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On residual and stable coordinates

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ABSTRACT

In a recent paper [10], M.E. Kahoui and M. Ouali have proved that over an algebraically closed field k of characteristic zero, residual coordinates in $k[X][Z_1, \dots, Z_n]$ are one-stable coordinates. In this paper we extend their result to the case of an algebraically closed field k of arbitrary characteristic. In fact, we show that the result holds when $k[X]$ is replaced by any one-dimensional seminormal domain R which is affine over an algebraically closed field k. For our proof, we extend a result of S. Maubach in [11] giving a criterion for a polynomial of the form $a(X)W + P(X, Z_1, \dots, Z_n)$ to be a coordinate in $k[X][Z_1, \dots, Z_n, W]$. Kahoui and Ouali had also shown that over a Noetherian d-dimensional ring R containing $\mathbb{Q}$ any residual coordinate in $R[Z_1, \dots, Z_n]$ is an r-stable coordinate, where $r = (2^d - 1)n$. We will give a sharper bound for r when R is affine over an algebraically closed field of characteristic zero.

1. Introduction

We will assume all rings to be commutative containing unity. The notation $R^{[n]}$ will be used to denote any R-algebra isomorphic to a polynomial algebra in n variables over R. Unless otherwise stated, capital letters like $X_1, X_2, \dots, X_n, Z_1, Z_2, \dots, Z_m, X, Y, Z, W$ will be used as variables in the polynomial ring.

We will discuss results connecting coordinates, residual coordinates and stable coordinates in polynomial algebras (see 2.2 to 2.4 for definitions). The study was initiated for the case $n = 2$ by Bhatwadekar and Dutta ([4]) and later extended to $n > 2$ by Das and Dutta ([7]).

An important problem in the study of polynomial algebras is to find fibre conditions for a polynomial F in a polynomial ring $A = R^{[n+1]}$ to be a coordinate in A. In the case $R = k[X]$, where k is an algebraically closed field of characteristic zero, S. Maubach gave the following useful result for polynomials linear in one of the variables ([11, Theorem 4.5]).

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Theorem 1.1. *Let k be an algebraically closed field of characteristic zero, $P(X, Z_1, \dots, Z_n)$ be an element in the polynomial ring $k[X, Z_1, \dots, Z_n]$ and $a(X) \in k[X] - \{0\}$. Suppose, for each root α of $a(X)$, $P(\alpha, Z_1, \dots, Z_n)$ is a coordinate in $k[Z_1, \dots, Z_n]$. Then, the polynomial F defined by $F := a(X)W + P(X, Z_1, \dots, Z_n)$ is a coordinate in $k[X, Z_1, \dots, Z_n, W]$, along with X.*

In this paper, we will show that Maubach's result holds when $k[X]$ is replaced by *any* one-dimensional ring R which is affine over an algebraically closed field k such that either the characteristic of k is zero or R_{red} is seminormal and $a(X)$ is replaced by a non-zerodivisor a in R for which the image of P becomes a coordinate over R/aR (see Theorem 3.2).

As an application of Theorem 1.1, Kahoui and Ouali have recently proved the following two results on the connection between residual coordinates and stable coordinates ([10, Theorem 1.1 and Theorem 1.2]).

Theorem 1.2. *Let k be an algebraically closed field of characteristic zero, $R = k[X]$ and $A = R[Z_1, \dots, Z_n]$. Then every residual coordinate in A is a 1-stable coordinate in A.*

Theorem 1.3. *Let R be a Noetherian d-dimensional ring containing $\mathbb{Q}$ and $A = R[Z_1, \dots, Z_n]$. Then every residual coordinate in A is a $(2^d - 1)n$-stable coordinate in A.*

Using our generalization of Maubach's result (Theorem 3.2) and the concept of exponential maps (see Definition 2.13 and Proposition 2.14) we will generalize Theorem 1.2 to the case when $k[X]$ is replaced by a one-dimensional ring R which is affine over an algebraically closed field k such that either the characteristic of k is zero or R_{red} is seminormal (see Theorem 3.4). Next, we will show (Theorem 3.6) that the condition "R contains $\mathbb{Q}$" can be dropped from Theorem 1.3. We will also show that when R is affine over an algebraically closed field of characteristic zero, the bound $(2^d - 1)n$ given in Theorem 1.3 can be reduced to $2^{d-1}(n+1) - n$ (see Theorem 3.7).

2. Preliminaries

In this section we recall a few definitions and some well-known results.

Definition 2.1. A reduced ring R is said to be *seminormal* if it satisfies the condition: for $b, c \in R$ with $b^3 = c^2$, there is an $a \in R$ such that $a^2 = b$ and $a^3 = c$.

Definition 2.2. Let $A = R[X_1, \dots, X_n]$ and $F \in A$. F is said to be a *coordinate in A* if there exist $F_2, \dots, F_n \in A$ such that $A = R[F, F_2, \dots, F_n]$.

Definition 2.3. Let $A = R[X_1, \dots, X_n]$, $F \in A$ and $m \geq 0$. F is said to be an *m-stable coordinate in A* if F is a coordinate in $A^{[m]}$.

Definition 2.4. Let $A = R[X_1, \dots, X_n]$ and $F \in A$. F is said to be a *residual coordinate in A* if, for each prime ideal p of R, $A \otimes_R k(p) = k(p)[\overline{F}]^{[n-1]}$, where $\overline{F}$ denotes the image of F in $A \otimes_R k(p)$ and $k(p) := \dfrac{R_p}{pR_p}$ is the residue field of R at p.

Definition 2.5. Let $A \longrightarrow B$ be an extension of Noetherian rings and M a B-module. M is said to be extended from A if $M \cong N \otimes_A B$ for some A-module N.

Now, we state some known results connecting coordinates, residual coordinates and stable coordinates in polynomial algebras. First, we state an elementary result ([9, Lemma 1.1.9]).

Lemma 2.6. *Let R be a ring, $nil(R)$ denote the nilradical of R and $F \in R[Z_1, \dots, Z_n]$. Let $\overline{R} := \dfrac{R}{nil(R)}$ and $\overline{F}$ denote the image of F in $\overline{R}[Z_1, \dots, Z_n]$. Then F is a coordinate in $R[Z_1, \dots, Z_n]$ if and only if $\overline{F}$ is a coordinate in $\overline{R}[Z_1, \dots, Z_n]$.*

The following result has been proved by Kahoui and Ouali in [10, Lemma 3.4].

Proposition 2.7. *Let R be an Artinian ring and $A = R[Z_1, \dots, Z_n]$. Then every residual coordinate in A is a coordinate in A.*

The following result on residual coordinates was proved by Bhatwadekar and Dutta ([4, Theorem 3.2]).

Theorem 2.8. *Let R be a Noetherian ring such that either R contains $\mathbb{Q}$ or R_{red} is seminormal. Let $A = R^{[2]}$ and $F \in A$. If F is a residual coordinate in A, then F is a coordinate in A.*

Now, we state a result on stable coordinates due to J. Berson, J.W. Bikker and A. van den Essen ([3, Proposition 5.3]); the following version was observed by Kahoui and Ouali in [10].

Theorem 2.9. *Let R be a ring, a be a non-zerodivisor of R and $P \in R[Z_1, \dots, Z_n]$. Suppose, the image of P is an m-stable coordinate in $\dfrac{R}{aR}[Z_1, \dots, Z_n]$. Then the polynomial F defined by $F := aW + P$ is a $(2m + n - 1)$-stable coordinate in $R[Z_1, \dots, Z_n, W]$.*

The following result on linear planes over a discrete valuation ring was proved by S.M. Bhatwadekar and A.K. Dutta in [5, Theorem 3.5].

Theorem 2.10. *Let R be a discrete valuation ring with parameter π, $K = R[\frac{1}{\pi}]$, $k = \frac{R}{\pi R}$ and $F = aW - b \in R[Y, Z, W]$, where $a(\neq 0), b \in R[Y, Z]$. Suppose that $\dfrac{R[Y, Z, W]}{(F)} = R^{[2]}$. Let, for each $G \in R[Y, Z, W]$, $\overline{G}$ denote the image of G in $k[Y, Z, W]$. Then there exists an element $Y_0 \in R[Y, Z]$ such that $a \in R[Y_0]$, $\overline{Y_0} \notin k$ and $K[Y, Z] = K[Y_0]^{[1]}$. Moreover, if $dim(k[\overline{F}, \overline{Y_0}]) = 2$, then F is a coordinate in $R[Y, Z, W]$.*

Now, we define $\mathbb{A}^r$-fibration and state a theorem of A. Sathaye ([13, Theorem 1]) on the triviality of $\mathbb{A}^2$-fibration over a discrete valuation ring containing $\mathbb{Q}$.

Definition 2.11. An R-algebra A is said to be an $\mathbb{A}^r$*-fibration over* R if the following hold:

(i) A is finitely generated over R,
(ii) A is flat over R,
(iii) $A \otimes_R k(p) = k(p)^{[r]}$, for each prime ideal p of R.

Theorem 2.12. *Let R be a discrete valuation ring containing $\mathbb{Q}$. Let A be an $\mathbb{A}^2$-fibration over R. Then $A = R^{[2]}$.*

Next, we define exponential maps and see it's correspondence with locally finite iterative higher derivation (see [6]). We also record a basic result on exponential maps.

Definition 2.13. Let R be a ring and A be an R-algebra. Let $\delta : A \longrightarrow A^{[1]}$ be an R-algebra homomorphism. We write $\delta = \delta_W : A \longrightarrow A[W]$ if we wish to emphasize an indeterminate W. We say δ is an R-linear *exponential map* if

(i) $\epsilon_0\delta_W$ is the identity map on A, where $\epsilon_0 : A[W] \longrightarrow A$ is the A-algebra homomorphism defined by $\epsilon_0(W) = 0$.
(ii) $\delta_V\delta_W = \delta_{V+W}$, where δ_V is extended to a homomorphism of $A[W]$ into $A[V, W]$ by setting $\delta_V(W) = W$.

For an exponential map $\delta : A \longrightarrow A^{[1]}$, we get a sequence of maps $\delta^{(i)} : A \longrightarrow A$ as follows: for $a \in A$, set $\delta^{(i)}(a)$ to be the coefficient of W^i in $\delta_W(a)$, i.e., for $\delta_W : A \longrightarrow A[W]$, we have

$$\delta_W(a) = \sum \delta^{(i)}(a)W^i.$$

Since $\delta_W(a)$ is an element in $A[W]$, the sequence $\{\delta^{(i)}(a)\}_{i\geq 0}$ has only finitely many nonzero elements for each $a \in A$. Since δ_W is a ring homomorphism, we see that $\delta^{(i)} : A \longrightarrow A$ is linear for each i and that the Leibnitz Rule:

$$\delta^{(n)}(ab) = \sum_{i+j=n} \delta^{(i)}(a)\delta^{(j)}(b) \qquad (*)$$

holds for all n and for all $a, b \in A$.

The properties (i) and (ii) of the exponential map δ_W translate into the following properties:

(i)′ $\delta^{(0)}$ is the identity map on A.
(ii)′ The "iterative property" $\delta^{(i)}\delta^{(j)} = \binom{i+j}{j}\delta^{(i+j)}$ holds for all $i, j \geq 0$.

The sequence $\{\delta^{(i)}\}_{i\geq 0}$ with only finitely many nonzero $\delta^{(i)}(a)$ for each $a \in A$ and satisfying $(*)$, (i)′, (ii)′ is called a *locally finite iterative higher derivation on A*.

Now, we state and prove a well known result on exponential maps which follows from a straightforward application of properties (i) and (ii) stated above.

Proposition 2.14. *Let R be a ring and A an R-algebra. Let $\delta_W : A \longrightarrow A[W]$ be an R-linear exponential map and $\{\delta^{(i)}\}_{i\geq 0}$ the sequence of maps on A defined above. Then the extension of δ_W to $\widetilde{\delta_W} : A[W] \longrightarrow A[W]$, defined by setting $\widetilde{\delta_W}(W) := W$, is an $R[W]$-automorphism of $A[W]$.*

Proof. Let $\delta_{-W} : A \longrightarrow A[W]$ be the R-linear exponential map defined by $\delta_{-W}(a) = \sum \delta^{(i)}(a)(-W)^i$ and $\widetilde{\delta_{-W}} : A[W] \longrightarrow A[W]$ be the extension of δ_{-W} defined by setting $\widetilde{\delta_{-W}}(W) := W$. If $\varphi, \psi : A[W, V] \longrightarrow A[W]$ are the $A[W]$-algebra homomorphisms defined by $\varphi(V) = -W$ and $\psi(V) = W$ respectively, then by properties (i) and (ii) of exponential maps, both $\varphi \circ \delta_V \circ \delta_W$ and $\psi \circ \delta_V \circ \delta_{-W}$ happen to be the inclusion map from A into $A[W]$. Since for each $a \in A$, $\varphi \circ \delta_V(a) = \delta_{-W}(a)$ and $\psi \circ \delta_V(a) = \delta_W(a)$, it follows that both $\widetilde{\delta_{-W}} \circ \widetilde{\delta_W}$ and $\widetilde{\delta_W} \circ \widetilde{\delta_{-W}}$ are the identity map on $A[W]$. Thus, $\widetilde{\delta_W}$ is an $R[W]$-automorphism of $A[W]$. □

Next, we quote some famous results which will be needed later in this paper. First, we state Bass's cancellation theorem ([1, Theorem 9.3]).

Theorem 2.15. *Let R be a Noetherian d-dimensional ring and Q be a finitely generated projective R-module whose rank at each localization at a prime ideal is at least $d + 1$. Let M be a finitely generated projective R-module such that $M \oplus Q \cong M \oplus N$ for some R-module N. Then $Q \cong N$.*

Now, we state Quillen's local-global theorem ([12, Theorem 1]).

Theorem 2.16. *Let R be a Noetherian ring, $D = R^{[1]}$ and M be a finitely generated D-module. Suppose, for each maximal ideal $\mathfrak{m}$ of R, $M_{\mathfrak{m}}$ is extended from $R_{\mathfrak{m}}$. Then M is extended from R.*

For convenience, we record a basic result on symmetric algebras of finitely generated modules ([8, Lemma 1.3]).

Lemma 2.17. *Let R be a ring and M, N two finitely generated R-modules. If $Sym_R(M)$ and $Sym_R(N)$ denote the respective symmetric algebras then $Sym_R(M) \cong Sym_R(N)$ as R-algebras if and only if $M \cong N$ as R-modules.*

Finally, we quote the theorem on the triviality of locally polynomial algebras proved by Bass-Connell-Wright ([2]) and independently by Suslin ([14]).

Theorem 2.18. *Let A be a finitely presented R-algebra. Suppose that for each maximal ideal $\mathfrak{m}$ of R, the $R_{\mathfrak{m}}$-algebra $A_{\mathfrak{m}}$ is $R_{\mathfrak{m}}$-isomorphic to the symmetric algebra of some $R_{\mathfrak{m}}$-module. Then A is R-isomorphic to the symmetric algebra of a finitely generated projective R-module.*

3. Main results

In this section we prove our main results. First, we will extend Theorem 1.1 (see Theorem 3.2). For convenience, we record below a local-global result.

Lemma 3.1. *Let R be a one-dimensional ring, $n \geq 2$, $A = R[Z_1, \dots, Z_n]$ and $F \in A$. Suppose that for each maximal ideal $\mathfrak{m}$ of R, $A_{\mathfrak{m}} = R_{\mathfrak{m}}[F]^{[n-1]}$. Then F is a coordinate in A.*

Proof. Let $D := R[F]$, $\mathfrak{n}$ be an arbitrary maximal ideal of D, $\mathfrak{p} := \mathfrak{n} \cap R$ and $\mathfrak{m}_0$ a maximal ideal of R such that $\mathfrak{p} \subseteq \mathfrak{m}_0$. From the natural maps $R_{\mathfrak{m}_0} \longrightarrow R_{\mathfrak{p}} \longrightarrow D_{\mathfrak{n}}$, we see that $A_{\mathfrak{p}} = R_{\mathfrak{p}}[F]^{[n-1]}$ and hence $A_{\mathfrak{n}} = {D_{\mathfrak{n}}}^{[n-1]}$. By Theorem 2.18, there exists a projective D-module Q' of rank $(n-1)$ such that $A \cong Sym_D(Q')$. Since, for each maximal ideal $\mathfrak{m}$ of R, $A_{\mathfrak{m}} = R_{\mathfrak{m}}[F]^{[n-1]} \cong Sym_{D_{\mathfrak{m}}}((D_{\mathfrak{m}})^{n-1})$, by Theorem 2.16, we have $Q'_{\mathfrak{m}} \cong (D_{\mathfrak{m}})^{n-1} \cong (R_{\mathfrak{m}})^{n-1} \otimes_R D$. Thus, Q' is locally extended from R and hence by Theorem 2.16, Q' is extended from R, i.e., there exists a projective R-module Q of rank $(n-1)$ such that $Q' = Q \otimes_R D$. Therefore, $A \cong Sym_R(Q) \otimes_R D \cong Sym_R(Q) \otimes_R Sym_R(R) \cong Sym_R(Q \oplus R)$. Since $A = R[Z_1, \dots, Z_n] \cong Sym_R(R^n)$, by Lemma 2.17, we have $Q \oplus R \cong R^n$. Hence, by Theorem 2.15, Q is a free R-module of rank $(n-1)$. Therefore, Q' is a free D-module of rank $(n-1)$. Hence, $A = R[F]^{[n-1]}$. $\square$

We now extend Theorem 1.1.

Theorem 3.2. *Let k be an algebraically closed field and R a one-dimensional affine k-algebra. Let a be a non-zerodivisor in R and $P(Z_1, \dots, Z_n) \in R[Z_1, \dots, Z_n]$ be such that the image of P is a coordinate in $\frac{R}{aR}[Z_1, \dots, Z_n]$. If R_{red} is seminormal or if the characteristic of k is zero then the polynomial F defined by $F := aW + P$ is a coordinate in $R[Z_1, \dots, Z_n, W]$.*

Proof. By Lemma 3.1, it is enough to consider the case when R is a local ring with unique maximal ideal $\mathfrak{m}$. Since R is an affine algebra over the algebraically closed field k, the residue field $\frac{R}{\mathfrak{m}}$ is k (by Hilbert's Nullstellensatz). Let η denote the canonical map $R[Z_1, \dots, Z_n, W] \longrightarrow k[Z_1, \dots, Z_n, W] (\subset R[Z_1, \dots, Z_n, W])$.

If $a \notin \mathfrak{m}$, then a is a unit in R and hence F is a coordinate in $R[Z_1, \dots, Z_n, W]$. So, we assume that $a \in \mathfrak{m}$. Then $\eta(F) = \eta(P)$. Since the image of P is a coordinate in $\frac{R}{aR}[Z_1, \dots, Z_n]$, hence $g := \eta(P) (= \eta(F))$ is a coordinate in $k[Z_1, \dots, Z_n]$ and hence in $R[Z_1, \dots, Z_n]$. Thus, there exist $g_2, \dots, g_n \in k[Z_1, \dots, Z_n]$ such that $k[Z_1, \dots, Z_n] = k[g, g_2, \dots, g_n]$ and hence $R[Z_1, \dots, Z_n] = R[g, g_2, \dots, g_n]$.

Set $A := R[g_2, \dots, g_n]$ and $B := R[Z_1, \dots, Z_n, W] = A[g, W] (= A^{[2]})$. We now show that F is a residual coordinate in $B = A[g, W]$. Let Q be an arbitrary prime ideal of A and $\mathfrak{p} = Q \cap R$. If $\mathfrak{p} = \mathfrak{m}$, then

$k = k(p) \hookrightarrow k(Q)$ and the image of F in $B \otimes_A k(Q)$, being the image of $g(= \eta(F))$ in $B \otimes_A k(Q)$, is a coordinate in $B \otimes_A k(Q)$. If $p \neq m$, then p is a minimal prime ideal of the one-dimensional ring R and hence $p \in Ass(R)$. Therefore, $a \notin p$. Hence a is a unit in $k(p)$ and therefore the image of F in $B \otimes_A k(Q)$ is a coordinate in $B \otimes_A k(Q)$. Thus, F is a residual coordinate in $A[g, W]$. Hence, by Theorem 2.8, F is a coordinate in $A[g, W](= R[Z_1, \ldots, Z_n, W])$. □

Now, we extend Theorem 1.2 to any algebraically closed field of arbitrary characteristic. First, we prove an easy lemma on existence of exponential maps which is a straightforward generalization of Lemma 3.1 in [10].

Lemma 3.3. *Let R be a ring and $A = R[Z_1, \ldots, Z_n]$. Let S be a multiplicatively closed subset of R consisting of non-zerodivisors in R and $P \in R[Z_1, \ldots, Z_n]$. If P is a coordinate in $S^{-1}A$ then there exists an R-linear exponential map $\varphi_W : A \to A[W]$ such that $\varphi_W(P) = aW + P$, for some $a \in S$.*

Proof. Since P is a coordinate in $S^{-1}A$, there exist $c \in S$ and $g_2, \ldots, g_n \in A$ such that

$$A_c = R_c[g_2, \ldots, g_n, P] \ (= R_c[g_2, \ldots, g_n]^{[1]}),$$

where A_c and R_c denote the localisation of the rings A and R respectively at the multiplicative set $\{1, c, c^2, \ldots\}$. Let $Z_i = \sum a_{i,j_1,\ldots,j_n} P^{j_1} g_2^{j_2} \ldots g_n^{j_n}$, where $a_{i,j_1,\ldots,j_n} \in R_c$, for all $i, 1 \le i \le n$ and $j_1, \ldots, j_n \ge 0$. Then there exists $m \ge 0$ such that $c^m a_{i,j_1,\ldots,j_n} \in R$, for all $i, 1 \le i \le n$ and $j_1, \ldots, j_n \ge 0$. Define an $R_c[g_2, \ldots, g_n]$-algebra homomorphism $\psi_W : A_c \to A_c[W]$ by setting $\psi_W(P) := P + c^m W$. Clearly, ψ_W is an exponential map and $\psi_W(Z_i) = Z_i + h_i$, for some $h_i \in A[W]$. Now, if we set a to be c^m and define $\varphi_W := \psi_W|_A$ then φ_W is our desired exponential map. □

We now generalize Theorem 1.2.

Theorem 3.4. *Let k be an algebraically closed field, R a one-dimensional affine k-algebra such that either the characteristic of k is zero or R_{red} is seminormal. Then, every residual coordinate in $A := R[Z_1, \ldots, Z_n], n \ge 3$, is a 1-stable coordinate.*

Proof. By Lemma 2.6, it is enough to consider the case when R is a reduced ring. Let $P(Z_1, \ldots, Z_n)$ be a residual coordinate in A and S be the set of all non-zerodivisors in R. Since $S^{-1}R$ is Artinian, hence by Proposition 2.7, P is a coordinate in $S^{-1}A$. Therefore, by Lemma 3.3, there exists an R-linear exponential map $\varphi_W : A \to A[W]$ such that $\varphi_W(P) = aW + P$, for some $a \in S$. Now, by Theorem 3.2, $aW + P$ is a coordinate in $A[W]$. Since by Proposition 2.14, the extension of φ_W to $A[W]$ is an R-automorphism of $A[W]$ which maps P to $aW + P$, P is a 1-stable coordinate in A. □

Remark 3.5. Recall that if R is as in Theorem 3.4 then a residual coordinate in $R[Z_1, Z_2]$ is actually a coordinate ([4, Theorem 3.2]).

Now, using Lemma 3.3, we will show that the condition "R contains $\mathbb{Q}$" can be dropped from Theorem 1.3.

Theorem 3.6. *Let R be a Noetherian d-dimensional ring. Then every residual coordinate in $R[Z_1, \ldots, Z_n]$ is a $(2^d - 1)n$-stable coordinate.*

Proof. We prove the result by induction on d. If $d = 0$, the result follows from Proposition 2.7. Now, let $d \ge 1$ and P a residual coordinate in $A := R[Z_1, \ldots, Z_n]$. We show that P is a $(2^d - 1)n$-stable coordinate in A.

By Lemma 2.6, we may assume that R is a reduced ring. Let S be the set of all non-zerodivisors of R. Then, as $S^{-1}R$ is an Artinian ring, by the case $d = 0$, P is a coordinate in $S^{-1}A$. Hence, by Lemma 3.3, there exists a non-zerodivisor a in R and an exponential map $\varphi_W : A \longrightarrow A[W]$ such that $\varphi_W(P) = aW + P$. Now, we observe that the image of P is a residual coordinate in $\frac{R}{aR}[Z_1, \dots, Z_n]$ and $\frac{R}{aR}$ is a $(d-1)$-dimensional ring. So, by induction hypothesis, P is a $(2^{d-1}-1)n$-stable coordinate in $\frac{R}{aR}[Z_1, \dots, Z_n]$. Hence, by Theorem 2.9, $aW + P$ is an r-stable coordinate in $A[W]$, where $r = 2n(2^{d-1}-1) + n - 1 = (2^d - 1)n - 1$. Since by Proposition 2.14, the extension of φ_W to $A[W]$ is an R-automorphism of $A[W]$ which maps P to $aW + P$, P is an $r + 1$ $(= (2^d - 1)n)$-stable coordinate in A. □

Next, using Theorem 3.2 and Lemma 3.3, we will show that under the additional hypothesis that R is affine over an algebraically closed field of characteristic zero, we can get a sharper bound in Theorem 3.6.

Theorem 3.7. *Let k be an algebraically closed field of characteristic zero and R a finitely generated k-algebra of dimension d. Then every residual coordinate in $R[Z_1, \dots, Z_n]$ is an r-stable coordinate, where $r = (2^d - 1)n - 2^{d-1}(n - 1) = 2^{d-1}(n + 1) - n$.*

Proof. We prove the result by induction on d. If $d = 1$, the result follows from Theorem 3.2. Let $d \geq 2$ and P a residual coordinate in $A := R[Z_1, \dots, Z_n]$. We show that P is a $(2^{d-1}(n + 1) - n)$-stable coordinate in A.

By Lemma 2.6, we may assume that R is a reduced ring. Let S be the set of all non-zerodivisors of R. Since $S^{-1}R$ is an Artinian ring, by Proposition 2.7, P is a coordinate in $S^{-1}A$. Hence, by Lemma 3.3, there exists a non-zerodivisor a in R and an exponential map $\varphi_W : A \longrightarrow A[W]$ such that $\varphi_W(P) = aW + P$. Now, we observe that the image of P is a residual coordinate in $\frac{R}{aR}[Z_1, \dots, Z_n]$ and $\frac{R}{aR}$ is a $(d-1)$-dimensional ring containing an algebraically closed field of characteristic zero. So, by induction hypothesis, P is a $(2^{d-2}(n + 1) - n)$-stable coordinate in $\frac{R}{aR}[Z_1, \dots, Z_n]$. Now, arguing as in Theorem 3.6, the result follows from Theorem 2.9 and Proposition 2.14. □

The following question asks whether Theorem 3.2 (and thereby Theorem 3.4) can be extended to an affine algebra of *any* dimension over *any* field (not necessarily algebraically closed) of arbitrary characteristic.

Question 3.8. *Let k be a field, R an affine k-algebra, a be a non-zerodivisor in R and $P(Z_1, \dots, Z_n) \in R[Z_1, \dots, Z_n]$ be such that the image of P is a coordinate in $\frac{R}{aR}[Z_1, \dots, Z_n]$. Suppose, R_{red} is seminormal or the characteristic of k is zero. Then, is $F := aW + P$ a coordinate in $R[Z_1, \dots, Z_n, W]$?*

The following result shows that for $n = 2$, the above question has an affirmative answer when R is a Dedekind domain containing a field of characteristic zero.

Proposition 3.9. *Let R be a Dedekind domain containing $\mathbb{Q}$, $a \in R - \{0\}$ and $F = aW + P(Y, Z) \in R[Y, Z, W]$. If the image of P is a coordinate in $\frac{R}{aR}[Y, Z]$, then F is a coordinate in $R[Y, Z, W]$.*

Proof. By Lemma 3.1, it is enough to assume that R is a discrete valuation ring with parameter t. Let $k = \frac{R}{tR}$, $K = R[\frac{1}{t}]$, $A = \frac{R[Y, Z, W]}{(F)}$ and $\overline{P}$ denote the image of P in $k[Y, Z]$. Note that a is a unit in K and hence F is a coordinate in $K[Y, Z, W]$; in particular $A[\frac{1}{t}] = K^{[2]}$.

If $a \notin tR$, then a is a unit in R and hence F is a coordinate in $R[Y, Z, W]$. We now consider the case $a \in tR$. Considering the natural map $\frac{R}{aR} \to \frac{R}{tR} (= k)$, we see that $\overline{P}$ is a coordinate in $k[Y, Z]$ and hence

$\dfrac{A}{tA} = k^{[2]}$. It also follows that $P \notin tR[Y, Z]$ and hence a and P are coprime in $R[Y, Z]$. Thus, $F(= aW + P)$ is irreducible in $R[Y, Z, W]$. So, A is a torsion free module over the discrete valuation ring R and hence A is flat over R. Thus, A is an $\mathbb{A}^2$-fibration over R and hence by Theorem 2.12, $A = R^{[2]}$. As $\overline{P}$ is a coordinate in $k[Y, Z]$, we see that $\overline{P} \notin k[Y] \cap k[Z](= k)$. Hence at least one of the rings $k[Y, \overline{P}]$ and $k[Z, \overline{P}]$ is of dimension two. Now, the result follows from Theorem 2.10. $\square$

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