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A Kunz-type characterization of regular rings via alterations

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ABSTRACT

We prove that a local domain R, essentially of finite type over a field, is regular if and only if for every regular alteration $\pi : X \longrightarrow \operatorname{Spec} R$, we have that $\mathbf{R}\pi_*\mathcal{O}_X$ has finite (equivalently zero in characteristic zero) projective dimension.

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1. Introduction

In [10] Kunz proved that a Noetherian ring of characteristic $p > 0$ is regular if and only if the e-th iterated Frobenius map

$$\begin{aligned} R &\longrightarrow R \\ x &\longmapsto x^{p^e} \end{aligned}$$

is flat for some, or equivalently every, $e > 0$. This is generalized in [17]: the condition $\operatorname{fd}_R R^{(e)} < \infty$ (for some, or equivalently every, $e > 0$) implies R is regular, where $R^{(e)}$ denotes the target of the e-th Frobenius map. Moreover, the direct limit of $R^{(e)}$ is the perfection R^∞ of R. Kunz's theorem can also be generalized using perfection: R is regular if and only if $\operatorname{fd}_R R^\infty < \infty$ [1,3].

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However, in all these characterizations, the Frobenius map plays a prominent role and hence they do not extend to characteristic zero. In this paper, motivated by the connections between multiplier ideals and test ideals [4], we prove the following characterization of regularity using alterations.

Main Theorem. *Suppose* $(R, \mathfrak{m})$ *is a local domain essentially of finite type over a field. Then the following conditions are equivalent.*

(a) *R is regular.*
(b) *For every regular alteration* $\pi : X \longrightarrow \operatorname{Spec} R$, $\operatorname{pd}_R \mathbf{R}\pi_*\mathcal{O}_X < \infty$ *(i.e., the derived image of the structure sheaf has finite projective dimension).*

Moreover, when R has characteristic 0*, the above are also equivalent to*

(c) *For every regular alteration* $\pi : X \longrightarrow \operatorname{Spec} R$, $\operatorname{pd}_R \mathbf{R}\pi_*\mathcal{O}_X = 0$.

This result is also motivated by the fact that there is close connection between big Cohen-Macaulay algebras and $\mathbf{R}\pi_*\mathcal{O}_X$ (recall that the latter is a Cohen-Macaulay complex, [15]).

In fact, this theorem in characteristic $p > 0$ essentially follows from a result of Bhatt on killing cohomology using finite covers [2] and the characterization of regularity using R^+, see [1,3]. Our main contribution is the characteristic zero case. Meanwhile, it is quite natural to ask whether the same characterization of regularity holds in mixed characteristic:

Question 1.1. Suppose $(R, \mathfrak{m})$ is an excellent local domain essentially of finite type over the integers. If for every regular alteration $\pi : X \longrightarrow \operatorname{Spec} R$, we have that $\operatorname{pd}_R \mathbf{R}\pi_*\mathcal{O}_X < \infty$, then is R regular?

See Remark 3.7 for some additional discussion. Finally, one should also compare with the classical result that if $R \subseteq S$ is a flat local extension and S is regular, then so is R, see [13, Theorem 23.7].

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2. Preliminaries

Suppose R is a Noetherian ring. For a bounded above cochain complex of R-modules C $(\dots \longrightarrow C^{-1} \longrightarrow C^0 \longrightarrow C^1 \longrightarrow \dots)$, we define the *projective dimension of* C, denoted $\operatorname{pd}_R C$ to be

$$\inf\{\sup\{i \mid P^{-i} \neq 0\} \mid P^\bullet \text{ is a projective resolution of } C\}.$$

Here a projective resolution is a cochain complex of projective modules quasi-isomorphic to C. If $(R, \mathfrak{m})$ is additionally local, we also define

$$\operatorname{depth} C = \inf\{i \mid H^i_{\mathfrak{m}}(C) \neq 0\}.$$

These are natural extensions of the projective dimension and the depth of a finitely generated R-module.

We will need the following result of Foxby and Iyengar, which is a vast generalization of the classical Auslander-Buchsbaum formula.

Theorem 2.1 *([6]). Let $(R, \mathfrak{m})$ be a local ring and let M and P be complexes of R-modules. If $\operatorname{pd}_R P$ is finite and $\mathrm{H}(P)$ is nonzero and finitely generated, then*

$$\operatorname{depth}_R M = \operatorname{depth}_R(M \otimes^{\mathbf{L}}_R P) + \operatorname{pd}_R P.$$

We can now prove our main result in positive characteristic.

Proof of Main Theorem in characteristic $p > 0$. First of all by Theorem 2.1,

$$\operatorname{pd}_R \mathbf{R}\pi_*\mathcal{O}_X + \operatorname{depth} \mathbf{R}\pi_*\mathcal{O}_X = \operatorname{depth} R.$$

Since $\operatorname{depth} \mathbf{R}\pi_*\mathcal{O}_X \geqslant 0$ (as $\mathbf{R}\pi_*\mathcal{O}_X$ lives in positive degree), we know that for all regular alteration, $\operatorname{pd}_R \mathbf{R}\pi_*\mathcal{O}_X \leqslant \operatorname{depth} R$.

By [2, Theorem 1.5], for every regular alteration $\pi\colon X \longrightarrow \operatorname{Spec} R$, there exists another regular alteration $\pi'\colon Y \xrightarrow{f} X \longrightarrow \operatorname{Spec} R$ such that the map $\tau_{\geqslant 1}\mathbf{R}\pi_*\mathcal{O}_X \longrightarrow \tau_{\geqslant 1}\mathbf{R}\pi'_*\mathcal{O}_Y$, induced by the diagram of triangles below, is 0.

$$\begin{tikzcd}
\pi_*\mathcal{O}_X \arrow[r] \arrow[d] & \mathbf{R}\pi_*\mathcal{O}_X \arrow[r] \arrow[d] & \tau_{\geqslant 1}\mathbf{R}\pi_*\mathcal{O}_X \arrow[r, "+1"] \arrow[d, "0"] & {} \\
\pi'_*\mathcal{O}_Y \arrow[r] & \mathbf{R}\pi'_*\mathcal{O}_Y \arrow[r] & \tau_{\geqslant 1}\mathbf{R}\pi'_*\mathcal{O}_Y \arrow[r, "+1"'] & {}
\end{tikzcd}$$

Tensoring with $k = R/\mathfrak{m}$ and taking cohomology, for all $i > \operatorname{depth} R$ we get:

$$\begin{tikzcd}
0 = H^{-i-1}(\mathbf{R}\pi_*\mathcal{O}_X \otimes^{\mathbf{L}} k) \arrow[r] & H^{-i-1}(\tau_{\geqslant 1}\mathbf{R}\pi_*\mathcal{O}_X \otimes^{\mathbf{L}} k) \arrow[r] \arrow[d, "0"] & H^{-i}(\pi_*\mathcal{O}_X \otimes^{\mathbf{L}} k) \arrow[r] \arrow[d] & 0 = H^{-i}(\mathbf{R}\pi_*\mathcal{O}_X \otimes^{\mathbf{L}} k) \\
0 = H^{-i-1}(\mathbf{R}\pi'_*\mathcal{O}_Y \otimes^{\mathbf{L}} k) \arrow[r] & H^{-i-1}(\tau_{\geqslant 1}\mathbf{R}\pi_*\mathcal{O}_X \otimes^{\mathbf{L}} k) \arrow[r] & H^{-i}(\pi'_*\mathcal{O}_Y \otimes^{\mathbf{L}} k) \arrow[r] & 0 = H^{-i}(\mathbf{R}\pi_*\mathcal{O}_X \otimes^{\mathbf{L}} k)
\end{tikzcd}$$

An easy diagram chasing shows that for all regular alteration X, we can find another regular alteration Y such that $H^{-i}(\pi_*\mathcal{O}_X \otimes^{\mathbf{L}} k) \longrightarrow H^{-i}(\pi'_*\mathcal{O}_Y \otimes^{\mathbf{L}} k)$ is 0. By writing R^+ as the colimit over finite domain extensions of R, $R^+ = \lim_{S \supseteq R} S$, we see that $H^{-i}(R^+ \otimes^{\mathbf{L}} k) = \operatorname{Tor}_i^R(R^+, k) = 0$ for all $i > \operatorname{depth} R$. Now by [1, Corollary 3.5] or [3, Theorem 4.13], R is regular. $\square$

We recall the following very useful result of Corso-Huneke-Katz-Vasconcelos.

Theorem 2.2 *([5, Corollary 3.3]). Suppose that $(R, \mathfrak{m})$ is a Noetherian local ring and that I is integrally closed and $\mathfrak{m}$-primary. Then M has projective dimension less than t if and only if $\operatorname{Tor}_t^R(R/I, M) = 0$.*

We specialize it in the following corollary that we will use in the next section.

Corollary 2.3. *Suppose that $(R, \mathfrak{m})$ is a Noetherian local ring and that I is an integrally closed $\mathfrak{m}$-primary ideal of finite projective dimension. Then R is regular.*

Proof. Since I has finite projective dimension, we see that $\operatorname{Tor}_i^R(R/I, k) = 0$ for $i \gg 0$. But now taking $M = k = R/\mathfrak{m}$ in the statement of Theorem 2.2 we see that k has finite projective dimension since I is integrally closed and $\mathfrak{m}$-primary. The result follows. $\square$

2.1. Multiplier ideals and multiplier submodules

For references in this section, see [4,11,19].

Definition 2.4 *(Multiplier submodules).* Suppose that $\pi : X \longrightarrow \operatorname{Spec} R$ is a resolution of singularities. Then the *multiplier submodule* of R, denoted $\mathcal{J}(\omega_R)$ is just $\pi_*\omega_X \subseteq \omega_R$. Here ω_R (respectively ω_X) is the first nonzero cohomology of the dualizing complex.

We now generalize this a bit. Suppose R is a normal domain, $\Gamma \geqslant 0$ is a $\mathbb{Q}$-Cartier divisor, and π is a log resolution of (X, Γ). Then we define $\mathcal{J}(\omega_R, \Gamma) = \pi_*\mathcal{O}_X(\lceil K_X - \pi^*\Gamma \rceil)$. If we choose $0 \neq f \in R$ and $t \in \mathbb{Q}_{\geqslant 0}$, then we set $\mathcal{J}(\omega_R, \Gamma, f^t) = \mathcal{J}(\omega_R, \Gamma + t\operatorname{div}(f))$. Finally, if $\mathfrak{a} \subseteq R$ is an ideal and π is a log resolution of $(R, \Gamma, \mathfrak{a}^t)$ with $\mathfrak{a} \cdot \mathcal{O}_X = \mathcal{O}_X(-G)$, then we define

$$\mathcal{J}(\omega_R, \Gamma, \mathfrak{a}^t) = \pi_*\mathcal{O}_X(\lceil K_X - \pi^*\Gamma - tG \rceil) \subseteq \omega_R.$$

All of this is independent of the choice of resolution.

In the above, if Γ is ever left out, it is treated as zero.

Definition 2.5 *(Multiplier ideals).* Suppose that R is a normal domain, $\Delta \geqslant 0$ is a $\mathbb{Q}$-divisor such that $K_R + \Delta$ is $\mathbb{Q}$-Cartier, $\mathfrak{a} \subseteq R$ is an ideal and $t \in \mathbb{Q}_{\geqslant 0}$, then we define the *multiplier ideal*

$$\mathcal{J}(R, \Delta, \mathfrak{a}^t) = \pi_*\mathcal{O}_X(\lceil K_X - \pi^*(K_R + \Delta) - tG \rceil)$$

where $\pi : X \longrightarrow \operatorname{Spec} R$ is a log resolution of $(R, \Delta, \mathfrak{a}^t)$ and $\mathfrak{a} \cdot \mathcal{O}_X = \mathcal{O}_X(-G)$. Again, this is independent of the choice of resolution.

If Δ is left off, then it is treated as zero and if $\mathfrak{a}$ is left off, it is treated as R.

3. The main result in characteristic zero

We begin with the "easy" direction.

Theorem 3.1. *Suppose that $(R, \mathfrak{m})$ is a regular local ring essentially of finite type over a field of characteristic zero. If $\pi : X \longrightarrow \operatorname{Spec} R$ is a regular alteration, then $\operatorname{pd}_R \mathbf{R}\pi_*\mathcal{O}_X = 0$.*

Proof. Since R is regular, the bounded complex $\mathbf{R}\pi_*\mathcal{O}_X$ has finite projective dimension. By Theorem 2.1, taking $M = R$ and $P = \mathbf{R}\pi_*\mathcal{O}_X$ we have that

$$\operatorname{pd}_R \mathbf{R}\pi_*\mathcal{O}_X + \operatorname{depth}_R(\mathbf{R}\pi_*\mathcal{O}_X) = \dim R.$$

By the Matlis-dual version of Grauert-Riemenschneider vanishing [7], $H^i_{\mathfrak{m}}(\mathbf{R}\pi_*\mathcal{O}_X) = 0$ for all $i < \dim R$ and hence $\operatorname{depth}_R(\mathbf{R}\pi_*\mathcal{O}_X) \geqslant \dim R$. Thus $\operatorname{pd}_R \mathbf{R}\pi_*\mathcal{O}_X \leqslant 0$. But clearly $\operatorname{pd}_R \mathbf{R}\pi_*\mathcal{O}_X \geqslant 0$ since $H^0(\mathbf{R}\pi_*\mathcal{O}_X) \neq 0$. The result follows. □

Lemma 3.2. *Suppose $(R, \mathfrak{m})$ is a local domain essentially of finite type over a field of characteristic zero and that $\pi : X \longrightarrow \operatorname{Spec} R$ is a resolution of singularities. If $\operatorname{pd}_R \mathbf{R}\pi_*\mathcal{O}_X < \infty$, then R is Cohen-Macaulay.*

Proof. Since $\operatorname{pd}_R \mathbf{R}\pi_*\mathcal{O}_X < \infty$, we see that the injective dimension of the Grothendieck dual, $\mathbf{R}\pi_*\omega_X^{\bullet} \cong \pi_*\omega_X[\dim R]$ (by Grauert-Riemenschneider vanishing), is finite. But then $\pi_*\omega_X$ is a finitely generated R-module of finite injective dimension and so R is Cohen-Macaulay by Bass' question [8,14,16]. □

Alternate proof. By the Matlis-dual version of the Grauert-Riemenschneider vanishing, we see that $H^i_{\mathfrak{m}}(\mathbf{R}\pi_*\mathcal{O}_X) = 0$ for all $i < \dim R$. Hence $\operatorname{depth} \mathbf{R}\pi_*\mathcal{O}_X = \dim R$. Note also that $\operatorname{pd}_R \mathbf{R}\pi_*\mathcal{O}_X \geqslant 0$ since $H^0(\mathbf{R}\pi_*\mathcal{O}_X) \neq 0$. Thus we have

$$\operatorname{depth} R = \operatorname{depth}(\mathbf{R}\pi_*\mathcal{O}_X) + \operatorname{pd}_R(\mathbf{R}\pi_*\mathcal{O}_X) \geqslant \dim R$$

by Theorem 2.1 and hence R is Cohen-Macaulay. □

We are ready to prove the following characterization of rational singularities, this result is an important step towards proving the main theorem and is interesting in its own right.

Theorem 3.3. *Suppose $(R, \mathfrak{m})$ is a local domain essentially of finite type over a field of characteristic zero. Let $\pi : X \longrightarrow \operatorname{Spec} R$ be a resolution of singularities. Then R has rational singularities if and only if $\operatorname{pd}_R \mathbf{R}\pi_*\mathcal{O}_X < \infty$.*

Proof. If R has rational singularities then obviously $\operatorname{pd}_R \mathbf{R}\pi_*\mathcal{O}_X < \infty$ since $\mathbf{R}\pi_*\mathcal{O}_X \cong R$. We now assume that $\operatorname{pd}_R \mathbf{R}\pi_*\mathcal{O}_X < \infty$. We already see that R is Cohen-Macaulay by Lemma 3.2. Hence, it is sufficient to show that $\pi_*\omega_X = \omega_R$.

So we suppose $\pi_*\omega_X \neq \omega_R$. By choosing a minimal prime P of $\operatorname{Supp}(\omega_R/\pi_*\omega_X)$ and replacing R by R_P, we may assume R has rational singularities on the punctured spectrum (i.e., $\omega_R/\pi_*\omega_X$ has finite length). Since $\pi_*\omega_X$ has finite injective dimension (see the proof of Lemma 3.2), by [18, Theorem 2.9], $\operatorname{Hom}_R(\omega_R, \pi_*\omega_X)$ has finite projective dimension. But

$$\operatorname{Hom}_R(\omega_R, \pi_*\omega_X) = \pi_* \mathscr{H}om_{\mathcal{O}_X}(\pi^*\omega_R, \omega_X).$$

Now $\mathscr{H}om_{\mathcal{O}_X}(\pi^*\omega_R, \omega_X)$ is a rank 1 reflexive sheaf on X. Since X is regular, $\mathscr{H}om_{\mathcal{O}_X}(\pi^*\omega_R, \omega_X)$ is locally free and so its pushforward, which is isomorphic to

$$\operatorname{Hom}_R(\omega_R, \pi_*\omega_X) \subseteq \operatorname{Hom}_R(\omega_R, \omega_R) \subseteq R,$$

is an integrally closed ideal. Since our assumption is $0 \neq \omega_R/\pi_*\omega_X$ has finite length, it follows that $\operatorname{Hom}_R(\omega_R, \pi_*\omega_X) \neq R$ is an $\mathfrak{m}$-primary integrally closed ideal. But then by Corollary 2.3, $\operatorname{pd}_R \operatorname{Hom}_R(\omega_R, \pi_*\omega_X) < \infty$ already implies R is regular and thus $\pi_*\omega_X = \omega_R$ which is a contradiction. □

Remark 3.4. Bhargav Bhatt communicated to us an alternate proof of Theorem 3.3, which we now sketch. Since $\mathbf{R}\pi_*\mathcal{O}_X$ is a perfect complex, there exists a trace map

$$\mathbf{R}\operatorname{Hom}_R(\mathbf{R}\pi_*\mathcal{O}_X, \mathbf{R}\pi_*\mathcal{O}_X) \longrightarrow R.$$

On the other hand, we have the map $\mathbf{R}\pi_*\mathcal{O}_X \longrightarrow \operatorname{Hom}_R(\mathbf{R}\pi_*\mathcal{O}_X, \mathbf{R}\pi_*\mathcal{O}_X)$ coming from $\mathcal{O}_X$'s left multiplication action on itself. We have the composition

$$R \longrightarrow \mathbf{R}\pi_*\mathcal{O}_X \longrightarrow \mathbf{R}\operatorname{Hom}_R(\mathbf{R}\pi_*\mathcal{O}_X, \mathbf{R}\pi_*\mathcal{O}_X) \longrightarrow R$$

which is an isomorphism generically (on the open set where π is an isomorphism), hence an isomorphism. But then R has rational singularities by [9] (note that that result still utilizes Grauert-Riemenschneider vanishing).

3.1. An aside on multiplier ideals

We assume the following is essentially well known to experts, but we do not know a reference.

Proposition 3.5. *Suppose $(R, \mathfrak{m})$ is a normal local domain essentially of finite type over a field of characteristic zero. Suppose $0 \neq f \in R$ such that $\mathrm{div}_R(f)$ is reduced. Fix $N \geqslant 0$ and let $S = R[f^{\frac{1}{N+1}}]$ be the normal cyclic cover. Then $\mathcal{J}(\omega_S)$ has an R-summand isomorphic to $\mathcal{J}(\omega_R, f^{\frac{N}{N+1}})$.*

Proof. Since $\mathrm{div}_R(f)$ is reduced, S is regular in codimension 1 and hence S is normal. Choose $-K_R$ effective. By [11, Theorem 9.5.42] (see also [4, Theorem 8.1]) we see that

$$\mathcal{J}(R, -K_R + \frac{N}{N+1}\,\mathrm{div}_R f) = R \cap \mathcal{J}(S, -\mathrm{Ram}_{S/R} - \rho^* K_R + \frac{N}{N+1}\,\mathrm{div}_S f).$$

Again since $\mathrm{div}_R f$ is reduced, we see that $\mathrm{Ram}_{S/R} = \frac{N}{N+1}\,\mathrm{div}_S f$ and hence

$$\mathcal{J}(R, -K_R + \frac{N}{N+1}\,\mathrm{div}_R f) \subseteq \mathcal{J}(S, -\rho^* K_R).$$

On the other hand, by [4, Theorem 8.1], we have a splitting (up to scalars)

$$\begin{aligned} &\mathrm{Tr}(\mathcal{J}(S, -\rho^* K_R)) \\ = {} &\mathrm{Tr}(\mathcal{J}(S, -\rho^* K_R - \mathrm{Ram}_{S/R} + \tfrac{N}{N+1}\,\mathrm{div}_S f)) \\ = {} &\mathcal{J}(R, -K_R + \tfrac{N}{N+1}\,\mathrm{div}_R f) \\ = {} &\mathcal{J}(\omega_R, f^{\frac{N}{N+1}}). \end{aligned}$$

But we have

$$\begin{aligned} &\mathcal{J}(S, -\rho^* K_R) \\ = {} &\mathcal{J}(S, -\rho^* K_R - \mathrm{Ram}_{S/R} + \tfrac{N}{N+1}\,\mathrm{div}_S f) \\ = {} &\mathcal{J}(S, -K_S + \tfrac{N}{N+1}\,\mathrm{div}_S f) \\ = {} &\mathcal{J}(\omega_S, f^{\frac{N}{N+1}}). \end{aligned}$$

We have just shown that $\mathcal{J}(\omega_S, f^{\frac{N}{N+1}})$ has an R-summand isomorphic to $\mathcal{J}(\omega_R, f^{\frac{N}{N+1}})$. But even as an S-module $\mathcal{J}(\omega_S, f^{\frac{N}{N+1}}) = f^{\frac{N}{N+1}}\mathcal{J}(\omega_S) \cong \mathcal{J}(\omega_S)$, and hence $\mathcal{J}(\omega_S)$ has an R-summand isomorphic to $\mathcal{J}(\omega_R, f^{\frac{N}{N+1}})$. $\square$

3.2. Proof of Main Theorem in characteristic zero

We now complete the proof of our main result in characteristic zero.

Theorem 3.6. *Suppose $(R, \mathfrak{m})$ is a local domain essentially of finite type over a field of characteristic zero. Suppose that for every regular alteration $\pi : X \longrightarrow \mathrm{Spec}\, R$, $\mathrm{pd}_R\, \mathbf{R}\pi_* \mathcal{O}_X < \infty$. Then R is regular.*

Proof. By Theorem 3.3, we already know that R has rational singularities. Choose $N > 0$ so that $\mathcal{J}(\omega_R, \mathfrak{m}^N) \neq \omega_R$. Then choose a general $f \in \mathfrak{m}^{N+1}$ and by [11, Proposition 9.2.28] we know that $\mathcal{J}(R, -K_R + \frac{N}{N+1}\,\mathrm{div}_R f) = \mathcal{J}(R, -K_R, \mathfrak{m}^N) = \mathcal{J}(\omega_R, \mathfrak{m}^N)$.

Consider the normal cyclic cover $S = R[f^{\frac{1}{N+1}}]$. Since f is general, $\mathrm{div}_R(f)$ is reduced and by Proposition 3.5, we know that $\mathcal{J}(\omega_S)$ has an R-summand isomorphic to $\mathcal{J}(\omega_R, f^{\frac{N}{N+1}})$.

Next consider a resolution of singularities $\pi\colon X \longrightarrow \operatorname{Spec} S$, then the composition $X \longrightarrow \operatorname{Spec} S \longrightarrow \operatorname{Spec} R$ is a regular alteration. Moreover, $\pi_*\omega_X = \mathcal{J}(\omega_S)$ has finite injective dimension over R (because $\pi_*\omega_X[\dim R]$ is the Grothendieck dual of $\mathbf{R}\pi_*\mathcal{O}_X$), so does its direct summand $\mathcal{J}(\omega_R, f^{\frac{N}{N+1}})$. Therefore by [18, Theorem 2.9],

$$\operatorname{Hom}_R(\omega_R, \mathcal{J}(\omega_R, f^{\frac{N}{N+1}})) \subseteq \operatorname{Hom}_R(\omega_R, \omega_R) \cong R$$

has finite projective dimension. Since $\mathcal{J}(\omega_R, f^{\frac{N}{N+1}}) = \mathcal{J}(\omega_R, \mathfrak{m}^N)$ agrees with ω_R except at the origin (where it *does not* agree). Thus $\operatorname{Hom}_R(\omega_R, \mathcal{J}(\omega_R, f^{\frac{N}{N+1}}))$ lacks the identity map $\omega_R \longrightarrow \omega_R$ and hence it is identified with an $\mathfrak{m}$-primary ideal of R.

Next we show that $\operatorname{Hom}_R(\omega_R, \mathcal{J}(\omega_R, f^{\frac{N}{N+1}})) \subseteq R$ is an integrally closed ideal. Take a log resolution of singularities $\pi : X \longrightarrow \operatorname{Spec} R$ of $(R, \operatorname{div}_R(f))$. By definition we have $\mathcal{J}(\omega_R, f^{\frac{N}{N+1}}) = \pi_*\mathcal{O}_X(\lceil K_X - \frac{N}{N+1}\operatorname{div}_X(f)\rceil)$. Thus

$$\begin{aligned} &\operatorname{Hom}_R(\omega_R, \mathcal{J}(\omega_R, f^{\frac{N}{N+1}})) \\ = \; &\operatorname{Hom}_R(\omega_R, \pi_*\mathcal{O}_X(\lceil K_X - \tfrac{N}{N+1}\operatorname{div}_X(f)\rceil)) \\ = \; &\pi_* \mathscr{H}om_{\mathcal{O}_X}(\pi^*\omega_R, \mathcal{O}_X(\lceil K_X - \tfrac{N}{N+1}\operatorname{div}_X(f)\rceil)). \end{aligned}$$

As in the proof of Theorem 3.3, since $\mathscr{L} := \mathscr{H}om_{\mathcal{O}_X}(\pi^*\omega_R, \mathcal{O}_X(\lceil K_X - \frac{N}{N+1}\operatorname{div}_X(f)\rceil))$ is a rank 1 reflexive sheaf and X is regular, $\mathscr{L}$ is invertible. Thus $\operatorname{Hom}_R(\omega_R, \pi_*\mathcal{O}_X(\lceil K_X - \frac{N}{N+1}\operatorname{div}_X(f)\rceil))$ is an integrally closed $\mathfrak{m}$-primary ideal of finite projective dimension. Therefore R is regular by Corollary 2.3. □

Remark 3.7. We believe that the above proof can be run (essentially without change) for excellent surfaces even in mixed characteristic. The key facts we need are that Grauert-Riemenschneider still holds for excellent surfaces [12, Corollary 2.10] and that we can choose a general element f in $\mathfrak{m}^{N+1}$ so that $\mathcal{J}(R, f^{N/N+1})$ is $\mathfrak{m}$-primary [20,21] (using that R is regular outside of the origin since we may reduce to the case that R is normal).

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