



# Verschiebung maps among $K$ -groups of truncated polynomial algebras



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## ABSTRACT

Let  $p$  be a prime number, and let  $A$  be a ring in which  $p$  is nilpotent. In this paper, we consider the maps

$$K_{q+1}(A[x]/(x^m), (x)) \rightarrow K_{q+1}(A[x]/(x^{mn}), (x)),$$

induced by the ring homomorphism  $A[x]/(x^m) \rightarrow A[x]/(x^{mn})$ ,  $x \mapsto x^n$ . We evaluate these maps, up to extension, for general  $A$  in terms of topological Hochschild homology, and for regular  $\mathbb{F}_p$ -algebras  $A$ , in terms of groups of de Rham-Witt forms. After the evaluation, we give a calculation of the relative  $K$ -group of  $\mathcal{O}_K/p\mathcal{O}_K$  for certain perfectoid fields  $K$ .

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## 1. Introduction

There is a natural map from algebraic  $K$ -theory to topological cyclic homology called the cyclotomic trace map. By the famous theorem of Dundas-Goodwillie-McCarthy [1, Theorem 7.0.0.2] about this map, relative topological cyclic homology and relative algebraic  $K$ -theory coincide for the algebras we are studying. In [4, Theorem 4.2.10], Hesselholt and Madsen constructed the following exact sequence and gave an explicit presentation [4, Theorem A] of the algebraic  $K$ -theory of a truncated polynomial algebra over a perfect field  $k$  of positive characteristic by the Witt vectors:

$$0 \longrightarrow \mathbb{W}_m(k) \xrightarrow{V_n} \mathbb{W}_{mn}(k) \longrightarrow \mathrm{TC}_{2m-1}(k[x]/(x^n), (x)) \longrightarrow 0,$$

where  $\mathrm{TC}$  denotes topological cyclic homology and  $\mathbb{W}_l$  denotes the big Witt vectors of length  $l$ . Furthermore they also show that such  $\mathrm{TC}$ -groups in even degrees are trivial.

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Using such stable homotopy theoretical results, in this paper, we consider the map of relative algebraic  $K$ -groups

$$v_n: K_{q+1}(A[x]/(x^m), (x)) \rightarrow K_{q+1}(A[x]/(x^{mn}), (x))$$

induced by the ring homomorphism

$$A[x]/(x^m) \rightarrow A[x]/(x^{nm}), \quad x \mapsto x^n.$$

We also consider the colimit over these  $v_n$ 's indexed by the filtered category  $(\mathbb{N}, *)$  of natural numbers under multiplication. One of our main results is the following (Theorem 4.1). The notation in the diagram below will be introduced in Section 4.

**Theorem 1.1.** *Let  $A$  be a ring in which  $p$  is nilpotent. There is a map of long exact sequences*

$$\begin{array}{ccc}
 \vdots & & \vdots \\
 \downarrow & & \downarrow \\
 \lim_R \mathrm{TR}_{q-\lambda_{\lfloor (i-1)/k \rfloor}}^{i/k}(A) & \xrightarrow{\mathrm{id}} & \lim_R \mathrm{TR}_{q-\lambda_{\lfloor (i-1)/kn \rfloor}}^{i/kn}(A) \\
 \downarrow V_{k*} & & \downarrow V_{kn*} \\
 \lim_R \mathrm{TR}_{q-\lambda_{\lfloor (i-1)/k \rfloor}}^i(A) & \xrightarrow{V_{n*}} & \lim_R \mathrm{TR}_{q-\lambda_{\lfloor (i-1)/kn \rfloor}}^i(A) \\
 \downarrow & & \downarrow \\
 K_{q+1}(A[x]/(x^k), (x)) & \xrightarrow{v_n} & K_{q+1}(A[x]/(x^{nk}), (x)) \\
 \downarrow & & \downarrow \\
 \vdots & & \vdots
 \end{array},$$

where  $\lfloor t \rfloor$  is the integer part of  $t$ ,  $V$ 's denote the maps induced by Verschiebung maps, the maps in the limits are restriction maps, and the maps  $v_n$  are induced by  $x \mapsto x^n$ .

One of the remarkable facts about topological Hochschild homology is that its homotopy groups correspond to the groups of de Rham-Witt forms. Via this correspondence, stable homotopy theory and  $p$ -adic Hodge theory have been getting closer to each other. Considering this, we also produce a diagram similar to the one above for regular algebras over  $\mathbb{F}_p$  in terms of de Rham-Witt groups via the translation between topological Hochschild homology and the de Rham-Witt complex given by [2, §1, §2, §5]. See also [3].

**Theorem 1.2.** *Let  $A$  be a regular  $\mathbb{F}_p$ -algebra. There is a map of long exact sequences*

$$\begin{array}{ccc}
 \vdots & & \vdots \\
 \downarrow & & \downarrow \\
 \bigoplus_{l \geq 0} \mathbb{W}_{l+1} \Omega_A^{q-2l} & \xrightarrow{\text{id}} & \bigoplus_{l \geq 0} \mathbb{W}_{l+1} \Omega_A^{q-2l} \\
 \downarrow V_{k*} & & \downarrow V_{kn*} \\
 \bigoplus_{l \geq 0} \mathbb{W}_{k(l+1)} \Omega_A^{q-2l} & \xrightarrow{V_{n*}} & \bigoplus_{l \geq 0} \mathbb{W}_{kn(l+1)} \Omega_A^{q-2l} \\
 \downarrow & & \downarrow \\
 K_{q+1}(A[x]/(x^k), (x)) & \xrightarrow{v_n} & K_{q+1}(A[x]/(x^{nk}), (x)) \\
 \downarrow & & \downarrow \\
 \vdots & & \vdots
 \end{array},$$

where the subscript  $m(l+1)$  denotes the truncation set  $\{1, 2, \dots, m(l+1)\}$  and  $\mathbb{W}_{(-)} \Omega_A^*$  denotes the big de Rham-Witt complex over  $A$ .

As an application of the above theorems, we calculate some relative  $K$ -groups. Let  $k$  be a perfect field of characteristic  $p$ . We define

$$\mathcal{O}_K := (\text{colim}_n W(k)[p^{1/p^n}])^\wedge$$

with  $\mathfrak{m}_K \subset \mathcal{O}_K$  the maximal ideal, and

$$\mathcal{O}_L := (\text{colim}_n W(k)[\zeta_{p^n}])^\wedge$$

with  $\mathfrak{m}_L \subset \mathcal{O}_L$  the maximal ideal. Let  $K$  and  $L$  denote the quotient fields  $\mathcal{O}_K[1/p]$  and  $\mathcal{O}_L[1/p]$  respectively. Note that these  $K$  and  $L$  are perfectoid fields.

With these notations, we have

$$K_{2j-1}(\mathcal{O}_K/p\mathcal{O}_K, \mathfrak{m}_K/p\mathcal{O}_K) = \text{colim}_n \mathbb{W}_{p^n j}(k)/V_{p^n} \mathbb{W}_j(k),$$

$$K_{2j-1}(\mathcal{O}_L/p\mathcal{O}_L, \mathfrak{m}_L/p\mathcal{O}_L) = \text{colim}_n \mathbb{W}_{p^{n-1}(p-1)j}(k)/V_{p^{n-1}(p-1)} \mathbb{W}_j(k),$$

where the colimits are indexed by the category of natural numbers under addition and  $\mathbb{W}$  denotes the big Witt vectors. Moreover, the relative  $K$ -groups in even degrees are zero.

## 2. The cyclic bar-construction

We recall some notations from [2] or [4]. For a positive integer  $k$ , we let  $\Pi_k := \{0, 1, x^1, \dots, x^{k-1}\}$  denote the finite commutative pointed monoid defined by  $x^n x^m = x^{m+n}$ ,  $0x^n = 0$ ,  $x^0 = 1$  and  $x^k = 0$ . We let  $\mathbb{T}$  denote the circle group and  $\lambda_d := \mathbb{C}(d) \oplus \dots \oplus \mathbb{C}(1)$ , where  $\mathbb{C}(j) = \mathbb{C}$  as linear spaces and it has the  $\mathbb{T}$ -action defined by  $\mathbb{T} \times \mathbb{C}(j) \rightarrow \mathbb{C}(j)$ ,  $(z, w) \mapsto z^j w$ , for  $1 \leq j \leq d$ .

We let  $N^{\text{cy}}(\Pi_k)[-]$  denote the cyclic set which is the cyclic bar construction of  $\Pi_k$  and  $N^{\text{cy}}(\Pi_k)$  denote its geometric realization. For  $i \in \mathbb{N}$ , we also let  $N^{\text{cy}}(\Pi_k, i)[-]$  denote the cyclic subset of  $N^{\text{cy}}(\Pi_k)[-]$  generated

by the  $(i-1)$ -simplex  $x \wedge \dots \wedge x$  ( $i$  factors) and  $N^{\text{cy}}(\Pi_k, i)$  denote its geometric realization. Since these gives a canonical decomposition of cyclic sets

$$N^{\text{cy}}(\Pi_k)[-] \cong \bigvee_{i \geq 0} N^{\text{cy}}(\Pi_k, i)[-],$$

we have the canonical decomposition of pointed spaces after geometric realization

$$N^{\text{cy}}(\Pi_k) \cong \bigvee_{i \geq 0} N^{\text{cy}}(\Pi_k, i).$$

See for example [4] for more detail.

In [4, §3], the homotopy classes of the following maps of pointed  $C_i$ -spaces are defined; for  $kd < i < k(d+1)$ ,

$$\theta_{i,k} : \Delta^{i-1}/C_i \cdot \Delta^{i-k} \rightarrow S^{\lambda_d},$$

and for  $i = k(d+1)$ ,

$$\theta_{i,k} : \Delta^{i-1}/C_i \cdot \Delta^{i-k} \rightarrow (S^0 * C_k) \wedge S^{\lambda_d},$$

where  $C_m$  is the  $m$ -th cyclic group,  $S^{\lambda_d}$  is the one point compactification of  $\lambda_d$  and  $*$  denotes the join.

In [4], for any positive integer  $i$ , the following cofibration sequence is constructed using  $\theta$ ;

$$\mathbb{T}_+ \wedge_{C_{i/k}} S^{\lambda_{d_i}} \xrightarrow{\text{pr}} \mathbb{T}_+ \wedge_{C_i} S^{\lambda_{d_i}} \longrightarrow N^{\text{cy}}(\Pi_k, i) \longrightarrow \Sigma \mathbb{T}_+ \wedge_{C_{i/k}} S^{\lambda_{d_i}},$$

where  $\mathbb{T}_+ \wedge_{C_{i/k}} S^{\lambda_{d_i}}$  is trivial when  $k$  does not divide  $i$  and  $d_i = \lfloor (i-1)/k \rfloor$  is the largest natural number less than or equal to  $(i-1)/k$ . We briefly recall the construction.

Let  $\mathbb{R}[C_i]$  denote the regular representation and  $\Delta^{i-1} \subset \mathbb{R}[C_i]$  denote the convex hull of the elements of  $C_i$ . By permutation  $C_i$  acts on  $\mathbb{R}[C_i]$  and the action restricts on  $\Delta^{i-1}$ . Let  $\xi_i$  denote the generator of  $C_i$  and  $\Delta^{i-m}$  the convex hull of  $1, \xi_i, \dots, \xi_i^{i-m}$ . Then the canonical decomposition of  $\mathbb{R}[C_i]$  induces the projection map ([2, p.97])

$$\pi_d : \mathbb{R}[C_i] \rightarrow \lambda_d,$$

if  $2d < i$ .

We first consider the case  $md < i < m(d+1)$ . In [4, §3], it is proved that  $0 \notin \pi_d(C_i \cdot \Delta^{i-m}) \subset \lambda_d$ . Using the radial projection away from 0, we get a  $C_i$ -equivariant map

$$\theta_{i,m} : \Delta^{i-1}/C_i \cdot \Delta^{i-m} \rightarrow S^{\lambda_d}.$$

We next consider the case  $i = m(d+1)$ . It is also proved that in [4, §3]  $0 \notin \pi_{d+1}(C_i \cdot \Delta^{i-m}) \subset \lambda_{d+1}$ . Furthermore, that proves

$$\pi_{d+1}(C_i \cdot \Delta^{i-m}) \cap \lambda_d^\perp = C'_m,$$

where  $\lambda_d^\perp$  is the orthogonal completion of the image of the canonical inclusion  $\lambda_d \xrightarrow{\iota} \lambda_{d+1}$  and  $C'_m$  is the pre-image of  $C_m$  by the isomorphism  $\lambda_d^\perp \rightarrow \mathbb{C}(d+1)$  induced by  $\iota$ . Picking a small ball  $B \subset \lambda_{d+1} \setminus C'_m$  around a point in the sphere  $S(\lambda_d^\perp)$ , we define  $U := (C_i \cdot B) \cap S(\lambda_{d+1})$ . If  $B$  is small enough, the projection  $\pi_{d+1}$  and radial projection define a  $C_i$ -equivariant map

$$\theta'_d : \Delta^{i-1}/C_i \cdot \Delta^{i-m} \rightarrow D(\lambda_{d+1})/(S(\lambda_{d+1}) \setminus U),$$

where  $D(\lambda_{d+1})$  denotes the disk in  $\lambda_{d+1}$ . Then [2, §3] shows that there is a strong deformation retract of  $C_i$ -spaces

$$(S^0 * C_m) \wedge S^{\lambda_d} \rightarrow D(\lambda_{d+1})/(S^{\lambda_{d+1}} \setminus U).$$

Therefore we get a homotopy class of  $C_i$ -equivariant maps

$$\theta_{i,m} : \Delta^{i-1}/C_i \cdot \Delta^{i-m} \rightarrow (S^0 * C_m) \wedge S^{\lambda_d}.$$

We now also recall some known theorems about topological Hochschild homology THH. For  $A$  a commutative ring, Hesselholt-Madsen shows that, in [5, Theorem 7.1], there is an equivalence

$$\mathrm{THH}(A[x]/(x^k)) \simeq \mathrm{THH}(A) \otimes \mathrm{N}^{\mathrm{cy}}(\Pi_k). \quad (\text{a})$$

This equivalence gives rise to

$$\mathrm{THH}(A[x]/(x^k), (x)) \simeq \bigoplus_{i>0} \mathrm{THH}(A) \otimes \mathrm{N}^{\mathrm{cy}}(\Pi_k, i).$$

Here is a corollary of the famous theorem by Dundas-Goodwillie-McCarthy. For a ring  $A$  in which  $p$  is nilpotent, the cyclotomic trace map

$$K(A[x]/(x^k), (x)) \rightarrow \mathrm{TC}(A[x]/(x^k), (x)) \quad (\text{b})$$

is an equivalence.

In the present paper, using above theorems, we study the map

$$K(A[x]/(x^k), (x)) \rightarrow K(A[x]/(x^{nk}), (x)),$$

induced by  $x \mapsto x^n$ , for an  $\mathbb{F}_p$ -algebra  $A$ .

### 3. The geometric Verschiebung map

In order to study the map  $K(A[x]/(x^k), (x)) \rightarrow K(A[x]/(x^{nk}), (x))$ , we use two pointed commutative monoids  $\Pi_k$  and  $\Pi_{nk}$  and their geometric realizations of cyclic bar constructions,  $\mathrm{N}^{\mathrm{cy}}(\Pi_k)$  and  $\mathrm{N}^{\mathrm{cy}}(\Pi_{nk})$  respectively, and construct a map between corresponding cofibration sequences.

In [5, 7.2], Hesselholt and Madsen defined an isomorphism between the geometric realization  $|\Lambda[n][\cdot]|$  of the standard  $n$ -th cyclic set and the product topological space  $\mathbb{T} \times \Delta^n$  of the circle  $\mathbb{R}/\mathbb{Z}$  and the standard  $n$ -simplex as follows.

In [6, Theorem 3.4], Jones constructed an homeomorphism between  $|\Lambda[n][\cdot]|$  and  $\mathbb{T} \times \Delta^n$  and defined an action of  $C_{n+1}$  on  $\mathbb{T} \times \Delta^n$  by

$$\tau_n \cdot (x; u_0, \dots, u_n) := (x - u_0; u_1, \dots, u_n, u_0).$$

However, Hesselholt and Madsen consider another action of  $C_{n+1}$  on  $\mathbb{T} \times \Delta^n$  given by

$$\tau_n * (x; u_0, \dots, u_n) := (x - 1/(n+1); u_1, \dots, u_n, u_0),$$

and defined an  $\mathbb{T} \times C_{n+1}$ -equivariant homeomorphism  $F_n : \mathbb{T} \times \Delta^n \rightarrow \mathbb{T} \times \Delta^n$  by

$$F_n(x; u_0, \dots, u_n) := (x - f_n(u_0, \dots, u_n); u_0, \dots, u_n)$$

with an affine map  $f_n : \Delta^n \rightarrow \mathbb{R}$  subject to the equations

$$\begin{aligned} f_n(u_1, \dots, u_n, u_0) - f_n(u_0, \dots, u_n) &= 1/(n+1) - u_0, \\ f_n(1, 0, \dots, 0) &= 0. \end{aligned}$$

Here, the action on the domain of  $F_n$  is given by  $\cdot$  and the action on the codomain is given by  $*$ . This map  $F_n$  is indeed equivariant. By definition, we have

$$\begin{aligned} F_n(\tau_n \cdot (x; u_0, \dots, u_n)) &= F_n((x - u_0; u_1, \dots, u_n, u_0)) \\ &= (x - u_0 - f_n(u_1, \dots, u_n, u_0); u_1, \dots, u_n, u_0), \end{aligned}$$

and

$$\begin{aligned} \tau_n * F_n(x; u_0, \dots, u_n) &= \tau_n * (x - f_n(u_0, \dots, u_n); u_0, \dots, u_n) \\ &= (x - f_n(u_0, \dots, u_n) - 1/(n+1); u_1, \dots, u_n, u_0). \end{aligned}$$

Thus that we need to check is

$$x - u_0 - f_n(u_1, \dots, u_n, u_0) = x - f_n(u_0, \dots, u_n) - 1/(n+1),$$

which follows from the definition of the affine map  $f_n$ . See [5] for more detail.

Note that by definition the restriction  $F_n|_{\Delta^n}$  is the identity map. We identify  $|\Lambda[n][-]|$  with  $\mathbb{T} \times \Delta^n$  via this isomorphism  $F_n$  and Jones isomorphism. We define a map

$$e_{i,n} : \Delta^{i-1} \rightarrow \Delta^{in-1},$$

which sends the vertex  $(0, \dots, 0, 1, 0, \dots, 0)$ , where the  $(m+1)$ -th coordinate is 1, to the vertex  $(0, \dots, 0, 1, 0, \dots, 0)$ , where the  $(mn+1)$ -th coordinate is 1, for every  $m \in \{0, \dots, i-1\}$ . In other words,  $e_{i,n}(\xi_i^j) = \xi_{in}^{nj}$  for  $0 \leq j \leq i-1$ , where  $\xi_i$ , respectively  $\xi_{in}$ , is the generator of  $C_i$ , respectively  $C_{in}$ .

**Lemma 3.1.** *The map  $e_{i,n}$  induces the map  $g_{i,n} : N^{\text{cy}}(\Pi_k, i) \rightarrow N^{\text{cy}}(\Pi_{kn}, in)$ ,  $a \mapsto b^n$ , via the isomorphisms defined above, where  $a$  (respectively  $b$ ) is the generator of  $\Pi_k$  (respectively  $\Pi_{kn}$ ).*

**Proof.** We denote by  $\alpha$  the map  $\Lambda[i-1][-] \rightarrow N^{\text{cy}}(\Pi_k, i)[-]$  representing the  $i-1$ -simplex  $a \wedge \dots \wedge a$  ( $i$  factors) and by  $\beta$  the map  $\Lambda[in-1][-] \rightarrow N^{\text{cy}}(\Pi_{kn}, in)[-]$  representing the  $in-1$ -simplex  $b \wedge \dots \wedge b$  ( $in$  factors).

We also define  $\Psi : \Lambda[i-1][-] \rightarrow \Lambda[in-1][-]$  to be the composite  $d_{in-1}d_{in-2}\dots d_2d_1$  except  $d_{nj}$  for all  $j \in \{1, \dots, i-1\}$ . Then we have the following commutative diagram

$$\begin{array}{ccc} N^{\text{cy}}(\Pi_k, i)[-] & \xrightarrow{g'_{i,n}} & N^{\text{cy}}(\Pi_{kn}, in)[-] \\ \alpha \uparrow & & \uparrow \beta \\ \Lambda[i-1][-] & \xrightarrow{\Psi} & \Lambda[in-1][-], \end{array}$$

where  $g'_{i,n}$  is the map of cyclic sets that is given by  $a \mapsto b^n$  and induces  $g_{i,n}$  via the geometric realization by definition. The geometric realization of  $\Psi$  with Hesselholt-Madsen's isomorphism mentioned above is given by

$$\begin{aligned} \mathbb{T} \times \Delta^{i-1} &\rightarrow \mathbb{T} \times \Delta^{in-1}, \\ (t, (u_0, u_1, \dots, u_{i-1})) &\mapsto (t, (u_0, 0, 0, \dots, 0, u_1, 0, \dots, 0, u_{i-1}, 0, 0, \dots, 0)), \end{aligned}$$

where there are  $n-1$  zeros between  $u_{s-1}$  and  $u_s$ . By definition, it is the map  $\text{id}_{\mathbb{T}} \times e_{i,n}$ . In other words, we have the commutative diagram

$$\begin{array}{ccc} \mathbb{T} \times \Delta^{i-1} & \xrightarrow{\text{id}_{\mathbb{T}} \times e_{i,n}} & \mathbb{T} \times \Delta^{in-1} \\ \uparrow \cong & & \uparrow \cong \\ |\Lambda[i-1][ - ]| & \xrightarrow{|\Psi|} & |\Lambda[in-1][ - ]|. \end{array} \quad \square$$

We now study the relation between the map  $g_{i,n}$  and the cofibration sequences above. More precisely, we have two cofibration sequences for every  $i > 0$ ,

$$\begin{aligned} \mathbb{T}_+ \wedge_{C_{i/k}} S^{\lambda_{d_i}} &\xrightarrow{\text{pr}} \mathbb{T}_+ \wedge_{C_i} S^{\lambda_{d_i}} \longrightarrow \text{N}^{\text{cy}}(\Pi_k, i), \\ \mathbb{T}_+ \wedge_{C_{in/kn}} S^{\lambda_{d_i}} &\xrightarrow{\text{pr}} \mathbb{T}_+ \wedge_{C_{in}} S^{\lambda_{d_i}} \longrightarrow \text{N}^{\text{cy}}(\Pi_{kn}, in), \end{aligned}$$

and are comparing them using  $g_{i,n} : \text{N}^{\text{cy}}(\Pi_k, i) \rightarrow \text{N}^{\text{cy}}(\Pi_{kn}, in)$ .

**Proposition 3.2.** (i) For  $kd < i < k(d+1)$ , the following diagram of  $C_i$ -spaces commutes

$$\begin{array}{ccc} \Delta^{i-1}/C_i \cdot \Delta^{i-k} & \xrightarrow{\theta_{i,k}} & S^{\lambda_d} \\ \downarrow e_{i,n*} & & \downarrow \text{id} \\ \Delta^{in-1}/C_{in} \cdot \Delta^{n(i-k)} & \xrightarrow{\theta_{in,kn}} & S^{\lambda_d}. \end{array}$$

(ii) For  $i = k(d+1)$ , the following diagram of  $C_i$ -spaces commutes

$$\begin{array}{ccc} \Delta^{i-1}/C_i \cdot \Delta^{i-k} & \xrightarrow{\theta_{i,k}} & (S^0 * C_k) \wedge S^{\lambda_d} \\ \downarrow e_{i,n*} & & \downarrow \\ \Delta^{in-1}/C_{in} \cdot \Delta^{n(i-k)} & \xrightarrow{\theta_{in,kn}} & (S^0 * C_{nk}) \wedge S^{\lambda_d}, \end{array}$$

where the right hand side vertical map is induced by the inclusion  $C_k \rightarrow C_{kn}$ ,  $\xi_k \mapsto \xi_{nk}^n$ , and the identity map on  $S^{\lambda_d}$ .

**Proof.** We prove (i). The same argument holds for (ii). First, by construction, the image  $\theta_{i,k}(\xi_i^j)$  of the generator  $\xi_i^j$  under the equivariant map  $\theta_{i,k}$  is  $(\xi_i^j, \xi_i^{2j}, \dots, \xi_i^{dj}) \in S^{\lambda_d}$ . Likewise,  $\theta_{in,kn}(\xi_{in}^j)$  is  $(\xi_{in}^j, \xi_{in}^{2j}, \dots, \xi_{in}^{dj})$ . By the definition of  $e_{i,n}$ , we have  $e_{i,n}(\xi_i^j) = \xi_{in}^{jn}$ . Lastly, in the complex plane  $\mathbb{C}$ ,  $\xi_i^j = \xi_{in}^{jn}$ .  $\square$

By this proposition, we get the following map of cofiber sequences.

**Corollary 3.3.** *There is a homotopy commutative diagram of cofibration sequences*

$$\begin{array}{ccccc} \mathbb{T}_+ \wedge_{C_{i/k}} S^{\lambda_d} & \xrightarrow{\text{pr}} & \mathbb{T}_+ \wedge_{C_i} S^{\lambda_d} & \longrightarrow & N^{\text{cy}}(\Pi_k, i) \\ \downarrow \text{id} & & \downarrow \text{pr} & & \downarrow g_{i,n} \\ \mathbb{T}_+ \wedge_{C_{in/nk}} S^{\lambda_d} & \xrightarrow{\text{pr}} & \mathbb{T}_+ \wedge_{C_{in}} S^{\lambda_d} & \longrightarrow & N^{\text{cy}}(\Pi_{nk}, ni), \end{array}$$

where  $d = \lfloor (i-1)/k \rfloor$  and  $\mathbb{T}_+ \wedge_{C_{i/k}} S^{\lambda_d}$  and  $\mathbb{T}_+ \wedge_{C_{in/nk}} S^{\lambda_d}$  are trivial when  $k$  does not divide  $i$ .

**Proof.** Again by [4, (3.1.1)], the map  $\Lambda[j-1][-] \rightarrow N^{\text{cy}}(\Pi_m, j)[-]$  representing  $y \wedge y \wedge \dots \wedge y$  ( $j$  factors) with the generator  $y$  of  $\Pi_m$  induces a  $\mathbb{T}$ -equivariant homeomorphism

$$\mathbb{T}_+ \wedge_{C_j} (\Delta^{j-1}/C_j \cdot \Delta^{j-m}) \rightarrow N^{\text{cy}}(\Pi_m, j).$$

We can get two cofibration sequences

$$\mathbb{T}_+ \wedge_{C_{i/k}} S^{\lambda_d} \xrightarrow{\text{pr}} \mathbb{T}_+ \wedge_{C_i} S^{\lambda_d} \longrightarrow N^{\text{cy}}(\Pi_k, i)$$

and

$$\mathbb{T}_+ \wedge_{C_{in/nk}} S^{\lambda_d} \xrightarrow{\text{pr}} \mathbb{T}_+ \wedge_{C_{in}} S^{\lambda_d} \longrightarrow N^{\text{cy}}(\Pi_{nk}, in),$$

applying  $\mathbb{T}_+ \wedge_{C_i} (-)$  and  $\mathbb{T}_+ \wedge_{C_{in}} (-)$  respectively to the diagrams in 3.2. The inclusion map  $C_i \rightarrow C_{ni}$ ,  $\xi_i^j \mapsto \xi_{in}^{nj}$ , induces the maps  $\text{id}$ ,  $\text{pr}$  and  $g_{i,n}$ , which make the diagram commutative.  $\square$

#### 4. Proofs of theorems

For a ring  $A$ , the topological Hochschild homology  $\text{THH}(A)$  is a cyclotomic spectrum. See, for example, [7, III.4, III.5]. For a finite dimensional orthogonal  $\mathbb{T}$ -representation  $\lambda$ , we let  $S^\lambda$  denote its one point compactification. We define

$$\text{TR}_{q-\lambda}^n(A) := [S^q \otimes (\mathbb{T}/C_n)_+, S^\lambda \otimes \text{THH}(A)]_{\mathbb{T}}$$

to be the abelian group of maps in the  $\mathbb{T}$ -stable homotopy category ([2, p. 92–93]).

As explained at page 93 of [2], for positive integers  $s$ , there are maps  $F_s$  called Frobenius,  $V_s$  called Verschiebung and  $R_s$  called restriction between TRs.

Using the diagram in Corollary 3.3 and these notions from stable homotopy theory, we get a map of long exact sequences to study commutative rings.



**Theorem 4.1.** *Let  $A$  be a ring in which  $p$  is nilpotent. Then there is a map of long exact sequences*

$$\begin{array}{ccc}
 \vdots & & \vdots \\
 \downarrow & & \downarrow \\
 \lim_R \mathrm{TR}_{q-\lambda_{\lfloor (i-1)/k \rfloor}}^{i/k}(A) & \xrightarrow{\mathrm{id}} & \lim_R \mathrm{TR}_{q-\lambda_{\lfloor (i-1)/kn \rfloor}}^{i/kn}(A) \\
 \downarrow V_{k*} & & \downarrow V_{kn*} \\
 \lim_R \mathrm{TR}_{q-\lambda_{\lfloor (i-1)/k \rfloor}}^i(A) & \xrightarrow{V_{n*}} & \lim_R \mathrm{TR}_{q-\lambda_{\lfloor (i-1)/kn \rfloor}}^i(A) \\
 \downarrow & & \downarrow \\
 K_{q+1}(A[x]/(x^k), (x)) & \xrightarrow{v_n} & K_{q+1}(A[x]/(x^{nk}), (x)) \\
 \downarrow & & \downarrow \\
 \vdots & & \vdots
 \end{array},$$

where  $V$  denotes the map induced by Verschiebung maps and the maps in the limits are restriction maps.

**Proof.** The same argument in the proof of [2, 2.1] holds. More precisely, we first take the infinite coproduct of the diagram in Corollary 3.3 to get the following map of cofibration sequences

$$\begin{array}{ccccc}
 \bigvee_{i \geq 0} \mathbb{T}_+ \wedge_{C_{i/k}} S^{\lambda_{\lfloor (i-1)/k \rfloor}} & \xrightarrow{\mathrm{pr}} & \bigvee_{i \geq 0} \mathbb{T}_+ \wedge_{C_i} S^{\lambda_{\lfloor (i-1)/k \rfloor}} & \longrightarrow & \mathrm{N}^{\mathrm{cy}}(\Pi_k) \\
 \downarrow \mathrm{id} & & \downarrow \mathrm{pr} & & \downarrow g_n \\
 \bigvee_{i \geq 0} \mathbb{T}_+ \wedge_{C_{i/kn}} S^{\lambda_{\lfloor (i-1)/kn \rfloor}} & \xrightarrow{\mathrm{pr}} & \bigvee_{i \geq 0} \mathbb{T}_+ \wedge_{C_i} S^{\lambda_{\lfloor (i-1)/kn \rfloor}} & \longrightarrow & \mathrm{N}^{\mathrm{cy}}(\Pi_{kn}),
 \end{array}$$

where  $g_n$  denotes the map induced by  $\Pi_k \rightarrow \Pi_{kn}$ ,  $a \mapsto b^n$ . Next we smash  $\mathrm{THH}(A)$  with the cofibration sequences above and use (a) in section 2. Lastly we take homotopy limits along Frobenius maps and homotopy limits along restriction maps. Then by (b) in section 2, we get the desired diagram.  $\square$

Taking the directed colimit of the diagram in the above theorem, we get the following.

**Corollary 4.2.** *Let  $A$  be an  $\mathbb{F}_p$ -algebra and  $\tilde{A}_k := \mathrm{colim}_n (A[x]/(x^{nk}), (x))$ . Then there is a long exact sequence*

$$\begin{aligned}
 \cdots &\longrightarrow \lim_R \mathrm{TR}_{q-\lambda_{\lfloor (i-1)/k \rfloor}}^{i/k}(A) \longrightarrow \mathrm{colim}_n \lim_R \mathrm{TR}_{q-\lambda_{\lfloor (i-1)/kn \rfloor}}^i(A) \\
 &\longrightarrow K_{q+1}(\tilde{A}_k) \longrightarrow \cdots
 \end{aligned}$$

**Proof.** The colimit is filtered. Therefore, we get a long exact sequence by taking the colimit of the long exact sequences obtained in the above theorem and  $\mathrm{colim}_n K_*(A[x]/(x^{nk}), (x))$  is canonically isomorphic to  $K_*(\tilde{A}_k)$ .  $\square$

For regular  $\mathbb{F}_p$ -algebras  $A$ , in [2, §5] Hesselholt gave a calculation of topological Hochschild homology in terms of big de Rham-Witt complex  $\mathbb{W}_{(-)}\Omega_A^*$ . Using this calculation, we immediately get the following from Theorem 4.1

**Corollary 4.3.** *Let  $A$  be a regular  $\mathbb{F}_p$ -algebra. There is a map of long exact sequences*

$$\begin{array}{ccc}
 \vdots & & \vdots \\
 \downarrow & & \downarrow \\
 \bigoplus_{l \geq 0} \mathbb{W}_{l+1} \Omega_A^{q-2l} & \xrightarrow{\text{id}} & \bigoplus_{l \geq 0} \mathbb{W}_{l+1} \Omega_A^{q-2l} \\
 \downarrow V_{k*} & & \downarrow V_{kn*} \\
 \bigoplus_{l \geq 0} \mathbb{W}_{k(l+1)} \Omega_A^{q-2l} & \xrightarrow{V_{n*}} & \bigoplus_{l \geq 0} \mathbb{W}_{kn(l+1)} \Omega_A^{q-2l} \\
 \downarrow & & \downarrow \\
 K_{q+1}(A[x]/(x^k), (x)) & \xrightarrow{v_n} & K_{q+1}(A[x]/(x^{nk}), (x)) \\
 \downarrow & & \downarrow \\
 \vdots & & \vdots
 \end{array},$$

where the subscript  $m(l+1)$  indicates the truncation set  $\{1, 2, \dots, m(l+1)\}$  and  $V$ 's denote the maps induced by Verschiebung maps.

By [4, Theorem A] and this corollary, we have the following commutative diagram of short exact sequences for any  $j$  and for a perfect field  $k$  with positive characteristic,

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \mathbb{W}_j(k) & \xrightarrow{V_m} & \mathbb{W}_{jm}(k) & \longrightarrow & K_{2j-1}(k[x]/(x^m), (x)) \longrightarrow 0 \\
 & & \downarrow \text{id} & & \downarrow V_n & & \downarrow \\
 0 & \longrightarrow & \mathbb{W}_j(k) & \xrightarrow{V_{mn}} & \mathbb{W}_{jmn}(k) & \longrightarrow & K_{2j-1}(k[x]/(x^{nm}), (x)) \longrightarrow 0,
 \end{array}$$

where  $\mathbb{W}$  denotes the big Witt vectors, and the right-hand side vertical map is induced by the map  $x \mapsto x^n$ . In the rest of this paper, we consider applications of this diagram.

Let  $k$  be a perfect field with characteristic  $p > 0$  and let

$$W(k)[p^{1/p^\infty}] := \text{colim}_n W(k)[p^{1/p^n}],$$

where the structure maps are given by  $p^{1/p^n} \mapsto (p^{1/p^{n+1}})^p$ . We consider the completion  $\mathcal{O}_K := W(k)[p^{1/p^\infty}]^\wedge$  with quotient field  $K := \mathcal{O}_K[1/p]$  and residue field  $k$ . For example, if  $k = \mathbb{F}_p$  then  $\mathcal{O}_K = \mathbb{Z}_p[p^{1/p^\infty}]^\wedge$  with quotient field  $\mathcal{O}_K[1/p] = \mathbb{Q}_p(p^{1/p^\infty})^\wedge$ . Using  $W(k)/p = k$  and  $W(k)[x]/(x^{p^n} - p) = W(k)[p^{1/p^n}]$ , we have

$$\mathcal{O}_K/p\mathcal{O}_K = \text{colim}_n k[x]/(x^{p^n})$$

with structure maps given by  $x \mapsto x^p$ . We let  $\mathfrak{m}_K \subset \mathcal{O}_K$  denote the maximal ideal. Taking the colimit of the diagram above, we obtain the following.

**Corollary 4.4.** *With the notation above, we have a canonical isomorphism*

$$K_{2j-1}(\mathcal{O}_K/p\mathcal{O}_K, \mathfrak{m}_K/p\mathcal{O}_K) \cong \text{colim}_n \mathbb{W}_{jp^n}(k)/V_{p^n} \mathbb{W}_j(k),$$

where the colimit is indexed by the category of natural numbers under addition. Moreover, the relative  $K$ -groups in even degrees are zero.

Let  $k$  again be a perfect field with characteristic  $p > 0$  and let

$$W(k)[\zeta_{p^\infty}] := \operatorname{colim}_n W(k)[\zeta_{p^n}],$$

where  $\zeta_{p^n}$  denotes a primitive  $p^n$ -th root of unity and we choose these to satisfy  $\zeta_{p^n}^p = \zeta_{p^{n-1}}$ . We consider the completion  $\mathcal{O}_L := W(k)[\zeta_{p^\infty}]^\wedge$  with quotient field  $L = \mathcal{O}_L[1/p]$  and residue field  $k$ . For example, if  $k = \mathbb{F}_p$  then  $\mathcal{O}_L = \mathbb{Z}_p[\zeta_{p^\infty}]^\wedge$  with quotient field  $\mathcal{O}_L[1/p] = \mathbb{Q}_p(\zeta_{p^\infty})^\wedge$ . Let us write  $L_0 = W(k)[1/p]$  and  $L_n = L_0(\zeta_{p^n})$ . Since  $|L_n : L_0| = p^{n-1}(p-1)$  and  $\zeta_{p^n} - 1$  is a uniformizer, the map

$$k[x]/(x^{p^{n-1}(p-1)}) \rightarrow W(k)(\zeta_{p^n})/p$$

given by  $x \mapsto \zeta_{p^n} - 1$  is an isomorphism. Moreover, with these isomorphisms, the following diagram

$$\begin{array}{ccc} W(k)(\zeta_{p^n})/p & \xleftarrow{\cong} & k[x]/(x^{p^{n-1}(p-1)}) \\ \uparrow & & \uparrow \\ W(k)(\zeta_{p^{n-1}})/p & \xleftarrow{\cong} & k[x]/(x^{p^{n-2}(p-1)}) \end{array}$$

commutes, where the left vertical map is given by  $\zeta_{p^{n-1}} - 1 \mapsto \zeta_{p^n}^p - 1$  and the right vertical map is given by  $x \mapsto x^p$ . By this construction, we have

$$\mathcal{O}_L/p\mathcal{O}_L = \operatorname{colim}_n k[x]/(x^{p^{n-1}(p-1)}).$$

We let  $\mathfrak{m}_L \subset \mathcal{O}_L$  denote the maximal ideal. Taking the colimit of the diagram above, we obtain the following.

**Corollary 4.5.** *With the notation above, we have a canonical isomorphism*

$$K_{2j-1}(\mathcal{O}_L/p\mathcal{O}_L, \mathfrak{m}_L/p\mathcal{O}_L) \cong \operatorname{colim}_n \mathbb{W}_{j p^{n-1}(p-1)}(k)/V_{p^{n-1}(p-1)} \mathbb{W}_j(k),$$

where the colimit is indexed by the category of natural numbers under addition. Moreover, the relative  $K$ -groups in even degrees are zero.

The  $p$ -typical decomposition of the right-hand sides in Corollary 4.7 and Corollary 4.8 is explained in page 90–91 of [2].

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