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Gerstenhaber bracket on Hopf algebra and Hochschild cohomologies ☆

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ABSTRACT

We calculate the Gerstenhaber bracket on Hopf algebra and Hochschild cohomologies of the Taft algebra T_p for any integer $p > 2$ which is a nonquasi-triangular Hopf algebra. We show that the bracket is indeed zero on Hopf algebra cohomology of T_p, as in all known quasi-triangular Hopf algebras. This example is the first known bracket computation for a nonquasi-triangular algebra. Also, we find a general formula for the bracket on Hopf algebra cohomology of any Hopf algebra with bijective antipode on the bar resolution that is reminiscent of Gerstenhaber's original formula for Hochschild cohomology.

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1. Introduction

Gerstenhaber brackets were originally defined on Hochschild cohomology by M. Gerstenhaber himself [3, Section 1.1]. In 2002, A. Farinati and A. Solotar showed that for any Hopf algebra A, Hopf algebra cohomology $\mathrm{H}^*(A) := \mathrm{Ext}^*_A(k,k)$ is a Gerstenhaber algebra [2]. Hence, we can define a Gerstenhaber bracket on Hopf algebra cohomology. In the same year, R. Taillefer used a different approach and found a bracket on Hopf algebra cohomology [9] which is equivalent to the bracket constructed by A. Farinati and A. Solotar. The category of A-modules and the category of A^e-modules are examples of strong exact monoidal categories. In 2016, Reiner Hermann [5, Theorem 6.3.12, Corollary 6.3.15] proved that if the strong exact monoidal category is lax braided, then the bracket is constantly zero. Therefore, the Gerstenhaber bracket on the Hopf algebra cohomology of a quasi-triangular Hopf algebra is trivial. However, we do not know the bracket structure for a nonquasi-triangular Hopf algebra. Taft algebras are nice examples of nonquasi-triangular Hopf algebras. In this paper, we show that the Gerstenhaber bracket on the Hochschild

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cohomology of a Taft Algebra is nontrivial. However, the bracket structure on Hopf algebra cohomology of a Taft algebra is constantly zero. Also, we take the Gerstenhaber bracket formula on Hochschild cohomology and find a general formula for Gerstenhaber bracket on Hopf algebra cohomology.

We start by giving some basic definitions and some tools to calculate the bracket on Hochschild cohomology in Section 2. Then, we compute the Gerstenhaber bracket on the Hochschild cohomology of $A = k[x]/(x^p)$ where the field k has characteristic 0 and the integer $p > 2$ in Section 3. We use the technique introduced by C. Negron and S. Witherspoon [6] and note that they computed the bracket on Hochschild cohomology of A for the case that k has positive characteristic p [6, Section 5].

In Section 4, we compute the Gerstenhaber bracket for the Taft algebra T_p which is a nonquasi-triangular Hopf algebra. We use a similar technique as in [6] to calculate the bracket on Hochschild cohomology of T_p. It is also known that the Hopf algebra cohomology of any Hopf algebra with a bijective antipode can be embedded in the Hochschild cohomology of the algebra [10, Theorem 9.4.5 and Corollary 9.4.7]. Since all finite dimensional Hopf algebras (also most of known infinite dimensional Hopf algebras) have bijective antipode, we can embed the Hopf algebra cohomology of T_p into the Hochschild cohomology of T_p. Then, we use this explicit embedding and find the bracket on the Hopf algebra cohomology of T_p. As a result of our calculation, we obtain that the bracket on Hopf algebra cohomology of T_p is also trivial.

In the last section, we derive a general expression for the bracket on Hopf algebra cohomology of any Hopf algebra A with bijective antipode. We first consider a specific resolution that agrees with the bar resolution of A and find a bracket formula for it. Then, we use the composition of various isomorphisms and an embedding from Hopf algebra cohomology into Hochschild cohomology in order to discover the bracket formula on Hopf algebra cohomology.

2. Gerstenhaber bracket on Hochschild cohomology

Let k be a field, A be a k-algebra, and $A^e = A \otimes_k A^{op}$ where A^{op} is the opposite algebra with reverse multiplication. For simplicity, we write $\otimes$ instead of $\otimes_k$. The following resolution $B(A)$ is a free resolution of the A^e-module A, called the *bar resolution*,

$$B(A) : \cdots \xrightarrow{d_3} A^{\otimes 4} \xrightarrow{d_2} A^{\otimes 3} \xrightarrow{d_1} A^{\otimes 2} \xrightarrow{\pi} A \longrightarrow 0, \tag{2.1}$$

where

$$d_n(a_0 \otimes a_1 \otimes \cdots \otimes a_{n+1}) = \sum_{i=0}^{n} (-1)^i a_0 \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}$$

and π is multiplication.

Consider the following complex that is derived by applying $\mathrm{Hom}_{A^e}(-, A)$ to the bar resolution $B(A)$

$$0 \longrightarrow \mathrm{Hom}_{A^e}(A^{\otimes 2}, A) \xrightarrow{d_1^*} \mathrm{Hom}_{A^e}(A^{\otimes 3}, A) \xrightarrow{d_2^*} \mathrm{Hom}_{A^e}(A^{\otimes 4}, A) \xrightarrow{d_3^*} \cdots \tag{2.2}$$

where $d_n^*(f) = f d_n$. The *Hochschild cohomology* of the algebra A is the cohomology of the cochain complex (2.1), i.e.

$$\mathrm{HH}^*(A, A) = \bigoplus_{n \geq 0} \mathrm{Ext}^n_{A^e}(A, A).$$

We also define the *Hopf algebra cohomology* of the Hopf algebra A over the field k as

$$\mathrm{H}^*(A, k) = \bigoplus_{n \geq 0} \mathrm{Ext}^n_A(k, k)$$

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under the cup product.

Let $f \in \mathrm{Hom}_k(A^{\otimes m}, A)$ and $g \in \mathrm{Hom}_k(A^{\otimes n}, A)$. The Hochschild cohomology of A is an algebra with the following cup product and the Gerstenhaber bracket structures. The *cup product* $f \smile g \in \mathrm{Hom}_k(A^{\otimes(m+n)}, A)$ is defined by

$$(f \smile g)(a_1 \otimes \cdots \otimes a_{m+n}) := (-1)^{mn} f(a_1 \otimes \cdots \otimes a_m) g(a_{m+1} \otimes \cdots a_{m+n})$$

for all $a_1, \cdots, a_{m+n} \in A$, and the *Gerstenhaber bracket* $[f, g]$ is an element of $\mathrm{Hom}_k(A^{\otimes(m+n-1)}, A)$ given by

$$[f, g] := f \circ g - (-1)^{(m-1)(n-1)} g \circ f$$

where the circle product $f \circ g$ is

$$(f \circ g)(a_1 \otimes \cdots \otimes a_{m+n-1}) := \sum_{i=1}^{m} (-1)^{(n-1)(i-1)} f(a_1 \otimes \cdots a_{i-1} \otimes g(a_i \otimes \cdots a_{i+n-1}) \otimes a_{i+n} \otimes \cdots \otimes a_{m+n-1})$$

for all $a_1, \cdots, a_{m+n-1} \in A$. We note that these definitions directly come from the bar resolution.

There is an identity between cup product and bracket [3, Section 1]:

$$[f^* \smile g^*, h^*] = [f^*, h^*] \smile g^* + (-1)^{|f^*|(|h^*|-1)} f^* \smile [g^*, h^*], \tag{2.3}$$

where f^*, g^*, and h^* are the images (in Hochschild cohomology) of the cocyles f, g, and h, respectively.

Computing the bracket on the bar resolution is not an ideal method. Instead, we can use another resolution, $\mathbb{A} \xrightarrow{\mu} A$, satisfying the following hypotheses [6, (3.1) and Lemma 3.4.1]:

(a) $\mathbb{A}$ admits an embedding $\iota : \mathbb{A} \to B(A)$ of complexes of A-bimodules for which the following diagram commutes

$$\begin{array}{ccc} \mathbb{A} & \xrightarrow{\iota} & B(A) \\ & \searrow & \downarrow \\ & & A \end{array}$$

(b) The embedding ι admits a section $\pi : B \to \mathbb{A}$, i.e. an A^e-chain map π with $\pi\iota = id_{\mathbb{A}}$.

(c) There is a diagonal map that satisfies $\Delta^{(2)}_{\mathbb{A}} = (\pi \otimes_A \pi \otimes_A \pi)\Delta^{(2)}_{B(A)}\iota$ where $\Delta_{B(A)} : B(A) \to B(A) \otimes_A B(A)$ is a diagonal map on $B(A)$ and $\Delta^{(2)} = (id \otimes \Delta)\Delta$.

We give the following theorem which is the combination of [6, Theorem 3.2.5] and [6, Lemma 3.4.1] that allows us to use a different resolution for the bracket calculation.

Theorem 2.4. *Suppose $\mathbb{A} \xrightarrow{\mu} A$ is a projective A-bimodule resolution of A that satisfies the hypotheses (a)-(c). Let $\phi : \mathbb{A} \otimes_A \mathbb{A} \to \mathbb{A}$ be any contracting homotopy for the chain map $F_{\mathbb{A}} : \mathbb{A} \otimes_A \mathbb{A} \to \mathbb{A}$ defined by $F_{\mathbb{A}} := (\mu \otimes_A id_{\mathbb{A}} - id_{\mathbb{A}} \otimes_A \mu)$, i.e.*

$$d(\phi) := d_{\mathbb{A}}\phi + \phi d_{\mathbb{A} \otimes_A \mathbb{A}} = F_{\mathbb{A}}. \tag{2.5}$$

Then for cocycles f and g in $\mathrm{Hom}_{A^e}(\mathbb{A}, A)$, the bracket given by

$$[f, g]_\phi = f \circ_\phi g - (-1)^{(|f|-1)(|g|-1)} g \circ_\phi f \tag{2.6}$$

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where the circle product is

$$f \circ_\phi g = f\phi(id_{\mathbb{A}} \otimes_A g \otimes_A id_{\mathbb{A}})\Delta^{(2)} \tag{2.7}$$

agrees with the Gerstenhaber bracket on cohomology.

In general, it is not easy to calculate the map ϕ by the formula (2.5). We use alternative way to find ϕ.

Let h be any k-linear contracting homotopy for the identity map on the extended complex $\mathbb{A} \to A \to 0$ where $\mathbb{A}$ is free. A contracting homotopy $\phi_i : (\mathbb{A} \otimes_A \mathbb{A})_i \longrightarrow \mathbb{A}_{i+1}$ in Theorem 2.4 is constructed by the following formula [6, Lemma 3.3.1]:

$$\phi_i = h_i((F_{\mathbb{A}})_i - \phi_{i-1} d_{(\mathbb{A}\otimes_A \mathbb{A})_i}). \tag{2.8}$$

3. Bracket on Hochschild cohomology of $A = k[x]/(x^p)$

Let $A = k[x]/(x^p)$ where k is a field of characteristic 0 and $p > 2$ is an integer. We compute the Lie bracket on Hochschild cohomology of A by Theorem 2.4. We work on a smaller resolution of A than the bar resolution of A. Consider the following A^e-module resolution of A:

$$\mathbb{A} : \cdots \xrightarrow{v.} A^e \xrightarrow{u.} A^e \xrightarrow{v.} A^e \xrightarrow{u.} A^e \xrightarrow{\pi} A \longrightarrow 0, \tag{3.1}$$

where $u = x \otimes 1 - 1 \otimes x$, $v = x^{p-1} \otimes 1 + x^{p-2} \otimes x + \cdots + x \otimes x^{p-2} + 1 \otimes x^{p-1}$, and π is the multiplication.

The bracket on A where k is a field with positive characteristic, is calculated by C. Negron and S. Witherspoon [6, Section 5]. We adopt the contracting homotopy h for the identity map from that calculation and obtain a new map h for our setup. Let ξ_i be the element $1\otimes 1$ of $\mathbb{A}_i$. The following maps $h_n : \mathbb{A}_n \longrightarrow \mathbb{A}_{n+1}$ form a contracting homotopy for identity map, as we can see by direct calculation:

$$\begin{aligned}
h_{-1}(x^i) &= \xi_0 x^i, \\
h_0(x^i \xi_0 x^j) &= \sum_{l=0}^{i-1} x^l \xi_1 x^{i+j-1-l}, \\
h_1(x^i \xi_1 x^j) &= \delta_{i,p-1} x^j \xi_2, \\
h_{2n}(x^i \xi_{2n} x^j) &= -\sum_{l=0}^{j-1} x^{i+j-1-l} \xi_{2n+1} x^l \ (n \geq 1)\ , \\
h_{2n+1}(x^i \xi_{2n+1} x^j) &= \delta_{j,p-1} x^i \xi_{2n+2} \ (n \geq 1)\ .
\end{aligned} \tag{3.2}$$

Then, we take $\phi_{-1} = 0$ and construct the following A^e-linear maps $\phi_i : (\mathbb{A} \otimes_A \mathbb{A})_i \longrightarrow \mathbb{A}_{i+1}$ for degree 1 and 2 by (2.8):

$$\begin{aligned}
\phi_0(\xi_0 \otimes_A x^i \xi_0) &= \sum_{l=0}^{i-1} x^l \xi_1 x^{i-1-l}, \\
\phi_1(\xi_1 \otimes_A x^i \xi_0) &= -\delta_{i,p-1} \xi_2, \\
\phi_1(\xi_0 \otimes_A x^i \xi_1) &= \delta_{i,p-1} \xi_2.
\end{aligned} \tag{3.3}$$

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Lastly, we form the following diagonal map $\Delta : \mathbb{A} \longrightarrow \mathbb{A} \otimes_A \mathbb{A}$:

$$\begin{aligned}
&\Delta_0(\xi_0) = \xi_0 \otimes_A \xi_0,\\
&\Delta_1(\xi_1) = \xi_1 \otimes_A \xi_0 + \xi_0 \otimes_A \xi_1,\\
&\Delta_{2n}(\xi_{2n}) = \sum_{i=0}^{n} \xi_{2i} \otimes_A \xi_{2n-2i} + \sum_{i=0}^{n-1} \sum_{a+b+c=p-2} x^a \xi_{2i+1} \otimes_A x^b \xi_{2n-2i-1} x^c, \text{ for } n \geq 1 \qquad (3.4)\\
&\Delta_{2n+1}(\xi_{2n+1}) = \sum_{i=0}^{2n+1} \xi_i \otimes_A \xi_{2n+1-i}, \text{ for } n \geq 1.
\end{aligned}$$

It can be seen that the map Δ is a chain map lifting the canonical isomorphism $A \xrightarrow{\sim} A \otimes_A A$ by direct calculation.

Now, we are ready to calculate the brackets on cohomology in low degrees. By applying $\mathrm{Hom}_{A^e}(-, A)$ to $\mathbb{A}$, we see that the differentials are all 0 in odd degrees and $(px^{p-1})\cdot$ in even degrees. In each degree, the term in the Hom complex is the free A-module $\mathrm{Hom}_{A^e}(A^e, A) \cong A$. Moreover, since p is not divisible by the characteristic of k, we deduce $\mathrm{HH}^0(A) \cong A, \mathrm{HH}^{2i+1}(A) \cong (x),$ and $\mathrm{HH}^{2i}(A) \cong A/(x^{p-1})$ [10, Section 1.1].

Let $x^j \xi_i^* \in \mathrm{Hom}_{A^e}(A^e, A)$ denote the function that takes ξ_i to x^j. Since the characteristic of k does not divide p, the Hochschild cohomology as an A-algebra is generated by ξ_1^* and ξ_2^* [10, Example 2.2.2]. We only calculate the brackets of the elements of degrees 1 and 2 which can be extended to higher degrees by the formula (2.3). Hence, we have the following calculations:

The bracket of the elements of degrees 1 and 1:

$$\begin{aligned}
&(x^i \xi_1^* \circ_\phi x^j \xi_1^*)(\xi_1)\\
&= x^i \xi_1^* \phi(1 \otimes_A x^j \xi_1^* \otimes_A 1)\Delta^{(2)}(\xi_1)\\
&= x^i \xi_1^* \phi(1 \otimes_A x^j \xi_1^* \otimes_A 1)(\xi_1 \otimes_A \xi_0 \otimes_A \xi_0 + \xi_0 \otimes_A \xi_1 \otimes_A \xi_0 + \xi_0 \otimes_A \xi_0 \otimes \xi_1)\\
&= x^i \xi_1^* \phi(\xi_0 \otimes_A x^j \xi_0)\\
&= x^i \xi_1^* (\xi_1 x^{j-1} + x\xi_1 x^{j-2} + \cdots + x^{j-1}\xi_1)\\
&= j x^{i+j-1}
\end{aligned}$$

and by symmetry $(x^j \xi_1^* \circ_\phi x^i \xi_1^*)(\xi_1) = i x^{i+j-1}$. Therefore, we have

$$[x^i \xi_1^*, x^j \xi_1^*] = (j-i) x^{i+j-1} \xi_1^*.$$

The bracket of the elements of degrees 1 and 2:

$$\begin{aligned}
&(x^i \xi_1^* \circ_\phi x^j \xi_2^*)(\xi_2)\\
&= x^i \xi_1^* \phi(1 \otimes_A x^j \xi_2^* \otimes_A 1)\Delta^{(2)}(\xi_2)\\
&= x^i \xi_1^* \phi(1 \otimes_A x^j \xi_2^* \otimes_A 1)(\xi_0 \otimes_A \xi_0 \otimes_A \xi_2 + \xi_0 \otimes_A \xi_2 \otimes_A \xi_0 + \xi_2 \otimes_A \xi_0 \otimes_A \xi_0\\
&+ \xi_0 \otimes_A \sum_{\substack{a+b+c\\=p-2}} (x^a \xi_1 \otimes_A x^b \xi_1 x^c) + \sum_{\substack{a+b+c\\=p-2}} x^a \xi_1 \otimes_A x^b (\xi_0 \otimes_A \xi_1 + \xi_1 \otimes_A x_0) x^c)\\
&= x^i \xi_1^* \phi(\xi_0 \otimes_A x^j \xi_0) = x^i \xi_1^* (\xi_1 x^{j-1} + x\xi_1 x^{j-2} + \cdots + x^{j-1}\xi_1) = j x^{i+j-1}.
\end{aligned}$$

The circle product in the reverse order is

$$
\begin{aligned}
&(x^j\xi_2^* \circ_\phi x^{p-1}\xi_1^*)(\xi_2)\\
&= x^j\xi_2^*\phi(1\otimes_A x^{p-1}\xi_1^*\otimes_A 1)\Delta^{(2)}(\xi_2)\\
&= x^j\xi_2^*\phi(1\otimes_A x^{p-1}\xi_1^*\otimes_A 1)(\xi_0\otimes_A\xi_0\otimes_A\xi_2+\xi_0\otimes_A\xi_2\otimes_A\xi_0+\xi_2\otimes_A\xi_0\otimes_A\xi_0\\
&+\xi_0\otimes_A\sum_{\substack{a+b+c\\=p-2}}(x^a\xi_1\otimes_A x^b\xi_1x^c)+\sum_{\substack{a+b+c\\=p-2}}x^a\xi_1\otimes_A x^b(\xi_0\otimes_A\xi_1+\xi_1\otimes_A x_0)x^c)\\
&= x^j\xi_2^*\phi(\sum_{\substack{a+b+c\\=p-2}}(\xi_0\otimes_A x^{a+b+i}\xi_1x^c-x^a\xi_1\otimes_A x^{b+i}\xi_0x^c))\\
&= x^j\xi_2^*(\sum_{\substack{a+b+c\\=p-2}}(\delta_{a+b+i,p-1}\xi_2x^c+x^a\delta_{b+i,p-1}\xi_2x^c))\\
&= x^j\xi_2^*((p-i)\xi_2x^{i-1}+\sum_{\substack{a+c\\=i-1}}x^a\xi_2x^c)\\
&= (p-i)x^{i+j-1}+\sum_{\substack{a+c\\=i-1}}x^{a+c+j}=(p-i)x^{i+j-1}+ix^{i+j-1}=px^{i+j-1}.
\end{aligned}
$$

Therefore, we obtain

$$[x^i\xi_1^*, x^j\xi_2^*] = (j-p)x^{i+j-1}\xi_2^*.$$

Lastly, the bracket of the elements of degrees 2 and 2:

$$
\begin{aligned}
(x^i\xi_2^*\circ_\phi x^j\xi_2^*)(\xi_3) &= x^i\xi_2^*\phi(1\otimes_A x^j\xi_2^*\otimes_A 1)\Delta^{(2)}(\xi_3)\\
&= x^i\xi_2^*\phi(\xi_1\otimes_A x^j\xi_0+\xi_0\otimes_A x^j\xi_1) = x^i\xi_2^*(0) = 0
\end{aligned}
$$

and by symmetry $(x^j\xi_2^*\circ_\phi x^i\xi_2^*)(\xi_3)=0$. Therefore, we have

$$[(x^i\xi_2^*, x^j\xi_2^*)] = 0.$$

As a consequence, the brackets for the elements of degrees 1 and 2 are

$$
\begin{aligned}
[(x^i\xi_1^*, x^j\xi_1^*)] &= (j-i)x^{i+j-1}\xi_1^*,\\
[(x^i\xi_1^*, x^j\xi_2^*)] &= (j-p)x^{i+j-1}\xi_2^*,\\
[(x^i\xi_2^*, x^j\xi_2^*)] &= 0.
\end{aligned}
$$

Brackets in higher degrees can be determined from these and the identity (2.3) since the Hochschild cohomology is generated as an A-algebra under the cup product in degrees 1 and 2.

A similar computation to the calculation in this section is made by L. Grimley, V. C. Nguyen, and S. Witherspoon [4] for Gerstenhaber brackets on Hochschild cohomology of a twisted tensor product of $k[x]/(x^2)$. S. Sanchez-Flores [8] also calculated the bracket on group algebras of a cyclic group over a field of positive characteristic which is isomorphic to $A = k[x]/(x^p)$. C. Negron and S. Witherspoon [6] calculated the bracket on group algebras of a cyclic group over a field of positive characteristic as well with the same h, ϕ, and Δ maps. They assume the field with characteristic p which makes the coefficient j instead of $j-p$ in the calculation of $[(x^i\xi_1^*, x^j\xi_2^*)]$ as p vanishes. Hence our calculation agrees with those except slightly different $[(x^i\xi_1^*, x^j\xi_2^*)]$.

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4. Bracket on Hopf algebra cohomology of a Taft algebra

The Taft algebra T_p with $p > 2$ is a k-algebra generated by g and x satisfying the relations: $g^p = 1, x^p = 0,$ and $xg = \omega gx$ where ω is a primitive p-th root of unity. It is a Hopf algebra with the structure:

- $\Delta(g) = g \otimes g$, $\Delta(x) = 1 \otimes x + x \otimes g$
- $\varepsilon(g) = 1, \varepsilon(x) = 0$
- $S(g) = g^{-1}, S(x) = -xg^{-1}$.

Note that as an algebra, T_p is a skew group algebra $A \rtimes kG$ where $A = k[x]/(x^p)$ and $G =< g \mid g^p = 1 >$. The action of G on A is given by ${}^g x = \omega x$.

In this section, our main goal is to calculate the bracket on Hochschild cohomology of T_p with the same technique in Section 3 and find the bracket on Hopf algebra cohomology of T_p by using the embedding of $\mathrm{H}^*(T_p, k)$ into $\mathrm{HH}^*(T_p, T_p)$.

We first find the bracket on Hochschild cohomology of T_p. Let $\mathcal{D}$ be the skew group algebra $A^e \rtimes G$ where the action of G on A^e is diagonal, i.e. ${}^g(a \otimes b) = ({}^g a) \otimes ({}^g b)$. Then, there is the following isomorphism [1, Section 2]

$$\mathcal{D} = A^e \rtimes G \cong \bigoplus_{g \in G} Ag \otimes Ag^{-1} \subset T_p^e.$$

Hence $\mathcal{D}$ is isomorphic to a subalgebra of T_p^e via $a_1 \otimes a_2 \otimes g \mapsto a_1 g \otimes ({}^{g^{-1}} a_2 g^{-1})$. Moreover, A is a $\mathcal{D}$-module under the following left and right action [1, Section 4]:

$$(a_1 g \otimes a_2 g^{-1}) a_3 = a_1 g a_3 a_2 g^{-1} = a_1({}^g(a_3 a_2))$$
$$a_3(a_1 g \otimes a_2 g^{-1}) = a_2 g^{-1} a_3 a_1 g = a_2({}^{g^{-1}}(a_3 a_1)).$$

Remember the resolution (3.1)

$$\mathbb{A} : \cdots \xrightarrow{v.} A^e \xrightarrow{u.} A^e \xrightarrow{v.} A^e \xrightarrow{u.} A^e \xrightarrow{\pi} A \longrightarrow 0.$$

This is also a $\mathcal{D}$-projective resolution of A and the action of G on A^e is given by

- $g \cdot (a_1 \otimes a_2) = ({}^g a_1) \otimes ({}^g a_2)$ in even degrees,
- $g \cdot (a_1 \otimes a_2) = \omega({}^g a_1) \otimes ({}^g a_2)$ in odd degrees.

From the resolution $\mathbb{A}$, we construct the following T_p^e resolution of T_p:

$$T_p^e \otimes_{\mathcal{D}} \mathbb{A} : \cdots \longrightarrow T_p^e \otimes_{\mathcal{D}} A^e \longrightarrow T_p^e \otimes_{\mathcal{D}} A^e \longrightarrow T_p^e \otimes_{\mathcal{D}} A^e \longrightarrow T_p^e \otimes_{\mathcal{D}} A \longrightarrow 0. \tag{4.1}$$

It is known that, $T_p \cong T_p^e \otimes_{\mathcal{D}} A$ as T_p-bimodules via the map sending $x^i \otimes g^k$ to $(1 \otimes g^k) \otimes_{\mathcal{D}} x^i$ [10, Section 3.5]. Then we have $A \otimes T_p \cong T_p^e \otimes_{\mathcal{D}} A^e$ with the T_p-bimodule isomorphism given by

$$\kappa(x^i \otimes (x^j \otimes g^k)) = (1 \otimes g^k) \otimes_{\mathcal{D}} (x^i \otimes x^j). \tag{4.2}$$

Then, we obtain the following resolution $\tilde{\mathbb{A}}$ which is isomorphic to the resolution (4.1), i.e.

$$\tilde{\mathbb{A}} : \cdots \xrightarrow{\tilde{u}.} A \otimes T_p \xrightarrow{\tilde{v}.} A \otimes T_p \xrightarrow{\tilde{u}.} A \otimes T_p \xrightarrow{\tilde{\pi}.} T_p \longrightarrow 0 \tag{4.3}$$

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where $\tilde{v} = v \otimes id_{kG}, \tilde{u} = u \otimes id_{kG}$, and $\tilde{\pi} = \pi \otimes id_{kG}$.

The following lemma gives us a contracting homotopy for the identity map on the resolution $\tilde{\mathbb{A}}$.

Lemma 4.4. *Let h_n be a contracting homotopy in (3.2). Then $\tilde{h}_n = h_n \otimes 1_{kG}$ forms a contracting homotopy for the identity map on $\tilde{\mathbb{A}}$.*

Proof. For $n \geq 0$, the domain of $h_n \otimes 1_{kG}$ is $A \otimes A \otimes kG$ which is $A \otimes T_p$ as a vector space. Moreover, by definition of contracting homotopy, h_n satisfy

$$h_{i-1}d_i + d_{i+1}h_i = id_{\mathbb{A}_i}.$$

Then,

$$\begin{aligned}\tilde{h}_{i-1}\tilde{d}_i + \tilde{d}_{i+1}\tilde{h}_i &= (h_{i-1} \otimes id_{kG})(d_i \otimes id_{kG}) + (d_{i+1} \otimes id_{kG})(h_i \otimes id_{kG}) \\ &= (h_{i-1}d_i \otimes id_{kG}) + (d_{i+1}h_i \otimes id_{kG}) = (h_{i-1}d_i + d_{i+1}h_i) \otimes id_{kG} \\ &= id_{\mathbb{A}_i} \otimes id_{kG} = id_{\tilde{\mathbb{A}}_i}\end{aligned}$$

and that implies $\tilde{h}_n$ is a contracting homotopy for $\tilde{\mathbb{A}}$. The proof is similar for $n = -1$. □

We abbreviate $a_1 \otimes a_2 \otimes g \in A \otimes T_p$ by $a_1 \otimes a_2 g$. By the Lemma 4.4, we obtain

$$\begin{aligned}\tilde{h}_{-1}(x^i g) &= \xi_0 x^i g, \\ \tilde{h}_0(x^i \xi_0 x^j g) &= \sum_{l=0}^{i-1} x^l \xi_1 x^{i+j-1-l} g, \\ \tilde{h}_1(x^i \xi_1 x^j g) &= \delta_{i,p-1} x^j \xi_2 g, \\ \tilde{h}_{2n}(x^i \xi_{2n} x^j g) &= -\sum_{l=0}^{j-1} x^{i+j-1-l} \xi_{2n+1} x^l g, \\ \tilde{h}_{2n+1}(x^i \xi_{2n+1} x^j g) &= \delta_{j,p-1} x^i \xi_{2n+2} g.\end{aligned}$$

We need a lemma to have the linear maps $\tilde{\phi}_i : (\tilde{\mathbb{A}} \otimes_{T_p} \tilde{\mathbb{A}})_i \longrightarrow \tilde{\mathbb{A}}_{i+1}$. However, we first mention that there is an isomorphism from $(A \otimes T_p) \otimes_{T_p} (A \otimes T_p)$ to $(A \otimes A) \otimes_A (A \otimes A) \otimes kG$ as T_p^e-modules given by

$$\psi((x^{i_1} \otimes x^{j_1} g^{k_1}) \otimes_{T_p} (x^{i_2} \otimes x^{j_2} g^{k_2})) = \omega^{k_1(i_2+j_2)}(x^{i_1} \otimes x^{j_1}) \otimes_A (x^{i_2} \otimes x^{j_2}) g^{(k_1+k_2)}. \tag{4.5}$$

Lemma 4.6. *Let $F_{\mathbb{A}} = (\pi \otimes_A id_{\mathbb{A}} - id_{\mathbb{A}} \otimes_A \pi)$ be the chain map for the resolution $\mathbb{A}$ in (3.1) which is used for calculation of ϕ in (3.3). Then $F_{\tilde{\mathbb{A}}} : \tilde{\mathbb{A}} \otimes_{T_p} \tilde{\mathbb{A}} \to \tilde{\mathbb{A}}$ defined by $(\tilde{\pi} \otimes_{T_p} id_{\tilde{\mathbb{A}}} - id_{\tilde{\mathbb{A}}} \otimes_{T_p} \tilde{\pi})$ is exactly $(F_{\mathbb{A}} \otimes id_{kG})\psi$. Moreover $\tilde{\phi} := (\phi \otimes id_{kG})\psi$ is a contracting homotopy for $F_{\tilde{\mathbb{A}}}$.*

Proof. Let $(x^{i_1} \otimes x^{j_1} g^{k_1}) \otimes_{T_p} (x^{i_2} \otimes x^{j_2} g^{k_2}) \in (A \otimes T_p) \otimes_{T_p} (A \otimes T_p)$. Note that $F_{\tilde{\mathbb{A}}}$ is zero if degrees of $(x^{i_1} \otimes x^{j_1} g^{k_1})$ and $(x^{i_2} \otimes x^{j_2} g^{k_2})$ are both nonzero since $\tilde{\pi}$ is only defined on degree zero. Also remember that $\tilde{\pi} = \pi \otimes id_{kG}$ for the resolution $\tilde{\mathbb{A}}$.

We check the case that the degree of $(x^{i_1} \otimes x^{j_1} g^{k_1})$ is zero and the degree of $(x^{i_2} \otimes x^{j_2} g^{k_2})$ is nonzero. By using definition of $F_{\tilde{\mathbb{A}}}$, we obtain

$$\begin{aligned}F_{\tilde{\mathbb{A}}}((x^{i_1} \otimes x^{j_1} g^{k_1}) \otimes_{T_p} (x^{i_2} \otimes x^{j_2} g^{k_2})) &= (x^{i_1+j_1} g^{k_1}) \otimes_{T_p} (x^{i_2} \otimes x^{j_2} g^{k_2}) \\ &= \omega^{k_1(i_2+j_2)} x^{i_1+i_2+j_1} \otimes x^{j_2} g^{k_1+k_2}.\end{aligned}$$

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On the other hand, we also have

$$\begin{aligned}&(F_{\mathbb{A}} \otimes id_{kG})\psi((x^{i_1} \otimes x^{j_1}g^{k_1}) \otimes_{T_p} (x^{i_2} \otimes x^{j_2}g^{k_2}))\\ &= (F_{\mathbb{A}} \otimes id_{kG})(\omega^{k_1(i_2+j_2)}(x^{i_1} \otimes x^{j_1}) \otimes_A (x^{i_2} \otimes x^{j_2})g^{k_1+k_2})\\ &= \omega^{k_1(i_2+j_2)}x^{i_1+i_2+j_1} \otimes x^{j_2}g^{k_1+k_2}.\end{aligned}$$

The proofs for other cases are similar. Hence $F_{\tilde{\mathbb{A}}}$ and $(F_{\mathbb{A}} \otimes id_{kG})\psi$ are identical.

In order to prove $\tilde{\phi} := (\phi \otimes id_{kG})\psi$ is a contracting homotopy for $F_{\tilde{\mathbb{A}}}$, we need to show that

$$\tilde{d}_{\tilde{\mathbb{A}}}\tilde{\phi} + \tilde{\phi}\tilde{d}_{\tilde{\mathbb{A}}\otimes_{T_p}\tilde{\mathbb{A}}} = F_{\tilde{\mathbb{A}}}.$$

It is clear that

$$\tilde{d}_{\tilde{\mathbb{A}}}\tilde{\phi} = (d_{\mathbb{A}} \otimes id_{kG})(\phi \otimes id_{kG})\psi = (d_{\mathbb{A}}\phi \otimes id_{kG})\psi. \tag{4.7}$$

We now claim that

$$\psi\tilde{d}_{\tilde{\mathbb{A}}\otimes_{T_p}\tilde{\mathbb{A}}} = (d_{\mathbb{A}\otimes_A\mathbb{A}} \otimes id_{kG})\psi. \tag{4.8}$$

By definition

$$\tilde{d}_{\tilde{\mathbb{A}}\otimes_{T_p}\tilde{\mathbb{A}}} = \tilde{d}_{\tilde{\mathbb{A}}} \otimes_{T_p} id_{T_p} + (-1)^* id_{T_p} \otimes_{T_p} \tilde{d}_{\tilde{\mathbb{A}}}$$

where $*$ is the degree of the element in left $A \otimes T_p$. Moreover, $(A \otimes T_p) \otimes_{T_p} (A \otimes T_p)$ is generated by $\xi_m 1_G \otimes_{T_p} x^i\xi_n 1_G$ as T_p-bimodule. Without loss of generality, assume m and n are odd. Then we have the following calculation:

$$\begin{aligned}&\psi\tilde{d}_{\tilde{\mathbb{A}}\otimes_{T_p}\tilde{\mathbb{A}}}(\xi_m 1_G \otimes_{T_p} x^i\xi_n 1_G)\\ &= \psi((x\xi_m 1_G - \xi_m x 1_G) \otimes_{T_p} x^i\xi_n 1_G - \xi_m 1_G \otimes_{T_p} (x^{i+1}\xi_n 1_G - x^i\xi_n x 1_G))\\ &= (x\xi_m - \xi_m x) \otimes_A x^i\xi_n 1_G - \xi_m \otimes_A (x^{i+1}\xi_n - x^i\xi_n x)1_G\end{aligned}$$

and

$$\begin{aligned}&(d_{\mathbb{A}\otimes_A\mathbb{A}} \otimes id_{kG})\psi(\xi_m 1_G \otimes_{T_p} x^i\xi_n 1_G)\\ &= (d_{\mathbb{A}\otimes_A\mathbb{A}} \otimes id_{kG})(\xi_m \otimes_A x^i\xi_n 1_G)\\ &= (x\xi_m - \xi_m x) \otimes_A x^i\xi_n 1_G - \xi_m \otimes_A (x^{i+1}\xi_n - x^i\xi_n x)1_G.\end{aligned}$$

The calculation is similar for the other cases of m and n. Therefore,

$$\tilde{\phi}\tilde{d}_{\tilde{\mathbb{A}}\otimes_{T_p}\tilde{\mathbb{A}}} = (\phi \otimes id_{kG})\psi\tilde{d}_{\tilde{\mathbb{A}}\otimes_{T_p}\tilde{\mathbb{A}}} = (\phi \otimes id_{kG})(d_{\mathbb{A}\otimes_A\mathbb{A}} \otimes id_{kG})\psi = (\phi d_{\mathbb{A}\otimes_A\mathbb{A}} \otimes id_{kG})\psi. \tag{4.9}$$

By combining (4.7) and (4.9), we obtain

$$\tilde{d}_{\tilde{\mathbb{A}}}\tilde{\phi} + \tilde{\phi}\tilde{d}_{\tilde{\mathbb{A}}\otimes_{T_p}\tilde{\mathbb{A}}} = ((d_{\mathbb{A}}\phi + \phi d_{\mathbb{A}\otimes_A\mathbb{A}}) \otimes id_{kG})\psi = (F_{\mathbb{A}} \otimes id_{kG})\psi = F_{\tilde{\mathbb{A}}}$$

whence $\tilde{\phi} = (\phi \otimes id_{kG})\psi$ is a contracting homotopy for $F_{\tilde{\mathbb{A}}}$. □

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We use the Lemma 4.6 and find the following T_p^e-linear maps $\tilde{\phi}_i : (\tilde{\mathbb{A}} \otimes_{T_p} \tilde{\mathbb{A}})_i \longrightarrow \tilde{\mathbb{A}}_{i+1}$:

$$\begin{aligned}
\tilde{\phi}_0(\xi_0 1_G \otimes_{T_p} x^i \xi_0 1_G) &= \sum_{l=0}^{i-1} x^l \xi_1 x^{i-1-l} 1_G, \\
\tilde{\phi}_1(\xi_1 1_G \otimes_{T_p} x^i \xi_0 1_G) &= -\delta_{i,p-1} \xi_2 1_G, \\
\tilde{\phi}_1(\xi_0 1_G \otimes_{T_p} x^i \xi_1 1_G) &= \delta_{i,p-1} \xi_2 1_G.
\end{aligned}$$

Next, we give a lemma to find the diagonal map.

Lemma 4.10. *The map $\tilde{\Delta} := \psi^{-1}(\Delta \otimes id_{kG})$ is a diagonal map on $\tilde{\mathbb{A}}$ where Δ is in (3.4).*

Proof. We need to check that $\tilde{\Delta}$ is a chain map. The following equations are straightforward by considering the fact that Δ is a chain map and (4.8):

$$\begin{aligned}
\tilde{d}_{\tilde{\mathbb{A}} \otimes_{T_p} \tilde{\mathbb{A}}} \tilde{\Delta} &= \tilde{d}_{\tilde{\mathbb{A}} \otimes_{T_p} \tilde{\mathbb{A}}} \psi^{-1}(\Delta \otimes id_{kG}) = \psi^{-1}(d_{\mathbb{A} \otimes_A \mathbb{A}} \otimes id_{kG})(\Delta \otimes id_{kG}) \\
&= \psi^{-1}(d_{\mathbb{A} \otimes_A \mathbb{A}} \Delta \otimes id_{kG}) = \psi^{-1}(\Delta d_{\mathbb{A}} \otimes id_{kG}) = \psi^{-1}(\Delta \otimes id_{kG})(d_{\mathbb{A}} \otimes id_{kG}) \\
&= \tilde{\Delta} \tilde{d}_{\tilde{\mathbb{A}}}. \quad \square
\end{aligned}$$

Lemma 4.10 allows us to compute the T_p-linear map $\tilde{\Delta} : \tilde{\mathbb{A}}_{i+1} \longrightarrow (\tilde{\mathbb{A}} \otimes_{T_p} \tilde{\mathbb{A}})_i$ as follows:

$$\begin{aligned}
\tilde{\Delta}_0(\xi_0 1_G) &= \xi_0 1_G \otimes_{T_p} \xi_0 1_G, \\
\tilde{\Delta}_1(\xi_1 1_G) &= \xi_1 1_G \otimes_{T_p} \xi_0 1_G + \xi_0 1_G \otimes_{T_p} \xi_1 1_G, \\
\tilde{\Delta}_{2n}(\xi_{2n} 1_G) &= \sum_{i=0}^{n} \xi_{2i} 1_G \otimes_{T_p} \xi_{2n-2i} 1_G \\
&\quad + \sum_{i=0}^{n-1} \sum_{\substack{a+b+c \\ =p-2}} x^a \xi_{2i+1} 1_G \otimes_{T_p} x^b \xi_{2n-2i-1} x^c 1_G, \text{ for } n \geq 1 \\
\tilde{\Delta}_{2n+1}(\xi_{2n+1} 1_G) &= \sum_{i=0}^{2n+1} \xi_i 1_G \otimes_{T_p} \xi_{2n+1-i} 1_G, \text{ for } n \geq 1.
\end{aligned}$$

Before computing the bracket on Hochschild cohomology of T_p, we need to find a basis of $\mathrm{Hom}_{T_p^e}(\tilde{\mathbb{A}}, T_p)$. In particular, we must find a basis of $\mathrm{Hom}_{T_p^e}(A \otimes T_p, T_p)$ as it is an invariant in each degree.

It is known that

$$\mathrm{HH}^*(T_p) := \mathrm{Ext}^*_{T_p^e}(T_p, T_p) \cong \mathrm{Ext}^*_{\mathcal{D}}(A, T_p) \cong \mathrm{Ext}^*_{A^e}(A, T_p)^G.$$

The Eckmann-Shapiro Lemma (Lemma 5.3) and (4.2) imply the first isomorphism and see [10, Theorem 3.6.2] for the second isomorphism.

Consider the following resolution

$$\mathrm{Hom}_{A^e}(\mathbb{A}, T_p)^G : 0 \longrightarrow \mathrm{Hom}_{A^e}(A^e, T_p)^G \longrightarrow \mathrm{Hom}_{A^e}(A^e, T_p)^G \longrightarrow \cdots \tag{4.11}$$

where the action of G on $\mathrm{Hom}_{A^e}(A^e, T_p)$ is defined by

$$g \cdot f(a_1 \otimes a_2) = {}^g f({}^{g^{-1}}(a_1 \otimes a_2)). \tag{4.12}$$

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This resolution is clearly isomorphic to

$$0 \longrightarrow T_p^G \longrightarrow T_p^G \longrightarrow T_p^G \longrightarrow \cdots \tag{4.13}$$

with the correspondence

$$f_t \mapsto t \text{ where } f_t(\xi_*) = t \text{ for all } t \in T_p. \tag{4.14}$$

We claim that $\mathrm{Hom}_{T_p^e}(A \otimes T_p, T_p) \cong T_p^G$. Suppose $x^i g^j \in T_p^G$. Then, we have $f_{x^i g^j} \in \mathrm{Hom}_{A^e}(A^e, T_p)^G$ defined by $f_{x^i g^j}(x^k \otimes x^l) := x^{k+l+i} g^j$ where $x^* \in A$. Now observe that, $f_{x^i g^j} \in \mathrm{Hom}_{A^e}(A^e, T_p)^G$ is a $\mathcal{D}$-module homomorphism since

$$\begin{aligned} f_{x^i g^j}((x^k \xi_* x^l g)(a_1 \otimes a_2)) &= f_{x^i g^j}((x^k \xi_* x^l 1_G) g(a_1 \otimes a_2)) = (x^k \xi_* x^l 1_G) f_{x^i g^j}(g(a_1 \otimes a_2)) \\ &= (x^k \xi_* x^l 1_G) g f_{x^i g^j}(a_1 \otimes a_2) = (x^k \xi_* x^l g) f_{x^i g^j}(a_1 \otimes a_2) \end{aligned}$$

where $x^k \xi_* x^l g \in \mathcal{D}, a_1 \otimes a_2 \in A^e$. Moreover, if $f \in \mathrm{Hom}_{\mathcal{D}}(A^e, T_p)$, then f is G-invariant as

$$g \cdot f(a_1 \otimes a_2) = {}^g f({}^{g^{-1}}(a_1 \otimes a_2)) = {}^{(gg^{-1})} f(a_1 \otimes a_2) = f(a_1 \otimes a_2)$$

where $g \in G, a_1 \otimes a_2 \in A^e$. Hence, the isomorphism from $\mathrm{Hom}_{A^e}(A^e, T_p)^G$ to $\mathrm{Hom}_{\mathcal{D}}(A^e, T_p)$ is the identity, so that $f_{x^i g^j}$ is also in $\mathrm{Hom}_{\mathcal{D}}(A^e, T_p)$. We next use the Eckmann-Shapiro lemma (Lemma 5.3) which implies that $\mathrm{Ext}^*_{\mathcal{D}}(A, T_p) \cong \mathrm{Ext}^*_{T_p^e}(T_p^e \otimes_{\mathcal{D}} A, T_p)$ and the isomorphism is given by

$$\begin{aligned} \sigma(f_{x^i g^j})(x^m g^s \otimes x^n g^r \otimes_{\mathcal{D}} x^k \otimes x^l) &= x^m g^s \otimes x^n g^r f_{x^i g^j}(x^k \otimes x^l) = x^m g^s \otimes x^n g^r (x^{k+l+i} g^j) \\ &= (x^m g^s)(x^{k+l+i} g^j)(x^n g^r) \\ &= ((x^m ({}^{g^s} x^{k+l+i})) g^{s+j})(x^n g^r) \\ &= \omega^{s(k+l+i)} (x^{m+k+l+i} g^{s+j})(x^n g^r) \\ &= \omega^{s(k+l+i)} (x^{m+k+l+i} ({}^{g^{s+j}} x^n)) g^{j+s+r} \\ &= \omega^{s(k+l+i+n)+jn} x^{i+k+l+m+n} g^{j+s+r}. \end{aligned}$$

Hence, $\sigma(f_{x^i g^j})$ is in $\mathrm{Hom}_{T_p^e}(T_p^e \otimes_{\mathcal{D}} A^e, T_p)$. Lastly, recall that $T_p^e \otimes_{\mathcal{D}} A^e \cong A \otimes T_p$ via κ (4.2); so that,

$$\kappa^*(\sigma(f_{x^i g^j}))(x^k \otimes x^l g^r) = \sigma(f_{x^i g^j})((1_{T_p} \otimes \xi_* g^r) \otimes_{\mathcal{D}} x^k \otimes x^l) = x^{i+k+l} g^{j+r}$$

which implies $\kappa^*(\sigma(f_{x^i g^j})) \in \mathrm{Hom}_{T_p^e}(A \otimes T_p, T_p)$. For simplicity, we define $\tilde{f}_{x^i g^j} := \kappa^*(\sigma(f_{x^i g^j}))$.

The action of G on T_p given by (4.12) and (4.14) depends on degree. Since T_p^G is spanned by $\{1, g, \cdots, g^{p-1}\}$ in even degrees and $\{x, xg, \cdots, xg^{p-1}\}$ in odd degrees [7, Section 8.2], we have $\{\tilde{f}_1, \tilde{f}_g, \cdots, \tilde{f}_{g^{p-1}}\}$ in even degrees and $\{\tilde{f}_x, \tilde{f}_{xg}, \cdots, \tilde{f}_{xg^{p-1}}\}$ in the odd degrees as a basis of $\mathrm{Hom}_{T_p^e}(A \otimes T_p, T_p)$.

We only calculate the bracket in degree 1 and 2 as before so we can extend it to higher degrees by the relation between cup product and the bracket. Since $A \otimes T_p \cong A^e \otimes kG$ as vector spaces, $\xi_i 1_G$ generates $A \otimes T_p$ as a T_p-bimodule. Through the calculation, id represents $id_{A \otimes T_p}$ and $\otimes$ represents $\otimes_{T_p}$.

The circle product of two elements in degree one is

$$\begin{aligned} (\tilde{f}_{xg^i} \circ_{\tilde{\phi}} \tilde{f}_{xg^j})(\xi_1 1_G) &= \tilde{f}_{xg^i} \tilde{\phi}(id \otimes \tilde{f}_{xg^j} \otimes id) \tilde{\Delta}^{(2)}(\xi_1 1_G) \\ &= \tilde{f}_{xg^i} \tilde{\phi}(id \otimes \tilde{f}_{xg^j} \otimes id)(\xi_0 1_G \otimes \xi_0 1_G \otimes \xi_1 1_G + \xi_0 1_G \otimes \xi_1 1_G \otimes \xi_0 1_G \end{aligned}$$

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$$+\xi_1 1_G \otimes \xi_0 1_G \otimes \xi_0 1_G)$$
$$= \tilde{f}_{xg^i}\tilde{\phi}(\xi_0 1_G \otimes x\xi_0 g^j) = \tilde{f}_{xg^i}(\xi_1 g^j) = xg^{i+j}.$$

Because of the symmetry, $(\tilde{f}_{xg^j} \circ_{\tilde{\phi}} \tilde{f}_{xg^i})(\xi_1 1_G) = xg^{i+j}$. Therefore

$$[\tilde{f}_{xg^i}, \tilde{f}_{xg^j}](\xi_1 1_G) = xg^{i+j} - (-1)^0 xg^{i+j} = 0.$$

The circle product of the elements of degrees 1 and 2:

$$\begin{aligned}(\tilde{f}_{xg^i} \circ_{\tilde{\phi}} \tilde{f}_{g^j})(\xi_2 1_G) &= \tilde{f}_{xg^i}\tilde{\phi}(id \otimes \tilde{f}_{g^j} \otimes id)\tilde{\Delta}^{(2)}(\xi_2 1_G) = \tilde{f}_{xg^i}\tilde{\phi}(id \otimes \tilde{f}_{g^j} \otimes id)\\ &(\xi_0 1_G \otimes \xi_0 1_G \otimes \xi_2 1_G + \xi_0 1_G \otimes \xi_2 1_G \otimes \xi_0 1_G\\ &+ \xi_0 1_G \otimes \sum_{\substack{a+b+c\\=p-2}} (x^a \xi_1 1_G \otimes x^b \xi_1 x^c 1_G) + \xi_2 1_G \otimes \xi_0 1_G \otimes \xi_0 1_G\\ &+ \sum_{\substack{a+b+c\\=p-2}} (x^a \xi_1 1_G \otimes (x^b \xi_0 1_G \otimes \xi_1 x^c 1_G + x^b \xi_1 1_G \otimes \xi_0 x^c 1_G)))\\ &= \tilde{f}_{xg^i}\tilde{\phi}(\xi_0 1_G \otimes \xi_0 g^j) = 0.\end{aligned}$$

And the circle product on the reverse order:

$$\begin{aligned}(\tilde{f}_{g^j} \circ_{\tilde{\phi}} \tilde{f}_{xg^i})(\xi_2 1_G) &= \tilde{f}_{g^j}\tilde{\phi}(id \otimes \tilde{f}_{xg^i} \otimes id)\tilde{\Delta}^{(2)}(\xi_2 1_G) = \tilde{f}_{g^j}\tilde{\phi}(id \otimes \tilde{f}_{xg^i} \otimes id)\\ &(\xi_0 1_G \otimes \xi_0 1_G \otimes \xi_2 1_G + \xi_0 1_G \otimes \xi_2 1_G \otimes \xi_0 1_G\\ &+ \xi_0 1_G \otimes \sum_{\substack{a+b+c\\=p-2}} (x^a \xi_1 1_G \otimes x^b \xi_1 x^c 1_G) + \xi_2 1_G \otimes \xi_0 1_G \otimes \xi_0 1_G\\ &+ \sum_{\substack{a+b+c\\=p-2}} (x^a \xi_1 1_G \otimes (x^b \xi_0 1_G \otimes \xi_1 x^c 1_G + x^b \xi_1 1_G \otimes \xi_0 x^c 1_G)))\\ &= \tilde{f}_{g^j}\tilde{\phi}(\sum_{\substack{a+b+c\\=p-2}} \omega^{i(b+c)} \xi_0 1_G \otimes x^{a+b+1} \xi_1 x^c g^i + \omega^{ic} x^a \xi_1 1_G \otimes x^{b+1} \xi_0 x^c g^i)\\ &= \tilde{f}_{g^j}(\sum_{\substack{a+b+c\\=p-2}} \omega^{i(b+c)} \delta_{a+b+1,p-1} x^c \xi_2 g^i - \omega^{ic} \delta_{b+1,p-1} x^{a+c} \xi_2 g^i)\\ &= \tilde{f}_{g^j}(\sum_{b=0}^{p-2} \omega^{ib} \xi_2 g^i) - \tilde{f}_{g^j}(\xi_2 g^i)\\ &= \begin{cases} (p-2)g^j, & \text{for } i = 0\\ -(\omega^{-i}+1)g^{i+j}, & \text{for } i \neq 0 \end{cases}.\end{aligned}$$

Therefore, we obtain

$$[\tilde{f}_{xg^i}, \tilde{f}_{g^j}] = \begin{cases} -(p-2)g^j, & \text{for } i = 0\\ (\omega^{-i}+1)g^{i+j}, & \text{for } i \neq 0 \end{cases}.$$

Lastly, the bracket of the elements of degrees 2 and 2:

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$$
\begin{aligned}
(\tilde{f}_{g^i} \circ_{\tilde{\phi}} \tilde{f}_{g^j})(\xi_3 1_G) &= \tilde{f}_{g^i}\tilde{\phi}(id \otimes \tilde{f}_{g^j} \otimes id)\tilde{\Delta}^{(2)}(\xi_3 1_G) = \tilde{f}_{g^i}\tilde{\phi}(id \otimes \tilde{f}_{g^j} \otimes id) \\
&\quad (\xi_0 1_G \otimes \xi_0 1_G \otimes \xi_3 1_G + \xi_0 1_G \otimes \xi_1 1_G \otimes \xi_2 1_G + \xi_0 1_G \otimes \xi_2 1_G \otimes \xi_1 1_G \\
&\quad + \xi_0 1_G \otimes \xi_3 1_G \otimes \xi_0 1_G + \xi_1 1_G \otimes \xi_2 1_G \otimes \xi_0 1_G + \xi_1 1_G \otimes \xi_0 1_G \otimes \xi_2 1_G \\
&\quad + \xi_2 1_G \otimes \xi_1 1_G \otimes \xi_0 1_G + \xi_2 1_G \otimes \xi_0 1_G \otimes \xi_1 1_G + \xi_3 1_G \otimes \xi_0 1_G \otimes \xi_0 1_G) \\
&\quad = \tilde{f}_{g^i}\tilde{\phi}(\xi_0 1_G \otimes \xi_1 g^j + \xi_1 1_G \otimes \xi_0 g^j) = 0
\end{aligned}
$$

and by symmetry $(\tilde{f}_{g^j} \circ_{\tilde{\phi}} \tilde{f}_{g^i})(\xi_3 1_G) = 0$. Therefore, we have $[\tilde{f}_{g^i}, \tilde{f}_{g^j}] = 0$. As a consequence, the bracket for the elements of degree 1 and 2 are

$$
[\tilde{f}_{xg^i}, \tilde{f}_{xg^j}] = 0, [\tilde{f}_{xg^i}, \tilde{f}_{g^j}] = \begin{cases} -(p-2)g^j, & \text{for } i = 0 \\ (\omega^{-i}+1)g^{i+j}, & \text{for } i \neq 0 \end{cases}, [\tilde{f}_{g^i}, \tilde{f}_{g^j}] = 0.
$$

By the identity (2.3), brackets in higher degrees can be determined, since the Hochschild cohomology is generated as an algebra under cup product in degrees 1 and 2.

Hopf algebra cohomology of T_p and Hochschild cohomology of T_p were calculated before by V. C. Nguyen [7, Section 8] as the Hopf algebra cohomology

$$
\mathrm{H}^n(T_p, k) = \begin{cases} k & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd,} \end{cases}
$$

and the Hochschild cohomology

$$
\mathrm{HH}^n(T_p, T_p) = \begin{cases} k & \text{if } n \text{ is even,} \\ Span_k\{x\} & \text{if } n \text{ is odd.} \end{cases}
$$

It is known that for any Hopf algebra with bijective antipode, the Hopf algebra cohomology can be embedded into the Hochschild cohomology [10, Theorem 9.4.5 and Corollary 9.4.7]. Since any finite dimensional Hopf algebra has a bijective antipode, the Taft algebra T_p is also a Hopf algebra with a bijective antipode. The embedding of $\mathrm{H}^n(T_p, k)$ into $\mathrm{HH}^n(T_p, T_p)$ turns out to be the map that is identity in even degrees and zero on odd degrees. Then, the corresponding bracket in Hopf algebra cohomology is

$$
[\tilde{f}_{g^i}, \tilde{f}_{g^j}] = 0,
$$

so that, the bracket on Hopf algebra cohomology for the elements of all degrees is 0 by the identity (2.3). We summarize all the work in this section with the following theorem:

Theorem 4.15. *The graded Lie algebra structure on Hopf algebra cohomology of a Taft algebra is abelian.*

This is the first example of the Gerstenhaber bracket on the Hopf algebra cohomology of a nonquasi-triangular Hopf algebra and our calculation shows that the bracket on Hopf algebra cohomology of a Taft algebra is zero as it is on the Hopf algebra cohomology of any quasi-triangular algebra. A natural question that arises whether the bracket structure on the Hopf algebra cohomology is always trivial. In the next section, we explore a general expression for the bracket on the Hopf algebra cohomology that may help us to approach this question with a more theoretical perspective in the future researches.

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5. Gerstenhaber bracket for Hopf algebras

In this section, we want to explore an expression for Gerstenhaber bracket on a Hopf algebra A with a bijective antipode S.

We give the following lemma which helps us to define the Gerstenhaber bracket on an equivalent resolution to the bar resolution of A as an A-bimodule.

Lemma 5.1. *Let A be a Hopf algebra with bijective antipode. Let $P_\bullet$ be the bar resolution of k as a left A-module:*

$$P_\bullet : \cdots \xrightarrow{d_3} A^{\otimes 3} \xrightarrow{d_2} A^{\otimes 2} \xrightarrow{d_1} A \xrightarrow{\varepsilon} k \longrightarrow 0,$$

with differentials

$$d_n(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = \sum_{i=0}^{n-1} (-1)^i a_0 \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n + (-1)^n \varepsilon(a_n) a_0 \otimes \cdots \otimes a_{n-1}$$

Then $X_\bullet = A^e \otimes_A P_\bullet$ is equivalent to the bar resolution of A as an A-bimodule.

Proof. Since S is bijective [10, Lemma 9.2.9], A^e is projective as a right A-module. Also there is an A^e-module isomorphism $\rho : A \to A^e \otimes_A k$ defined by $\rho(a) = a \otimes 1 \otimes 1$ for all $a \in A$ [10, Lemma 9.4.2].

For each n, define $\theta_n : X_n \to A^{\otimes(n+2)}$ by

$$\theta_n((a \otimes b) \otimes_A (1 \otimes c^1 \otimes c^2 \otimes \cdots \otimes c^n)) = \sum a \otimes c_1^1 \otimes c_1^2 \otimes \cdots \otimes c_1^n \otimes S(c_2^1 c_2^2 \cdots c_2^n) b$$

for all $a, b, c^1, \cdots c^n \in A$.

Now, we show that θ is a chain map:

$$\begin{aligned}
&\theta_{n-1} d_n((a \otimes b) \otimes_A (1 \otimes c^1 \otimes c^2 \otimes \cdots \otimes c^n)) \\
&= \theta_{n-1}((a \otimes b) \otimes_A (c^1 \otimes c^2 \otimes \cdots \otimes c^n) \\
&\quad + \sum_{i=1}^{n-1} (-1)^i (a \otimes b) \otimes_A (1 \otimes c^1 \otimes c^2 \otimes \cdots \otimes c^i c^{i+1} \otimes \cdots \otimes c^n) \\
&\quad + (-1)^n (a \otimes b) \otimes_A (\varepsilon(c^n) \otimes c^1 \otimes c^2 \otimes \cdots \otimes c^{n-1})) \\
&= \theta_{n-1}(\sum (a c_1^1 \otimes S(c_2^1) b) \otimes_A (1 \otimes c^2 \otimes \cdots \otimes c^n) \\
&\quad + \sum_{i=1}^{n-1} (-1)^i (a \otimes b) \otimes_A (1 \otimes c^1 \otimes c^2 \otimes \cdots \otimes c^i c^{i+1} \otimes \cdots \otimes c^n) \\
&\quad + (-1)^n (\varepsilon(c^n) a \otimes b) \otimes_A (1 \otimes c^1 \otimes c^2 \otimes \cdots \otimes c^{n-1})) \\
&= \sum a c_1^1 \otimes c_1^2 \otimes \cdots \otimes c_1^n \otimes S(c_2^2 \cdots c_2^n) S(c_2^1) b \\
&\quad + \sum_{i=1}^{n-1} (-1)^i \sum a \otimes c_1^1 \otimes \cdots \otimes c_1^i c_1^{i+1} \otimes \cdots \otimes c_1^n \otimes S(c_2^1 \cdots c_2^n) b \\
&\quad + \sum (-1)^n a \otimes c_1^1 \otimes \cdots \otimes c_1^{n-1} \otimes \varepsilon(c^n) S(c_2^1 \cdots c_2^{n-1}) b
\end{aligned}$$

and

$$
\begin{aligned}
&d_n\theta_n((a\otimes b)\otimes_A(1\otimes c^1\otimes c^2\otimes\cdots\otimes c^n))\\
&=d_n(\sum a\otimes c_1^1\otimes c_1^2\otimes\cdots\otimes c_1^n\otimes S(c_2^1c_2^2\cdots c_2^n)b)\\
&=\sum ac_1^1\otimes c_1^2\otimes\cdots\otimes c_1^n\otimes S(c_2^1c_2^2\cdots c_2^n)b\\
&\quad+\sum\sum_{i=1}^{n-1}(-1)^i a\otimes c_1^1\otimes\cdots\otimes c_1^ic_1^{i+1}\otimes\cdots\otimes c_1^n\otimes S(c_2^1\cdots c_2^n)b\\
&\quad+\sum(-1)^n a\otimes c_1^1\otimes\cdots\otimes c_1^{n-1}\otimes c_1^nS(c_2^1\cdots c_2^n)b
\end{aligned}
$$

where $\Delta(c)=\sum c_1\otimes c_2$ known as Sweedler's notation.

Since S is an algebra anti-homomorphism that is convolution inverse to the identity map,

$$\sum c_1^nS(c_2^1\cdots c_2^n)=\sum c_1^nS(c_2^n)S(c_2^{n-1})\cdots S(c_2^1)=\sum\varepsilon(c^n)S(c_2^1\cdots c_2^{n-1})$$

and

$$S(c_2^2\cdots c_2^n)S(c_2^1)=S(c_2^1c_2^2\cdots c_2^n)$$

so that the two expressions are equal which follows θ is a chain map.

Lastly, one can see that the A^e-module homomorphism

$$\psi_n(a\otimes c^1\otimes c^2\otimes\cdots c^n\otimes b)=\sum(a\otimes c_2^1c_2^2\cdots c_2^nb)\otimes_A(1\otimes c_1^1\otimes c_1^2\otimes\cdots\otimes c_1^n)$$

is the inverse of θ_n by using the property that S is an algebra anti-homomorphism that is convolution inverse to the identity map. □

Let $f_x\in\mathrm{Hom}_{A^e}(X_m,A)$ and $g_x\in\mathrm{Hom}_{A^e}(X_n,A)$. Then we define the X-bracket $[f_x,g_x]_X\in\mathrm{Hom}_{A^e}(X_{m+n-1},A)$ to be a composition $X\xrightarrow{\theta}B(A)\xrightarrow{[\psi^*f_x.\psi^*g_x]}A$; so that, we have

$$[f_x,g_x]_X=[\psi^*f_x,\psi^*g_x]\theta=(\psi^*f_x\circ\psi^*g_x)\theta-(-1)^{(m-1)(n-1)}(\psi^*g_x\circ\psi^*f_x)\theta$$

where

$$
\begin{aligned}
&(\psi^*f_x\circ\psi^*g_x)\theta_{m+n-1}((a\otimes b)\otimes_A 1\otimes c^1\otimes\cdots\otimes c^{m+n-1})\\
&=(\psi^*f_x\circ\psi^*g_x)(\sum a\otimes c_1^1\otimes c_1^2\otimes\cdots\otimes c_1^{m+n-1}\otimes S(c_2^1c_2^2\cdots c_2^{m+n-1})b)\\
&=\sum\sum_{i=1}^{m}(-1)^{(n-1)(i-1)}f_x\psi_m(a\otimes c_1^1\otimes\cdots\otimes c_1^{i-1}\otimes g_x\psi_n(1\otimes c_1^i\otimes\cdots\otimes c_1^{i+n-1}\otimes 1)\\
&\quad\otimes c_1^{i+n}\otimes\cdots\otimes c_1^{m+n-1}\otimes S(c_2^1c_2^2\cdots c_2^{m+n-1})b)\\
&=\sum\sum_{i=1}^{m}(-1)^{(n-1)(i-1)}f_x\psi_m(a\otimes c_1^1\otimes\cdots\otimes c_1^{i-1}\\
&\quad\otimes\sum g_x(1\otimes c_2^ic_2^{i+1}\cdots c_2^{i+n-1}\otimes_A 1\otimes c_1^i\otimes c_1^{i+1}\otimes\cdots\otimes c_1^{i+n-1})\\
&\quad\otimes c_1^{i+n}\otimes\cdots\otimes c_1^{m+n-1}\otimes S(c_2^1\cdots c_2^{i-1}c_3^i\cdots c_3^{i+n-1}c_2^{i+n}\cdots c_2^{m+n-1})b)\\
&=\sum\sum_{i=1}^{m}(-1)^{(n-1)(i-1)}f_x(a\otimes c_2^1c_2^2\cdots c_2^{i-1}c_2^*c_2^{i+n}\cdots c_2^{m+n-1}S(c_3^1c_3^2\cdots c_3^{m+n-1})b\\
&\quad\otimes_A 1\otimes c_1^1\otimes c_1^2\otimes\cdots\otimes c_1^{i-1}\otimes c_1^*\otimes c_1^{i+n}\otimes\cdots\otimes c_1^{m+n-1})
\end{aligned}
$$

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where

$$\Delta(c) = \sum c_1 \otimes c_2, \Delta^{(2)}(c) = \sum c_1 \otimes c_2 \otimes c_3, \Delta(c^*) = \sum c_1^* \otimes c_2^* \text{ and}$$
$$c^* = \sum g_x(1 \otimes c_2^i c_2^{i+1} \cdots c_2^{i+n-1} \otimes_A 1 \otimes c_1^i \otimes c_1^{i+1} \otimes \cdots \otimes c_1^{i+n-1}).$$

This is the general expression of the Gerstenhaber bracket on Hochschild cohomology of A. Next, we start with the following theorem [10, Theorem 9.4.5] to construct an embedding from $\mathrm{H}^*(A,k)$ into $\mathrm{HH}^*(A)$.

Theorem 5.2. *Let A be a Hopf algebra over k with bijective antipode. Then*

$$\mathrm{HH}^*(A) \cong \mathrm{H}^*(A, A^{ad}).$$

In this theorem A^{ad} is an A-module A under left adjoint action, given by $a \cdot b = \sum a_1 b S(a_2)$ for all $a, b \in A$. To find explicit isomorphism between $\mathrm{HH}^*(A)$ and $\mathrm{H}^*(A, A^{ad})$, we give the Eckmann-Shapiro lemma.

Lemma 5.3 *(Eckmann-Shapiro). Let A be a ring and let B be a subring of A such that A is projective as a right B-module. Let M be an A-module and N be a B-module. Then*

$$\mathrm{Ext}_B^n(N, M) \cong \mathrm{Ext}_A^n(A \otimes_B N, M).$$

Proof. Let $P_\bullet \to N$ be a B projective resolution of N. Then $A \otimes_B P_n$ is projective as A-module so that $A \otimes_B P_\bullet \to A \otimes_B N$ is a projective resolution of $A \otimes_B N$ as an A-module. Let

$$\sigma : \mathrm{Hom}_B(P_n, M) \to \mathrm{Hom}_A(A \otimes_B P_n, M) \text{ defined by } \sigma(f)(a \otimes_B p) = af(p),$$
$$\tau : \mathrm{Hom}_A(A \otimes_B P_n, M) \to \mathrm{Hom}_B(P_n, M) \text{ defined by } \tau(g)(p) = g(1 \otimes_B p)$$

where $a \in A, p \in P_n, f \in \mathrm{Hom}_B(P_n, M), g \in \mathrm{Hom}_A(A \otimes_B P_n, M)$. Since σ and τ are inverse of each other and they are homomorphisms, $\mathrm{Hom}_A(A \otimes_B P_n, M) \cong \mathrm{Hom}_B(P_n, M)$. □

If we replace A with A^e, B with A and take $M = A, N = k$ in the Eckmann-Shapiro lemma, we have the isomorphism $\mathrm{Ext}_{A^e}^n(A^e \otimes_A k, A) \cong \mathrm{Ext}_A^n(k, A^{ad})$. We also know that $A \cong A^e \otimes_A k$ [10, Lemma 9.4.2] and the isomorphism is given by $\rho(a) = a \otimes 1 \otimes 1$ for all $a \in A$. Therefore $\mathrm{Ext}_{A^e}^n(A, A) \cong \mathrm{Ext}_{A^e}^n(A^e \otimes_A k, A) \cong \mathrm{Ext}_A^n(k, A^{ad})$.

We already have the Gerstenhaber bracket $[,]_X$ on $\mathrm{Ext}_{A^e}^n(A^e \otimes_A k, A)$. Hence we can use the isomorphisms σ and τ in Eckmann-Shapiro Lemma and find the bracket expression on $\mathrm{H}^*(A, A^{ad})$. Now let $\tilde{f} \in \mathrm{Hom}_A(P_m, A^{ad})$ and $\tilde{g} \in \mathrm{Hom}_A(P_n, A^{ad})$. Then $[\tilde{f}, \tilde{g}]_P \in \mathrm{Hom}_A(P_{m+n-1}, A^{ad})$ and we have

$$\begin{aligned}[\tilde{f}, \tilde{g}]_P &= \tau[\sigma(\tilde{f}), \sigma(\tilde{g})]_X \\ &= \tau((\psi^*(\sigma(\tilde{f})) \circ \psi^*(\sigma(\tilde{g})))\theta) - (-1)^{(m-1)(n-1)}\tau((\psi^*(\sigma(\tilde{g})) \circ \psi^*(\sigma(\tilde{f})))\theta).\end{aligned}$$

For simplification we define

$$\tilde{f} \circ_P \tilde{g} := \tau((\psi^*(\sigma(\tilde{f})) \circ \psi^*(\sigma(\tilde{g})))\theta).$$

Then by using previous circle product formula we obtain:

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$$
\begin{aligned}
&\tilde{f} \circ_P \tilde{g}(1 \otimes c^1 \otimes c^2 \otimes \cdots \otimes c^{m+n-1}) \\
&= \tau((\psi^*(\sigma(\tilde{f})) \circ \psi^*(\sigma(\tilde{g})))\theta)(1 \otimes c^1 \otimes c^2 \otimes \cdots \otimes c^{m+n-1}) \\
&= (\psi^*(\sigma(\tilde{f})) \circ \psi^*(\sigma(\tilde{g})))\theta((1 \otimes 1) \otimes_A 1 \otimes c^1 \otimes c^2 \otimes \cdots \otimes c^{m+n-1}) \\
&= \sum \sum_{i=1}^{m} (-1)^{(n-1)(i-1)} \sigma(\tilde{f})(1 \otimes c_2^1 c_2^2 \cdots c_2^{i-1} c_2^* c_2^{i+n} \cdots c_2^{m+n-1} S(c_3^1 c_3^2 \cdots c_3^{m+n-1}) \\
&\quad \otimes_A 1 \otimes c_1^1 \otimes c_1^2 \otimes \cdots \otimes c_1^{i-1} \otimes c_1^* \otimes c_1^{i+n} \otimes \cdots \otimes c_1^{m+n-1}) \\
&= \sum \sum_{i=1}^{m} (-1)^{(n-1)(i-1)} \tilde{f}(1 \otimes c_1^1 \otimes c_1^2 \otimes \cdots \otimes c_1^{i-1} \otimes c_1^* \otimes c_1^{i+n} \otimes \cdots \otimes c_1^{m+n-1}) \\
&\quad c_2^1 c_2^2 \cdots c_2^{i-1} c_2^* c_2^{i+n} \cdots c_2^{m+n-1} S(c_3^1 c_3^2 \cdots c_3^{m+n-1}))
\end{aligned}
$$

with $\Delta(c^*) = \sum c_1^* \otimes c_2^*$ and

$$
\begin{aligned}
c^* &= \sum \sigma(\tilde{g})(1 \otimes c_2^i c_2^{i+1} \cdots c_2^{i+n-1} \otimes_A 1 \otimes c_1^i \otimes c_1^{i+1} \otimes \cdots \otimes c_1^{i+n-1}) \\
&= \sum (1 \otimes c_2^i c_2^{i+1} \cdots c_2^{i+n-1}) \tilde{g}(1 \otimes c_1^i \otimes c_1^{i+1} \otimes \cdots \otimes c_1^{i+n-1}) \\
&= \sum \tilde{g}(1 \otimes c_1^i \otimes c_1^{i+1} \otimes \cdots \otimes c_1^{i+n-1}) c_2^i c_2^{i+1} \cdots c_2^{i+n-1}.
\end{aligned}
$$

We now have the Lie bracket $[,]_P$ on $\mathrm{H}^*(A, A^{ad})$. Next, we embed $\mathrm{H}^*(A, k)$ into $\mathrm{H}^*(A, A^{ad})$ [10, Corollary 9.4.7] via the unit map

$$\eta_* : \mathrm{Hom}_A(P_\bullet, k) \to \mathrm{Hom}_A(P_\bullet, A^{ad}).$$

Let $f \in \mathrm{Hom}_A(P_m, k)$ and $g \in \mathrm{Hom}_A(P_n, k)$. Then by using counit map

$$\varepsilon_* : \mathrm{Hom}_A(P_\bullet, A) \to \mathrm{Hom}_A(P_\bullet, k),$$

η_* and bracket on $\mathrm{H}^*(A, A^{ad})$, we derive the formula for $[f, g] \in \mathrm{Hom}_A(P_{m+n-1}, k)$:

$$[f, g] = \varepsilon_*[\eta_*(f), \eta_*(g)]_P = \varepsilon_*(\eta_*(f) \circ_P \eta_*(g)) - (-1)^{(m-1)(n-1)} \varepsilon_*(\eta_*(g) \circ_P \eta_*(f)) \tag{5.4}$$

where

$$
\begin{aligned}
&\varepsilon_*((\eta_*(f) \circ_P \eta_*(g))(1 \otimes c^1 \otimes c^2 \otimes \cdots \otimes c^{m+n-1})) \\
&= \varepsilon(\sum \sum_{i=1}^{m} (-1)^{(n-1)(i-1)} \eta(f(1 \otimes c_1^1 \otimes c_1^2 \otimes \cdots \otimes c_1^{i-1} \otimes c_1^* \otimes c_1^{i+n} \otimes \cdots \otimes c_1^{m+n-1})) \\
&\quad c_2^1 c_2^2 \cdots c_2^{i-1} c_2^* c_2^{i+n} \cdots c_2^{m+n-1} S(c_3^1 c_3^2 \cdots c_3^{m+n-1}))
\end{aligned}
$$

with

$$
\begin{aligned}
\Delta(c^*) &= \sum c_1^* \otimes c_2^* \text{ and} \\
c^* &= \sum \eta(g(1 \otimes c_1^i \otimes c_1^{i+1} \otimes \cdots \otimes c_1^{i+n-1})) c_2^i c_2^{i+1} \cdots c_2^{i+n-1}.
\end{aligned}
$$

Therefore, the formula (5.4) is a general expression of the Gerstenhaber bracket on a Hopf algebra cohomology which is indeed inherited from the formula of the bracket on Hochschild cohomology.

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