



## Umbral calculus in Ore extensions

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## ABSTRACT

The aim of the paper is to show the existence of some ingredients for an umbral calculus on some Ore extensions, in a manner analogous to Rota's classical umbral calculus which deals with a univariate polynomial ring on a field of characteristic zero. For that, we introduce the notion of a quasi-derivation in order to specify Ore extensions on which building up this umbral calculus is possible. This allows in particular to define an action of the Ore extension on tensor products of modules. We develop also a Pincherle calculus for operators and we define a coalgebra structure on the Ore extension.

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## 0. Introduction

Using operator and symbolic methods has proved essential in finding and interpreting many formulae of combinatorics. In particular, the umbral calculus developed by Rota and his collaborators [24] made it possible to obtain or explain many properties of polynomial sequences [6,13], such as the sequence of Bernoulli polynomials, that of Abel polynomials, and many other analogs. The fundamental idea of Rota's umbral calculus is to consider such a sequence as the image by a suitable operator of the sequence  $(x^n)$  of powers of the indeterminate  $x$ . The properties of this operator allow to account for the properties of the sequence in question. An important limitation of Rota's classical umbral calculus is that it works only for polynomial sequences over a field of characteristic zero. Besides, Ore's work [17] led to the definition of skew polynomial rings (also called Ore extensions) that generalize polynomial rings. Our objective in this paper

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is to study the natural problem of laying the foundations of an umbral calculus to deal with the properties of sequences of elements in these Ore extensions. As it has been recognized that the existence of classical umbral calculus depends on a coalgebra structure on the polynomial rings, we wish in particular to define a comultiplication on these Ore extensions.

Our exposition rests upon three new ideas that we now describe.

First, as it is well-known, Ore extensions are constructed using a base ring  $R$ , an endomorphism  $\flat$  of  $R$  and a  $\flat$ -derivation  $\partial$  of  $R$ . If we want to define a comultiplication  $\Delta$  on the Ore extension  $R[x, \flat, \partial]$  in this general framework, a difficulty arises immediately, because the action of  $\flat$  on  $R$  cannot be represented by an element of the Ore extension; this forbids to define a comultiplication in a manner which is compatible with the multiplication of the Ore extension. Indeed, as we have the identity  $\partial(ab) = \flat(a)\partial(b) + \partial(a)b$ , we should put  $\Delta(x) = \flat \otimes x + x \otimes 1$ , but  $\flat$  cannot be interpreted as the action of an element of the Ore extension  $R[x, \flat, \partial]$ . The first new idea in this paper is to overcome this difficulty by introducing the new notion of a *quasi-derivation* of a ring  $R$ . Thus we restrict our study to the case where there exists between  $\flat$  and  $\partial$  a relation of the type  $\flat = \text{Id}_R + c\partial$  for a fixed element  $c$  of the base ring, called the *parameter* of the quasi-derivation  $\partial$ . Many common examples satisfy this condition, so that is actually not very restrictive.

Second, for any quasi-derivation  $\partial$  of a ring  $R$ , with parameter  $c$ , denoting by  $\flat$  the endomorphism  $\text{Id}_R + c\partial$ , given a finite sequence of left  $R[x, \flat, \partial]$ -modules, we show how to define on the tensor product (on the base ring  $R$ ) of these modules a structure of left  $R[x, \flat, \partial]$ -module. This construction extends the one that is used in the context of the differential algebra [12, page 81], corresponding to the case where the parameter is null.

Our third innovation is the generalization of Pincherle calculus of operators, as defined in the “classical” framework by [24], to the broader context we are studying. Recall that the word operator refers, according to the terminology of the specialists of the umbral calculus [24], to an  $R$ -linear endomorphism of the  $R$ -module  $R[x]$ , and that the Pincherle derivation designates the map that to such an operator  $T$  associates the commutator of  $T$  with the multiplication by  $x$ . In the case of the null parameter ( $c = 0$ ), the generalization of this Pincherle calculus has already been realized in the work [9], which recognized the fact that the non commutativity of the corresponding Ore extension imposes to split the Pincherle derivative into two distinct derivations. In the context of quasi-derivations of any kind, we have sought in the same way to define two distinct pseudo-derivations, having properties similar to those already identified in the situation where  $c = 0$ . This led us to restrict the notion of operator by requiring the stabilization of a particular filtration of the ring  $R[x, \flat, \partial]$ . This condition of stabilization is satisfied by the analogs of many operators studied classically. We intend to return to this point in the future, limiting us here to the study of two very special cases: firstly, translation operators generalizing the mappings  $p(x) \mapsto p(x + a)$  of the classical case, and secondly, left or right multiplications by fixed elements.

To obtain the best results, and especially for the development of Pincherle calculus, we have to use the additional hypothesis according to which the endomorphism  $\flat$  is bijective. We show that this hypothesis corresponds exactly to the existence of a second quasi-derivation linked to the first by a simple relation which turns out to be symmetrical: we speak about conjugate quasi-derivations. In fact, if  $\flat'$  is the inverse isomorphism of  $\flat$ , the conjugate quasi-derivation of  $\partial$  is simply  $\partial' = \partial \circ \flat'$ . This hypothesis allows us, when the ring  $R$  is commutative, to construct between the rings  $R[x, \flat, \partial]$  and  $R[x', \flat', \partial']$  an antimorphism that we call adjunction (because, in the case where  $c = 0$ , it amounts to transforming a linear ordinary differential equation into its adjoint equation). This adjunction makes it possible to identify the left  $R[x', \flat', \partial']$ -modules to the right  $R[x, \flat, \partial]$ -modules. The bijectivity of the endomorphism  $\flat$  also allows us to construct extensions of  $\flat$  to automorphisms of  $R[x, \flat, \partial]$ ; these extensions constitute an important ingredient for the development of Pincherle calculus of operators.

The present article thus leads to a double result: on the one hand, there is a Pincherle calculus of operators within the framework of the Ore extensions considered, as we have described above; on the other hand, one can build on these same Ore extensions a structure of coalgebra which in the “classical” case is the structure

of the binomial coalgebra which explains the existence of the umbral calculus [10, XI, p. 115]. About this latter point, we show (see Proposition 4.8) the following theorem.

**Theorem 0.1.** *Let  $\partial$  be a quasi-derivation of parameter  $c$  of a commutative ring  $R$ , and  $\mathcal{D} = R[x, \mathfrak{p}, \partial]$  the corresponding Ore extension. We denote by  ${}^\circ\mathcal{D}$  the  $R$ -module whose additive group is that of  $\mathcal{D}$  and in which  $R$  acts by left multiplication. Then there exists on  ${}^\circ\mathcal{D}$  a unique structure of  $R$ -coalgebra defined by a left  $\mathcal{D}$ -linear augmentation  $\varepsilon : \mathcal{D} \rightarrow R$  and a left  $\mathcal{D}$ -linear comultiplication  $\Delta : \mathcal{D} \rightarrow {}^\circ\mathcal{D} \otimes {}^\circ\mathcal{D}$  such that  $\varepsilon(1) = 1$  and  $\Delta(1) = 1 \otimes 1$ .*

Since this coalgebra is not isomorphic to the binomial coalgebra as soon as  $c \neq 0$  (indeed, it contains at least two distinct grouplike elements, namely 1 and  $1 + cx$ , whereas the binomial coalgebra has only one), this provides new examples that might interest the algebraists.

As can be seen, the approach of our work, inspired by the development of the umbral calculus, is distinct from that of [3] who has also used operator methods to clarify the meaning of  $q$ -identities. Indeed, the operators it uses are defined on the polynomial ring, while ours are defined on an Ore extension of a base ring. This base ring can possibly be a ring of polynomials, as in Example 1.10 of the Jackson derivation which is a basic tool in the development of  $q$ -calculus.

Thus our contribution, following [9] which concerns the particular case of the derivations ( $c = 0$ ), is in the course of the works which aim to extend the ideas of Rota and his collaborators [10, 19–24] about the umbral calculus to other contexts [5, 14, 15].

We emphasize that the analysis of Ore extensions on which we base our exposition has led us to elicit phenomena that do not seem to be documented until now. This is particularly true for the existence of the two endomorphisms that we call primary: the first primary endomorphism has already been introduced into the narrower framework of  $q$ -skew derivations [7, page 11], but to highlight these two endomorphisms has proved necessary for elaborating Pincherle calculus of operators.

## 1. Quasi-derivations

### 1.1. Definitions and first properties

All the rings that we consider are supposed to be unital.

**Definition 1.1.** Let  $R$  be a not necessarily commutative ring, and  $c$  be a central element in  $R$ . We call QUASI-DERIVATION OF THE RING  $R$  OF PARAMETER  $c$  any map  $\partial : R \rightarrow R$  satisfying the following identities for all elements  $r_1$  and  $r_2$  of  $R$ :

$$\begin{aligned}\partial(r_1 + r_2) &= \partial(r_1) + \partial(r_2) \\ \partial(r_1 r_2) &= r_1 \partial(r_2) + \partial(r_1) r_2 + \partial(r_1) c \partial(r_2) \\ \partial(1) &= 0.\end{aligned}$$

Let  $R$ ,  $c$  and  $\partial$  be as above: the triple  $(R, c, \partial)$  is called a quasi-differential ring. We say that an element  $a \in R$  of the quasi-differential ring  $(R, c, \partial)$  is a constant of  $\partial$  when  $\partial(a) = 0$ . It is clear that the set of constants of  $\partial$  is a subring of  $R$ .

**Examples 1.2.** 1. The DERIVATIONS are by definition the quasi-derivations of parameter 0.

2. If  $\partial$  is a quasi-derivation of parameter  $c \neq 0$  of the ring  $R$ , then  $c\partial$  is a quasi-derivation of  $R$  of parameter 1.

3. Let  $\partial$  be a quasi-derivation of parameter  $c$  of the ring  $R$ , and  $z$  be a central and invertible element of  $R$ . Then the map  $z\partial$  is a quasi-derivation of  $R$  of parameter  $z^{-1}c$ .

**Proposition 1.3.** *Let  $c$  be a central element of a ring  $R$ , and  $\partial$  a quasi-derivation of parameter  $c$  of  $R$ . Then the map  $\mathfrak{p} : R \rightarrow R$  defined by*

$$\forall r \in R, \quad \mathfrak{p}(r) = r + c\partial(r) \quad (1)$$

*is an endomorphism of the ring  $R$ , and  $\partial$  is a  $\mathfrak{p}$ -derivation [11, page 14]. Conversely, if  $\mathfrak{p}$  is an endomorphism of  $R$ , if  $c$  is a central element of  $R$ , and if  $\partial$  is a  $\mathfrak{p}$ -derivation such that  $\mathfrak{p}(a) = a + c\partial(a)$ , then  $\partial$  is a quasi-derivation of  $R$  of parameter  $c$ .*

**Proof.** Given a quasi-derivation  $\partial$  of parameter  $c$  of  $R$ , it is easy to check that the map  $\mathfrak{p} : R \rightarrow R$  defined by (1) satisfies the identities  $\mathfrak{p}(1) = 1$ ,  $\mathfrak{p}(r_1 + r_2) = \mathfrak{p}(r_1) + \mathfrak{p}(r_2)$ ,  $\mathfrak{p}(r_1 r_2) = \mathfrak{p}(r_1)\mathfrak{p}(r_2)$  and  $\partial(r_1 r_2) = \mathfrak{p}(r_1)\partial(r_2) + \partial(r_1)r_2$ . So we made sure that  $\mathfrak{p}$  is an endomorphism of the ring  $R$ , and that  $\partial$  is a  $\mathfrak{p}$ -derivation. Conversely, the identity  $\mathfrak{p}(a) = a + c\partial(a)$  leads for the  $\mathfrak{p}$ -derivation  $\partial$  to the equality  $\partial(r_1 r_2) = \mathfrak{p}(r_1)\partial(r_2) + \partial(r_1)r_2 = (r_1 + c\partial(r_1))\partial(r_2) + \partial(r_1)r_2 = r_1\partial(r_2) + \partial(r_1)r_2 + c\partial(r_1)\partial(r_2)$ , so that, the element  $c$  being supposed central, the map  $\partial : R \rightarrow R$  is indeed a quasi-derivation of parameter  $c$ .  $\square$

**Definition 1.4.** Let  $\partial$  be a quasi-derivation of parameter  $c$  of a ring  $R$ . Then the map  $\mathfrak{p} : R \rightarrow R$  defined by Equation (1) is called the ASSOCIATED ENDOMORPHISM OF THE QUASI-DERIVATION  $\partial$  FOR PARAMETER  $c$ . (If the value of  $c$  results clearly from the context, we speak simply of the associated endomorphism of the quasi-derivation  $\partial$ .)

**Remark 1.5.** Let  $R$  be a ring and  $c$  a central element of  $R$ . A map  $\partial : R \rightarrow R$  is a quasi-derivation of the ring  $R$  of parameter  $c$  if and only if it is a quasi-derivation of the opposite ring  $R^{op}$  of parameter  $c$ . In particular, any quasi-derivation  $\partial$  of  $R$  of parameter  $c$  is a  $\mathfrak{p}$ -derivation of the ring  $R^{op}$ , where  $\mathfrak{p}$  is the associated endomorphism of  $\partial$ . This means that it verifies the identity

$$\forall (a, b) \in R^2, \quad \partial(ab) = a\partial(b) + \partial(a)\mathfrak{p}(b), \quad (2)$$

as it is easy to verify directly.

**Proposition 1.6.** *Let  $\partial$  be a quasi-derivation of parameter  $c$  of the ring  $R$  and  $\mathfrak{p} : R \rightarrow R$  its associated endomorphism. Then the element  $1 + \partial(c)$  is a RUNG of the  $\mathfrak{p}$ -derivation  $\partial$ , which means*

$$\forall r \in R, \quad \partial(\mathfrak{p}(r)) = (1 + \partial(c))\mathfrak{p}(\partial(r)). \quad (3)$$

**Proof.** To lighten the notation, we denote by  $\dot{c}$  the element  $\partial(c) \in R$ . By definition of the associated endomorphism  $\mathfrak{p}$ , we have for every element  $r$  of  $R$  the identity  $\mathfrak{p}(r) = r + c\partial(r)$ . From this identity, using the fact that  $\partial$  is a quasi-derivation of parameter  $c$ , one draws out the relationship

$$\partial(\mathfrak{p}(r)) = (1 + \dot{c})(\partial(r) + c\partial^2(r)).$$

Now  $\mathfrak{p}(\partial(r)) = \partial(r) + c\partial^2(r)$ , whence the relationship to show.  $\square$

## 1.2. Examples of quasi-derivations

**Example 1.7.** Let  $R$  be a ring, and  $\mathfrak{p}$  be an endomorphism of  $R$ . Then the map  $\tilde{\partial} = \mathfrak{p} - \text{Id}_R$  is a quasi-derivation of parameter 1. The converse results from Proposition 1.3, so that the quasi-derivations of parameter 1 of the ring  $R$  correspond bijectively to the endomorphisms of  $R$ .

In particular, let  $A$  be a ring, and denote by  $\mathcal{S}(A)$  the set of sequences of elements in  $A$  indexed by  $\mathbb{N}$ . Provided with termwise addition and multiplication, the set  $\mathcal{S}(A)$  becomes a ring. It is equipped with the shift map  $\tau$  defined by

$$\forall u = (u(n))_{n \in \mathbb{N}}, \quad \tau(u) = (u(n+1))_{n \in \mathbb{N}}.$$

The map  $\tau$  is a surjective endomorphism of the ring  $\mathcal{S}(A)$  which is not injective. To obtain an automorphism, we consider the factor ring

$$R = \overline{\mathcal{S}}(A) = \mathcal{S}(A) / \cup_{j \in \mathbb{N}} \ker(\tau^j).$$

The endomorphism  $\tau$  induces an automorphism  $\bar{\tau}$  of  $R = \overline{\mathcal{S}}(A)$ . We consider  $\bar{\partial} : \overline{\mathcal{S}}(A) \rightarrow \overline{\mathcal{S}}(A)$  such that  $\bar{\partial} = \bar{\tau} - \text{Id}_{\overline{\mathcal{S}}(A)}$ . We thus obtain a quasi-differential ring  $(\overline{\mathcal{S}}(A), 1, \bar{\partial})$ . We call it the quasi-differential ring of sequences over  $A$ . The ring of constants of this quasi-differential ring is precisely the ring of classes of constant sequences, which explains the terminology.

**Example 1.8.** Let  $A$  be a commutative ring, and  $R = A[x]/(x^2)$  the  $A$ -algebra of dual numbers. We denote by  $X$  the class in  $R$  of the indeterminate  $x \in A[x]$ . For any element  $u$  of  $A$ , we can define a mapping  $\bar{\partial}_u : R \rightarrow R$  by writing

$$\forall (a, b) \in A^2, \quad \bar{\partial}_u(a + bX) = buX.$$

The map  $\bar{\partial}_u$  is a quasi-derivation of parameter  $c$  for any  $c \in R$ . We thus see that situations exist where a given quasi-derivation admits several parameters.

**Proposition 1.9.** *In a commutative ring  $R$ , we denote by  $R^*$  the group of invertible elements of  $R$ . If the  $\mathfrak{p}$ -derivation  $\bar{\partial} : R \rightarrow R$  is such that  $\text{im}(\bar{\partial}) \cap R^* \neq \emptyset$ , then  $\bar{\partial}$  is a quasi-derivation.*

**Proof.** By hypothesis, there is  $x \in R$  such that  $\bar{\partial}(x) \in R^*$ . We put  $c = \bar{\partial}(x)^{-1}(\mathfrak{p}(x) - x) \in R$ . Then we have for all  $a \in R$  :

$$c\bar{\partial}(a) + a = \bar{\partial}(x)^{-1}(\mathfrak{p}(x)\bar{\partial}(a) - \bar{\partial}(a)x + \bar{\partial}(x)a) = \bar{\partial}(x)^{-1}(\bar{\partial}(ax) - x\bar{\partial}(a)) = \mathfrak{p}(a),$$

which proves by Proposition 1.3 that the  $\mathfrak{p}$ -derivation  $\bar{\partial}$  is a quasi-derivation of parameter  $c$ .  $\square$

**Example 1.10.** Let  $A$  be a ring and  $q$  an element of the center of  $A$ . For a natural integer  $m$ , we define the symbol  $\{m\}_q$  by putting

$$\{m\}_q = \sum_{k=0}^{m-1} q^k.$$

We observe that for any ordered pair of integers  $(m, n)$ , we have

$$(q-1)\{m\}_q = q^m - 1 \quad \text{and} \quad q^n \{m\}_q + \{n\}_q = \{m+n\}_q.$$

Let  $R = A[t]$  be the ring of polynomials in one variable, where the indeterminate  $t$  commutes with the elements of  $R$ . The Jackson  $q$ -derivation is defined as the map  $\bar{\partial}_q : R \rightarrow R$  which is left  $A$ -linear such that  $\bar{\partial}_q(1) = 0$  and

$$\forall n \in \mathbb{N}^*, \quad \bar{\partial}_q(t^n) = \{n\}_q t^{n-1}.$$

We can easily verify that the Jackson  $q$ -derivation is a quasi-derivation of parameter  $qt - t = (q - 1)t$ . The associated endomorphism is the map  $\mathfrak{p}_q = \text{Id}_R + (qt - t)\partial_q$  which is left  $A$ -linear and characterized by

$$\forall n \in \mathbb{N}, \quad \mathfrak{p}_q(t^n) = q^n t^n.$$

### 1.3. Conjugate quasi-derivations

**Definition 1.11.** Let  $R$  be a ring. For  $c \in R$ , we denote by  $(c \cdot) : R \rightarrow R$  the left multiplication  $r \mapsto cr$  by  $c$ . Two quasi-derivations  $\partial, \partial'$  of  $R$ , of respective parameters  $c$  and  $c'$ , are called CONJUGATE if we have the relations

$$c + c' = 0, \quad \text{and} \quad \partial - \partial' = \partial \circ (c \cdot) \circ \partial' = \partial' \circ (c \cdot) \circ \partial.$$

**Proposition 1.12.** Two quasi-derivations  $\partial$  and  $\partial'$  of  $R$  are conjugate if and only if the two quasi-derivations  $\partial'$  and  $\partial$  are also.

**Proof.** Indeed, we have  $c' + c = 0$ , whereas, for every element  $r$  of  $R$ , we immediately check the equalities

$$\partial'(r) - \partial(r) = -\partial'(c\partial(r)) = \partial'(c'\partial(r)) = \partial(c'\partial'(r)). \quad \square$$

**Example 1.13.** If  $\partial$  is a derivation, and if  $\partial'$  is conjugate to  $\partial$ , then  $\partial = \partial'$ .

**Example 1.14.** In the quasi-differential ring  $R = \overline{\mathcal{S}}(A)$  of sequences over the ring  $A$ , we define  $\partial : R \rightarrow R$  and  $\partial' : R \rightarrow R$  as following. If  $r \in R$  is the equivalence class of the sequence  $u = (u(n))_{n \in \mathbb{N}}$ , then  $\partial(r)$  is the equivalence class of the sequence  $\tau(u) - u = (u(n+1) - u(n))_{n \in \mathbb{N}}$ , while  $\partial'(r)$  is the equivalence class of the sequence  $v$  such that

$$v(n) = \begin{cases} 0 & \text{if } n = 0 \\ u(n) - u(n-1) & \text{if } n \geq 1 \end{cases}.$$

**Example 1.15.** In a ring  $A$ , let  $q$  be a central and invertible element such that  $q - 1$  does not divide zero in  $A$ . The Jackson  $q$ -derivation  $\partial_q$  of the ring  $R = A[t]$  is a quasi-derivation of parameter  $qt - t$ . It is conjugate to the quasi-derivation  $\partial'_q : R \rightarrow R$  such that

$$\forall f \in R = A[t], \quad \partial'_q(f) = (qt - t)^{-1}(f(t) - f(q^{-1}t)).$$

**Proposition 1.16.** If the two quasi-derivations  $\partial$  and  $\partial'$  of the ring  $R$  are conjugate, then their associated endomorphisms are two mutually inverse bijections. In addition, the quasi-derivations  $\partial$  and  $\partial'$  are related to their respective associated endomorphisms  $\mathfrak{p}$  and  $\mathfrak{p}'$  by the relations

$$\partial' = \partial \circ \mathfrak{p}' \quad \text{and} \quad \partial = \partial' \circ \mathfrak{p}. \quad (4)$$

**Proof.** Let  $c$  be the parameter of the quasi-derivation  $\partial$ . Denoting by  $\mathfrak{p}$  and  $\mathfrak{p}'$  the endomorphisms associated respectively with the quasi-derivations  $\partial$  and  $\partial'$ , we have by definition, for every element  $r$  of  $R$ , the equalities  $\mathfrak{p}(r) = r + c\partial(r)$  and  $\mathfrak{p}'(r) = r - c\partial'(r)$ . It follows that

$$(\mathfrak{p}' \circ \mathfrak{p})(r) = \mathfrak{p}'(\mathfrak{p}(r)) = r + c(\partial(r) - \partial'(r) - \partial'(c\partial(r))) = r,$$

which proves that  $\mathfrak{p}' \circ \mathfrak{p} = \text{Id}_R$ . Using Proposition 1.12 that allows to exchange the quasi-derivations  $\partial$  and  $\partial'$ , we also show that  $\mathfrak{p} \circ \mathfrak{p}' = \text{Id}_R$ .

Moreover, we have  $(\partial \circ \mathfrak{p}')(r) = \partial(\mathfrak{p}'(r)) = \partial(r - c\partial'(r)) = \partial(r) - \partial(c\partial'(r))$ . Now, since the quasi-derivations  $\partial$  and  $\partial'$  are conjugate, we have  $\partial(r) - \partial(c\partial'(r)) = \partial'(r)$ . Hence the equality  $\partial' = \partial \circ \mathfrak{p}'$ , that we can compose to the right with  $\mathfrak{p}$  to get  $\partial = \partial' \circ \mathfrak{p}$ .  $\square$

**Corollary 1.17.** *Given a quasi-derivation  $\partial$  of parameter  $c$  of a ring  $R$ , the quasi-derivation of  $R$  conjugate to  $\partial$ , if it exists, is unique.*

We stress that the uniqueness property of the conjugate quasi-derivation guaranteed by the preceding Corollary holds only for the given value of  $c$ .

The following proposition is the converse of Proposition 1.16.

**Proposition 1.18.** *Given an element  $c$  in the center of  $R$ , let  $\partial$  be a quasi-derivation of parameter  $c$  of the ring  $R$ . If the associated endomorphism  $\mathfrak{p}$  of the quasi-derivation  $\partial$  is a bijection, with inverse bijection  $\mathfrak{p}'$ , then  $\partial \circ \mathfrak{p}'$  is a quasi-derivation of  $R$  conjugated to  $\partial$ .*

**Proof.** The inverse  $\mathfrak{p}'$  of  $\mathfrak{p}$  being an automorphism of the ring  $R$ , we see immediately that  $\partial \circ \mathfrak{p}'$  is an endomorphism of the additive group of  $R$  which sends 1 to 0. For every element  $r \in R$ , we have by definition  $\mathfrak{p}(r) = r + c\partial(r)$ , from which we find, substituting  $\mathfrak{p}'(r)$  for  $r$ , the relation  $\mathfrak{p}'(r) = r - c\partial(\mathfrak{p}'(r))$ . We then easily check that  $\partial' = \partial \circ \mathfrak{p}'$  is a quasi-derivation of  $R$  of parameter  $c' = -c$ . By computing the images by the quasi-derivation  $\partial$  of the two sides of the identity  $\mathfrak{p}'(r) = r - c\partial'(r)$ , we obtain  $\partial'(r) = \partial(r) - \partial(c\partial'(r))$ , which is the equality  $\partial - \partial' = \partial \circ (c \cdot) \circ \partial'$ . Similarly, by taking the images by  $\partial'$  of the two sides of  $\mathfrak{p}(r) = r + c\partial(r)$ , we get the equality  $\partial - \partial' = \partial' \circ (c \cdot) \circ \partial$ . So  $\partial'$  is conjugate to  $\partial$ .  $\square$

**Corollary 1.19.** *Let  $\partial$  be a quasi-derivation of parameter  $c$  of a ring  $R$ . For the existence of a quasi-derivation of  $R$  conjugate to  $\partial$ , it is necessary and sufficient that the endomorphism of the ring  $R$  associated with  $\partial$  is bijective.*

**Proposition 1.20.** *Let  $\partial$  be a quasi-derivation of the ring  $R$  of parameter  $c$ . If  $\partial$  admits a quasi-derivation conjugate  $\partial'$ , then the ring  $q = 1 + \partial(c)$  is invertible in  $R$ , and its inverse is the element  $1 - \mathfrak{p}(\partial'(c))$ , where  $\mathfrak{p} : R \rightarrow R$  is the endomorphism associated with the quasi-derivation  $\partial$ .*

**Proof.** We calculate

$$(1 + \partial(c))(1 - \mathfrak{p}(\partial'(c))) = 1 + \partial(c) - \mathfrak{p}(\partial'(c)) - \partial(c)\mathfrak{p}(\partial'(c)).$$

Since quasi-derivations  $\partial$  and  $\partial'$  are conjugate, we have  $\partial(c) = \partial'(c) + \partial(c\partial'(c))$ , hence

$$(1 + \partial(c))(1 - \mathfrak{p}(\partial'(c))) = 1 + \partial'(c) + \partial(c\partial'(c)) - \mathfrak{p}(\partial'(c)) - \partial(c)\mathfrak{p}(\partial'(c)).$$

Using Equation (2) which expresses the fact that the map  $\partial$  is a  $\mathfrak{p}$ -derivation of the opposite ring of  $R$ , we have  $\partial(c\partial'(c)) = c\partial(\partial'(c)) + \partial(c)\mathfrak{p}(\partial'(c))$ , so

$$(1 + \partial(c))(1 - \mathfrak{p}(\partial'(c))) = 1 + \partial'(c) + c\partial(\partial'(c)) - \mathfrak{p}(\partial'(c)).$$

By definition of the endomorphism  $\mathfrak{p}$  associated with quasi-derivation  $\partial$ , we have  $\mathfrak{p}(\partial'(c)) = \partial'(c) + c\partial(\partial'(c))$ . So it remains

$$(1 + \partial(c))(1 - \mathfrak{p}(\partial'(c))) = 1,$$

that is to say that the element  $1 + \partial(c)$  of  $R$  is right invertible in  $R$ . Now  $\partial$  and  $\partial'$  are two quasi-derivations of respective parameters  $c$  and  $c'$  of the opposite ring  $R^{op}$  of  $R$ , and, since the element  $c$  is central in  $R$ , they are also conjugate as quasi-derivations of  $R^{op}$ . We conclude that  $1 + \partial(c)$  is right invertible in  $R^{op}$ , that is,  $1 + \partial(c)$  is also left invertible in  $R$ .  $\square$

## 2. The ring of formal quasi-differential operators

### 2.1. The Ore extension $R[x, \flat, \partial]$

#### 2.1.1. Essential properties

In many sources, such as books [8,16,25], one finds the construction of the Ore extension  $R[x, \flat, \partial]$ , for any pseudo-derivation  $\partial$  with respect to any endomorphism  $\flat$  of a ring  $R$  not necessarily commutative. This Ore extension has an element  $x$  that satisfies the following properties.

**Property 2.1.** *The ring  $R$  is a subring of  $R[x, \flat, \partial]$ , so that  $R[x, \flat, \partial]$  is provided with an  $R$ -bimodule structure by the left operation  $(r, p) \in R \times R[x, \flat, \partial] \mapsto rp$  and by the right operation  $(p, r) \in R[x, \flat, \partial] \times R \mapsto pr$ .*

**Property 2.2.** *The sequence  $(x^n)_{n \in \mathbb{N}}$  of powers of  $x$  is a basis of the left  $R$ -module  $R[x, \flat, \partial]$ . Therefore, every element  $p$  of  $R[x, \flat, \partial]$  can be expressed in a unique way under the form*

$$p = \sum_{n \in \mathbb{N}} a_n x^n, \quad (5)$$

where the sequence  $(a_n)_{n \in \mathbb{N}} \in R^{\mathbb{N}}$  has only a finite number of non-zero terms.

**Property 2.3.** *For every element  $a$  of  $R$ , we have the equality  $xa = \flat(a)x + \partial(a)$ .*

Properties 2.1 and 2.2 have the following immediate consequences.

**Corollary 2.4.** *The ring  $R[x, \flat, \partial]$  is generated by the set  $R \cup \{x\}$ .*

**Corollary 2.5.** *In the ring  $R[x, \flat, \partial]$ , the element  $x$  is right cancellable.*

From this, we immediately deduce the following lemma, which will be used as needed.

**Lemma 2.6.** *For any non-zero element  $p$  of  $R[x, \flat, \partial]$ , there exists a unique integer  $i$  such that  $p$  is of the form  $dx^i$ , where  $d \in R[x, \flat, \partial]$ , but not of the form  $ex^{i+1}$ .*

#### 2.1.2. Filtration by degree

It is obviously possible [16] to define the DEGREE of an element  $p$  of  $R[x, \flat, \partial]$ .

**Proposition 2.7.** *For all elements  $d$  and  $e$  of  $R[x, \flat, \partial]$ , we have*

$$\deg(p + q) \leq \max(\deg(d), \deg(e)), \quad \deg(de) \leq \deg(d) + \deg(e).$$

Moreover, when  $d$  and  $e$  are both non-zero, let  $a$  and  $b$  be their respective leading coefficients; we have the inequality

$$\deg(de - a\flat^{\deg(d)}(b)x^{\deg(d)+\deg(e)}) < \deg(d) + \deg(e). \quad (6)$$



**Proof.** The upper bound on the degree of the sum  $d + e$  is obvious, that on the degree of the product clearly results from the inequality (6). By Property 2.2, this last inequality is deduced from the particular case where  $d = x^j$ , for which it can be established by induction on  $j$ .  $\square$

### 2.1.3. Notation and vocabulary

In the present work, we consider only the case of quasi-derivations. For a quasi-derivation  $\partial$  of parameter  $c$  of a ring  $R$ , we denote by  $\mathcal{D}_{c,\partial}$ , or more simply  $\mathcal{D}$  if there is no risk of confusion, the Ore extension  $R[x, \mathfrak{p}, \partial]$ , where  $\mathfrak{p}$  is the endomorphism of  $R$  associated with the quasi-derivation  $\partial$ . In this context, Property 2.3 provides the equation

$$\forall a \in R, \quad xa = ax + \partial(a)(1 + cx). \quad (7)$$

We shall also use the notations  ${}^\diamond\mathcal{D}$ ,  $\mathcal{D}^\diamond$  and  ${}^\diamond\mathcal{D}^\diamond$  for the structures of  $\mathcal{D}$  as respectively left  $R$ -module, right  $R$ -module, and  $R$ -bimodule.

**Definition 2.8.** A FORMAL QUASI-DIFFERENTIAL OPERATOR is an element of the ring  $\mathcal{D}_{c,\partial}$ .

We shall often abbreviate formal quasi-differential operator as *f. q.-d. o.* The ring  $\mathcal{D}$  is denominated the *ring of formal quasi-differential operators*.

In the sequel, we often use the hypothesis that there is a quasi-derivation  $\partial'$  of  $R$  conjugate to the quasi-derivation  $\partial$ , which, as we have seen (Propositions 1.16 and 1.18), is equivalent to bijectivity of the endomorphism  $\mathfrak{p}$  of  $R$  associated with quasi-derivation  $\partial$ . In this context, we use the notation  $\mathcal{D}_{c',\partial'}$  or more simply  $\mathcal{D}'$  to designate the Ore extension  $R^{op}[x', \mathfrak{p}', \partial']$ , where  $\mathfrak{p}'$  is the inverse automorphism of  $\mathfrak{p}$ .

**Definition 2.9.** An ADJOINT QUASI-DIFFERENTIAL OPERATOR is an element of the ring  $\mathcal{D}_{c',\partial'}$ .

### 2.1.4. A universal property of the ring of formal quasi-differential operators

Let  $R$  be a not necessarily commutative ring equipped with a quasi-derivation  $\partial : R \rightarrow R$  of parameter  $c$ , where  $c$  is a central element of  $R$ . The ring  $\mathcal{D}$  of formal quasi-differential operators can be characterized by the following universal property [8, page 54].

**Property 2.10.** Let  $E$  be a ring,  $y$  an element of  $E$ , and  $\mathbf{f} : R \rightarrow E$  a ring homomorphism. If we have the identity

$$y\mathbf{f}(a) = \mathbf{f}(a)y + \mathbf{f}(\partial(a))(1 + \mathbf{f}(c)y)$$

for any  $a \in R$ , then there is a unique ring homomorphism  $\mathbf{g} : \mathcal{D} \rightarrow E$  such that  $\mathbf{g}(x) = y$  and which extends  $\mathbf{f}$ .

## 2.2. Operations of the ring of f. q.-d. o.

The use of some structures of left  $\mathcal{D}$ -modules will be essential in the following.

### 2.2.1. Operation of the ring of f. q.-d. o. on the base ring

**Proposition 2.11.** There is a unique ring homomorphism  $\mathbf{g} : \mathcal{D} \rightarrow \text{End}_{\mathbb{Z}}(R)$  such that  $\mathbf{g}(x) = \partial$  and satisfying the identity

$$\forall (a, r) \in R^2, \quad \mathbf{g}(a)(r) = ar.$$

**Proof.** The proposition results from the universal Property 2.10 of  $\mathcal{D}$  applied to the homomorphism  $\mathbf{f}$  of  $R$  in the ring  $E$  of endomorphisms of the additive group of  $R$  defined by

$$\forall(a, r) \in R^2, \quad \mathbf{f}(a)(r) = ar. \quad \square$$

**Notation 2.12.** If  $p$  is a f. q.-d. o. and if  $r$  is an element of  $R$ , we denote by  $p \cdot r$  the element  $\mathbf{g}(p)(r)$  where  $\mathbf{g}$  is the homomorphism of Proposition 2.11.

If the f. q.-d. o. is written  $p = \sum_{i \in \mathbb{N}} a_i x^i$ , we have the formula

$$p \cdot r = \sum_{i \in \mathbb{N}} a_i \partial^i(r).$$

We have thus defined on  $R$  a structure of left  $\mathcal{D}$ -module. We shall use the notation  $R_0$  to designate this left  $\mathcal{D}$ -module.

**Example 2.13.** In the left  $\mathcal{D}$ -module  $R_0$ , we have  $(1 + cx) \cdot a = \mathbf{p}(a)$  for every element  $a \in R$ .

### 2.2.2. Operation of the ring of f. q.-d. o. on a tensor product

Suppose that the base ring  $R$  is commutative. Given an integer  $n \geq 2$ , let  $(M_1, \dots, M_n)$  be a finite list of left  $\mathcal{D}$ -modules. We denote by  $\times_{j=1}^n M_j = M_1 \times \dots \times M_n$  their cartesian product. For any ordered pair of integers  $(i, j)$  such that  $1 \leq \min(i, j)$  and  $\max(i, j) \leq n$ , and for any element  $r$  of the base ring  $R$ , we denote by  $r_{i,j}$  the element of  $R$  that is equal to  $r$  if  $i = j$  and to 1 if  $i \neq j$ .

**Definition 2.14.** Given any abelian group  $\mathcal{A}$  and an  $n$ -additive mapping  $\Theta : \times_{j=1}^n M_j \rightarrow \mathcal{A}$ , we say that  $\Theta$  is BALANCED when

$$\forall i \in [2..n], \forall r \in R, \forall (p_j)_{1 \leq j \leq n} \in \times_{j=1}^n M_j, \quad \Theta((r_{i,j} p_j)_{1 \leq j \leq n}) = \Theta((r_{1,j} p_j)_{1 \leq j \leq n}).$$

Consider the tensor product  $\mathcal{T} = M_1 \otimes_R \dots \otimes_R M_n$  on the base ring  $R$  of modules  $M_1, \dots, M_n$ . It is known [1] to be an abelian group with a balanced  $n$ -additive mapping

$$(p_1, \dots, p_n) = (p_j)_{1 \leq j \leq n} \mapsto p_1 \otimes \dots \otimes p_n = \otimes_{j=1}^n p_j$$

from  $\times_{j=1}^n M_j$  to  $\mathcal{T}$  satisfying the following universal property: for any abelian group  $\mathcal{A}$  and for any balanced  $n$ -additive map  $\Theta : \times_{j=1}^n M_j \rightarrow \mathcal{A}$ , there exists a unique additive map  $u : \mathcal{T} \rightarrow \mathcal{A}$  such that  $u(\otimes_{j=1}^n p_j) = \Theta(p_1, \dots, p_n)$  for any  $(p_j)_{1 \leq j \leq n}$  in  $\times_{j=1}^n M_j$ . Moreover, since the ring  $R$  is supposed to be commutative, we know that this abelian group  $\mathcal{T}$  is equipped with an  $R$ -module structure such that

$$\forall(r, p_1, \dots, p_n) \in R \times M_1 \times \dots \times M_n, \forall i \in [2..n], \quad r(\otimes_{1 \leq j \leq n} p_j) = \otimes_{1 \leq j \leq n} r_{1,j} p_j = \otimes_{1 \leq j \leq n} r_{i,j} p_j.$$

For any integer  $i \in [2..n]$ , let  $\rho_i$  be the permutation of the set of integers  $[1..n]$  that exchanges 1 and  $i$  and leaves all other elements fixed; we then denote by  $\text{voc}_i$  the map from  $\times_{j=1}^n M_{\rho_i(j)}$  into  $\times_{j=1}^n M_j$  such that

$$\forall(p_{\rho_i(j)})_{1 \leq j \leq n} \in \times_{j=1}^n M_{\rho_i(j)}, \quad \text{voc}_i((p_{\rho_i(j)})_{1 \leq j \leq n}) = (p_j)_{1 \leq j \leq n}. \quad (8)$$

For any integer  $i \in [2..n]$ , it is immediate, using the universal property of the tensor product, to show the existence of an  $R$ -linear mapping  $\text{vot}_i$  from the  $R$ -module  $\mathcal{T}_i = \otimes_{j=1}^n M_{\rho_i(j)}$  to the  $R$ -module  $\mathcal{T} = \otimes_{j=1}^n M_j$  such that

$$\forall (p_{\rho_i(j)})_{1 \leq j \leq n} \in \times_{j=1}^n M_{\rho_i(j)}, \quad \text{vot}_i(\otimes_{j=1}^n p_{\rho_i(j)}) = \otimes_{j=1}^n p_j. \quad (9)$$

We will use the following notations. The symbol  $\mathcal{P}_n^\bullet$  stands for the set of non-empty subsets of the set  $[1..n]$  of the first  $n$  positive integers. For an element  $I$  of  $\mathcal{P}_n^\bullet \cup \{\emptyset\}$ , we denote by  $|I|$  the number of elements of  $I$ . Finally,  $(I, -) : [1..n] \rightarrow \mathbb{N}$  is the indicator function of the subset  $I$ , so that  $(I, j) = 1$  if and only if  $j$  belongs to the subset  $I$ , and that  $(I, j) = 0$  is equivalent to  $j \notin I$ .

**Lemma 2.15.** *There is a unique endomorphism  $\Gamma$  of the additive group  $\mathcal{T}$  such that*

$$\forall (p_j)_{1 \leq j \leq n} \in \times_{j=1}^n M_j, \quad \Gamma(\otimes_{j=1}^n p_j) = \sum_{I \in \mathcal{P}_n^\bullet} c^{|I|-1} \otimes_{j=1}^n x^{(I,j)} p_j.$$

Moreover, the endomorphism  $\Gamma$  of  $\mathcal{T}$  satisfies the relationship

$$\forall (r, b) \in R \times \mathcal{T}, \quad \Gamma(rb) = r\Gamma(b) + \partial(r)(b + c\Gamma(b)). \quad (10)$$

**Proof.** For any sequence  $\mathcal{M} = (M_1, \dots, M_n)$  of  $n$  left-modules on the ring  $\mathcal{D}$ , we introduce the map  $\Theta_{\mathcal{M}}$  of  $\times_{j=1}^n M_j$  in  $\mathcal{T}_{\mathcal{M}} = \otimes_{j=1}^n M_j$  defined by

$$\forall (p_j)_{1 \leq j \leq n} \in \times_{j=1}^n M_j, \quad \Theta_{\mathcal{M}}((p_j)_{1 \leq j \leq n}) = \sum_{I \in \mathcal{P}_n^\bullet} c^{|I|-1} \otimes_{j=1}^n x^{(I,j)} p_j.$$

The maps  $\Theta_{\mathcal{M}}$  are obviously  $n$ -additive. We will need three results concerning these mappings.

First, we easily check that, for any sequence  $\mathcal{M} = (M_1, \dots, M_n)$  of  $n$  left modules on the ring  $\mathcal{D}$ , we have the equalities

$$\forall i \in [2..n], \quad \text{vot}_i \circ \Theta_i = \Theta_{\mathcal{M}} \circ \text{voc}_i, \quad (11)$$

where  $\Theta_i$  is the map  $\Theta_{\rho_i(\mathcal{M})}$  defined by the sequence  $\rho_i(\mathcal{M}) = (M_{\rho_i(j)})_{1 \leq j \leq n}$ .

A second preliminary result on the maps  $\Theta_{\mathcal{M}}$  is the identity

$$\Theta_{\mathcal{M}}((r_{1,j}p_j)_{1 \leq j \leq n}) = r\Theta_{\mathcal{M}}((p_j)_{1 \leq j \leq n}) + \partial(r)(\otimes_{j=1}^n p_j + c\Theta_{\mathcal{M}}((p_j)_{1 \leq j \leq n})). \quad (12)$$

In order to verify (12), we calculate  $\Theta_{\mathcal{M}}((r_{1,j}p_j)_{1 \leq j \leq n})$ , using Equation (7) and the  $n$ -additivity of the tensor product, which gives after simplification

$$\Theta_{\mathcal{M}}((r_{1,j}p_j)_{1 \leq j \leq n}) = \sum_{I \in \mathcal{P}_n^\bullet} c^{|I|-1} \otimes_{j=1}^n r_{1,j} x^{(I,j)} p_j + b,$$

where the first term  $\sum_{I \in \mathcal{P}_n^\bullet} c^{|I|-1} \otimes_{j=1}^n r_{1,j} x^{(I,j)} p_j$  is none other than  $r\Theta_{\mathcal{M}}((p_j)_{1 \leq j \leq n})$ . In the second term

$$b = \sum_{I \in \mathcal{P}_n^\bullet} c^{|I|-1} (\partial(r)(I, 1)(1 + cx)p_1 \otimes (\otimes_{j=2}^n r_{1,j} x^{(I,j)} p_j)),$$

the only subsets  $I$  having a non-zero contribution are of the form  $\{1\} \cup J$ , with  $J$  in the set  $\mathcal{P}_n^*$  of subsets of  $[2..n]$ . This leads to

$$b = \partial(r) \left( \sum_{J \in \mathcal{P}_n^*} c^{|J|} p_1 \otimes \left( \otimes_{j=2}^n x^{(J,j)} p_j \right) + \sum_{J \in \mathcal{P}_n^*} c^{|J|+1} x p_1 \otimes \left( \otimes_{j=2}^n x^{(J,j)} p_j \right) \right).$$

Separating the term corresponding to  $J = \emptyset$  from the first sum in the right-hand side, we obtain

$$\flat = \partial(r) \left( \otimes_{j=1}^n p_j + c \left( \sum_{J \in \mathcal{P}_n^* \setminus \{\emptyset\}} c^{|J|-1} p_1 \otimes \left( \otimes_{j=2}^n x^{(J,j)} p_j \right) + \sum_{J \in \mathcal{P}_n^*} c^{|J|} x p_1 \otimes \left( \otimes_{j=2}^n x^{(J,j)} p_j \right) \right) \right).$$

In the second sum  $\sum_{J \in \mathcal{P}_n^*} c^{|J|} x p_1 \otimes (\otimes_{j=2}^n x^{(J,j)} p_j)$ , we make the change of indices  $J' = J \cup \{1\}$  which realizes a bijection between the set  $\mathcal{P}_n^*$  of subsets of  $[2..n]$  and the set  $\mathcal{P}_n^\bullet \setminus \mathcal{P}_n^*$  of subsets of  $[1..n]$  containing 1. By using the fact that the set  $\mathcal{P}_n^\bullet$  of non-empty subsets of  $[1..n]$  is the disjoint union of the set  $\mathcal{P}_n^* \setminus \{\emptyset\}$  of non-empty subsets of  $[2..n]$  with the set  $\mathcal{P}_n^\bullet \setminus \mathcal{P}_n^*$  of subsets of  $[1..n]$  containing 1, we finish to verify Equation (12).

The third fact to justify is that the mapping  $\Theta_{\mathcal{M}}$  is balanced, which can be established by a direct computation using Equations (11) and (12).

The universal property of the tensor product  $\mathcal{T} = \otimes_{j=1}^n M_j$ , applied to the balanced  $n$ -additive mapping  $\Theta_{\mathcal{M}}$ , then shows the existence and uniqueness of the additive endomorphism  $\Gamma$  of  $\mathcal{T}$  satisfying the required condition. To verify Equation (10), it suffices to observe that the two members of this identity are, when we arbitrarily fix an element  $r$  of  $R$ , two additive functions of the variable  $\flat$  which, according to Equation (12) and the definition of  $\Gamma$ , coincide on all elements of the form  $\otimes_{j=1}^n p_j$ . But such an additive mapping is unique according to the universal property of the tensor product  $\mathcal{T}$ .  $\square$

**Proposition 2.16.** *Let  $(R, c, \partial)$  be a commutative quasi-differential ring. For any finite sequence  $\mathcal{M} = (M_1, \dots, M_n)$  of left  $\mathcal{D}$ -modules, whose tensor product on  $R$  is denoted by  $\mathcal{T}$ , there is a unique ring homomorphism  $\mathbf{g} : \mathcal{D} \rightarrow \text{End}_{\mathbb{Z}}(\mathcal{T})$  such that  $\mathbf{g}(x) = \Gamma$  and satisfying the identity*

$$\forall (a, p_1, \dots, p_n) \in R \times M_1 \times \dots \times M_n, \quad \mathbf{g}(a)(\otimes_{j=1}^n p_j) = a(\otimes_{j=1}^n p_j).$$

**Proof.** Let  $a$  be an element of  $R$ . Since  $R$  is supposed to be commutative, the mapping from  $\times_{j=1}^n M_j$  to  $\mathcal{T}$  that sends the element  $(p_j)_{1 \leq j \leq n}$  on the tensor  $a(\otimes_{j=1}^n p_j)$  is a balanced  $n$ -additive mapping. Therefore exists a unique map  $\mathbf{f}(a) \in \text{End}_{\mathbb{Z}}(\mathcal{T})$  such that

$$\forall (p_j)_{1 \leq j \leq n} \in \times_{j=1}^n M_j, \quad \mathbf{f}(a)(\otimes_{j=1}^n p_j) = a(\otimes_{j=1}^n p_j).$$

The map  $\mathbf{f} : R \rightarrow \text{End}_{\mathbb{Z}}(\mathcal{T})$  is then a ring homomorphism. Moreover Equation (10) shows that

$$\forall a \in R, \quad \Gamma \circ \mathbf{f}(a) = \mathbf{f}(a) \circ \Gamma + \mathbf{f}(\partial(a)) \circ (\text{Id}_{\mathcal{T}} + \mathbf{f}(c) \circ \Gamma).$$

Hence the desired result by using Property 2.10.  $\square$

So we have defined on every finite tensor product of left  $\mathcal{D}$ -modules a structure of left  $\mathcal{D}$ -module. This will be applied especially to the case where all these modules are equal to  $\mathcal{D}$  itself. In this case, we have a structure of left  $\mathcal{D}$ -module on the tensor power  $({}^\circ\mathcal{D})^{\otimes n}$ .

**Notation 2.17.** For any  $p$  in  $\mathcal{D}$  and  $\flat$  in the tensor power  $({}^\circ\mathcal{D})^{\otimes n}$ , we denote by  $p \triangleleft_n \flat$  the element  $\mathbf{g}(p)(\flat)$ , where  $\mathbf{g}$  is the ring homomorphism of Proposition 2.16. When  $n = 2$ , we simply write  $p \triangleleft \flat$  instead of  $p \triangleleft_2 \flat$ .

It is easy to show the following result about the functoriality of the  $\mathcal{D}$ -module structure on the tensor products.

**Proposition 2.18.** *Let  $(R, c, \partial)$  be a commutative quasi-differential ring. Given, for an integer  $n \geq 2$ , two lists  $\mathcal{M} = (M_1, \dots, M_n)$  and  $\mathcal{N} = (N_1, \dots, N_n)$  of  $n$  left modules on the ring  $\mathcal{D}$  of formal quasi-differential*

operators, denote by  $\mathcal{T}_{\mathcal{M}}$  (resp.  $\mathcal{T}_{\mathcal{N}}$ ) the tensor product  $M_1 \otimes_R \cdots \otimes_R M_n$  (resp.  $N_1 \otimes_R \cdots \otimes_R N_n$ ), equipped with the structure of left  $\mathcal{D}$ -module defined by Proposition 2.16. If we have  $n$  homomorphisms of left  $\mathcal{D}$ -module  $f_j : M_j \rightarrow N_j$  ( $1 \leq j \leq n$ ), their tensor product  $\otimes f_j : \mathcal{T}_{\mathcal{M}} \rightarrow \mathcal{T}_{\mathcal{N}}$  is again a homomorphism of left  $\mathcal{D}$ -modules.

We now consider the situation where, in addition to the list  $(M_1, \dots, M_n)$  of left  $\mathcal{D}$ -modules, we give a partition of the set  $[1..n]$  in two non-empty subsets that are described as images of two injections  $\alpha$  and  $\alpha'$  from respective sources  $[1..m]$  and  $[1..m']$ , with necessarily  $m + m' = n$ . In this context, three tensor products can be formed, namely  $\mathcal{T} = \otimes_{j=1}^n M_j$ ,  $\mathcal{U} = \otimes_{i=1}^m M_{\alpha(i)}$ , and  $\mathcal{U}' = \otimes_{i'=1}^{m'} M_{\alpha'(i')}$ . We obviously have a balanced  $n$ -additive map

$$\times_{j=1}^n M_j \rightarrow \mathcal{U} \otimes \mathcal{U}'$$

which to the sequence  $(p_j)_{1 \leq j \leq n}$  associates  $(\otimes_{i=1}^m p_{\alpha(i)}) \otimes (\otimes_{i'=1}^{m'} p_{\alpha'(i')})$ . By the universal property of  $\mathcal{T}$ , we get existence and uniqueness of the additive map

$$\mathbf{u} : \mathcal{T} \rightarrow \mathcal{U} \otimes \mathcal{U}'$$

such that

$$\mathbf{u}(\otimes_{j=1}^n p_j) = (\otimes_{i=1}^m p_{\alpha(i)}) \otimes (\otimes_{i'=1}^{m'} p_{\alpha'(i')}). \quad (13)$$

It is well known [1] that the map  $\mathbf{u}$  is bijective and  $R$ -linear. Concerning the  $\mathcal{D}$ -linearity, we have the following result.

**Proposition 2.19.** *The map  $\mathbf{u}$  is left  $\mathcal{D}$ -linear.*

**Proof.** By Corollary 2.4, we are reduced to justify that

$$\mathbf{u}(xb) = x\mathbf{u}(b)$$

for any element  $b \in \mathcal{T}$ . According to the universal property of the tensor product  $\mathcal{T}$ , we have to check for any choice of  $(p_j)_{1 \leq j \leq n}$  the equality between the image by  $\mathbf{u}$  of the element  $\sum_{I \in \mathcal{P}_{\bullet}^n} c^{|I|-1} \otimes_{j=1}^n x^{(I,j)} p_j$  and the sum  $\beth_1 + \beth_2 + \beth_3$  of the three tensors  $\beth_1 = \Gamma_{\mathcal{M}_{\alpha}}(\otimes_{i=1}^m p_{\alpha(i)}) \otimes (\otimes_{i'=1}^{m'} p_{\alpha'(i')})$ ,  $\beth_2 = (\otimes_{i=1}^m p_{\alpha(i)}) \otimes \Gamma_{\mathcal{M}_{\alpha'}}(\otimes_{i'=1}^{m'} p_{\alpha'(i')})$  and

$$\beth_3 = c\Gamma_{\mathcal{M}_{\alpha}}(\otimes_{i=1}^m p_{\alpha(i)}) \otimes \Gamma_{\mathcal{M}_{\alpha'}}(\otimes_{i'=1}^{m'} p_{\alpha'(i')}).$$

This equality can be obtained by writing the sum  $\sum_{I \in \mathcal{P}_{\bullet}^n} c^{|I|-1} \otimes_{j=1}^n x^{(I,j)} p_j$  as the sum of three terms: the first term groups the contributions of subsets of the form  $\alpha(J)$ , where  $J$  is a non-empty subset of  $[1..m]$ , the second the contributions of subsets of the form  $\alpha'(J')$ , where  $J'$  is a non-empty subset of  $[1..m']$ , and the third the contributions of subsets  $\alpha(J) \cup \alpha'(J')$ , where  $J$  and  $J'$  are non-empty subsets respectively of  $[1..m]$  and of  $[1..m']$ .  $\square$

### 2.3. Translation operators

In this subsection,  $\mathcal{D}$  designates the ring of f. q.-d. o. into a quasi-derivation  $\partial$  of a ring that is not necessarily commutative.

### 2.3.1. Definition of translation operators

**Proposition 2.20.** *If  $a$  is a central element of  $R$ , there are two endomorphisms  $E^a$  and  $F^a$  of the ring  $\mathcal{D}$  determined uniquely by the conditions*

$$\forall r \in R, \quad E^a(r) = F^a(r) = r, \quad E^a(x) = x + a + cax \quad \text{and} \quad F^a(x) = x + a + xca$$

**Proof.** Let  $\mathbf{f} : R \rightarrow \mathcal{D}$  be the canonical inclusion,  $y = x + a + cax \in \mathcal{D}$  and  $z = x + a + xca \in \mathcal{D}$ . For all  $r \in R$ , we easily verify, using Equation (7), that we have the identities  $\mathbf{f}(r)y + \mathbf{f}(\partial(r))(1 + \mathbf{f}(c)y) = y\mathbf{f}(r)$  and  $\mathbf{f}(r)z + \mathbf{f}(\partial(r))(1 + \mathbf{f}(c)z) = z\mathbf{f}(r)$ . By the universal property (Property 2.10) of  $\mathcal{D}$ , the existence and uniqueness of ring homomorphisms  $E^a : \mathcal{D} \rightarrow \mathcal{D}$  and  $F^a : \mathcal{D} \rightarrow \mathcal{D}$  result therefrom.  $\square$

**Definition 2.21.** For every element  $a$  of the center of the ring  $R$ , the endomorphisms  $E^a$  and  $F^a$  are called TRANSLATION OPERATORS OF  $a$ . The endomorphism  $E^a$  is more precisely called a translation operator of the FIRST TYPE, and the endomorphism  $F^a$  a translation operator of the SECOND TYPE.

**Remark 2.22.** Let  $q = 1 + \partial(c)$  be the rung of the quasi-derivation  $\partial$ . We easily check that  $E^{\mathfrak{b}(a)q}(x) = x + a + xca$ , so we actually have  $F^a = E^{\mathfrak{b}(a)q}$ .

### 2.3.2. Commutation of translation operators

Observe that translation operators commute between themselves.

**Property 2.23.** *Let  $a$  and  $b$  be two central elements of  $R$ . Then we have*

$$E^a \circ E^b = E^b \circ E^a, \quad E^a \circ F^b = F^b \circ E^a, \quad F^a \circ F^b = F^b \circ F^a.$$

**Property 2.24.** *Let  $a$  and  $b$  be two central elements of  $R$ . Then we have*

$$E^a \circ E^b = E^{a+b+cab}, \quad \text{as well as} \quad F^a \circ F^b = F^{a+b+cab}.$$

## 2.4. Adjunction

### 2.4.1. Definition of the adjunction

**Proposition 2.25.** *Let  $\{\partial, \partial'\}$  be a pair of conjugate quasi-derivations of a ring  $R$  and  $\mathfrak{p}, \mathfrak{p}'$  the endomorphisms of  $R$  associated respectively with  $\partial, \partial'$ . We put  $\mathcal{D} = R[x, \mathfrak{p}, \partial]$  and  $\mathcal{D}' = R^{op}[x', \mathfrak{p}', \partial']$ . Then there is a unique ring antimorphism  $\Lambda$  from  $\mathcal{D}$  to  $\mathcal{D}'$  such that  $\Lambda(r) = r$  for any  $r \in R$  and  $\Lambda(x) = -x'$ .*

**Proof.** Let  $c$  and  $c'$  be the respective parameters of the quasi-derivations  $\partial$  and  $\partial'$  and  $\mathfrak{p}$  and  $\mathfrak{p}'$  their associated automorphisms. We denote by  $\star_{op}$  the multiplicative operation of the ring  $\mathcal{D}'^{op}$ .

We consider the ring homomorphism  $\Omega$  from  $R$  into  $\mathcal{D}'^{op}$  such that  $\Omega(r) = r$  for any element  $r$  of  $R$ . By definition of  $x' \in \mathcal{D}'$  we have the relationship  $x'r = \mathfrak{p}'(r)x' + \partial'(r)$ . By substituting  $\mathfrak{p}(r)$  for  $r$  in this last identity, we obtain the equality  $x'\mathfrak{p}(r) = rx' + \partial'(\mathfrak{p}(r)) = rx' + \partial(r)$ . So we have

$$(-x') \star_{op} \Omega(r) = -rx' = -x'\mathfrak{p}(r) + \partial(r).$$

As  $\mathfrak{p}(r) = r + c\partial(r)$ , we find

$$(-x') \star_{op} \Omega(r) = -x'r - x'c\partial(r) + \partial(r) = \Omega(r) \star_{op} (-x') + \Omega(\partial(r)) \star_{op} (1 + \Omega(c) \star_{op} (-x')).$$

This identity allows us to apply Property 2.10 which shows the existence and uniqueness of the antimorphism  $\Lambda$ .  $\square$

**Definition 2.26.** The map  $\Lambda : \mathcal{D} \rightarrow \mathcal{D}'$  is called the ADJUNCTION.

If we write every element  $p \in \mathcal{D}$  in the form  $p = \sum_{n \in \mathbb{N}} a_n x^n$ , where  $a_n \in R$  for all  $n \in \mathbb{N}$ , and where the set of integers  $n$  such that  $a_n \neq 0$  is finite, we have  $\Lambda(p) = \sum_{n \in \mathbb{N}} (-1)^n x'^n a_n$ .

#### 2.4.2. Duality

We can swap in the statement of Proposition 2.25 the roles of the quasi-derivations  $\partial$  and  $\partial'$ . We thus build a unique antimorphism  $\Lambda' : \mathcal{D}' \rightarrow \mathcal{D}$  such that  $\Lambda'(r) = r$  for all  $r \in R$  and  $\Lambda'(x') = -x$ .

**Proposition 2.27.** The adjunction  $\Lambda : \mathcal{D} \rightarrow \mathcal{D}'$  is bijective, and  $\Lambda'$  is its inverse bijection.

**Proof.** As composition of two antimorphisms, the map  $\Lambda' \circ \Lambda$  is a ring endomorphism of  $\mathcal{D}$ . Since  $\Lambda$  and  $\Lambda'$  fix all elements of  $R$ , it is the same with their composition  $\Lambda' \circ \Lambda$ . In addition, an immediate computation gives

$$\Lambda' \circ \Lambda(x) = \Lambda'(-x') = x.$$

According to Corollary 2.4 (or according to the universal property of  $\mathcal{D}$ ), this is enough to prove that  $\Lambda' \circ \Lambda$  is the automorphism identity of  $\mathcal{D}$ .

By reversing the roles of  $\mathcal{D}$  and  $\mathcal{D}'$ , we also show that  $\Lambda \circ \Lambda'$  is the automorphism identity of  $\mathcal{D}'$ .  $\square$

For any left  $\mathcal{D}$ -module  $M$ , whose structure is given by a ring homomorphism  $\mathbf{g}$  from  $\mathcal{D}$  to the ring  $\text{End}_{\mathbb{Z}}(M)$  of endomorphisms of the additive group of  $M$ , we define  $\mathbf{J}(M)$  as the right  $\mathcal{D}'$ -module whose structure is given by the antimorphism  $\mathbf{g} \circ \Lambda'$  from  $\mathcal{D}'$  to  $\text{End}_{\mathbb{Z}}(M)$ . Similarly, for any right  $\mathcal{D}'$ -module  $M'$ , whose structure is given by an antimorphism  $\mathbf{g}'$  from  $\mathcal{D}'$  to the ring  $\text{End}_{\mathbb{Z}}(M')$  of endomorphisms of the additive group of  $M'$ , we define  $\mathbf{J}'(M')$  as the left  $\mathcal{D}$ -module whose structure is given by the homomorphism  $\mathbf{g}' \circ \Lambda$  from  $\mathcal{D}$  to  $\text{End}_{\mathbb{Z}}(M')$ . It is immediate to see that given two left  $\mathcal{D}$ -modules  $M$  and  $N$  (resp. two right  $\mathcal{D}'$ -modules  $M'$  and  $N'$ ), a map  $\mathbf{u}$  from  $M$  to  $N$  (resp.  $\mathbf{u}'$  from  $M'$  to  $N'$ ) is left  $\mathcal{D}$ -linear (resp. right  $\mathcal{D}'$ -linear) if and only if it is right  $\mathcal{D}'$ -linear (resp. left  $\mathcal{D}$ -linear) from  $\mathbf{J}(M)$  to  $\mathbf{J}(N)$  (resp. from  $\mathbf{J}'(M')$  to  $\mathbf{J}'(N')$ ). As a result, we can consider  $\mathbf{J}$  and  $\mathbf{J}'$  as two functors between the category of left  $\mathcal{D}$ -modules and the category of right  $\mathcal{D}'$ -modules.

**Proposition 2.28.** The functors  $\mathbf{J}$  and  $\mathbf{J}'$  are two mutually inverse functors, so that the category of left  $\mathcal{D}$ -modules is isomorphic to the category of right  $\mathcal{D}'$ -modules.

**Proof.** Indeed,  $\mathbf{J}'(\mathbf{J}(M)) = M$  because, if the structure of  $M$  as a left  $\mathcal{D}$ -module is given by the homomorphism  $\mathbf{g} : \mathcal{D} \rightarrow \text{End}_{\mathbb{Z}}(M)$ , then that of  $\mathbf{J}'(\mathbf{J}(M))$  is given by  $\mathbf{g} \circ \Lambda' \circ \Lambda = \mathbf{g}$ . We can also similarly verify that  $\mathbf{J}(\mathbf{J}'(M')) = M'$  for any right  $\mathcal{D}'$ -module  $M'$ .  $\square$

#### 2.5. Primary endomorphisms of the ring of f. q.-d. o.

##### 2.5.1. The first primary endomorphism

**Proposition 2.29.** Let  $R$  be a commutative ring equipped with a quasi-derivation  $\partial$  of parameter  $c$ . If the rung  $q = 1 + \partial(c)$  is invertible in  $R$ , then there is a unique ring endomorphism  $\mathbb{P}_1$  of  $\mathcal{D}$  such that  $\mathbb{P}_1(r) = r + c\partial(r)$  for any  $r \in R$  and  $\mathbb{P}_1(x) = q^{-1}x$ , where  $q^{-1}$  is the inverse of  $q$ .

**Proof.** Let  $\mathbf{f} : R \rightarrow \mathcal{D}$  be the ring homomorphism obtained by composition of the endomorphism  $\mathfrak{p}$  associated with the quasi-derivation  $\partial$  followed by the canonical inclusion from  $R$  in  $\mathcal{D}$ . With the notation  $\dot{c} = \partial(c)$ , we thus have  $\mathbf{f}(c) = c + c\dot{c} = cq$ . By virtue of (3), we have

$$\forall r \in R, \quad \partial(\mathbf{f}(r)) = q\mathbf{f}(\partial(r)). \quad (14)$$

By using the commutativity of the ring  $R$ , and Equations (7) and (14), we check the equality  $q^{-1}x\mathbf{f}(r) = \mathbf{f}(r)q^{-1}x + \mathbf{f}(\partial(r))(1 + \mathbf{f}(c)q^{-1}x)$ . By virtue of Property 2.10, this shows the existence and uniqueness of the ring homomorphism  $\mathbb{P}_1 : \mathcal{D} \rightarrow \mathcal{D}$  extending  $\mathbf{f}$  and such that  $\mathbb{P}_1(x) = q^{-1}x$ .  $\square$

**Definition 2.30.** The endomorphism  $\mathbb{P}_1$  of Proposition 2.29 takes the name of FIRST PRIMARY ENDOMORPHISM OF  $\mathcal{D}$ .

**Remark 2.31.** The hypothesis of Proposition 2.29 is in particular satisfied when the quasi-derivation  $\partial$  admits a conjugate quasi-derivation  $\partial'$ . In this case, Proposition 1.20 implies that the rung  $q = 1 + \partial(c)$  of the quasi-derivation  $\partial$  is linked to the rung  $q' = 1 + \partial'(c')$  of its conjugate by the relation

$$q\mathfrak{p}(q') = 1, \quad (15)$$

where  $\mathfrak{p}$  is associated with quasi derivation  $\partial$ , so we have  $q^{-1} = \mathfrak{p}(q')$ .

### 2.5.2. The second primary endomorphism

**Proposition 2.32.** Let  $R$  be a commutative ring equipped with a quasi-derivation  $\partial$  of parameter  $c$ . If the element  $q = 1 + \partial(c)$  is invertible in  $R$ , then there is a unique ring endomorphism  $\mathbb{P}_2$  of  $\mathcal{D}$  such that  $\mathbb{P}_2(r) = r + c\partial(r)$  for all  $r \in R$  and  $\mathbb{P}_2(x) = xq^{-1}$ .

**Proof.** Similar to the previous demonstration, it consists in using Property 2.10 to the same homomorphism  $\mathbf{f} : R \rightarrow \mathcal{D}$ . For this, one verifies the identity  $xq^{-1}\mathbf{f}(r) = \mathbf{f}(r)xq^{-1} + \mathbf{f}(\partial(r))(1 + \mathbf{f}(c)xq^{-1})$  for any element  $r \in R$ .  $\square$

**Definition 2.33.** The endomorphism  $\mathbb{P}_2$  of Proposition 2.32 takes the name of SECOND PRIMARY ENDOMORPHISM of  $\mathcal{D}$ .

**Proposition 2.34.** Let  $R$  be a commutative ring equipped with a quasi-derivation  $\partial$  of parameter  $c$ . If the rung  $q = 1 + \partial(c)$  is invertible in  $R$ , then the second primary endomorphism  $\mathbb{P}_2$  of  $\mathcal{D}$  is connected to the first primary endomorphism by the relationship

$$\mathbb{P}_2 = J_q \circ \mathbb{P}_1,$$

where  $J_q : \mathcal{D} \rightarrow \mathcal{D}$  is the interior automorphism of  $\mathcal{D}$  defined by

$$\forall p \in \mathcal{D}, \quad J_q(p) = qpq^{-1}.$$

**Proof.** It is immediate that the endomorphisms  $\mathbb{P}_2$  and  $J_q \circ \mathbb{P}_1$  coincide on  $R \cup \{x\}$ , thus on the ring  $\mathcal{D}$  generated by  $R \cup \{x\}$ .  $\square$



### 2.5.3. Bijectivity of primary endomorphisms

**Proposition 2.35.** *Let  $R$  be a commutative ring equipped with a quasi-derivation  $\partial$  of parameter  $c$ . If the endomorphism  $\mathfrak{p} : R \rightarrow R$  associated with the quasi-derivation  $\partial$  is bijective, then the primary endomorphisms of  $\mathcal{D}$  are both bijective.*

**Proof.** By Propositions 1.18 and 1.20, we know that the element  $q = 1 + \partial(c)$  is invertible in  $R$ , so that primary endomorphisms exist. One can show by induction, using Proposition 2.7, that the f. q.-d. o.  $(\mathbb{P}_1(x))^n$  is of degree  $n$  with an invertible leading coefficient in  $R$ . In particular, the degree of  $a(\mathbb{P}_1(x))^n$  is equal to  $n$  for any non-zero element  $a$  of  $R$ . Now consider, if it exists, a non-zero element  $p$  of the kernel of  $\mathbb{P}_1$ , and let  $d$  be its degree. We can write

$$p = \sum_{j=0}^d a_j x^j \quad \text{and so} \quad 0 = \mathbb{P}_1(p) = \sum_{j=0}^d \mathfrak{p}(a_j)(\mathbb{P}_1(x))^j$$

from where we find

$$\mathfrak{p}(a_d)(\mathbb{P}_1(x))^d = - \sum_{j=0}^{d-1} \mathfrak{p}(a_j)(\mathbb{P}_1(x))^j,$$

equation whose right-hand side is of degree at most  $d-1$ . Therefore  $\mathfrak{p}(a_d) = 0$ , which, since  $\mathfrak{p}$  is injective, implies that  $a_d = 0$ , and this contradicts the assertion  $\deg(p) = d$ . This contradiction establishes the injectivity of  $\mathbb{P}_1$ . Let  $p = \sum_{n \in \mathbb{N}} a_n x^n$  be any element of  $\mathcal{D}$ , where the sequence  $(a_n)_{n \in \mathbb{N}}$  of elements in  $R$  has only a finite number of non-zero terms. Since  $\mathfrak{p} : R \rightarrow R$  is supposed surjective, for any index  $n \in \mathbb{N}$ , there is an element  $b_n \in R$  such that  $\mathfrak{p}(b_n) = a_n$ . Similarly, the surjectivity of  $\mathfrak{p}$  leads to the existence of an element  $c^* \in R$  such that  $\mathfrak{p}(c^*) = \partial(c)$ . So, it is easy to check that

$$\mathbb{P}_1 \left( \sum_{n \in \mathbb{N}} b_n ((1 + c^*)x)^n \right) = p,$$

which shows the surjectivity of  $\mathbb{P}_1$ . From the bijectivity of  $\mathbb{P}_1$  follows that of  $\mathbb{P}_2$  by Proposition 2.34.  $\square$

### 2.5.4. Noteworthy identities

**Proposition 2.36.** *Let  $R$  be a commutative ring equipped with a quasi-derivation  $\partial$  of parameter  $c$ . We suppose that the rung  $q = 1 + \partial(c)$  is invertible in  $R$ , so that the primary endomorphisms  $\mathbb{P}_1$  and  $\mathbb{P}_2$  of  $\mathcal{D}$  exist. For every element  $p$  in  $\mathcal{D}$ , we have the commutation identities*

$$(1 + cx)p = \mathbb{P}_1(p)(1 + cx) \tag{16}$$

and

$$(1 + xc)p = \mathbb{P}_2(p)(1 + xc). \tag{17}$$

**Proof.** According to Corollary 2.4, it suffices to check the equalities (16) and (17), firstly if  $p$  is element of  $R$ , and then if  $p = x$ . When  $p = a$  is an element of  $R$ , this follows by an easy computation from Equation (7). For  $p = x$ , we use again (7) in the form  $xc = cx + \partial(c)(1 + cx) = cx + (q - 1)(1 + cx) = q - 1 + cqx$  to obtain  $\mathbb{P}_1(x)(1 + cx) = q^{-1}x(1 + cx) = q^{-1}x + q^{-1}(xc)x = x + cx^2 = (1 + cx)x$ , as well as  $\mathbb{P}_2(x)(1 + xc) = xq^{-1}(1 + xc) = xq^{-1} + xq^{-1}(q - 1 + cqx) = x + xc = (1 + xc)x$ .  $\square$

### 2.5.5. Application to the cancellation of $1 + cx$

**Proposition 2.37.** *Let  $R$  be a commutative ring equipped with a quasi-derivation  $\partial$  of parameter  $c$ . We assume that the endomorphism  $\flat$  associated with  $\partial$  is bijective. Then the element  $1 + cx$  is left cancellable and right cancellable in the ring  $\mathcal{D}$  of formal quasi-differential operators.*

**Proof.** According to Corollary 2.35, the map  $\mathbb{P}_1$  is a bijection. From Equation (16), it suffices to show that  $1 + cx$  is left cancellable. Suppose we have  $(1 + cx)p = 0$  for a  $p \in \mathcal{D}$ . If  $p \neq 0$ , by Lemma 2.6, there is an integer  $i \in \mathbb{N}$  such that  $x^i$  is a right divisor of  $p$  and  $x^{i+1}$  is not a right divisor of  $p$ . So we have  $p = dx^i$  and  $(1 + cx)dx^i = 0$ . As  $x$  is right cancellable (Corollary 2.5), we can deduce  $(1 + cx)d = 0$ . By Equation (16), we obtain  $\mathbb{P}_1(d)(1 + cx) = 0$ . We write  $\mathbb{P}_1(d)$  using Equation (5), so that

$$\mathbb{P}_1(d) = \sum_{n \in \mathbb{N}} a_n x^n = a_0 + ex$$

by putting  $e = \sum_{n \geq 1} a_n x^{n-1} \in \mathcal{D}$ . We then have  $(a_0 + ex)(1 + cx) = 0$ , hence developing  $a_0 + ex + a_0 cx + excx = 0$ , that is  $a_0 + (e + a_0 c + ex)c = 0$ . Using again (5), we write  $e + a_0 c + exc = \sum_{n \in \mathbb{N}} b_n x^n$ , where the sequence  $(b_n)_{n \in \mathbb{N}}$  of elements of  $R$  has only a finite number of non-zero terms. Thus  $a_0 + \sum_{n \in \mathbb{N}} b_n x^{n+1} = 0$ , hence  $a_0 = 0$  by Property 2.2. Therefore  $\mathbb{P}_1(d) = ex$ , from which  $d = \mathbb{P}_1^{-1}(e)\flat'(q)x$  and  $p = \mathbb{P}_1^{-1}(e)\flat'(q)x^{i+1}$  contrary to the definition of  $i$ . This contradiction proves that  $1 + cx$  is left cancellable.  $\square$

## 3. Operators

It is now assumed that the base ring  $R$  is commutative. We start with a quasi-derivation  $\partial : R \rightarrow R$  of parameter  $c$ , and we put  $\mathcal{D} = \mathcal{D}_{c,\partial}$  for the ring of formal quasi-differential operators attached to this quasi-derivation. We suppose also that the endomorphism  $\flat$  of  $R$  associated with the quasi-derivation  $\partial$  is bijective, so that there is a conjugate quasi-derivation  $\partial' : R \rightarrow R$  of  $\partial$ . We then denote by  $\mathcal{D}' = \mathcal{D}_{c',\partial'}$  the ring of adjoint quasi-differential operators. We recall the notation  $q = 1 + \partial(c)$  for the rung of the quasi-derivation  $\partial$ .

### 3.1. The algebra of operators

#### 3.1.1. The fundamental filtration of the ring of f. q.-d. o.

We introduce the subset  $\mathcal{K}$  of  $\mathcal{D}$  whose elements are those of the form  $d(1 + cx)$ , where  $d$  is an element of  $\mathcal{D}$ . By using (16), we immediately see that  $\mathcal{K}$  is in fact a two-sided ideal of  $\mathcal{D}$ . According to the same relation (16), we also verify that, for all  $k \in \mathbb{N}$ , the power  $\mathcal{K}^k$  of exponent  $k$  of the ideal  $\mathcal{K}$  is the subset of elements of the form  $d(1 + cx)^k$ , for  $d$  in  $\mathcal{D}$ .

**Definition 3.1.** The FUNDAMENTAL FILTRATION of  $\mathcal{D}$  is the decreasing sequence

$$\mathcal{D} = \mathcal{K}^0 \supseteq \mathcal{K} \supseteq \mathcal{K}^2 \supseteq \dots \supseteq \mathcal{K}^k \supseteq \mathcal{K}^{k+1} \supseteq \dots \quad (18)$$

of two-sided ideals of  $\mathcal{D}$ .

**Proposition 3.2.** *The fundamental filtration of  $\mathcal{D}$  is invariant by the primary endomorphisms of  $\mathcal{D}$ , in the sense that, for every integer  $k \geq 0$ , we have the equalities*

$$\mathbb{P}_1(\mathcal{K}^k) = \mathcal{K}^k \quad \text{and} \quad \mathbb{P}_2(\mathcal{K}^k) = \mathcal{K}^k.$$

**Proof.** The set  $\mathcal{K}^k$  is precisely the two-sided ideal of  $\mathcal{D}$  generated by  $(1 + cx)^k$ . Therefore, its image by an automorphism of the ring  $\mathcal{D}$  is the two-sided ideal generated by the image of  $(1 + cx)^k$ . And we have  $\mathbb{P}_1(1 + cx) = 1 + cx$  and  $\mathbb{P}_2(1 + cx) = q(1 + cx)q^{-1}$ , and so  $\mathbb{P}_1((1 + cx)^k) = (1 + cx)^k$  and  $\mathbb{P}_2((1 + cx)^k) = q(1 + cx)^kq^{-1}$ . As  $q$  and  $q^{-1}$  are invertible, the two-sided ideal generated by  $\mathbb{P}_2((1 + cx)^k)$  coincides with that generated by  $(1 + cx)^k$ .  $\square$

### 3.1.2. Definitions and first examples

It is recalled that  ${}^\diamond\mathcal{D}$  is the left  $R$ -module where  $R$  acts on  $\mathcal{D}$  by left multiplication.

**Definition 3.3.** We call OPERATOR any endomorphism of the  $R$ -module  ${}^\diamond\mathcal{D}$  which stabilizes all terms of the fundamental filtration.

In other words, an operator is a map  $T : \mathcal{D} \rightarrow \mathcal{D}$  which is additive (that is  $T(p + q) = T(p) + T(q)$  for any pair  $(p, q)$  of elements of  $\mathcal{D}$ ), and which satisfies the identity

$$\forall (r, p) \in R \times \mathcal{D}, \quad T(rp) = rT(p),$$

and such that we have  $T(\mathcal{K}^k) \subseteq \mathcal{K}^k$  for every integer  $k \in \mathbb{N}$ .

We notice that there are two meanings for the word operator: it can be used for either elements of  $\mathcal{D}$  or endomorphisms of  ${}^\diamond\mathcal{D}$ . In fact, to avoid any confusion, we will systematically designate, as we said in the previous section, the elements of  $\mathcal{D}$  by the term formal quasi-differential operator, abbreviated f. q.-d. o.

It is clear that the set  $\mathcal{O}$  of all operators is a  $R$ -algebra when it is provided with its usual  $R$ -module structure and with the composition of mappings.

**Example 3.4.** If an endomorphism  $T$  of the ring  $\mathcal{D}$ , fixing all elements of  $R$ , is such that  $T(1 + cx) \in \mathcal{K}$ , then  $T$  is an operator. In particular, the translation operators  $E^a$  and  $F^a$  of Definition 2.21 are actually operators, since we have  $E^a(1 + cx) = (1 + ca)(1 + cx)$  and  $F^a(1 + cx) = (1 + cqb(a))(1 + cx)$ .

**Example 3.5.** Let  $r$  be an element of  $R$ . LEFT MULTIPLICATION by  $r$  is the map  $L_r : \mathcal{D} \rightarrow \mathcal{D}$  such that  $L_r(p) = rp$  for every element  $p$  of  $\mathcal{D}$ . It is an operator because  $R$  is assumed to be commutative.

**Example 3.6.** Let  $d$  be an element of  $\mathcal{D}$ . RIGHT MULTIPLICATION by  $d$  is the map  $M_d : \mathcal{D} \rightarrow \mathcal{D}$  such that  $M_d(p) = pd$  for every element  $p$  of  $\mathcal{D}$ . Since we know that all terms of the fundamental filtration are two-sided ideals, the map  $M_d$  is clearly an operator.

**Example 3.7.** LEFT MULTIPLICATION by  $x$  is the map  $L_x : \mathcal{D} \rightarrow \mathcal{D}$  such that  $L_x(p) = xp$  for every element  $p$  of  $\mathcal{D}$ . In general, this is not an operator because it is not always  $R$ -linear.

**Example 3.8.** Let  $r$  be an invertible element of the base ring  $R$ . The CONJUGATION determined by  $r$  is the map  $J_r : \mathcal{D} \rightarrow \mathcal{D}$  such that  $J_r(p) = rpr^{-1}$  for every element  $p$  of  $\mathcal{D}$ . It is an operator.

**Example 3.9.** The conjugation  $J_r$  is the composition  $J_r = M_{r^{-1}} \circ L_r = L_r \circ M_{r^{-1}}$ , which gives another way to justify that  $J_r$  is an operator.

**Lemma 3.10.** Let  $T$  be an operator. Then its conjugate by the first primary endomorphism  $\mathbb{P}_1^{-1} \circ T \circ \mathbb{P}_1 : \mathcal{D} \rightarrow \mathcal{D}$  is also an operator.

**Proof.** Indeed, we know by Proposition 3.2 that all terms of the fundamental filtration are stabilized by the automorphism  $\mathbb{P}_1$  and its inverse  $\mathbb{P}_1^{-1}$ .  $\square$

**Table 1**  
Conjugate operators.

$T$	$\mathfrak{p}_1^{-1} \circ T \circ \mathfrak{p}_1$
$E^a \ (a \in R)$	$E^{\mathfrak{p}'(aq^{-1})}$
$F^a = E^{\mathfrak{p}(a)q} \ (a \in R)$	$F^{\mathfrak{p}'(aq^{-1})} = E^a$
$L_r \ (r \in R)$	$L_{\mathfrak{p}'(r)}$
$M_d \ (d \in \mathcal{D})$	$M_{\mathfrak{p}_1^{-1}(d)}$
$J_r \ (r \in R^*)$	$J_{\mathfrak{p}'(r)}$

**Example 3.11.** Table 1 gives the expression of the conjugate by the first primary endomorphism of some remarkable operators.

### 3.2. Co-operators

#### 3.2.1. The fundamental filtration of the ring of adjoint quasi-differential operators

Since  $\mathcal{D}' = \mathcal{D}_{c', \partial'}$ , there is also a fundamental filtration in  $\mathcal{D}'$ , whose terms are powers of the two-sided ideal  $\mathcal{K}' = \mathcal{D}'(1 + c'x')$  of  $\mathcal{D}'$ .

**Proposition 3.12.** Let  $\Lambda : \mathcal{D} \rightarrow \mathcal{D}'$  be the adjunction, and  $\Lambda' : \mathcal{D}' \rightarrow \mathcal{D}$  its inverse. For any integer  $k$ , we have equalities

$$\Lambda(\mathcal{K}^k) = \mathcal{K}'^k \quad \text{and} \quad \Lambda'(\mathcal{K}'^k) = \mathcal{K}^k$$

**Proof.** It is immediate to verify that  $\Lambda(1 + cx) = 1 + x'c' = q'(1 + c'x')$  and  $\Lambda'(1 + c'x') = q(1 + cx)$ . We deduce for every integer  $k$  the inclusions  $\Lambda(\mathcal{K}^k) \subseteq \mathcal{K}'^k$  and  $\Lambda'(\mathcal{K}'^k) \subseteq \mathcal{K}^k$ . Since  $\Lambda$  and  $\Lambda'$  are two mutually inverse bijections, we deduce the inclusions in the reverse directions  $\mathcal{K}^k \subseteq \Lambda'(\mathcal{K}'^k)$  and  $\mathcal{K}'^k \subseteq \Lambda(\mathcal{K}^k)$ .  $\square$

#### 3.2.2. Definition and examples

It is recalled that  $\mathcal{D}'^\diamond$  is the  $R$ -module  $\mathcal{D}'$  where  $R$  acts on  $\mathcal{D}'$  by right multiplication.

**Definition 3.13.** Let  $R$  be a commutative ring equipped with a pair  $\{\partial, \partial'\}$  of conjugate quasi-derivations. We call CO-OPERATOR every endomorphism of the  $R$ -module  $\mathcal{D}'^\diamond$  which stabilizes all terms of the fundamental filtration of  $\mathcal{D}'$ .

In other words, a co-operator is a map  $T' : \mathcal{D}' \rightarrow \mathcal{D}'$  that is additive (that is  $T'(p' + q') = T'(p') + T'(q')$  for any ordered pair  $(p', q')$  of elements in  $\mathcal{D}'$ ), that satisfies the identity

$$\forall (r, p') \in R \times \mathcal{D}', \quad T'(p'r) = T'(p')r$$

and such that we have  $T'(\mathcal{K}'^k) \subseteq \mathcal{K}'^k$  for every integer  $k$ .

The set of co-operators is denoted by  $\mathcal{O}'$ . This is obviously an  $R$ -algebra when it is provided with its usual  $R$ -module structure and the composition of maps.

**Example 3.14.** The translations  $E'^a : \mathcal{D}' \rightarrow \mathcal{D}'$  and  $F'^a : \mathcal{D}' \rightarrow \mathcal{D}'$  defined on  $\mathcal{D}'$  according to Definition 2.21 are co-operators.

**Example 3.15.** Let  $r$  be an element of  $R$ . The map  $M'_r : \mathcal{D}' \rightarrow \mathcal{D}'$  such that  $M'_r(p') = p'r$  for any element  $p'$  of  $\mathcal{D}'$ , is a co-operator because the ring  $R$  is assumed to be commutative.

**Example 3.16.** The RIGHT MULTIPLICATION by  $x'$  is the map  $M'_{x'} : \mathcal{D}' \rightarrow \mathcal{D}'$  such that  $M'_{x'}(p') = p'x'$  for any element  $p'$  of  $\mathcal{D}'$ . This is not usually a co-operator.

**Table 2**  
Caron transforms.

$T = T'^{\vee}$	$T' = T^{\vee}$
$E^a \ (a \in R)$	$F'^{-a}$
$F^a = E^{b(a)q} \ (a \in R)$	$E'^{-a}$
$L_d \ (d \in \mathcal{D})$	$M'_{\Lambda(d)}$
$M_d \ (d \in \mathcal{D})$	$L'_{\Lambda(d)}$
$J_r \ (r \in R^*)$	$J'^{-1}_r$
$P_1$	$P'^{-1}_2$
$P_2$	$P'^{-1}_1$

**Example 3.17.** Let  $d' \in \mathcal{D}'$ . The LEFT MULTIPLICATION by  $d'$  is the map  $L'_{d'} : \mathcal{D}' \rightarrow \mathcal{D}'$  such that  $L'_{d'}(p') = d'p'$  for any element  $p'$  of  $\mathcal{D}'$ . It is a co-operator.

**Example 3.18.** Let  $r$  be an invertible element of  $R$ . The conjugation of  $\mathcal{D}'$  determined by  $r$  is the map  $J'_r : \mathcal{D}' \rightarrow \mathcal{D}'$  such that  $J'_r(p') = rp'r^{-1}$  for any  $p'$  of  $\mathcal{D}'$ . It is a co-operator because the ring  $R$  is commutative.

Applying to the ring  $\mathcal{D}' = \mathcal{D}_{C', \mathcal{D}'}$  the construction of primary endomorphisms, we define in particular an endomorphism  $P'_2$  of this ring extending the endomorphism  $p'$  of  $R$  and such that  $P'_2(x') = x'q'^{-1}$ .

**Lemma 3.19.** For any co-operator  $T'$ , its conjugate  $P'_2 \circ T' \circ P'^{-1}_2$  is a co-operator.

**Proof.** Results from Proposition 3.2.  $\square$

### 3.2.3. The caron transformation

**Definition 3.20.** Let  $A : \mathcal{D} \rightarrow \mathcal{D}$  (respectively  $A' : \mathcal{D}' \rightarrow \mathcal{D}'$ ) be an additive map. The CARON TRANSFORM of  $A$  (respectively of  $A'$ ) is the map  $A^{\vee} : \mathcal{D}' \rightarrow \mathcal{D}'$  (respectively  $A'^{\vee} : \mathcal{D} \rightarrow \mathcal{D}$ ) such that  $A^{\vee} = \Lambda \circ A \circ \Lambda'$  (respectively  $A'^{\vee} = \Lambda' \circ A' \circ \Lambda$ ).

**Property 3.21.** (i) If  $A$  and  $B$  are two additive maps from  $\mathcal{D}$  to  $\mathcal{D}$ , or from  $\mathcal{D}'$  to  $\mathcal{D}'$ , we have  $(A \circ B)^{\vee} = A^{\vee} \circ B^{\vee}$ .

(ii) If  $A$  is an additive map from  $\mathcal{D}$  to  $\mathcal{D}$ , or from  $\mathcal{D}'$  to  $\mathcal{D}'$ , then  $(A^{\vee})^{\vee} = A$ .

(iii) The caron transform of an operator is a co-operator, and the caron transform of a co-operator is an operator.

**Proof.** For the demonstration of (iii) just use Proposition 3.12.  $\square$

**Example 3.22.** Table 2 gives the caron transforms of the most common additive maps. For any f. q.-d. o.  $d$ , we use the notation  $L_d$  for the map  $p \mapsto dp$  from  $\mathcal{D}$  to  $\mathcal{D}$ , and the notation  $M_d$  for the map  $p \mapsto pd$ . Similarly, for any element  $d'$  of the ring  $\mathcal{D}'$ , the maps  $L'_{d'} : \mathcal{D}' \rightarrow \mathcal{D}'$  and  $M'_{d'} : \mathcal{D}' \rightarrow \mathcal{D}'$  are defined by  $L'_{d'}(p') = d'p'$  and  $M'_{d'}(p') = p'd'$ .

It is also recalled that  $E^a$  and  $F^a$  are the translation operators defined in Subsection 2.3 of the above Section. The notations  $P_1$  and  $P_2$  represent the two primary endomorphisms of  $\mathcal{D}$  introduced in Subsection 2.5 of the same Section. In the same way, we define endomorphisms  $P'_1$  and  $P'_2$  of  $\mathcal{D}'$  which extends the endomorphism  $p'$  of  $R$ , and such that  $P'_1(x') = q'^{-1}x'$  and  $P'_2(x') = x'q'^{-1}$ .

### 3.3. Pincherle operator calculus

The Pincherle calculus was initiated in 1897 [18, p. 352] in the context of operators on an ordinary polynomial ring, but received no name until 1973 [24]. The case of operators on a ring of formal differential

operators has been dealt with by Héraoua [9, pp. 33–39]. In the latter context, due to the non-commutativity of the ring of formal differential operators, it has been necessary to split the Pincherle derivative into two different operations. In the new case studied in the present work, that is to say that of operators on a ring of formal quasi-differential operators, we will no longer be dealing with derivatives, but with pseudo-derivatives.

### 3.3.1. The two particular endomorphisms of the ring of operators

**Lemma 3.23.** *For every operator  $T$  and for every integer  $k$ , there exists a unique endomorphism  $U_k$  of  $R$ -module  ${}^\diamond\mathcal{D}$  such that  $T \circ M_{(1+cx)^k} = M_{(1+cx)^k} \circ U_k$ .*

*For any pair  $(k, h)$  of integers, we have the equality  $U_k \circ M_{(1+cx)^h} = M_{(1+cx)^h} \circ U_{k+h}$ .*

*Moreover, the endomorphism  $U_k$  is an operator.*

**Proof.** The uniqueness of the endomorphism  $U_k$  stems from the fact that the multiplication operator  $M_{(1+cx)^k}$  is injective, that is to say that the element  $(1+cx)^k$  of  $\mathcal{D}$  is right cancellable, which is deduced immediately from the property of  $1+cx$  stated in Proposition 2.37.

Let us show now the existence of  $U_k$ . By hypothesis, the map  $T$  is an operator, and therefore  $T(\mathcal{K}^k) \subseteq \mathcal{K}^k$ . As a result, for every integer  $n$ , there exists in  $\mathcal{D}$  an element  $p_{k,n}$  such that  $T(x^n(1+cx)^k) = p_{k,n}(1+cx)^k$ . Since the sequence  $(x^n)_{n \in \mathbb{N}}$  is a basis of the  $R$ -module  ${}^\diamond\mathcal{D}$ , we deduce that there exists a unique endomorphism  $U_k$  of  ${}^\diamond\mathcal{D}$  such that  $U_k x^n = p_{k,n}$ . The endomorphisms  $T \circ M_{(1+cx)^k}$  and  $M_{(1+cx)^k} \circ U_k$ , taking the same values in all elements  $x^n$  of the basis, are therefore equal. This completes the proof of the first statement of our Lemma.

To justify the second statement, the composition  $T \circ M_{(1+cx)^{k+h}}$  is calculated in the two following ways. On the one hand, by definition of  $U_{k+h}$ , we have

$$T \circ M_{(1+cx)^{k+h}} = M_{(1+cx)^{k+h}} \circ U_{k+h} = M_{(1+cx)^k} \circ M_{(1+cx)^h} \circ U_{k+h}.$$

On the other hand, we have

$$T \circ M_{(1+cx)^{k+h}} = T \circ M_{(1+cx)^k} \circ M_{(1+cx)^h} = M_{(1+cx)^k} \circ U_k \circ M_{(1+cx)^h}.$$

We deduce  $M_{(1+cx)^k} \circ U_k \circ M_{(1+cx)^h} = M_{(1+cx)^k} \circ M_{(1+cx)^h} \circ U_{k+h}$ . As the map  $M_{(1+cx)^k}$  is injective, we get the desired equality.

The third statement clearly follows from the second, since, for every integer  $h$ , the two-sided ideal  $\mathcal{K}^h$  is exactly the image of the map  $M_{(1+cx)^h}$ .  $\square$

**Definition 3.24.** The FIRST PARTICULAR ENDOMORPHISM of the ring  $\mathcal{O}$  is the map  $\psi_1 : \mathcal{O} \rightarrow \mathcal{O}$  sending any operator  $T$  to operator  $U_1$  of Lemma 3.23.

In other words,  $\psi_1(T)$  is the unique endomorphism of the  $R$ -module  ${}^\diamond\mathcal{D}$  such that

$$T \circ M_{(1+cx)} = M_{(1+cx)} \circ \psi_1(T) \quad (19)$$

It is easy to make sure that  $\psi_1$  is an endomorphism of the ring  $\mathcal{O}$  of operators.

On the other hand, we know from Lemma 3.10 that, whatever the operator  $T$ , the map  $P_1^{-1} \circ \psi_1(T) \circ P_1 : \mathcal{D} \rightarrow \mathcal{D}$  is an operator. We are thus led to the following definition.

**Definition 3.25.** The SECOND PARTICULAR ENDOMORPHISM of the ring  $\mathcal{O}$  is the map  $\psi_2 : \mathcal{O} \rightarrow \mathcal{O}$  mapping any operator  $T$  to the operator  $\psi_2(T) = P_1^{-1} \circ \psi_1(T) \circ P_1$ .

**Table 3**

The two particular endomorphisms.

$T$	$\Psi_1(T)$	$\Psi_2(T)$
$E^a$ ( $a \in R$ )	$M_{1+ca} \circ E^a$	$L_{\mathfrak{p}'(1+ca)} \circ E^a = M_{\mathfrak{p}'(1+ca)} \circ E^{\mathfrak{p}'(aq^{-1})}$
$F^a = E^{\mathfrak{p}(a)q}$ ( $a \in R$ )	$M_{1+cq\mathfrak{p}(a)} \circ F^a$	$L_{1+ca} \circ F^a = M_{1+ca} \circ E^a$
$L_r$ ( $r \in R$ )	$L_r$	$L_{\mathfrak{p}'(r)}$
$M_d$ ( $d \in \mathcal{D}$ )	$M_{\mathfrak{p}_1(d)}$	$M_d$
$J_r$ ( $r \in R^*$ )	$L_r \circ M_{\mathfrak{p}(r^{-1})}$	$L_{\mathfrak{p}'(r)} \circ M_{r^{-1}}$

It is immediate that  $\psi_2$  is an endomorphism of the ring  $\mathcal{O}$ . We will now give a characterization of the operator  $\psi_2(T)$  analogous to Equation (19). For this, we introduce the left multiplication  $L_{1+cx} : \mathcal{D} \rightarrow \mathcal{D}$  defined by

$$\forall p \in \mathcal{D}, \quad L_{1+cx}(p) = (1+cx)p.$$

According to Equation (16), we have

$$L_{1+cx} = M_{1+cx} \circ \mathfrak{p}_1 \quad (20)$$

In particular, we see that  $L_{1+cx}$  is injective.

**Proposition 3.26.** *Let  $T \in \mathcal{O}$  be an operator. Then  $\psi_2(T)$  is the unique map from  $\mathcal{D}$  to  $\mathcal{D}$  such that*

$$T \circ L_{1+cx} = L_{1+cx} \circ \psi_2(T). \quad (21)$$

**Proof.** Uniqueness is deduced from the injectivity of  $L_{1+cx}$ . We can easily check (21) from (20).  $\square$

**Example 3.27.** Table 3 lists the images by the two particular endomorphisms of some remarkable operators.

**Proposition 3.28.** *The two particular endomorphisms of the ring  $\mathcal{O}$  of operators commute, in the sense that*

$$\forall T \in \mathcal{O}, \quad \psi_1(\psi_2(T)) = \psi_2(\psi_1(T)).$$

**Proof.** This results from the fact that the maps  $M_{1+cx}$  and  $L_{1+cx}$  commute, using Equations (19) and (21), as well as the injectivity of the two mappings  $M_{1+cx}$  and  $L_{1+cx}$ .  $\square$

### 3.3.2. The two Pincherle pseudo-derivatives

We recall the definition of the two maps  $M_x$  and  $L_x$  from  $\mathcal{D}$  to  $\mathcal{D}$  defined by  $M_x(d) = dx$  and  $L_x(d) = xd$  for any element  $d \in \mathcal{D}$ . We have seen that  $M_x$  is an operator, but that  $L_x$  generally is not an operator.

**Definition 3.29.** Let  $T$  be an operator. The FIRST PINCHERLE PSEUDO-DERIVATIVE of  $T$  is the operator  $\partial_1(T) = T \circ M_x - M_x \circ \psi_1(T)$ .

It is clear that  $\partial_1(T)$  is an operator, because it is the difference of two compositions of operators.

We obviously have  $M_{1+cx} \circ \partial_1(T) = M_{1+cx} \circ T \circ M_x - M_{x(1+cx)} \circ \psi_1(T)$ . Since  $x(1+cx) = (1+cx)\mathfrak{p}'(q)x$  by Equation (16), we find that the operator  $\partial_1(T)$  is characterized by the formula

$$M_{1+cx} \circ \partial_1(T) = M_{1+cx} \circ T \circ M_x - M_{\mathfrak{p}'(q)x} \circ T \circ M_{1+cx}. \quad (22)$$

**Proposition 3.30.** *For any operator  $T$ , the map  $L_x \circ \psi_2(T) - T \circ L_x$  is an operator.*

**Table 4**  
The two Pincherle pseudo-derivatives.

$T$	$\partial_1(T)$	$\partial_2(T)$
$E^a$ ( $a \in R$ )	$M_a \circ E^a$	$L_{-\mathfrak{p}'(aq^{-1})} \circ E^a$
$F^a = E^{\mathfrak{b}(a)q}$ ( $a \in R$ )	$M_{\mathfrak{b}(a)q} \circ F^a$	$L_{-a} \circ F^a$
$L_r$ ( $r \in R$ )	0	$L_{\partial'(r)}$
$M_d$ ( $d \in \mathcal{D}$ )	$M_{xd - \mathfrak{p}_1(d)x}$	0
$J_r$ ( $r \in R^*$ )	$L_r \circ M_{\partial(r^{-1})}$	$L_{\partial'(r)} \circ M_{r^{-1}}$

**Proof.** The operators  $T$  and  $\psi_2(T)$  stabilize all terms  $\mathcal{K}^k$  of the fundamental filtration; the map  $L_x$  has the same property since  $\mathcal{K}^k$  is a two-sided ideal of  $\mathcal{D}$ . Therefore, the map  $L_x \circ \psi_2(T) - T \circ L_x$  also stabilizes these sets  $\mathcal{K}^k$ . Moreover, one verifies the  $R$ -linearity of this map by a computation using the  $R$ -linearity of  $\Psi_2(T)$  and Equations (7) and (21).  $\square$

**Definition 3.31.** Let  $T$  be an operator. The SECOND PINCHERLE PSEUDO-DERIVATIVE of  $T$  is the operator  $\partial_2(T) = L_x \circ \psi_2(T) - T \circ L_x$ .

Composing the two members of the equality  $\partial_2(T) = L_x \circ \psi_2(T) - T \circ L_x$  with the map  $L_{1+cx}$ , we obtain

$$L_{1+cx} \circ \partial_2(T) = L_{1+cx} \circ L_x \circ \psi_2(T) - L_{1+cx} \circ T \circ L_x.$$

By Equation (16), we have  $(1+cx)x = q^{-1}x(1+cx)$  and so  $L_{1+cx} \circ L_x = L_{q^{-1}x} \circ L_{1+cx}$ . By Equation (21), we deduce that, for every operator  $T$ , we have

$$L_{1+cx} \circ \partial_2(T) = L_{q^{-1}x} \circ T \circ L_{1+cx} - L_{1+cx} \circ T \circ L_x. \quad (23)$$

**Proposition 3.32.** The maps  $\partial_1 : \mathcal{O} \rightarrow \mathcal{O}$  and  $\partial_2 : \mathcal{O} \rightarrow \mathcal{O}$  are pseudo-derivations of the opposite ring of the ring  $\mathcal{O}$ , with respect to  $\psi_1$  and  $\psi_2$  respectively.

**Proof.** The map  $\partial_1$  is, by its very definition, an inner  $\psi_1$ -derivation [2, Subsection 1.1] of the ring  $\mathcal{O}^{op}$  opposite to the ring  $\mathcal{O}$ .

Regarding  $\partial_2$ , this is not an inner  $\psi_2$ -derivation of the opposite ring  $\mathcal{O}^{op}$ , because  $L_x$  is not usually an operator. However, by straightforward computation, it is possible to verify that  $\partial_2(T \circ_{op} U) = \psi_2(T) \circ_{op} \partial_2(U) + \partial_2(T) \circ_{op} U$ .  $\square$

**Proposition 3.33.** The two pseudo-derivations  $\partial_1$  and  $\partial_2$  commute, that is to say that

$$\partial_1(\partial_2(T)) = \partial_2(\partial_1(T)) \quad (24)$$

for any operator  $T$ .

**Proof.** Since any left multiplication commutes with any right multiplication, using (22) and (23), we check that  $L_{1+cx} \circ M_{1+cx} \circ \partial_1(\partial_2(T)) = M_{1+cx} \circ L_{1+cx} \circ \partial_2(\partial_1(T))$ . Since the maps  $L_{1+cx}$  and  $M_{1+cx}$  both commute and are injective, we deduce the desired result.  $\square$

**Example 3.34.** Table 4 gives the Pincherle pseudo-derivatives of some remarkable operators.



#### 4. The coalgebra of formal quasi-differential operators

##### 4.1. The coalgebra structure

##### 4.1.1. The augmentation

**Definition 4.1.** The AUGMENTATION of  $\mathcal{D}$  is the unique map  $\varepsilon : \mathcal{D} \rightarrow R$  that is left  $\mathcal{D}$ -linear from  $\mathcal{D}$  to  $R_0$  (cf. Section 2, 2.2.1) and such that  $\varepsilon(1) = 1$ .

In particular, if  $n$  is an integer, we have  $\varepsilon(x^n) = \varepsilon(x^n 1) = x^n \cdot 1 = \partial^n(1)$ , hence

$$\varepsilon(x^n) = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n \geq 1 \end{cases}$$

We have the identity

$$\forall p \in \mathcal{D}, \forall r \in R, \quad \varepsilon(pr) = p \cdot r \quad (25)$$

which translates the definition of  $\varepsilon$ . In particular, the equation

$$\varepsilon(p) = p \cdot 1,$$

could also serve as a definition for the map  $\varepsilon : \mathcal{D} \rightarrow R$ .

##### 4.1.2. The diagonalization

**Definition 4.2.** The DIAGONALIZATION (or DIAGONAL MAP) of  $\mathcal{D}$  is the unique map  $\Delta : \mathcal{D} \rightarrow {}^\circ\mathcal{D} \otimes {}^\circ\mathcal{D}$  that is left  $\mathcal{D}$ -linear from  $\mathcal{D}$  into  $(\mathcal{D} \otimes {}^\circ\mathcal{D}, \triangleleft)$  and such that  $\Delta(1) = 1 \otimes 1$ .

In other words,  $\Delta$  is the map of source  $\mathcal{D}$  and target  $\mathcal{D} \otimes {}^\circ\mathcal{D}$  defined by

$$\forall p \in \mathcal{D}, \quad \Delta(p) = p \triangleleft (1 \otimes 1).$$

Recall here the definition [4] of *Gaussian binomial coefficients*  $\begin{bmatrix} n \\ k \end{bmatrix}_q$  where  $q \in R$  and  $n$  and  $k \leq n$  are integers. These symbols are recursively defined by

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_q = q^k \begin{bmatrix} n \\ k \end{bmatrix}_q + \begin{bmatrix} n \\ k-1 \end{bmatrix}_q$$

with  $\begin{bmatrix} n \\ 0 \end{bmatrix}_q = 1 = \begin{bmatrix} n \\ n \end{bmatrix}_q$ . By induction, we immediately see that  $\begin{bmatrix} n \\ k \end{bmatrix}_q$  is a polynomial function of  $q$ , whose degree is  $k(n-k)$ .

**Proposition 4.3.** If  $\partial^2(c) = 0$ , let  $q = 1 + \partial(c)$ . For all integers  $n$ ,

$$\Delta(x^n) = \sum_{j=0}^n \begin{bmatrix} n \\ j \end{bmatrix}_q (cx + 1)^{n-j} x^j \otimes x^{n-j}.$$

**Proof.** By induction on  $n$ .  $\square$

**Corollary 4.4.** *If  $\partial(c) = 0$ , then, for any integer  $n$ , we have*

$$\Delta(x^n) = \sum_{j=0}^n \binom{n}{j} (cx + 1)^{n-j} x^j \otimes x^{n-j}.$$

**Proposition 4.5.** *The diagonalization  $\Delta$  is coassociative.*

**Proof.** Let  $I$  be the identity map of  $\mathcal{D}$ . We identify the two modules  $({}^\circ\mathcal{D} \otimes {}^\circ\mathcal{D}) \otimes {}^\circ\mathcal{D}$  and  ${}^\circ\mathcal{D} \otimes ({}^\circ\mathcal{D} \otimes {}^\circ\mathcal{D})$  to the tensor cube  $({}^\circ\mathcal{D})^{\otimes 3}$ , and we want to show equality  $(\Delta \otimes I) \circ \Delta = (I \otimes \Delta) \circ \Delta$ , or more exactly

$$\mathbf{v}_1 \circ (\Delta \otimes I) \circ \Delta = \mathbf{v}_2 \circ (I \otimes \Delta) \circ \Delta, \quad (26)$$

where the isomorphism  $\mathbf{v}_1$  goes from  $({}^\circ\mathcal{D} \otimes {}^\circ\mathcal{D}) \otimes {}^\circ\mathcal{D}$  to  $({}^\circ\mathcal{D})^{\otimes 3}$ , while isomorphism  $\mathbf{v}_2$  goes from  ${}^\circ\mathcal{D} \otimes ({}^\circ\mathcal{D} \otimes {}^\circ\mathcal{D})$  to  $({}^\circ\mathcal{D})^{\otimes 3}$ . We equip  $\mathcal{D}$  with its usual left  $\mathcal{D}$ -module structure, and we provide the abelian groups  ${}^\circ\mathcal{D} \otimes {}^\circ\mathcal{D}$ ,  $({}^\circ\mathcal{D} \otimes {}^\circ\mathcal{D}) \otimes {}^\circ\mathcal{D}$ ,  ${}^\circ\mathcal{D} \otimes ({}^\circ\mathcal{D} \otimes {}^\circ\mathcal{D})$ ,  $({}^\circ\mathcal{D})^{\otimes 3}$ , with their left  $\mathcal{D}$ -module structures which are deduced from that of  $\mathcal{D}$  according to Proposition 2.16. By definition,  $\Delta$  is  $\mathcal{D}$ -linear for these structures. By Proposition 2.18, it is the same for the maps  $\Delta \otimes I$  and  $I \otimes \Delta$ . By Proposition 2.19, the isomorphisms  $\mathbf{v}_1$  and  $\mathbf{v}_2$  are  $\mathcal{D}$ -linear too. Therefore, the members of Equation (26) are both homomorphisms of left  $\mathcal{D}$ -modules. Since they give to the element 1 the same image  $1 \otimes 1 \otimes 1$ , we conclude that they are equal, which shows that  $\Delta$  is coassociative.  $\square$

**Proposition 4.6.** *The diagonalization  $\Delta$  is cocommutative.*

**Proof.** Let  $\Upsilon : {}^\circ\mathcal{D} \otimes {}^\circ\mathcal{D} \rightarrow {}^\circ\mathcal{D} \otimes {}^\circ\mathcal{D}$  be the  $R$ -linear map such that  $\Upsilon(d \otimes e) = e \otimes d$  for every ordered pair  $(d, e) \in \mathcal{D}^2$ . We want to show that  $\Upsilon \circ \Delta = \Delta$ . Since  $(\Upsilon \circ \Delta)(1) = \Delta(1)$ , it suffices to show that the map  $\Upsilon$  is left  $\mathcal{D}$ -linear. According to Corollary 2.4, we have simply to show that the biggest subring  $E$  of  $\mathcal{D}$  such that  $\Upsilon$  is  $E$ -linear includes the element  $x$ . We are thus reduced to verify the equality  $\Upsilon \circ \Gamma = \Gamma$ , where  $\Gamma$  is the unique endomorphism of the additive group of  ${}^\circ\mathcal{D} \otimes {}^\circ\mathcal{D}$  such that

$$\forall (p, q) \in \mathcal{D}^2, \quad \Gamma(p \otimes q) = xp \otimes q + p \otimes xq + cxp \otimes xq.$$

By the universal property of tensor product, it is therefore sufficient to verify that  $(\Upsilon \circ \Gamma)(d \otimes e) = \Gamma(d \otimes e)$  for all ordered pair  $(d, e) \in \mathcal{D}^2$ , which is immediate.  $\square$

#### 4.1.3. The coalgebra of formal quasi-differential operators

Recall that the notation  $I$  designates the identity mapping of  $\mathcal{D}$ . We denote by  $\mathbf{f} : \mathcal{D} \rightarrow {}^\circ\mathcal{D} \otimes_R R$  and  $\mathbf{g} : \mathcal{D} \rightarrow R \otimes_R {}^\circ\mathcal{D}$  the canonical isomorphisms such as

$$\forall p \in \mathcal{D}, \quad \mathbf{f}(p) = p \otimes 1 \quad \text{and} \quad \mathbf{g}(p) = 1 \otimes p.$$

**Proposition 4.7.** *We have the equalities  $\mathbf{f} = (I \otimes \varepsilon) \circ \Delta$  and  $\mathbf{g} = (\varepsilon \otimes I) \circ \Delta$ .*

**Proof.** It is enough to see that the two members of each of these two equalities are left  $\mathcal{D}$ -linear (this results from Proposition 2.18) and send 1 to the same image.  $\square$

We thus see that the augmentation  $\varepsilon$  is a counity for the diagonalization  $\Delta$ . We can therefore sum up the preceding exposition by the following statement, which provides us with a proof of Theorem 0.1.

**Proposition 4.8.** *The triple  $({}^\diamond\mathcal{D}, \Delta, \varepsilon)$  is a cocommutative  $R$ -coalgebra.*

**Notation 4.9.** The coalgebra  $({}^\diamond\mathcal{D}, \Delta, \varepsilon)$  is denoted by  $\mathcal{D}_{cog}$ . It is called the COALGEBRA OF FORMAL QUASI-DIFFERENTIAL OPERATORS.

#### 4.1.4. Dualization

We now consider the situation where the quasi-derivation  $\partial$  admits a conjugate quasi-derivation  $\partial'$ . In this case, we have built in Section 2, Subsection 2.4, two mutually inverse antimorphisms  $\Lambda : \mathcal{D} \rightarrow \mathcal{D}'$  and  $\Lambda' : \mathcal{D}' \rightarrow \mathcal{D}$  between the ring  $\mathcal{D}$  of formal quasi-differential operators and the ring  $\mathcal{D}'$  of adjoint quasi-differential operators. Since  $\Lambda$  leaves all elements of  $R$  fixed, it establishes an isomorphism of  $R$ -modules, between, on the one hand, the  $R$ -module  ${}^\diamond\mathcal{D}$  obtained by making the elements of  $R$  act on  $\mathcal{D}$  by left multiplication, and on the other hand, the  $R$ -module  $\mathcal{D}'^\diamond$  obtained by making the elements of  $R$  act on  $\mathcal{D}'$  by right multiplication. It follows that  $\Lambda \otimes \Lambda$  is a homomorphism of  ${}^\diamond\mathcal{D} \otimes {}^\diamond\mathcal{D}$  over  $\mathcal{D}'^\diamond \otimes \mathcal{D}'^\diamond$ , whose inverse isomorphism is  $\Lambda' \otimes \Lambda'$ . We define on  $\mathcal{D}'$  the diagonalization  $\Delta' : \mathcal{D}' \rightarrow \mathcal{D}'^\diamond \otimes \mathcal{D}'^\diamond$  by putting

$$\Delta' = (\Lambda \otimes \Lambda) \circ \Delta \circ \Lambda',$$

and augmentation  $\varepsilon' : \mathcal{D}' \rightarrow R$  by putting

$$\varepsilon' = \varepsilon \circ \Lambda'.$$

**Lemma 4.10.** *The maps  $\Delta' : \mathcal{D}' \rightarrow \mathcal{D}'^\diamond \otimes \mathcal{D}'^\diamond$  and  $\varepsilon' : \mathcal{D}' \rightarrow R$  are homomorphisms of right  $\mathcal{D}'$ -modules.*

**Proof.** We equip  $\mathcal{D}'$  with its usual structure of right  $\mathcal{D}'$ -module, and thus of left  $\mathcal{D}$ -module (cf. Proposition 2.28). This structure of left  $\mathcal{D}$ -module allows, according to the process described in Section 2, Subsubsection 2.2.2, to put on the tensor product  $\mathcal{D}'^\diamond \otimes \mathcal{D}'^\diamond$  a structure of left  $\mathcal{D}$ -module, and thus of right  $\mathcal{D}'$ -module, given by a ring antimorphism  $\mathbf{g}'$  from  $\mathcal{D}'$  into the ring  $\text{End}_{\mathbb{Z}}(\mathcal{D}'^\diamond \otimes \mathcal{D}'^\diamond)$  of endomorphisms of the abelian group  $\mathcal{D}'^\diamond \otimes \mathcal{D}'^\diamond$ . According to the constructions described in Section 2, this homomorphism  $\mathbf{g}'$  is such that

$$\forall a \in R, \forall (p', q') \in \mathcal{D}'^2, \quad \mathbf{g}'(a)(p' \otimes q') = p'a \otimes q' = p' \otimes q'a$$

and

$$\forall (p', q') \in \mathcal{D}'^2, \quad \mathbf{g}'(x')(p' \otimes q') = p'x' \otimes q' + p' \otimes q'x' + p'x'c' \otimes q'x',$$

where  $c' = -c$  is the parameter of the conjugate quasi-derivation  $\partial'$ . Since it is immediate that  $\Lambda$  is a homomorphism of left  $\mathcal{D}$ -modules from  $\mathcal{D}$  into  $\mathcal{D}'$ , it follows from Proposition 2.18 that  $\Lambda \otimes \Lambda$  is left  $\mathcal{D}$ -linear from  ${}^\diamond\mathcal{D} \otimes {}^\diamond\mathcal{D}$  to  $\mathcal{D}'^\diamond \otimes \mathcal{D}'^\diamond$ . Moreover, the inverse  $\Lambda'$  of  $\Lambda$  is a left  $\mathcal{D}$ -linear map from  $\mathcal{D}'$  into  $\mathcal{D}$ . Therefore, the diagonalization  $\Delta'$  is, as composition of three left  $\mathcal{D}$ -linear maps, a homomorphism of right  $\mathcal{D}'$ -modules. Similarly the map  $\varepsilon'$  is a homomorphism of right  $\mathcal{D}'$ -modules as composition of two homomorphisms of left  $\mathcal{D}$ -modules.  $\square$

**Proposition 4.11.** *The triple  $(\mathcal{D}'^\diamond, \Delta', \varepsilon')$  is a cocommutative  $R$ -coalgebra.*

**Proof.** We denote by  $I'$  the identity map of  $\mathcal{D}'$ ,  $\mathbf{v}'_1$  the usual isomorphism from  $(\mathcal{D}'^\diamond \otimes \mathcal{D}'^\diamond) \otimes \mathcal{D}'^\diamond$  to the tensor cube  $(\mathcal{D}'^\diamond)^{\otimes 3}$ ,  $\mathbf{v}'_2$  that from  $\mathcal{D}'^\diamond \otimes (\mathcal{D}'^\diamond \otimes \mathcal{D}'^\diamond)$  to  $(\mathcal{D}'^\diamond)^{\otimes 3}$ ,  $\mathbf{f}'$  the homomorphism from  $\mathcal{D}'$  to  $\mathcal{D}'^\diamond \otimes R$  which sends  $p'$  to  $p' \otimes 1$ ,  $\mathbf{g}'$  the homomorphism from  $\mathcal{D}'$  to  $R \otimes \mathcal{D}'^\diamond$  which sends  $p'$  to  $1 \otimes p'$ , and finally  $\Upsilon'$  the endomorphism of the tensor square  $\mathcal{D}'^\diamond \otimes \mathcal{D}'^\diamond$  which sends  $p' \otimes q'$  to  $q' \otimes p'$ . We have to show the relationships

$$v'_1 \circ (\Delta' \otimes I') \circ \Delta' = v'_2 \circ (I' \otimes \Delta') \circ \Delta',$$

which express that diagonalization  $\Delta'$  is coassociative, as well as

$$f' = (I' \otimes \varepsilon') \circ \Delta' \quad \text{and} \quad g' = (\varepsilon' \otimes I') \circ \Delta',$$

which mean that  $\varepsilon'$  is a counity relative to comultiplication  $\Delta'$ , and finally

$$\Upsilon' \circ \Delta' = \Delta',$$

translating the cocommutativity of the comultiplication  $\Delta'$ . All these relations are verified by observing that their two members are on the one hand homomorphisms of right  $\mathcal{D}'$ -modules from  $\mathcal{D}'$  towards a same right  $\mathcal{D}'$ -module (this can be justified by Propositions 2.18 and 2.19, considering the category isomorphism of Proposition 2.28), and on the other hand send 1 to the same image.  $\square$

**Notation 4.12.** The coalgebra  $(\mathcal{D}'^\diamond, \Delta', \varepsilon')$  is denoted by  $\mathcal{D}'_{cog}$ . It is called the COALGEBRA OF ADJOINT QUASI-DIFFERENTIAL OPERATORS.

**Proposition 4.13.** The adjunction  $\Lambda$  is an isomorphism from the coalgebra  $\mathcal{D}_{cog}$  to the coalgebra  $\mathcal{D}'_{cog}$ .

**Proof.** We know that the map  $\Lambda$  is bijective, and that its inverse is  $\Lambda'$ . The very definitions of the diagonalization  $\Delta'$  and of the augmentation  $\varepsilon'$  mean that

$$\Delta' \circ \Lambda = (\Lambda \otimes \Lambda) \circ \Delta, \quad \text{and} \quad \varepsilon' \circ \Lambda = \varepsilon,$$

identities which express that  $\Lambda$  is a coalgebra homomorphism from  $\mathcal{D}_{cog}$  to  $\mathcal{D}'_{cog}$ .  $\square$

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