]) & (b = \pm 1) \\ ([b+1, -2] \cup \{-c\}, \{c\} \cup [2, a-1]) & (b \leq -2) \end{cases}.$$

Theorem 4.10. *Let $w \in W$. Then, the semibrick $S(w)$ is $\bigoplus_{d \in \mathsf{des}(w)} S_d$, where S_d is the brick whose abbreviated description as in Corollary 3.3 is given as follows.*

- *Set R_d as in Proposition 4.8, and $a_d := w(d)$, $b_d := w(d+1)$,*

$$r_d := \max\{k \geq 0 \mid [1,k] \subset \pm R_d\}, \quad c_d := \begin{cases} w(|w^{-1}(1)|) & (r_d \geq 1) \\ 1 & (r_d = 0) \end{cases},$$

$$((V_-)_d, (V_+)_d) := \begin{cases} (V_-(-b_d, -a_d, c_d), V_+(-b_d, -a_d, c_d)) & ((\mathrm{A})) \\ (V_-(a_d, b_d, c_d), V_+(a_d, b_d, c_d)) & ((\mathrm{B})) \end{cases},$$

$$V_d := (V_+)_d \amalg (V_-)_d.$$

- *The brick S_d has a K-basis $(\langle i \rangle_d)_{i \in V_d}$, where $\langle i \rangle_d$ belongs to $e_i S_d$ if $i \geq -1$, and $e_{|i|} S_d$ if $i \leq -2$.*
- *For each $i \in V_d$, place a symbol i denoting the K-vector subspace $K\langle i \rangle_d$.*
- *We write the following arrows.*
  - (i) *For each $i \in (V_+)_d \setminus \{\max(V_+)_d\}$, draw an arrow $i \to |i|+1$ if $|i|+1 \in R_d$; and $i \leftarrow |i|+1$ otherwise.*
  - (ii) *For each $i \in (V_-)_d \setminus \{\min(V_-)_d\}$, draw an arrow $i \leftarrow -(|i|+1)$ if $-(|i|+1) \in R_d$; and $i \to -(|i|+1)$ otherwise.*
  - (iii) *If $r_d \geq 1$, then for each $i \in (V_-)_d$ with $|i| \leq r_d$, draw an arrow $-i \leftarrow -(|i|+1)$ if $|i|+1 \in R_d$; and $i \to |i|+1$ otherwise.*
  - (iv) *If $r_d = 0$, then draw an arrow $-c \leftarrow 2$ if $c \leftarrow 2$ exists in* (i), *and an arrow $c \to -2$ if $-c \to -2$ exists in* (ii).

Proof. Apply Corollary 3.10 to the element $w_d \in \mathsf{j\text{-}irr}\, W$ defined in Proposition 4.8 for each $d \in \mathsf{des}(w)$. $\square$

Example 4.11. Let $n = 9$ and $w = (5, 3, -7, 4, -6, 8, 9, -1, 2)$ as in Example 4.9. Then, the semibrick $S(w)$ is the direct sum of the following bricks:

$$S_1 = \quad 3 \longrightarrow 4 \quad ,$$

$$S_2 = \begin{tikzcd} 1 \arrow[r] & -2 \arrow[dl] \arrow[dr] & & & & \\ -1 \arrow[r] & 2 & 3 \arrow[l] \arrow[r] & 4 \arrow[r] & 5 & 6 \arrow[l] \end{tikzcd},$$

$$S_4 = \begin{tikzcd} 1 \arrow[r] & -2 \arrow[r] \arrow[dl] \arrow[dr] & -3 \arrow[r] & -4 \arrow[r] & -5 \\ -1 \arrow[r] & 2 & 3 \arrow[l] & & \end{tikzcd},$$

$$S_5 = \begin{tikzcd} 6 \arrow[r] & 7 \end{tikzcd},$$

$$S_7 = \begin{tikzcd} -1 \arrow[r] & 2 & 3 \arrow[l] & 4 \arrow[l] & 5 \arrow[l] & 6 \arrow[l] & 7 \arrow[l] & 8 \arrow[l] \end{tikzcd}.$$

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Appendix A. Example: the bricks over the preprojective algebra of type $\mathbb{D}_5$

In this section, we give the list of bricks over the preprojective algebra of type $\mathbb{D}_5$.

For the preparation, we first define two notions denoted by $\sigma(w)$ and $\chi(w)$ associated to each join-irreducible element $w \in \mathsf{j\text{-}irr}\, W$ in the Coxeter group $W = W(\mathbb{D}_n)$ of type $\mathbb{D}_n$ (it is not needed to assume $n = 5$ here), and then list all the join-irreducible elements and the corresponding bricks by using these notions in the case $n = 5$.

First, we define $\sigma(w)$. Recall that we have defined the integers a, b, r in Subsection 3.2 for w. By using these integers, we define $\sigma(w)$ of w as the triple $(a, b, r') \in \mathbb{Z}^3$, where

$$r' := \begin{cases} 0 & (b \geq -1) \\ \mathsf{min}\{r, |b| - 1\} & (b \leq -2) \end{cases},$$

and call $\sigma(w)$ the *shape* of w. For any $\sigma \in \mathbb{Z}^3$, we write $(\mathsf{j\text{-}irr}\, W)_\sigma \subset \mathsf{j\text{-}irr}\, W$ for the subset of elements in $\mathsf{j\text{-}irr}\, W$ whose shapes are σ. It is easy to see that $(\mathsf{j\text{-}irr}\, W)_\sigma \neq \emptyset$ if and only if σ is a triple (a, b, r') satisfying one of the following conditions (a), (b), (c):

(a) $2 \leq a \leq n$, $-1 \leq b < a$, $b \neq 0$, $r' = 0$; or
(b) $2 \leq a \leq n$, $-a < b \leq -2$, $0 \leq r' \leq |b| - 1$; or
(c) $2 \leq a \leq n$, $-n \leq b < -a$, $0 \leq r' \leq |a| - 2$.

Next, we define the other notion $\chi(w)$ by using R defined in Subsection 3.2 for w. We set $\chi(w)$ as the sequence $(x(1), x(2), \dots, x(n)) \in \{0, 1, 2\}^n$ whose terms are given by

$$x(i) := \begin{cases} 0 & (-j, j \notin R) \\ 1 & (-j \in R) \\ 2 & (j \in R) \end{cases}.$$

We have a map $\chi\colon \mathsf{j\text{-}irr}\, W \to \{0, 1, 2\}^n$, which is clearly injective. For $\sigma = (a, b, r')$ satisfying the condition above, we can straightforwardly check $\chi((\mathsf{j\text{-}irr}\, W)_\sigma) = \prod_{i=1}^n X_i$, where X_i is defined as follows in each of the three cases (a), (b), and (c):

(a)

i	$i<\lvert b\rvert$	$i=\lvert b\rvert$	$\lvert b\rvert<i<a$	$i=a$	$i>a$
X_i	$\{0\}$	$\{1\}$ if $b=-1$; $\{2\}$ otherwise	$\{0,2\}$	$\{0\}$	$\{2\}$

,

(b)

i	$i\le r'$	$i=r'+1\neq\lvert b\rvert$	$r'+1<i<\lvert b\rvert$	$i=\lvert b\rvert$	$\lvert b\rvert<i<a$	$i=a$	$i>a$
X_i	$\{1,2\}$	$\{0\}$	$\{0,1,2\}$	$\{1\}$	$\{0,2\}$	$\{0\}$	$\{2\}$

,

(c)

i	$i\le r'$	$i=r'+1$	$r'+1<i<a$	$i=a$	$a<i<\lvert b\rvert$	$i=\lvert b\rvert$	$i>a$
X_i	$\{1,2\}$	$\{0\}$	$\{0,1,2\}$	$\{0\}$	$\{1,2\}$	$\{1\}$	$\{2\}$

.

Therefore, by setting $x := \mathsf{max}\{a, |b|\}$ and $y := \mathsf{min}\{a, |b|\}$, we have

$$\#(\mathsf{j\text{-}irr}\, W)_\sigma = \begin{cases} 2^{x-y-1} & (b \geq -1) \\ 2^{r'} \cdot 3^{\mathsf{max}\{y-r'-2,0\}} \cdot 2^{x-y-1} & (b \leq -2) \end{cases}.$$

From now on, we consider $\mathbb{D}_5$, so let $n = 5$. For σ satisfying the condition above, the following lists show all the elements w in $(\mathsf{j\text{-}irr}\, W)_\sigma$ and the corresponding bricks $S(w)$ over the preprojective algebra $\mathit{\Pi}$ of type $\mathbb{D}_5$. The elements in $(\mathsf{j\text{-}irr}\, W)_\sigma$ are arranged so that w comes before w' if and only if $\chi(w) < \chi(w')$ in the lexicographical order of $\{0, 1, 2\}^n$, and each w is shortly denoted by a string $j_1 j_2 \cdots j_n$, where $j_i := w(i)$ if $w(i) > 0$; $j_i := \underline{w(i)}$ if $w(i) < 0$. For example, $\underline{1}2\underline{5}34$ means $(-1, 2, -5, 3, 4)$. The join-irreducible elements and the bricks are explicitly described as follows by Corollary 3.10:

- $\sigma = (2, -5, 0)$ (4 elements):

$$S(\underline{1}2\underline{543}) = \begin{tikzcd} -1 \arrow[r] & -2 & -3 \arrow[l] & -4 \arrow[l] \\ 1 \arrow[ur] \end{tikzcd}, \qquad S(12\underline{53}4) = \begin{tikzcd} -1 \arrow[r] & -2 & -3 \arrow[l] \arrow[r] & -4 \\ 1 \arrow[ur] \end{tikzcd},$$

$$S(12\underline{54}3) = \begin{tikzcd} -1 \arrow[r] & -2 \arrow[r] & -3 & -4 \arrow[l] \\ 1 \arrow[ur] \end{tikzcd}, \qquad S(\underline{1}2\underline{5}34) = \begin{tikzcd} -1 \arrow[r] & -2 \arrow[r] & -3 \arrow[r] & -4 \\ 1 \arrow[ur] \end{tikzcd};$$

- $\sigma = (2, -4, 0)$ (2 elements):

$$S(12\underline{43}5) = \begin{tikzcd} -1 \arrow[r] & -2 & -3 \arrow[l] \\ 1 \arrow[ur] \end{tikzcd}, \qquad S(\underline{1}2\underline{4}35) = \begin{tikzcd} -1 \arrow[r] & -2 \arrow[r] & -3 \\ 1 \arrow[ur] \end{tikzcd};$$

- $\sigma = (2, -3, 0)$ (1 element):

$$S(\underline{1}2\underline{3}45) = \begin{tikzcd} -1 \arrow[r] & -2 \\ 1 \arrow[ur] \end{tikzcd};$$

- $\sigma = (2, -1, 0)$ (1 element):

$$S(\underline{21}345) = \; -1 \;;$$

- $\sigma = (2, 1, 0)$ (1 element):

$$S(21345) = \; 1 \;;$$

- $\sigma = (3, -5, 0)$ (6 elements):

$$S(123\underline{54}) = \begin{tikzcd} -1 \arrow[r] & -2 \arrow[r] & -3 & -4 \arrow[l] \\ 1 \arrow[ur] & 2 \arrow[l] \arrow[ul] \end{tikzcd}, \qquad S(\underline{1}23\underline{5}4) = \begin{tikzcd} -1 \arrow[r] & -2 \arrow[r] & -3 \arrow[r] & -4 \\ 1 \arrow[ur] & 2 \arrow[l] \arrow[ul] \end{tikzcd},$$

$$S(\underline{1}3\underline{542}) = \begin{tikzcd} -1 & -2 \arrow[l] \arrow[r] & -3 & -4 \arrow[l] \\ 1 & 2 \arrow[l] \arrow[ul] \end{tikzcd}, \qquad S(13\underline{52}4) = \begin{tikzcd} -1 & -2 \arrow[l] \arrow[r] & -3 \arrow[r] & -4 \\ 1 & 2 \arrow[l] \arrow[ul] \end{tikzcd},$$

$$S(13\underline{54}2) = \begin{tikzcd} -1 \arrow[r] & -2 \arrow[r] & -3 & -4 \arrow[l] \\ 1 \arrow[r] \arrow[ur] & 2 \end{tikzcd}, \qquad S(\underline{1}3\underline{5}24) = \begin{tikzcd} -1 \arrow[r] & -2 \arrow[r] & -3 \arrow[r] & -4 \\ 1 \arrow[r] \arrow[ur] & 2 \end{tikzcd};$$

- $\sigma = (3, -5, 1)$ (4 elements):

$$S(\underline{2}3\underline{541}) = \begin{tikzcd} 1 \arrow[r] \arrow[dr] & -2 \arrow[r] & -3 & -4 \arrow[l] \\ -1 & 2 \arrow[l] \end{tikzcd}, \qquad S(23\underline{51}4) = \begin{tikzcd} 1 \arrow[r] \arrow[dr] & -2 \arrow[r] & -3 \arrow[r] & -4 \\ -1 & 2 \arrow[l] \end{tikzcd},$$

$$S(23\underline{54}1) = \begin{tikzcd} -1 \arrow[r] \arrow[dr] & -2 \arrow[r] & -3 & -4 \arrow[l] \\ 1 & 2 \arrow[l] \end{tikzcd}, \qquad S(\underline{2}3\underline{5}14) = \begin{tikzcd} -1 \arrow[r] \arrow[dr] & -2 \arrow[r] & -3 \arrow[r] & -4 \\ 1 & 2 \arrow[l] \end{tikzcd};$$

- $\sigma = (3, -4, 0)$ (3 elements):

$$S(\underline{1}23\underline{4}5) = \begin{tikzcd} -1 \arrow[r] & -2 \arrow[r] & -3 \\ 1 \arrow[ur] & 2 \arrow[l] \arrow[ul] \end{tikzcd}, \qquad S(13\underline{42}5) = \begin{tikzcd} -1 & -2 \arrow[l] \arrow[r] & -3 \\ 1 & 2 \arrow[l] \arrow[ul] \end{tikzcd},$$

$$S(\underline{1}3\underline{4}25) = \begin{tikzcd} -1 \arrow[r] & -2 \arrow[r] & -3 \\ 1 \arrow[r] \arrow[ur] & 2 \end{tikzcd};$$

- $\sigma = (3, -4, 1)$ (2 elements):

$$S(23\underline{41}5) = \begin{tikzcd} 1 \arrow[r] \arrow[dr] & -2 \arrow[r] & -3 \\ -1 & 2 \arrow[l] \end{tikzcd}, \qquad S(\underline{2}3\underline{4}15) = \begin{tikzcd} -1 \arrow[r] \arrow[dr] & -2 \arrow[r] & -3 \\ 1 & 2 \arrow[l] \end{tikzcd};$$

- $\sigma = (3, -2, 0)$ (1 element):

$$S(\underline{1}3\underline{2}45) = \begin{tikzcd} -1 & \\ 1 & 2 \arrow[l] \arrow[ul] \end{tikzcd};$$

- $\sigma = (3, -2, 1)$ (2 elements):

$$S(3\underline{21}45) = \begin{tikzcd} 1 \arrow[dr] & \\ -1 & 2 \arrow[l] \end{tikzcd}, \qquad S(\underline{32}145) = \begin{tikzcd} -1 \arrow[dr] & \\ 1 & 2 \arrow[l] \end{tikzcd};$$

- $\sigma = (3, -1, 0)$ (2 elements):

$$S(\underline{2}3\underline{1}45) = -1 \longleftarrow 2, \qquad S(\underline{31}245) = -1 \longrightarrow 2;$$

- $\sigma = (3, 1, 0)$ (2 elements):

$$S(23145) = 1 \longleftarrow 2, \qquad S(31245) = 1 \longrightarrow 2;$$

- $\sigma = (3, 2, 0)$ (1 element):

$$S(13245) = 2;$$

- $\sigma = (4, -5, 0)$ (9 elements):

$$S(\underline{1}234\underline{5}) = \begin{tikzcd} -1 \arrow[r] & -2 \arrow[r] & -3 \arrow[r] & -4 \\ 1 \arrow[ur] & 2 \arrow[l] \arrow[ul] & 3 \arrow[l] \end{tikzcd}, \qquad S(124\underline{53}) = \begin{tikzcd} -1 \arrow[r] & -2 & -3 \arrow[l] \arrow[r] & -4 \\ 1 \arrow[ur] & 2 \arrow[l] \arrow[ul] & 3 \arrow[l] \end{tikzcd},$$

$$S(\underline{1}24\underline{5}3) = \begin{tikzcd} -1 \arrow[r] & -2 \arrow[r] & -3 \arrow[r] & -4 \\ 1 \arrow[ur] & 2 \arrow[l] \arrow[ul] \arrow[r] & 3 \end{tikzcd}, \qquad S(134\underline{52}) = \begin{tikzcd} -1 & -2 \arrow[l] \arrow[r] & -3 \arrow[r] & -4 \\ 1 & 2 \arrow[l] \arrow[ul] & 3 \arrow[l] \end{tikzcd},$$

$$S(\underline{1}4\underline{532}) = \begin{tikzcd} -1 & -2 \arrow[l] & -3 \arrow[l] \arrow[r] & -4 \\ 1 & 2 \arrow[l] \arrow[ul] & 3 \arrow[l] \end{tikzcd}, \qquad S(14\underline{52}3) = \begin{tikzcd} -1 & -2 \arrow[l] \arrow[r] & -3 \arrow[r] & -4 \\ 1 & 2 \arrow[l] \arrow[ul] \arrow[r] & 3 \end{tikzcd},$$

$$S(\underline{1}34\underline{5}2) = \begin{tikzcd} -1 \arrow[r] & -2 \arrow[r] & -3 \arrow[r] & -4 \\ 1 \arrow[ur] \arrow[r] & 2 & 3 \arrow[l] \end{tikzcd}, \qquad S(14\underline{53}2) = \begin{tikzcd} -1 \arrow[r] & -2 & -3 \arrow[l] \arrow[r] & -4 \\ 1 \arrow[ur] \arrow[r] & 2 & 3 \arrow[l] \end{tikzcd},$$

$$S(\underline{1}4\underline{5}23) = \begin{tikzcd} -1 \arrow[r] & -2 \arrow[r] & -3 \arrow[r] & -4 \\ 1 \arrow[ur] \arrow[r] & 2 \arrow[r] & 3 \end{tikzcd};$$

- $\sigma = (4, -5, 1)$ (6 elements):

$$S(234\underline{51}) = \begin{tikzcd} 1 \arrow[r] \arrow[dr] & -2 \arrow[r] & -3 \arrow[r] & -4 \\ -1 & 2 \arrow[l] & 3 \arrow[l] \end{tikzcd}, \qquad S(\underline{2}4\underline{531}) = \begin{tikzcd} 1 \arrow[r] \arrow[dr] & -2 & -3 \arrow[l] \arrow[r] & -4 \\ -1 & 2 \arrow[l] & 3 \arrow[l] \end{tikzcd},$$

$$S(24\underline{51}3) = \begin{tikzcd} 1 \arrow[r] \arrow[dr] & -2 \arrow[r] & -3 \arrow[r] & -4 \\ -1 & 2 \arrow[l] \arrow[r] & 3 \end{tikzcd}, \qquad S(\underline{2}34\underline{5}1) = \begin{tikzcd} -1 \arrow[r] \arrow[dr] & -2 \arrow[r] & -3 \arrow[r] & -4 \\ 1 & 2 \arrow[l] & 3 \arrow[l] \end{tikzcd},$$

$$S(24\underline{53}1) = \begin{tikzcd} -1 \arrow[r] \arrow[dr] & -2 & -3 \arrow[l] \arrow[r] & -4 \\ 1 & 2 \arrow[l] & 3 \arrow[l] \end{tikzcd}, \qquad S(\underline{2}4\underline{5}13) = \begin{tikzcd} -1 \arrow[r] \arrow[dr] & -2 \arrow[r] & -3 \arrow[r] & -4 \\ 1 & 2 \arrow[l] \arrow[r] & 3 \end{tikzcd};$$

- $\sigma = (4, -5, 2)$ (4 elements):

$$S(\underline{3}4\underline{521}) = \begin{tikzcd} 1 \arrow[dr] & -2 \arrow[l] \arrow[r] \arrow[dr] & -3 \arrow[r] & -4 \\ -1 & 2 \arrow[l] & 3 \arrow[l] \end{tikzcd}, \qquad S(34\underline{51}2) = \begin{tikzcd} 1 \arrow[r] & -2 \arrow[r] \arrow[dl] \arrow[dr] & -3 \arrow[r] & -4 \\ -1 \arrow[r] & 2 & 3 \arrow[l] \end{tikzcd},$$

$$S(34\underline{52}1) = \begin{tikzcd} -1 \arrow[dr] & -2 \arrow[l] \arrow[r] \arrow[dr] & -3 \arrow[r] & -4 \\ 1 & 2 \arrow[l] & 3 \arrow[l] \end{tikzcd}, \qquad S(\underline{3}4\underline{5}12) = \begin{tikzcd} -1 \arrow[r] & -2 \arrow[r] \arrow[dl] \arrow[dr] & -3 \arrow[r] & -4 \\ 1 \arrow[r] & 2 & 3 \arrow[l] \end{tikzcd};$$

- $\sigma = (4, -3, 0)$ (3 elements):

$$S(\underline{1}24\underline{3}5) = \begin{tikzcd} -1 \arrow[r] & -2 \\ 1 \arrow[ur] & 2 \arrow[l] \arrow[ul] & 3 \arrow[l] \end{tikzcd}, \qquad S(14\underline{32}5) = \begin{tikzcd} -1 & -2 \arrow[l] \\ 1 & 2 \arrow[l] \arrow[ul] & 3 \arrow[l] \end{tikzcd},$$

$$S(\underline{1}4\underline{3}25) = \begin{tikzcd} -1 \arrow[r] & -2 \\ 1 \arrow[ur] \arrow[r] & 2 & 3 \arrow[l] \end{tikzcd};$$

- $\sigma = (4, -3, 1)$ (2 elements):

$$S(24\underline{31}5) = \begin{tikzcd} 1 \arrow[r] \arrow[dr] & -2 \\ -1 & 2 \arrow[l] & 3 \arrow[l] \end{tikzcd}, \qquad S(\underline{2}4\underline{3}15) = \begin{tikzcd} -1 \arrow[r] \arrow[dr] & -2 \\ 1 & 2 \arrow[l] & 3 \arrow[l] \end{tikzcd};$$

- $\sigma = (4, -3, 2)$ (4 elements):

$$S(\underline{4321}5) = \begin{tikzcd} 1 \arrow[dr] & -2 \arrow[l] \arrow[dr] & \\ -1 & 2 \arrow[l] & 3 \arrow[l] \end{tikzcd}, \qquad S(4\underline{31}25) = \begin{tikzcd} 1 \arrow[r] & -2 \arrow[dl] \arrow[dr] & \\ -1 \arrow[r] & 2 & 3 \arrow[l] \end{tikzcd},$$

$$S(4\underline{32}15) = \begin{tikzcd} -1 \arrow[dr] & -2 \arrow[l] \arrow[dr] & \\ 1 & 2 \arrow[l] & 3 \arrow[l] \end{tikzcd}, \qquad S(\underline{43}125) = \begin{tikzcd} -1 \arrow[r] & -2 \arrow[dl] \arrow[dr] & \\ 1 \arrow[r] & 2 & 3 \arrow[l] \end{tikzcd};$$

- $\sigma = (4, -2, 0)$ (2 elements):

$$S(\underline{1}34\underline{2}5) = \begin{tikzcd} -1 & & \\ 1 & 2 \arrow[l] \arrow[ul] & 3 \arrow[l] \end{tikzcd}, \qquad S(\underline{1}4\underline{2}35) = \begin{tikzcd} -1 & & \\ 1 & 2 \arrow[l] \arrow[ul] \arrow[r] & 3 \end{tikzcd};$$

- $\sigma = (4, -2, 1)$ (4 elements):

$$S(34\underline{21}5) = \begin{tikzcd} 1 \arrow[dr] & & \\ -1 & 2 \arrow[l] & 3 \arrow[l] \end{tikzcd}, \qquad S(4\underline{21}35) = \begin{tikzcd} 1 \arrow[dr] & & \\ -1 & 2 \arrow[l] \arrow[r] & 3 \end{tikzcd},$$

$$S(\underline{3}4\underline{2}15) = \begin{tikzcd} -1 \arrow[dr] & & \\ 1 & 2 \arrow[l] & 3 \arrow[l] \end{tikzcd}, \qquad S(\underline{42}135) = \begin{tikzcd} -1 \arrow[dr] & & \\ 1 & 2 \arrow[l] \arrow[r] & 3 \end{tikzcd};$$

- $\sigma = (4, -1, 0)$ (4 elements):

$$S(\underline{2}34\underline{1}5) = -1 \longleftarrow 2 \longleftarrow 3\ , \qquad S(\underline{2}4\underline{1}35) = -1 \longleftarrow 2 \longrightarrow 3\ ,$$

$$S(\underline{3}4\underline{1}25) = -1 \longrightarrow 2 \longleftarrow 3\ , \qquad S(\underline{41}235) = -1 \longrightarrow 2 \longrightarrow 3\ ;$$

- $\sigma = (4, 1, 0)$ (4 elements):

$$S(23415) = 1 \longleftarrow 2 \longleftarrow 3\ , \qquad S(24135) = 1 \longleftarrow 2 \longrightarrow 3\ ,$$

$$S(34125) = 1 \longrightarrow 2 \longleftarrow 3\ , \qquad S(41235) = 1 \longrightarrow 2 \longrightarrow 3\ ;$$

- $\sigma = (4, 2, 0)$ (2 elements):

$$S(13425) = 2 \longleftarrow 3\ , \qquad S(14235) = 2 \longrightarrow 3\ ;$$

- $\sigma = (4, 3, 0)$ (1 element):

$$S(12435) = 3\ ;$$

- $\sigma = (5, -4, 0)$ (9 elements):

$$S(\underline{1}235\underline{4}) = \begin{tikzcd} -1 \arrow[r] & -2 \arrow[r] & -3 & \\ 1 \arrow[ur] & 2 \arrow[l] \arrow[ul] & 3 \arrow[l] & 4 \arrow[l] \end{tikzcd}, \qquad S(125\underline{43}) = \begin{tikzcd} -1 \arrow[r] & -2 & -3 \arrow[l] & \\ 1 \arrow[ur] & 2 \arrow[l] \arrow[ul] & 3 \arrow[l] & 4 \arrow[l] \end{tikzcd},$$

$$S(\underline{1}25\underline{4}3) = \begin{tikzcd} -1 \arrow[r] & -2 \arrow[r] & -3 & \\ 1 \arrow[ur] & 2 \arrow[l] \arrow[ul] \arrow[r] & 3 & 4 \arrow[l] \end{tikzcd}, \qquad S(135\underline{42}) = \begin{tikzcd} -1 & -2 \arrow[l] \arrow[r] & -3 & \\ 1 & 2 \arrow[l] \arrow[ul] & 3 \arrow[l] & 4 \arrow[l] \end{tikzcd},$$

$$S(\underline{1}5\underline{432}) = \begin{tikzcd} -1 & -2 \arrow[l] & -3 \arrow[l] & \\ 1 & 2 \arrow[l] \arrow[ul] & 3 \arrow[l] & 4 \arrow[l] \end{tikzcd}, \qquad S(15\underline{42}3) = \begin{tikzcd} -1 & -2 \arrow[l] \arrow[r] & -3 & \\ 1 & 2 \arrow[l] \arrow[ul] \arrow[r] & 3 & 4 \arrow[l] \end{tikzcd},$$

$$S(\underline{1}35\underline{4}2) = \begin{tikzcd} -1 \arrow[r] & -2 \arrow[r] & -3 \\ 1 \arrow[ur] \arrow[r] & 2 & 3 \arrow[l] & 4 \arrow[l] \end{tikzcd}, \qquad S(15\underline{43}2) = \begin{tikzcd} -1 \arrow[r] & -2 & -3 \arrow[l] \\ 1 \arrow[ur] \arrow[r] & 2 & 3 \arrow[l] & 4 \arrow[l] \end{tikzcd},$$

$$S(\underline{1}5\underline{4}23) = \begin{tikzcd} -1 \arrow[r] & -2 \arrow[r] & -3 \\ 1 \arrow[ur] \arrow[r] & 2 \arrow[r] & 3 & 4 \arrow[l] \end{tikzcd};$$

- $\sigma = (5, -4, 1)$ (6 elements):

$$S(235\underline{41}) = \begin{tikzcd} 1 \arrow[r] \arrow[dr] & -2 \arrow[r] & -3 \\ -1 & 2 \arrow[l] & 3 \arrow[l] & 4 \arrow[l] \end{tikzcd}, \qquad S(\underline{2}5\underline{431}) = \begin{tikzcd} 1 \arrow[r] \arrow[dr] & -2 & -3 \arrow[l] \\ -1 & 2 \arrow[l] & 3 \arrow[l] & 4 \arrow[l] \end{tikzcd},$$

$$S(25\underline{41}3) = \begin{tikzcd} 1 \arrow[r] \arrow[dr] & -2 \arrow[r] & -3 \\ -1 & 2 \arrow[l] \arrow[r] & 3 & 4 \arrow[l] \end{tikzcd}, \qquad S(\underline{2}35\underline{4}1) = \begin{tikzcd} -1 \arrow[r] \arrow[dr] & -2 \arrow[r] & -3 \\ 1 & 2 \arrow[l] & 3 \arrow[l] & 4 \arrow[l] \end{tikzcd},$$

$$S(25\underline{43}1) = \begin{tikzcd} -1 \arrow[r] \arrow[dr] & -2 & -3 \arrow[l] \\ 1 & 2 \arrow[l] & 3 \arrow[l] & 4 \arrow[l] \end{tikzcd}, \qquad S(\underline{2}5\underline{4}13) = \begin{tikzcd} -1 \arrow[r] \arrow[dr] & -2 \arrow[r] & -3 \\ 1 & 2 \arrow[l] \arrow[r] & 3 & 4 \arrow[l] \end{tikzcd};$$

- $\sigma = (5, -4, 2)$ (4 elements):

$$S(\underline{3}5\underline{421}) = \begin{tikzcd} 1 \arrow[dr] & -2 \arrow[l] \arrow[r] \arrow[dr] & -3 \\ -1 & 2 \arrow[l] & 3 \arrow[l] & 4 \arrow[l] \end{tikzcd}, \qquad S(35\underline{41}2) = \begin{tikzcd} 1 \arrow[r] & -2 \arrow[r] \arrow[dl] \arrow[dr] & -3 \\ -1 \arrow[r] & 2 & 3 \arrow[l] & 4 \arrow[l] \end{tikzcd},$$

$$S(35\underline{42}1) = \begin{tikzcd} -1 \arrow[dr] & -2 \arrow[l] \arrow[r] \arrow[dr] & -3 \\ 1 & 2 \arrow[l] & 3 \arrow[l] & 4 \arrow[l] \end{tikzcd}, \qquad S(\underline{3}5\underline{4}12) = \begin{tikzcd} -1 \arrow[r] & -2 \arrow[r] \arrow[dl] \arrow[dr] & -3 \\ 1 \arrow[r] & 2 & 3 \arrow[l] & 4 \arrow[l] \end{tikzcd};$$

- $\sigma = (5, -4, 3)$ (8 elements):

$$S(5\underline{4321}) = \begin{tikzcd} 1 \arrow[dr] & -2 \arrow[l] \arrow[dr] & -3 \arrow[l] \arrow[dr] \\ -1 & 2 \arrow[l] & 3 \arrow[l] & 4 \arrow[l] \end{tikzcd}, \qquad S(\underline{5421}3) = \begin{tikzcd} 1 \arrow[dr] & -2 \arrow[l] \arrow[r] & -3 \arrow[dl] \arrow[dr] \\ -1 & 2 \arrow[l] \arrow[r] & 3 & 4 \arrow[l] \end{tikzcd},$$

$$S(\underline{5431}2) = \begin{tikzcd} 1 \arrow[r] & -2 \arrow[dl] \arrow[dr] & -3 \arrow[l] \arrow[dr] \\ -1 \arrow[r] & 2 & 3 \arrow[l] & 4 \arrow[l] \end{tikzcd}, \qquad S(5\underline{41}23) = \begin{tikzcd} 1 \arrow[r] & -2 \arrow[r] \arrow[dl] & -3 \arrow[dl] \arrow[dr] \\ -1 \arrow[r] & 2 \arrow[r] & 3 & 4 \arrow[l] \end{tikzcd},$$

$$S(\underline{5432}1) = \begin{tikzcd} -1 \arrow[dr] & -2 \arrow[l] \arrow[dr] & -3 \arrow[l] \arrow[dr] \\ 1 & 2 \arrow[l] & 3 \arrow[l] & 4 \arrow[l] \end{tikzcd}, \qquad S(5\underline{42}13) = \begin{tikzcd} -1 \arrow[dr] & -2 \arrow[l] \arrow[r] & -3 \arrow[dl] \arrow[dr] \\ 1 & 2 \arrow[l] \arrow[r] & 3 & 4 \arrow[l] \end{tikzcd},$$

$$S(5\underline{43}12) = \begin{tikzcd} -1 \arrow[r] & -2 \arrow[dl] \arrow[dr] & -3 \arrow[l] \arrow[dr] \\ 1 \arrow[r] & 2 & 3 \arrow[l] & 4 \arrow[l] \end{tikzcd}, \qquad S(\underline{54}123) = \begin{tikzcd} -1 \arrow[r] & -2 \arrow[r] \arrow[dl] & -3 \arrow[dl] \arrow[dr] \\ 1 \arrow[r] & 2 \arrow[r] & 3 & 4 \arrow[l] \end{tikzcd};$$

- $\sigma = (5, -3, 0)$ (6 elements):

$$S(\underline{1}245\underline{3}) = \begin{tikzcd} -1 \arrow[r] & -2 \\ 1 \arrow[ur] & 2 \arrow[l] \arrow[ul] & 3 \arrow[l] & 4 \arrow[l] \end{tikzcd}, \qquad S(\underline{1}25\underline{3}4) = \begin{tikzcd} -1 \arrow[r] & -2 \\ 1 \arrow[ur] & 2 \arrow[l] \arrow[ul] & 3 \arrow[l] \arrow[r] & 4 \end{tikzcd},$$

$$S(145\underline{32}) = \begin{tikzcd} -1 & -2 \arrow[l] \\ 1 & 2 \arrow[l] \arrow[ul] & 3 \arrow[l] & 4 \arrow[l] \end{tikzcd}, \qquad S(15\underline{32}4) = \begin{tikzcd} -1 & -2 \arrow[l] \\ 1 & 2 \arrow[l] \arrow[ul] & 3 \arrow[l] \arrow[r] & 4 \end{tikzcd},$$

$$S(\underline{1}45\underline{3}2) = \begin{tikzcd} -1 \arrow[r] & -2 \\ 1 \arrow[ur] \arrow[r] & 2 & 3 \arrow[l] & 4 \arrow[l] \end{tikzcd}, \qquad S(\underline{1}5\underline{3}24) = \begin{tikzcd} -1 \arrow[r] & -2 \\ 1 \arrow[ur] \arrow[r] & 2 & 3 \arrow[l] \arrow[r] & 4 \end{tikzcd};$$

- $\sigma = (5, -3, 1)$ (4 elements):

$$S(245\underline{31}) = \begin{tikzcd} 1 \arrow[r] \arrow[dr] & -2 \\ -1 & 2 \arrow[l] & 3 \arrow[l] & 4 \arrow[l] \end{tikzcd}, \qquad S(25\underline{31}4) = \begin{tikzcd} 1 \arrow[r] \arrow[dr] & -2 \\ -1 & 2 \arrow[l] & 3 \arrow[l] \arrow[r] & 4 \end{tikzcd},$$

$$S(\underline{2}45\underline{3}1) = \begin{tikzcd} -1 \arrow[r] \arrow[dr] & -2 \\ 1 & 2 \arrow[l] & 3 \arrow[l] & 4 \arrow[l] \end{tikzcd}, \qquad S(\underline{2}5\underline{3}14) = \begin{tikzcd} -1 \arrow[r] \arrow[dr] & -2 \\ 1 & 2 \arrow[l] & 3 \arrow[l] \arrow[r] & 4 \end{tikzcd};$$

- $\sigma = (5, -3, 2)$ (8 elements):

$$S(\underline{4}5\underline{321}) = \begin{tikzcd} 1 \arrow[dr] & -2 \arrow[l] \arrow[dr] \\ -1 & 2 \arrow[l] & 3 \arrow[l] & 4 \arrow[l] \end{tikzcd}, \qquad S(\underline{5321}4) = \begin{tikzcd} 1 \arrow[dr] & -2 \arrow[l] \arrow[dr] \\ -1 & 2 \arrow[l] & 3 \arrow[l] \arrow[r] & 4 \end{tikzcd},$$

$$S(45\underline{31}2) = \begin{tikzcd} 1 \arrow[r] & -2 \arrow[dl] \arrow[dr] \\ -1 \arrow[r] & 2 & 3 \arrow[l] & 4 \arrow[l] \end{tikzcd}, \qquad S(5\underline{31}24) = \begin{tikzcd} 1 \arrow[r] & -2 \arrow[dl] \arrow[dr] \\ -1 \arrow[r] & 2 & 3 \arrow[l] \arrow[r] & 4 \end{tikzcd},$$

$$S(45\underline{32}1) = \begin{tikzcd} -1 \arrow[dr] & -2 \arrow[l] \arrow[dr] \\ 1 & 2 \arrow[l] & 3 \arrow[l] & 4 \arrow[l] \end{tikzcd}, \qquad S(5\underline{32}14) = \begin{tikzcd} -1 \arrow[dr] & -2 \arrow[l] \arrow[dr] \\ 1 & 2 \arrow[l] & 3 \arrow[l] \arrow[r] & 4 \end{tikzcd},$$

$$S(\underline{4}5\underline{3}12) = \begin{tikzcd} -1 \arrow[r] & -2 \arrow[dl] \arrow[dr] \\ 1 \arrow[r] & 2 & 3 \arrow[l] & 4 \arrow[l] \end{tikzcd}, \qquad S(\underline{53}124) = \begin{tikzcd} -1 \arrow[r] & -2 \arrow[dl] \arrow[dr] \\ 1 \arrow[r] & 2 & 3 \arrow[l] \arrow[r] & 4 \end{tikzcd};$$

- $\sigma = (5, -2, 0)$ (4 elements):

$$S(\underline{1}345\underline{2}) = \begin{tikzcd} -1 \\ 1 & 2 \arrow[l] \arrow[ul] & 3 \arrow[l] & 4 \arrow[l] \end{tikzcd}, \qquad S(\underline{1}35\underline{2}4) = \begin{tikzcd} -1 \\ 1 & 2 \arrow[l] \arrow[ul] & 3 \arrow[l] \arrow[r] & 4 \end{tikzcd},$$

$$S(\underline{1}45\underline{2}3) = \begin{tikzcd} -1 \\ 1 & 2 \arrow[l] \arrow[ul] \arrow[r] & 3 & 4 \arrow[l] \end{tikzcd}, \qquad S(\underline{1}5\underline{2}34) = \begin{tikzcd} -1 \\ 1 & 2 \arrow[l] \arrow[ul] \arrow[r] & 3 \arrow[r] & 4 \end{tikzcd};$$

- $\sigma = (5, -2, 1)$ (8 elements):

$$S(345\underline{21}) = \begin{tikzcd} 1 \arrow[dr] \\ -1 & 2 \arrow[l] & 3 \arrow[l] & 4 \arrow[l] \end{tikzcd}, \qquad S(35\underline{21}4) = \begin{tikzcd} 1 \arrow[dr] \\ -1 & 2 \arrow[l] & 3 \arrow[l] \arrow[r] & 4 \end{tikzcd},$$

$$S(45\underline{21}3) = \begin{tikzcd} 1 \arrow[dr] \\ -1 & 2 \arrow[l] \arrow[r] & 3 & 4 \arrow[l] \end{tikzcd}, \qquad S(5\underline{21}34) = \begin{tikzcd} 1 \arrow[dr] \\ -1 & 2 \arrow[l] \arrow[r] & 3 \arrow[r] & 4 \end{tikzcd},$$

$$S(\underline{3}45\underline{2}1) = \begin{tikzcd} -1 \arrow[dr] \\ 1 & 2 \arrow[l] & 3 \arrow[l] & 4 \arrow[l] \end{tikzcd}, \qquad S(\underline{3}5\underline{2}14) = \begin{tikzcd} -1 \arrow[dr] \\ 1 & 2 \arrow[l] & 3 \arrow[l] \arrow[r] & 4 \end{tikzcd},$$

$$S(\underline{4}5\underline{2}13) = \begin{tikzcd} -1 \arrow[dr] \\ 1 & 2 \arrow[l] \arrow[r] & 3 & 4 \arrow[l] \end{tikzcd}, \qquad S(\underline{52}134) = \begin{tikzcd} -1 \arrow[dr] \\ 1 & 2 \arrow[l] \arrow[r] & 3 \arrow[r] & 4 \end{tikzcd};$$

- $\sigma = (5, -1, 0)$ (8 elements):

$$S(\underline{2}345\underline{1}) = -1 \leftarrow 2 \longleftarrow 3 \longleftarrow 4, \qquad S(\underline{2}35\underline{1}4) = -1 \leftarrow 2 \longleftarrow 3 \longrightarrow 4,$$

$$S(\underline{2}45\underline{1}3) = -1 \leftarrow 2 \longrightarrow 3 \longleftarrow 4, \qquad S(\underline{2}5\underline{1}34) = -1 \leftarrow 2 \longrightarrow 3 \longrightarrow 4,$$

$$S(\underline{3}45\underline{1}2) = -1 \longrightarrow 2 \longleftarrow 3 \longleftarrow 4\ , \qquad S(\underline{3}5\underline{1}24) = -1 \longrightarrow 2 \longleftarrow 3 \longrightarrow 4\ ,$$

$$S(\underline{4}5\underline{1}23) = -1 \longrightarrow 2 \longrightarrow 3 \longleftarrow 4\ , \qquad S(\underline{5}\underline{1}234) = -1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4\ ;$$

- $\sigma = (5, 1, 0)$ (8 elements):

$$S(23451) = 1 \longleftarrow 2 \longleftarrow 3 \longleftarrow 4\ , \qquad S(23514) = 1 \longleftarrow 2 \longleftarrow 3 \longrightarrow 4\ ,$$

$$S(24513) = 1 \longleftarrow 2 \longrightarrow 3 \longleftarrow 4\ , \qquad S(25134) = 1 \longleftarrow 2 \longrightarrow 3 \longrightarrow 4\ ,$$

$$S(34512) = 1 \longrightarrow 2 \longleftarrow 3 \longleftarrow 4\ , \qquad S(35124) = 1 \longrightarrow 2 \longleftarrow 3 \longrightarrow 4\ ,$$

$$S(45123) = 1 \longrightarrow 2 \longrightarrow 3 \longleftarrow 4\ , \qquad S(51234) = 1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4\ ;$$

- $\sigma = (5, 2, 0)$ (4 elements):

$$S(13452) = 2 \longleftarrow 3 \longleftarrow 4\ , \qquad S(13524) = 2 \longleftarrow 3 \longrightarrow 4\ ,$$

$$S(14523) = 2 \longrightarrow 3 \longleftarrow 4\ , \qquad S(15234) = 2 \longrightarrow 3 \longrightarrow 4\ ;$$

- $\sigma = (5, 3, 0)$ (2 elements):

$$S(12453) = 3 \longleftarrow 4\ , \qquad S(12534) = 3 \longrightarrow 4\ ;$$

- $\sigma = (5, 4, 0)$ (1 element):

$$S(12354) = 4\ .$$

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