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Factorizations in numerical semigroup algebras

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A B S T R A C T

We study a numerical semigroup ring as an algebra over another numerical semigroup ring. The complete intersection property of numerical semigroup algebras is investigated using factorizations of monomials into minimal ones. The goal is to study whether a flat rectangular algebra is a complete intersection. Along this direction, special types of algebras generated by few monomials are worked out in detail.

1. Introduction

Despite its simple definition, numerical semigroups provide a fertile ground for research [12]. Following [10], we investigate algebraic properties of an exponential counterpart of numerical semigroups from a relative point of view. Throughout this paper, κ is a field. A numerical semigroup ring is a complete local domain of the form $\kappa[\![\mathbf{u}^{s_1}, \dots, \mathbf{u}^{s_n}]\!]$, where $s_1, \dots, s_n$ are positive *rational* numbers. Note that, in the literature, $s_1, \dots, s_n$ are often assumed to be relatively prime positive integers. The objects we study are local homomorphisms $R \to R'$ of numerical semigroup rings. Through the homomorphism, we have an algebra R' over the coefficient ring R, which we denote by R'/R. Within Cohen–Macaulay homomorphisms, we are mainly interested in the property of complete intersection. We remark that the classical study of

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numerical semigroup rings is a special case of our relative situation. Indeed, a numerical semigroup ring can be considered as a numerical semigroup algebra over a Noether normalization.

Following the terminology of [6,7], a numerical semigroup algebra R'/R is called *Cohen–Macaulay* (resp. *complete intersection*) if the homomorphism $R \to R'$ is flat and its fibers are Cohen–Macaulay (resp. complete intersection) rings. There are only two fibers of the homomorphism. Both are zero dimensional and hence Cohen–Macaulay. Therefore the Cohen–Macaulay property for a numerical semigroup algebra is simply equivalent to flatness. The notion of complete intersection has been extended to arbitrary homomorphisms of Noetherian rings [1]. In this paper, flatness is included as a part of the definition of complete intersection so that the hierarchy of complete intersection inside Cohen–Macaulay automatically holds.

In the context of numerical semigroup algebras, our definition of complete intersection takes the form directly from the historical origin. Recall that, in algebraic geometry, a d-dimensional variety in the n-dimensional ambient space is complete intersection if it can be cut out by $n-d$ hypersurfaces. For a flat numerical semigroup algebra R'/R, we consider a local surjective R-algebra homomorphism

$$\hat{\pi}\colon R[\![Y_1,\dots,Y_n]\!] \to R'.$$

The power series ring $R[\![Y_1,\dots,Y_n]\!]$ has dimension $n+1$. The kernel of $\hat{\pi}$ needs at least n generators. If the kernel can be generated by n elements, the flat algebra R'/R is called *complete intersection*. In the language of [1],

$$R \to R[\![Y_1,\dots,Y_n]\!] \to R'$$

is a Cohen factorization of $R \to R'$. If the kernel of $\hat{\pi}$ is generated by a regular sequence, the homomorphism $R \to R'$ is called complete intersection at the maximal ideal of R'. If $R \to R'$ is complete intersection at the maximal ideal, it is also complete intersection at the zero ideal [13]. In [1], a homomorphism is called complete intersection if it is complete intersection at all prime ideals. Note that the kernel of $\hat{\pi}$ is generated by a regular sequence if and only if it is generated by n elements. Hence our definition of complete intersection for flat numerical semigroup algebras agrees with that of [1], and also with that of [7].

There are n candidates of the form $Y_i^{\beta_i} - \mathbf{u}^{\beta_{i0}} Y_1^{\beta_{i1}} \cdots Y_n^{\beta_{in}}$ for the set of generators of the kernel of $\hat{\pi}$, where $\mathbf{u}^{\beta_{i0}} \in R$ and $\beta_{ii} = 0$. For prescribed numerical invariants β_i, the image of the set $\{Y_1^{s_1} \cdots Y_n^{s_n} \mid 0 \le s_i < \beta_i\}$ in R' can be used to study the complete intersection property of R'/R. Each monomial in the set describes a factorization of its image. If an element of R' has factorizations from two distinct monomials in the set, the difference of the monomials creates an extra generator for the kernel of $\hat{\pi}$. This is the idea of α-rectangular, β-rectangular and γ-rectangular sets of Apéry numbers in the classical case [4,5]. In the relative situation, Apéry numbers are generalized to Apéry monomials; the role of these numerical invariants is replaced by the notion of rectangles to emphasize the “shape” rather than the “size” of the set of Apéry monomials. Investigating factorizations of a numerical semigroup algebra, we obtain a main result asserting that a flat numerical semigroup algebra is complete intersection if it has a nonsingular rectangle. See Theorem 4.9.

This paper is organized as follows. Section 2 starts with our broader definition of numerical semigroups. Notions related to Apéry monomials are defined. Section 3 reviews flatness of numerical semigroup algebras. A new criterion for flatness is given in terms of the number of Apéry monomials. In Section 4, we define rectangles with examples to clarify the notion. Sufficient conditions are given for a flat numerical semigroup algebra to be complete intersection. Section 5 consists of a detailed study of rectangular numerical semigroup algebras generated by few monomials. For flat algebras generated by three monomials, we show that rectangles must be non-singular with a special type resembling free numerical semigroups. For flat algebras generated by four monomials, we find a subclass of rectangular algebras, which consists only of complete intersection algebras.

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2. Apéry monomials

In the literature, a numerical semigroup is a submonoid of the set $\mathbb{N}$ of non-negative integers, whose greatest common divisor is 1. The condition on the greatest common divisor is equivalent to the statement that the complement of the semigroup in $\mathbb{N}$ consists of finitely many elements. In this classical definition, the multiplicity of the numerical semigroup is recognized as the smallest non-zero number in the semigroup. To study numerical semigroups, other submonoids of $\mathbb{N}$ appear without the condition on greatest common divisors. It is also observed that a numerical semigroup divided by its multiplicity naturally occurs in the study of the tangent cones of the numerical semigroups [8,9]. We will see that numerical semigroups in the following broader sense give rise to many flat numerical semigroup algebras.

Definition 2.1 *(numerical semigroup).* A numerical semigroup is a monoid generated by finitely many positive rational numbers.

Let S be a numerical semigroup. The numerical semigroup ring $R := \kappa[\![\mathbf{u}^S]\!]$ in the variable $\mathbf{u}$ consists of power series $\sum_{s\in S} a_s\mathbf{u}^s$ with $a_s \in \kappa$. An element $\mathbf{u}^s \in R$ is called a *monomial* of R. Note that, in the notation $\kappa[\![\mathbf{u}^S]\!]$ for R, the numerical semigroup ring comes with a choice of variable $\mathbf{u}$. We denote $S = \log_{\mathbf{u}} R$. We shall allow ourself to multiply S by a positive rational number t and study the numerical semigroup ring $\kappa[\![\mathbf{v}^{tS}]\!]$. Easily tracked from the relation $\mathbf{u} = \mathbf{v}^t$, two rings $\kappa[\![\mathbf{u}^S]\!]$ and $\kappa[\![\mathbf{v}^{tS}]\!]$ are essentially the same. If a numerical semigroup S is generated by $s_1, \dots, s_n$, we also write the numerical semigroup ring $\kappa[\![\mathbf{u}^S]\!]$ in terms of the chosen variable $\mathbf{u}$ as $\kappa[\![\mathbf{u}^{s_1}, \dots, \mathbf{u}^{s_n}]\!]$.

Let S and S' be numerical semigroups. For a positive rational number t satisfying $tS \subset S'$, the numerical semigroup ring $R' = \kappa[\![\mathbf{v}^{S'}]\!]$ has an algebra structure over $R = \kappa[\![\mathbf{u}^S]\!]$ through the relation $\mathbf{u} = \mathbf{v}^t$. We denote the algebra by R'/R and call it a *numerical semigroup algebra* with R as its *coefficient ring*. Note that there are different ways to embed R into R'. By scaling the semigroups, we may work in the situation where R and R' share the same variable $\mathbf{u}$. For such a situation, we say R'/R is a numerical semigroup algebra in the variable $\mathbf{u}$. If S' is generated by S together with rational numbers $s_1, \dots, s_n$, the numerical semigroup ring R' is also denoted by $R[\![\mathbf{u}^{S'}]\!]$ or $R[\![\mathbf{u}^{s_1}, \dots, \mathbf{u}^{s_n}]\!]$ to indicate its R-algebra structure. In this paper, the notation $R[\![\mathbf{u}^{s_1}, \dots, \mathbf{u}^{s_n}]\!]$ will always mean a numerical semigroup algebra with the numerical semigroup ring R as its coefficient ring.

The classical study of numerical semigroup rings fits in our relative situation by choosing Noether normalizations. More precisely, for a numerical semigroup ring $\kappa[\![\mathbf{u}^S]\!]$, we take an element $s \in S$ and obtain a Noether normalization $\kappa[\![\mathbf{u}^s]\!]$ of $\kappa[\![\mathbf{u}^S]\!]$. The numerical semigroup ring $\kappa[\![\mathbf{u}^S]\!]$ is Cohen–Macaulay, Gorenstein or complete intersection if and only if the algebra $\kappa[\![\mathbf{u}^S]\!]/\kappa[\![\mathbf{u}^s]\!]$ has the same property [10].

Apéry numbers are among the most important tools to study numerical semigroups. The notion can be extended to the relative situation.

Definition 2.2 *(Apéry monomial).* For a numerical semigroup algebra R'/R, a monomial p of R' is called an Apéry monomial if, whenever there are monomials p_1 of R and p_2 of R' such that $p_1p_2 = p$, then $p_1 = 1$.

In other words, a monomial is Apéry if it is not divisible by any non-trivial coefficient. We denote the set of Apéry monomials of R'/R by $\mathrm{Apr}(R'/R)$.

Example 2.3. $\mathrm{Apr}(\kappa[\![\mathbf{u}^3, \mathbf{u}^5]\!]/\kappa[\![\mathbf{u}^6]\!]) = \{\mathbf{u}^0, \mathbf{u}^3, \mathbf{u}^5, \mathbf{u}^8, \mathbf{u}^{10}, \mathbf{u}^{13}\}$.

Example 2.4. $\mathrm{Apr}(\kappa[\![\mathbf{u}^3, \mathbf{u}^5]\!]/\kappa[\![\mathbf{u}^6, \mathbf{u}^8]\!]) = \{\mathbf{u}^0, \mathbf{u}^3, \mathbf{u}^5, \mathbf{u}^{10}\}$.

Apéry numbers are the logarithmic form of Apéry monomials. Let R'/R be a numerical semigroup algebra in the variable $\mathbf{u}$. Let $S' = \log_{\mathbf{u}} R'$ and $S = \log_{\mathbf{u}} R$. Then an element $s \in S'$ such that $\mathbf{u}^s \in \mathrm{Apr}(R'/R)$

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is called an *Apéry number* of S' with respect to S. In the classical case, recall that an Apéry number of a numerical semigroup S' with respect to an element $m \in S'$ is defined to be an element $s \in S'$ such that $s - m \notin S'$. In our terminology, these numbers are exactly Apéry numbers of S' with respect to the numerical semigroup $m\mathbb{N}$.

For a numerical semigroup algebra R'/R, an Apéry monomial p not equal to 1 is called a *minimal monomial* if, whenever there are monomials p_1 and p_2 of R' such that $p_1 p_2 = p$, then one of p_1 and p_2 has to be p. In other words, minimal monomials of a numerical semigroup algebra are minimal elements among non-trivial Apéry monomials with respect to the partial order given by divisions. Let $\mathbf{u}^{s_1}, \ldots, \mathbf{u}^{s_n}$ be the minimal monomials of R'/R, then $R' = R[\![\mathbf{u}^{s_1}, \ldots, \mathbf{u}^{s_n}]\!]$.

The following two notions are central in the computational aspect of numerical semigroup algebras.

Definition 2.5 *(representation).* Let R'/R be a numerical semigroup algebra in the variable $\mathbf{u}$. A representation of a monomial $\mathbf{u}^s$ of R'/R is an expression $\mathbf{u}^s = \mathbf{u}^{s_0}\mathbf{u}^w$, where $\mathbf{u}^{s_0} \in R$ and $\mathbf{u}^w \in \operatorname{Apr}(R'/R)$.

Definition 2.6 *(factorization).* Let R'/R be a numerical semigroup algebra in the variable $\mathbf{u}$. Let $\mathbf{u}^{s_1}, \ldots, \mathbf{u}^{s_n}$ be the minimal monomials of R'/R. An expression $\mathbf{u}^s = \mathbf{u}^{s_0}(\mathbf{u}^{s_1})^{a_1} \cdots (\mathbf{u}^{s_n})^{a_n}$ for a monomial $\mathbf{u}^s$ of R', where $\mathbf{u}^{s_0} \in R$ and $a_i \in \mathbb{N}$, is called a *factorization* of $\mathbf{u}^s$.

Note that representations and factorizations always exist for monomials in a numerical semigroup algebra.

3. Flatness

Flatness is a homological notion for modules. It also has a characterization in terms of relations. Roughly speaking, a module is flat if all its module relations come from the relations of the underlying ring. See [11, Theorem 7.6] for the precise statement. For a finitely generated module over a Noetherian local ring, free and flat are equivalent properties. To emphasize the computational aspect of numerical semigroup algebras, we interpret this fact again as in [10] using the flatness criterion by relations.

Lemma 3.1. *A numerical semigroup algebra is flat if and only if every monomial has a unique representation.*

Proof. If every monomial has a unique representation, the numerical semigroup algebra is free and thus flat. Assume that the algebra $\kappa[\![\mathbf{u}^{S'}]\!]/\kappa[\![\mathbf{u}^S]\!]$ is flat. Consider two different representations $\mathbf{u}^{t_1}\mathbf{u}^{w_1} = \mathbf{u}^{t_2}\mathbf{u}^{w_2}$, where $\mathbf{u}^{t_1}, \mathbf{u}^{t_2}$ are coefficients and $\mathbf{u}^{w_1}, \mathbf{u}^{w_2}$ are Apéry. Assume that $t_1 < t_2$. Applying [11, Theorem 7.6] for the flat algebra, there are elements $f_{ij} \in \kappa[\![\mathbf{u}^S]\!]$ and $g_j \in \kappa[\![\mathbf{u}^{S'}]\!]$ such that

$$\mathbf{u}^{t_1} f_{1j} - \mathbf{u}^{t_2} f_{2j} = 0, \tag{1}$$

$$f_{i1}g_1 + \cdots + f_{in}g_n = \mathbf{u}^{w_i}, \tag{2}$$

where $i = 1, 2$ and $1 \le j \le n$ for some n. From (1), we see that the constant term of f_{1j} vanishes for each j. Since $\mathbf{u}^{w_1}$ is an Apéry monomial, this implies that the coefficient of $f_{11}g_1 + \cdots + f_{1n}g_n$ at $\mathbf{u}^{w_1}$ vanishes, contradicting (2). □

In particular, Apéry monomials form an R-module basis of a flat numerical semigroup algebra over R.

Example 3.2. The algebra $\kappa[\![\mathbf{u}]\!]/\kappa[\![\mathbf{u}^2, \mathbf{u}^3]\!]$ has two Apéry monomials $\mathbf{u}^0$ and $\mathbf{u}$. The algebra is not flat, since $\mathbf{u}^3$ has different representations $\mathbf{u}^2\mathbf{u}$ and $\mathbf{u}^3\mathbf{u}^0$.

Example 3.3. The algebra $\kappa[\![\mathbf{u}^2, \mathbf{u}^3]\!]/\kappa[\![\mathbf{u}^2]\!]$ has two Apéry monomials $\mathbf{u}^0$ and $\mathbf{u}^3$. All monomials with even exponents belong to the coefficient ring $\kappa[\![\mathbf{u}^2]\!]$, and all monomials in $\kappa[\![\mathbf{u}^2, \mathbf{u}^3]\!]$ with odd exponents are uniquely represented as $\mathbf{u}^{2t}\mathbf{u}^3$ for some integer $t \ge 0$. By Lemma 3.1, the algebra is flat.

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Example 3.4. The algebra $\kappa[\![\mathbf{u}^{12}, \mathbf{u}^{14}, \mathbf{u}^{16}, \mathbf{u}^{35}]\!]/\kappa[\![\mathbf{u}^{12}, \mathbf{u}^{16}]\!]$ has four Apéry monomials $\mathbf{u}^0, \mathbf{u}^{14}, \mathbf{u}^{35}, \mathbf{u}^{49}$. Since the difference of any two distinct elements in $\{0, 14, 35, 49\}$ is not the difference of any two elements in the numerical semigroup $\langle 12, 16\rangle$, any monomial of the algebra has a unique representation. The algebra is flat by Lemma 3.1.

Example 3.3 is part of a general phenomenon in the classical case. If the coefficients of a numerical semigroup algebra provide a Noether normalization, then the algebra is flat [10, Corollary 2.2]. Example 3.4 is a general phenomenon about gluing. Let S and T be numerical semigroups generated by integers, $q \in S$ and $p \in T$ be relatively prime numbers. Recall that the numerical semigroup $pS + qT$ is called a *gluing* of S and T, if p and q are not in the minimal set of generators of T and S, respectively. The algebra $\kappa[\![\mathbf{u}^{pS+qT}]\!]$ over $\kappa[\![\mathbf{u}^{pS}]\!]$ or over $\kappa[\![\mathbf{u}^{qT}]\!]$ is flat [10, Proposition 2.9]. See Example 3.6 for another flat numerical semigroup algebra beyond these general phenomena.

In this section, we provide two more criteria for flatness: A criterion counts Apéry monomials and another is expressed in terms of the set

$$\Delta_S(S') := \{a_1 - a_2 \mid \mathbf{u}^{a_1}, \mathbf{u}^{a_2} \in \mathrm{Apr}(\kappa[\![\mathbf{u}^{S'}]\!]/\kappa[\![\mathbf{u}^{S}]\!]) \text{ and } a_1 \geq a_2\}$$

for numerical semigroups $S \subset S'$.

Proposition 3.5. *Let $\kappa[\![\mathbf{u}^{S'}]\!]/\kappa[\![\mathbf{u}^{S}]\!]$ be a numerical semigroup algebra, where S' and S are subsets of $\mathbb{N}$ with greatest common divisors d' and d respectively. There are at least d/d' Apéry monomials. The following conditions are equivalent.*

- *The algebra is flat.*
- *The algebra has d/d' Apéry monomials.*
- $\Delta_S(S') \cap (d/d')\mathbb{Z} \subset S$.

In particular, if the algebra is flat, then $S = S' \cap (d/d')\mathbb{Z}$.

Proof. Dividing every number in S' by d', we may assume that $d' = 1$. With this assumption, there exists an Apéry number in each congruence class modulo d. So totally there are at least d Apéry monomials. If there are exactly d Apéry monomials, each congruence class consists of exactly one Apéry monomial. Consider two representations $\mathbf{u}^{s_1}\mathbf{u}^{w_1} = \mathbf{u}^{s_2}\mathbf{u}^{w_2}$, where $\mathbf{u}^{s_1}, \mathbf{u}^{s_2}$ are coefficients and $\mathbf{u}^{w_1}, \mathbf{u}^{w_2}$ are Apéry. Since s_1 and s_2 are both divisible by d, the exponents w_1 and w_2 are in the same congruence class. Hence $\mathbf{u}^{w_1} = \mathbf{u}^{w_2}$. In other words, every monomial has a unique representation. By Lemma 3.1, the algebra is flat. If there are more than d Apéry monomials, two different Apéry numbers w_1 and w_2 are in the same congruence class. Say $w_1 > w_2$. Choose s_1 and s_2 in S large enough so that $w_1 - w_2 = s_2 - s_1$. Then we have two different representations $\mathbf{u}^{s_1}\mathbf{u}^{w_1} = \mathbf{u}^{s_2}\mathbf{u}^{w_2}$ for a monomial. Consequently, the algebra is not flat.

The inclusion $\Delta_S(S') \cap d\mathbb{Z} \subset S$ is just another way to state that each congruence class contains exactly one Apéry number. The condition that there is only one Apéry number congruent to 0 modulo d can be stated as $S' \cap d\mathbb{Z} = S$, which is therefore a necessary condition for flatness. □

Example 3.6. For $\mathbf{u} = \mathbf{v}^6$, the algebra $\kappa[\![\mathbf{v}^4, \mathbf{v}^9]\!]/\kappa[\![\mathbf{u}^2, \mathbf{u}^3]\!]$ is flat, since it has six Apéry monomials $1, \mathbf{v}^4, \mathbf{v}^8, \mathbf{v}^9, \mathbf{v}^{13}, \mathbf{v}^{17}$. Note that the numerical semigroup $\langle 4, 9\rangle$ can not be written as a gluing of $\langle 2, 3\rangle$ and another numerical semigroup, because the semigroup obtained from such a gluing needs at least three generators [12, Lemma 9.8].

For a numerical semigroup algebra $R[\![\mathbf{u}^{s_1}, \ldots, \mathbf{u}^{s_n}]\!]$, we may choose $m \in \mathbb{N}$ such that $ms_i \in \log_{\mathbf{u}} R$ for all i. Therefore $R[\![\mathbf{u}^{s_1}, \ldots, \mathbf{u}^{s_n}]\!]$ can be considered as the algebra obtained from R by joining m-th roots of the monomials $\mathbf{u}^{ms_1}, \ldots, \mathbf{u}^{ms_n}$ in R. We may apply Proposition 3.5 to the case that only one root is joined.

Corollary 3.7. *Assume that $\log_{\mathbf{u}} R \subset \mathbb{N}$ and has greatest common divisor 1. Assume that $s, m \in \mathbb{N}$ are relatively prime. Then the algebra $R[\![\mathbf{u}^{s/m}]\!]$ is flat if and only if $s \in \log_{\mathbf{u}} R$.*

Proof. The set of Apéry numbers is $\{0, s/m, 2s/m, \ldots, (t-1)s/m\}$, where t is the smallest positive integer satisfying $ts/m \in \log_{\mathbf{u}} R$. The algebra is flat if and only if there are m Apéry numbers. If the algebra is flat, then $t = m$ and hence $s \in \log_{\mathbf{u}} R$. Conversely, if $s \in \log_{\mathbf{u}} R$, then $t \leq m$. There are at least m Apéry monomials. Hence $t = m$ and consequently the algebra is flat. □

Let $S \subset \mathbb{N}$ be a numerical semigroup with the greatest common divisor 1. As an application of Corollary 3.7, the algebra $\kappa[\![\mathbf{u}^S, \mathbf{u}^s]\!]/\kappa[\![\mathbf{u}^S]\!]$ is not flat for any $s \in \mathbb{N} \setminus S$. See Example 3.2. More generally, for any numerical semigroup S' satisfying $S \subsetneq S' \subset \mathbb{N}$, Proposition 3.5 implies that the algebra $\kappa[\![\mathbf{u}^{S'}]\!]/\kappa[\![\mathbf{u}^S]\!]$ is not flat. If we remove the condition on the greatest common divisor, plenty of flat algebras come out by adding monomials.

Example 3.8. Let $R = \kappa[\![\mathbf{u}^2, \mathbf{u}^3]\!]$. Since $3 \in \log_{\mathbf{u}} R$, the algebra $R[\![\mathbf{u}^{3/2}]\!]$ is flat by Corollary 3.7. In terms of the new variable $\mathbf{v} = \mathbf{u}^{1/2}$, we may write $R[\![\mathbf{u}^{3/2}]\!] = \kappa[\![\mathbf{v}^4, \mathbf{v}^6, \mathbf{v}^3]\!] = \kappa[\![\mathbf{v}^4, \mathbf{v}^3]\!]$.

Example 3.9. Joining the 4-th root of $\mathbf{u}^6$ and the square roots of $\mathbf{u}^5$ and $\mathbf{u}^7$ to $\kappa[\![\mathbf{u}^5, \mathbf{u}^6, \mathbf{u}^7]\!]$, we obtain $\kappa[\![\mathbf{v}^3, \mathbf{v}^5, \mathbf{v}^7]\!]$ in terms of the variable $\mathbf{v} = \mathbf{u}^{1/2}$. The algebra $\kappa[\![\mathbf{v}^3, \mathbf{v}^5, \mathbf{v}^7]\!]/\kappa[\![\mathbf{u}^5, \mathbf{u}^6, \mathbf{u}^7]\!]$ has more than two Apéry monomials, including three minimal monomials. By Proposition 3.5, the algebra is not flat.

If S' is generated by S and one more element in Proposition 3.5, the condition $S' \cap d\mathbb{Z} = S$ is also sufficient for flatness as stated in Proposition 2.5 in [10]. This is not true in general.

Example 3.10. In the algebra $\kappa[\![\mathbf{u}^5, \mathbf{u}^8, \mathbf{u}^9]\!]/\kappa[\![\mathbf{u}^9, \mathbf{u}^{15}, \mathbf{u}^{21}]\!]$, the monomial $\mathbf{u}^{23}$ has different representations $\mathbf{u}^{18}\mathbf{u}^5$ and $\mathbf{u}^{15}\mathbf{u}^8$. The algebra is not flat, although $\langle 5, 8, 9\rangle \cap 3\mathbb{Z} = \langle 9, 15, 21\rangle$.

Now, we give a necessary condition for a numerical semigroup algebra to be flat. For an element s in a numerical semigroup T, a *divisor* of s in T is an element $t \in T$ such that $s - t \in T$. The terminology is justified by its exponential counterpart, where $\mathbf{u}^t$ divides $\mathbf{u}^s$ in $\kappa[\![\mathbf{u}^T]\!]$. We call 0 the *trivial divisor* of s and any divisor not equal to s a *proper divisor*.

Proposition 3.11. *If a numerical semigroup algebra $R[\![\mathbf{u}^{s_1}, \ldots, \mathbf{u}^{s_n}]\!]$ is flat, then any two minimal generators of $\log_{\mathbf{u}} R$ belonging to $T := \langle s_1, \ldots, s_n\rangle$ have only trivial common divisor in T.*

Proof. Let $a_1, \ldots, a_m$ be the minimal generators of $\log_{\mathbf{u}} R$. Assume that $a_1 \in T$. We claim that any proper divisor a_1' of a_1 in T is an Apéry number. Write

$$a_1' = \sum \beta_i a_i + \sum \gamma_j s_j \tag{3}$$

and

$$a_1 - a_1' + \sum \gamma_j s_j = \sum \beta_i' a_i + \sum \gamma_j' s_j$$

for certain Apéry numbers $\sum \gamma_j s_j$ and $\sum \gamma_j' s_j$. Then

$$a_1 = \sum(\beta_i + \beta_i')a_i + \sum \gamma_j' s_j.$$

By the criterion of flatness in Lemma 3.1,

$$\sum(\beta_i + \beta_i')a_i = a_1$$

Since $a_1, \ldots, a_m$ are minimal generators, we have $\beta_1 + \beta_1' = 1$ and $\beta_i = \beta_i' = 0$ for $i > 1$. Since $a_1 > a_1'$, we have $\beta_1 = 0$ from (3). Hence a_1' is the Apéry number $\sum \gamma_j s_j$.

For $\alpha \in \mathbb{Z}$, we use the notation $\alpha^+ := \alpha$ if $\alpha > 0$ and $\alpha^+ := 0$ otherwise. To show the proposition, we assume the contrary that $a_1, a_2 \in T$ come with a common non-trivial divisor t. By joining t to the set $\{s_i\}$, we may assume $t = s_1$. Then there are expressions $a_1 = \sum \alpha_i s_i$ and $a_2 = \sum \alpha_i' s_i$ for positive α_1 and α_1'. By flatness, $\sum(\alpha_i' - \alpha_i)^+ s_i$ and $\sum(\alpha_i - \alpha_i')^+ s_i$ in the expressions

$$a_1 + \sum(\alpha_i' - \alpha_i)^+ s_i = a_2 + \sum(\alpha_i - \alpha_i')^+ s_i$$

cannot be both Apéry. Assume that the divisor $\sum(\alpha_i' - \alpha_i)^+ s_i$ of a_2 is not Apéry. By the claim in the previous paragraph, the number $(\alpha_1' - \alpha_1)^+$ can not be less than α_1'. This is impossible since $\alpha_1 > 0$. □

For $n = 2$ in Proposition 3.11, we have more precise information.

Corollary 3.12. *Let s_1 and s_2 be positive integers satisfying $\gcd(s_1, s_2) = 1$. The following statements are equivalent for a numerical semigroup algebra $R[\![\mathbf{u}^{s_1}, \mathbf{u}^{s_2}]\!]$ satisfying the condition $\log_{\mathbf{u}} R \subseteq \langle s_1, s_2\rangle$.*

- *The algebra $R[\![\mathbf{u}^{s_1}, \mathbf{u}^{s_2}]\!]$ is flat.*
- *$\log_{\mathbf{u}} R$ is either principal or is generated by $a_1 s_1$ and $a_2 s_2$ for some positive integers a_1 and a_2 such that a_1 divides s_2 and a_2 divides s_1.*

Proof. We work on the case that $\log_{\mathbf{u}} R$ is not principal. By Proposition 3.11, it is generated by two elements $a_1 s_1$ and $a_2 s_2$ with only trivial divisor in common. For $0 \le r_1 < a_1$ and $0 \le r_2 < a_2$, we claim that $r_1 s_1 + r_2 s_2$ is Apéry. Assume the contrary, by changing indices, we may write $r_1 s_1 + r_2 s_2 = a_1 s_1 + r_1' s_1 + r_2' s_2$. Then $a_2 s_2 = (a_1 - r_1 + r_1')s_1 + (a_2 - r_2 + r_2')s_2$ implies that s_1 is a divisor of $a_2 s_2$. This is impossible. For distinct pairs (r_1, r_2) and (r_1', r_2') satisfying $0 \le r_i < a_i$ and $0 \le r_i' < a_i$, we claim that $r_1 s_1 + r_2 s_2 \ne r_1' s_1 + r_2' s_2$. We may assume $r_1 < r_1'$. If $r_1 s_1 + r_2 s_2 = r_1' s_1 + r_2' s_2$, then $a_2 s_2 = (r_1' - r_1)s_1 + (a_2 - r_2 + r_2')s_2$ implies that s_1 is a divisor of $a_2 s_2$. This is impossible. Therefore there are $a_1 a_2$ Apéry numbers. By Proposition 3.5, the R-algebra $R[\![\mathbf{u}^{s_1}, \mathbf{u}^{s_2}]\!]$ is flat if and only if $\gcd(a_1 s_1, a_2 s_2) = a_1 a_2$, equivalently a_1 divides s_2 and a_2 divides s_1. □

Example 3.13. The algebra $\kappa[\![\mathbf{u}^3, \mathbf{u}^4]\!]/\kappa[\![\mathbf{u}^9, \mathbf{u}^{12}]\!]$ is not flat by Corollary 3.12. The algebra has Apéry monomials $1, \mathbf{u}^3, \mathbf{u}^4, \mathbf{u}^6, \mathbf{u}^7, \mathbf{u}^8, \mathbf{u}^{10}, \mathbf{u}^{11}, \mathbf{u}^{14}$. While $\gcd(9, 12) = 3$, one sees that the algebra is not flat also by counting Apéry monomials.

4. Rectangles

In this section, we denote by $\mathbf{u}^{s_1}, \ldots, \mathbf{u}^{s_n}$ the minimal monomials of a numerical semigroup algebra R'/R in the variable $\mathbf{u}$.

Definition 4.1 *(rectangle).* The set of the Apéry monomials of the algebra R'/R is called a rectangle of size $\beta_1 \times \cdots \times \beta_n$ if the following conditions hold.

- All Apéry monomials can be factored uniquely as $(\mathbf{u}^{s_1})^{\ell_1}\cdots(\mathbf{u}^{s_n})^{\ell_n}$, where $0 \le \ell_i < \beta_i$.
- All monomials $(\mathbf{u}^{s_1})^{\ell_1}\cdots(\mathbf{u}^{s_n})^{\ell_n}$ with $0 \le \ell_i < \beta_i$ are Apéry.

A numerical semigroup algebra is *rectangular* if the set of its Apéry monomials forms a rectangle.

As seen in the next example, the condition of uniqueness is essential in the definition of rectangles.

Example 4.2. The algebra $\kappa[\![\mathbf{u}^{14},\mathbf{u}^{21},\mathbf{u}^{22},\mathbf{u}^{33}]\!]/\kappa[\![\mathbf{u}^{22}]\!]$ is flat with 22 Apéry monomials. Since $22 \neq \beta_1 \times \beta_2 \times \beta_3$ for integers $\beta_i > 1$, the algebra is not rectangular. Note that the set of Apéry monomials can be described as $\{1,\mathbf{u}^{14},\mathbf{u}^{28},\mathbf{u}^{42}\} \times \{1,\mathbf{u}^{21},\mathbf{u}^{42}\} \times \{1,\mathbf{u}^{33}\}$. However this set is not a rectangle, because $\mathbf{u}^{42}$ and $\mathbf{u}^{75}$ have different expressions.

A rectangle of Apéry monomials may have different sizes.

Example 4.3. The algebra $\kappa[\![\mathbf{u}^{2},\mathbf{u}^{3}]\!]/\kappa[\![\mathbf{u}^{12}]\!]$ is rectangular. Its set of Apéry monomials can be described as a 4×3 rectangle $\{1,\mathbf{u}^{3},\mathbf{u}^{6},\mathbf{u}^{9}\} \times \{1,\mathbf{u}^{2},\mathbf{u}^{4}\}$ or a 2×6 rectangle $\{1,\mathbf{u}^{3}\} \times \{1,\mathbf{u}^{2},\mathbf{u}^{4},\mathbf{u}^{6},\mathbf{u}^{8},\mathbf{u}^{10}\}$.

A numerical semigroup algebra obtained by joining one root of a monomial is always rectangular, but it may not be flat, see Example 3.2. We are mainly interested in flat rectangular algebras. Since a rectangular algebra has a unique maximal Apéry monomial, flat rectangular algebras are Gorenstein by [10, Proposition 3.1]. Here is another non-flat rectangular algebra:

Example 4.4. The algebra $\kappa[\![\mathbf{u}^{14},\mathbf{u}^{21},\mathbf{u}^{22},\mathbf{u}^{33}]\!]/\kappa[\![\mathbf{u}^{14},\mathbf{u}^{22}]\!]$ is not flat, since $\mathbf{u}^{231}$ has two representations $(\mathbf{u}^{14})^{15}\mathbf{u}^{21} = (\mathbf{u}^{22})^{9}\mathbf{u}^{33}$. The algebra is rectangular and the set of its Apéry monomials is $\{1,\mathbf{u}^{21}\} \times \{1,\mathbf{u}^{33}\}$.

Certain flat numerical semigroup algebras are always rectangular. See the proof of Corollary 3.12.

Proposition 4.5. *Let R be a numerical semigroup ring which is not a power series ring. A flat algebra $R[\![\mathbf{u}^{s_1},\mathbf{u}^{s_2}]\!]$ satisfying $\log_{\mathbf{u}} R \subseteq \langle s_1, s_2\rangle \subseteq \mathbb{N}$ and $\gcd(s_1,s_2) = 1$ is rectangular.*

Free numerical semigroups [2] (not to be confused with the notion of free algebras given by numerical semigroups) provide examples of rectangular algebras in the classical case: Assume that $R = \kappa[\![\mathbf{u}^{s_0}]\!]$. Let S be the semigroup minimally generated by $s_0, s_1, \dots, s_n$. Assume that S is free in the sense that its Apéry numbers with respect to $s_0\mathbb{N}$ can be listed as $\sum_{i=1}^{n} \lambda_i s_i$ for $0 \le \lambda_i < \phi_i$ after rearranging the indices, where $\phi_i = \min\{h \in \mathbb{N} \mid hs_i \in \langle s_0, \dots, s_{i-1}\rangle\}$. By [12, Proposition 9.15], there are $\phi_1\phi_2\cdots\phi_n$ Apéry monomials. So the algebra $\kappa[\![\mathbf{u}^{s_0},\dots,\mathbf{u}^{s_n}]\!]/\kappa[\![\mathbf{u}^{s_0}]\!]$ is rectangular.

The notions of α-rectangular, β-rectangular and γ-rectangular Apéry sets for numerical semigroups also provide rectangular algebras. Indeed, there are strict implications

$$\alpha\text{-rectangular} \implies \beta\text{-rectangular} \implies \gamma\text{-rectangular} \implies \text{free}$$

for numerical semigroups [5, Theorem 2.13]. In particular, if S has an α-rectangular Apéry set, then the algebra $\kappa[\![\mathbf{u}^{S}]\!]/\kappa[\![\mathbf{u}^{s}]\!]$ is rectangular for some $s \in S$. However, the converse is not true. For instance, the semigroup $\langle 5,6,9\rangle$ does not have γ-rectangular Apéry set (hence not β-rectangular nor α-rectangular), but $\kappa[\![\mathbf{u}^{5},\mathbf{u}^{6},\mathbf{u}^{9}]\!]/\kappa[\![\mathbf{u}^{6}]\!]$ is a rectangular algebra with the set $\{1,\mathbf{u}^{9}\} \times \{1,\mathbf{u}^{5},\mathbf{u}^{10}\}$ of Apéry monomials.

Given a numerical semigroup S, the properties of Cohen–Macaulay, Gorenstein and complete intersection of the algebra $\kappa[\![\mathbf{u}^{S}]\!]/\kappa[\![\mathbf{u}^{s}]\!]$ are independent of the choice of an element $s \in S$. This is not the case for rectangular algebras.

Example 4.6. The Apéry monomials $1, \mathbf{u}^2, \mathbf{u}^3, \mathbf{u}^4, \mathbf{u}^6$ of $\kappa[\![\mathbf{u}^2, \mathbf{u}^3]\!]/\kappa[\![\mathbf{u}^5]\!]$ do not form a rectangle. An algebra with only one minimal monomial, for instance $\kappa[\![\mathbf{u}^2, \mathbf{u}^3]\!]/\kappa[\![\mathbf{u}^3]\!]$, is always rectangular.

If the set of Apéry monomials of R'/R form a rectangle of size $\beta_1 \times \cdots \times \beta_n$, every monomial can be factored as $\mathbf{u}^t(\mathbf{u}^{s_1})^{\ell_1} \cdots (\mathbf{u}^{s_n})^{\ell_n}$, where $\mathbf{u}^t \in R$ and $0 \le \ell_i < \beta_i$ for all i. Such a factorization is unique if and only if the algebra is flat. Under the flatness assumption, there is a unique factorization

$$\mathbf{u}^{s_i\beta_i} = \mathbf{u}^{t_i}(\mathbf{u}^{s_1})^{\beta_{i1}} \cdots (\mathbf{u}^{s_n})^{\beta_{in}}$$

such that $0 \le \beta_{ij} < \beta_i$ for all j. Note that $\beta_{ii} = 0$ in the above factorization. Let $\mathbf{Y}$ be the variables $Y_1, \ldots, Y_n$ and $\mathbf{Z}$ be the shorthand of $Z_1, \ldots, Z_n$, where $Z_\ell := Y_\ell^{\beta_\ell} Y_1^{-\beta_{\ell 1}} \cdots Y_n^{-\beta_{\ell n}}$. In terms of the matrix

$$\log_{\mathbf{Y}} \mathbf{Z} := \begin{pmatrix} \beta_1 & -\beta_{12} & \cdots & -\beta_{1n} \\ -\beta_{21} & \beta_2 & \cdots & -\beta_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -\beta_{n1} & -\beta_{n2} & \cdots & \beta_n \end{pmatrix}, \tag{4}$$

we have a relation

$$\log_{\mathbf{Y}} \mathbf{Z} \begin{pmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{pmatrix} = \begin{pmatrix} t_1 \\ t_2 \\ \vdots \\ t_n \end{pmatrix}. \tag{5}$$

Example 4.7. In the algebra $\kappa[\![\mathbf{u}^{32}, \mathbf{u}^{35}, \mathbf{u}^{38}, \mathbf{u}^{44}, \mathbf{u}^{48}, \mathbf{u}^{56}]\!]/\kappa[\![\mathbf{u}^{32}, \mathbf{u}^{48}]\!]$, the squares of minimal monomials are not Apéry. So all Apéry monomials are in the set $\{0, \mathbf{u}^{35}\} \times \{0, \mathbf{u}^{38}\} \times \{0, \mathbf{u}^{44}\} \times \{0, \mathbf{u}^{56}\}$. Since $\gcd(32, 48) = 16$, this set consists of all the 16 Apéry monomials. Therefore the algebra is flat with a rectangle of size $2 \times 2 \times 2 \times 2$. Relation (5) becomes

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 35 \\ 38 \\ 44 \\ 56 \end{pmatrix} = \begin{pmatrix} 32 \\ 32 \\ 32 \\ 2 \times 32 + 48 \end{pmatrix}.$$

If $\log_{\mathbf{Y}} \mathbf{Z}$ is invertible, we call the rectangle *non-singular*. In such a case, every monomial in $\mathbf{Y}$ can be written uniquely as $Z_1^{i_1} \cdots Z_n^{i_n}$ with rational exponents $i_1, \ldots, i_n$. In other words, row vectors of $\log_{\mathbf{Y}} \mathbf{Z}$ form a basis for the vector space $\mathbb{Q}^n$. If furthermore all entries of the inverse of $\log_{\mathbf{Y}} \mathbf{Z}$ are non-negative, every monomial in $\mathbf{Y}$ can be written as $Z_1^{i_1} \cdots Z_n^{i_n}$ with non-negative rational exponents. This statement can be also expressed in terms of vectors of exponents: If a vector in $\mathbb{Q}^n$ sits in the first orthant with respect to the standard basis, then the vector also sits in the first orthant with respect to the basis given by the row vectors of $\log_{\mathbf{Y}} \mathbf{Z}$.

Lemma 4.8. *Let*

$$B = \begin{pmatrix} \beta_1 & -\beta_{12} & \cdots & -\beta_{1n} \\ -\beta_{21} & \beta_2 & \cdots & -\beta_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -\beta_{n1} & -\beta_{n2} & \cdots & \beta_n \end{pmatrix}$$

be a matrix of real numbers satisfying the property that $\beta_j > \beta_{ij} \ge 0$ for all i and j. If $\beta_i s_i \ge \sum_{j \ne i} \beta_{ij} s_j$ for certain positive numbers $s_1, \ldots, s_n$ and for all i, then $\det B$ *and all the entries of the adjoint of B are non-negative.*

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Proof. We use induction on the size n of the matrix to prove the lemma. For $n = 1, 2$, the lemma is clearly true. To work on the case $n > 2$, we assume that the lemma holds for matrices of size less than n.

Deleting the i-th row and the j-th column from B, we obtain a matrix B_{ij} whose determinant multiplied by $(-1)^{i+j}$ is an entry of the adjoint of B. Note that B_{ii} still satisfies the conditions of the lemma. By the induction hypothesis, $\det B_{ii} \geq 0$. If $i < j$, we first perform $j - 2$ operations of switching rows so that the $(j - 1)$-th row of B_{ij} becomes the first row and other rows of B_{ij} keep the order; then we perform $i - 1$ operations of switching columns on the new matrix obtained so that i-th column becomes the first column and other columns keep the order. After these $i + j - 3$ operations, B_{ij} becomes a matrix B'_{ij} satisfying the following conditions.

- The first row consists of $\{-\beta_{j\ell}\}_{\ell \neq j}$.
- The first column consists of $\{-\beta_{\ell i}\}_{\ell \neq i}$.
- Replacing the entry $-\beta_{ji}$ at the upper left corner of B'_{ij} by β_i, the matrix B''_{ij} obtained satisfies the condition of the lemma.

If $i > j$, we perform $j - 1$ operations of switching rows and $i - 2$ operations of switching columns. After $i + j - 3$ operations of switching rows and columns, B_{ij} also becomes a matrix B'_{ij} satisfying the above three conditions. Now we compute $\det B'_{ij}$ using Laplace expansion on the first row. The first row of $\operatorname{adj}(B'_{ij})$ is the same as that of $\operatorname{adj}(B''_{ij})$. By the induction hypothesis, the entries of the first row of $\operatorname{adj}(B''_{ij})$ are non-negative. Since the first row of B'_{ij} consists of $\{-\beta_{j\ell}\}_{\ell \neq j}$, Laplace expansion shows $\det B'_{ij} \leq 0$. Taking the negative signs from switching rows and columns into account, $\det B'_{ij} = (-1)^{i+j-3} \det B_{ij}$. Hence the entry $(-1)^{i+j} \det B_{ij}$ of $\operatorname{adj}(B)$ is non-negative.

To prove $\det B \geq 0$, we may replace B by the matrix obtained by multiplying the i-th column by s_i for all i. In other words, we may assume that $s_1 = \cdots = s_n = 1$. Now, adding all other columns to the first column of B, we obtain a matrix whose entries in the first column are all non-negative. To compute the determinant of the new matrix, we use Laplace expansion on the first column. Since all entries of $\operatorname{adj}(B)$ are non-negative, $\det B \geq 0$ as well. □

Theorem 4.9. *A flat numerical semigroup algebra R'/R is a complete intersection, if its Apéry monomials form a non-singular rectangle.*

Proof. Let $\beta_1 \times \cdots \times \beta_n$ be the size of the non-singular rectangle. Consider the local R-algebra homomorphism $\hat{\pi} \colon R[\![Y_1, \ldots, Y_n]\!] \to R'$, where Y_ℓ maps to $\mathbf{u}^{s_\ell}$ for $1 \leq \ell \leq n$. The restriction $\pi \colon R[Y_1, \ldots, Y_n] \to R'$ of $\hat{\pi}$ is surjective. Since $\ker \hat{\pi}$ is generated by $\ker \pi$, it suffices to show that $\ker \pi$ is generated by n elements.

We claim that $f_\ell := Y_\ell^{\beta_\ell} - \mathbf{u}^{t_\ell} Y_1^{\beta_{\ell 1}} \cdots Y_n^{\beta_{\ell n}}$ generate $\ker \pi$, where $\mathbf{u}^{t_\ell} \in R$ and $0 \leq \beta_{\ell i} < \beta_i$ for all i. Let $\mathbf{Y}$ and $\mathbf{Z}$ be as in (4). Since $\log_{\mathbf{Y}} \mathbf{Z}$ is invertible, we may associate a non-negative number to a monomial $\mathbf{Y}^{\mathbf{i}} = Y_1^{i_1} \cdots Y_n^{i_n}$. Let $(j_1, \ldots, j_n) \in \mathbb{Q}^n$ be the vector given by $(j_1, \ldots, j_n) \log_{\mathbf{Y}} \mathbf{Z} = (i_1, \ldots, i_n)$. By Lemma 4.8, all j_ℓ are non-negative. We define

$$\| \log \mathbf{Y}^{\mathbf{i}} \| := j_1 + \cdots + j_n.$$

If $i_\ell \geq \beta_\ell$ for some ℓ, we replace the factor $Y_\ell^{\beta_\ell}$ of $\mathbf{Y}^{\mathbf{i}}$ by $\mathbf{u}^{t_\ell} Y_1^{\beta_{\ell 1}} \cdots Y_n^{\beta_{\ell n}}$ resulting $\mathbf{u}^{t_\ell} \mathbf{Y}^{\mathbf{i}} / Z_\ell$. Then

$$\mathbf{Y}^{\mathbf{i}} - \mathbf{u}^{t_\ell} \mathbf{Y}^{\mathbf{i}} / Z_\ell = f_\ell \mathbf{Y}^{\mathbf{i}} / Y_\ell^{\beta_\ell} \in \langle f_1, \ldots, f_n \rangle.$$

In the logarithmic form, the operation subtracts the ℓ-th row of $\log_{\mathbf{Y}} \mathbf{Z}$ from the vector $\mathbf{i} := (i_1, \ldots, i_n)$. Hence

$$\| \log(\mathbf{u}^{t_\ell} \mathbf{Y}^{\mathbf{i}} / Z_\ell) \| = \| \log \mathbf{Y}^{\mathbf{i}} \| - 1.$$

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After finitely many such operations, every monomial $Y_1^{i_1}\cdots Y_n^{i_n}$ can be changed so that the condition $i_\ell < \beta_\ell$ is satisfied for each ℓ. Applying these operations, every element of $R[Y_1,\dots,Y_n]$ can be written as the sum of an element of $\langle f_1,\dots,f_n\rangle$ and an R-linear combination of monomials $Y_1^{i_1}\cdots Y_n^{i_n}$ satisfying $i_\ell<\beta_\ell$. By flatness and the rectangular property, $f_1,\dots,f_n$ generate $\ker\pi$. $\square$

If S is a free numerical semigroup, we can find a minimal generator s such that the Apéry monomials of $\kappa[\![\mathbf{u}^S]\!]/\kappa[\![\mathbf{u}^s]\!]$ form a non-singular rectangle. The next example is observed already in [14, Lemma 3].

Example 4.10. Let a be an odd positive integer. With relations

$$\begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0\\ 0 & 2 & -1 & \cdots & 0 & 0\\ 0 & 0 & 2 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & 2 & -1\\ 0 & 0 & 0 & \cdots & 0 & 2 \end{pmatrix}\begin{pmatrix} 2^n+a\\ 2^n+2a\\ 2^n+4a\\ \vdots\\ 2^n+2^{n-2}a\\ 2^n+2^{n-1}a\end{pmatrix}=\begin{pmatrix} 2^n\\ 2^n\\ 2^n\\ \vdots\\ 2^n\\ 2^n(a+2)\end{pmatrix},$$

the flat rectangular algebra $\kappa[\![\mathbf{u}^{2^n},\mathbf{u}^{2^n+a},\dots,\mathbf{u}^{2^n+2^{n-1}a}]\!]/\kappa[\![\mathbf{u}^{2^n}]\!]$ is a complete intersection.

Let S and T be numerical semigroups generated by integers. If $\kappa[\![\mathbf{u}^S]\!]/\kappa[\![\mathbf{u}^p]\!]$ and $\kappa[\![\mathbf{u}^T]\!]/\kappa[\![\mathbf{u}^q]\!]$ are rectangular for some relatively prime numbers $p\in S$ and $q\in T$, then $\kappa[\![\mathbf{u}^{qS+pT}]\!]/\kappa[\![\mathbf{u}^{pq}]\!]$ is rectangular. Indeed,

$$\mathrm{Apr}(\kappa[\![\mathbf{u}^{qS+pT}]\!]/\kappa[\![\mathbf{u}^{pq}]\!]) =$$

$$\{\mathbf{u}^{qw_1+pw_2}\mid \mathbf{u}^{w_1}\in \mathrm{Apr}(\kappa[\![\mathbf{u}^S]\!]/\kappa[\![\mathbf{u}^p]\!]) \text{ and } \mathbf{u}^{w_2}\in\mathrm{Apr}(\kappa[\![\mathbf{u}^T]\!]/\kappa[\![\mathbf{u}^q]\!])\}.$$

See [10] and also [12, Proposition 9.11] for the case of gluing. Given rectangles of $\kappa[\![\mathbf{u}^S]\!]/\kappa[\![\mathbf{u}^p]\!]$ and $\kappa[\![\mathbf{u}^T]\!]/\kappa[\![\mathbf{u}^q]\!]$ with matrices $\log_{\mathbf{Y}_1}\mathbf{Z}_1$ and $\log_{\mathbf{Y}_2}\mathbf{Z}_2$, the algebra $\kappa[\![\mathbf{u}^{qS+pT}]\!]/\kappa[\![\mathbf{u}^{pq}]\!]$ has a rectangle with the matrix

$$\log_{\mathbf{Y}}\mathbf{Z}=\begin{pmatrix}\log_{\mathbf{Y}_1}\mathbf{Z}_1 & 0\\ 0 & \log_{\mathbf{Y}_2}\mathbf{Z}_2\end{pmatrix}.$$

Clearly, $\log_{\mathbf{Y}}\mathbf{Z}$ is invertible if and only if $\log_{\mathbf{Y}_1}\mathbf{Z}_1$ and $\log_{\mathbf{Y}_2}\mathbf{Z}_2$ are invertible. In the absolute case, the existence of rectangles depends on the choice of a Noether normalization. It is possible that $\kappa[\![\mathbf{u}^{qS+pT}]\!]/\kappa[\![\mathbf{u}^r]\!]$ is not rectangular for any $r\in qS+pT$, even though $\kappa[\![\mathbf{u}^S]\!]/\kappa[\![\mathbf{u}^s]\!]$ and $\kappa[\![\mathbf{u}^T]\!]/\kappa[\![\mathbf{u}^t]\!]$ are rectangular for some $s\in S$ and $t\in T$.

Example 4.11. Let $S=\langle 2,3\rangle$ and $T=\langle 3,4\rangle$. The algebras $\kappa[\![\mathbf{u}^S]\!]/\kappa[\![\mathbf{u}^{12}]\!]$ and $\kappa[\![\mathbf{u}^T]\!]/\kappa[\![\mathbf{u}^{24}]\!]$ have rectangles $\{1,\mathbf{u}^2,\mathbf{u}^4\}\times\{1,\mathbf{u}^3,\mathbf{u}^6,\mathbf{u}^9\}$ and $\{1,\mathbf{u}^4,\mathbf{u}^8\}\times\{1,\mathbf{u}^3,\mathbf{u}^6,\mathbf{u}^9,\mathbf{u}^{12},\mathbf{u}^{15},\mathbf{u}^{18},\mathbf{u}^{21}\}$, respectively. To see that $\kappa[\![\mathbf{u}^{7S+5T}]\!]/\kappa[\![\mathbf{u}^r]\!]$ is not rectangular for any $r\in 7S+5T$, we observe the relation $14+21=15+20$ in $7S+5T=\langle 14,21,15,20\rangle$. If $r\in\{14,15,20,21\}$, the product of two minimal monomials is not Apéry. If $r\notin\{14,15,20,21\}$, the product of two minimal monomials equals the product of the other two minimal monomials. Both cases can not happen for a rectangle.

To provide more examples, we present a class of flat rectangular algebras.

Proposition 4.12. *Let a, b and 4 be integers with greatest common divisor 1. Assume that $\mathbf{u}^a$ and $\mathbf{u}^b$ are the minimal monomials of the algebra $\kappa[\![\mathbf{u}^4,\mathbf{u}^a,\mathbf{u}^b]\!]/\kappa[\![\mathbf{u}^4]\!]$. Then the algebra is rectangular if and only if one of a or b is even.*

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Proof. If the algebra is rectangular, then its Apéry monomials are $1, \mathbf{u}^a, \mathbf{u}^b$ and $\mathbf{u}^{a+b}$. Assume $a < b$. Since $\mathbf{u}^a$ and $\mathbf{u}^b$ are minimal, the monomial $\mathbf{u}^{2a}$ is not Apéry. So $2a = 4r + sb$ for some non-negative integers r and s. Then $s \leq 1$. If $s = 0$, then $a = 2r$ is even. If $s = 1$, then b is even.

Now assume conversely that the algebra is not rectangular. Then $\mathbf{u}^{a+b}$ is not an Apéry monomial. Otherwise, the Apéry monomials form the rectangle $\{1, \mathbf{u}^a\} \times \{1, \mathbf{u}^b\}$. So $a + b = 4r + sa + tb$ for some non-negative integers r, s, t. As $\mathbf{u}^a$ and $\mathbf{u}^b$ are minimal, we have $s = t = 0$. So $a + b$ is even. Since a and b can not be both even, they have to be both odd. $\square$

Next section will provide a partial answer to the following questions.

Questions 4.13. *Is every flat rectangular algebra a complete intersection? Assume that the Apéry monomials of a flat numerical semigroup algebra form a rectangle. Is the rectangle always non-singular?*

5. Algebras with few minimal monomials

In this section, we work on a rectangle of size $\beta_1 \times \cdots \times \beta_n$ of a flat numerical semigroup algebra R'/R for the cases $n = 2, 3, 4$. Let $\mathbf{u}^{s_1}, \ldots, \mathbf{u}^{s_n}$ be minimal monomials of R'/R. We use the notation

$$\log_{\mathbf{Y}} \mathbf{Z} \begin{pmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{pmatrix} = \begin{pmatrix} t_1 \\ t_2 \\ \vdots \\ t_n \end{pmatrix}$$

as in (5), where $Z_\ell = Y_\ell^{\beta_\ell} Y_1^{-\beta_{\ell 1}} \cdots Y_n^{-\beta_{\ell n}}$ and $\mathbf{u}^{t_\ell} \in R$.

Consider the case $n = 2$. From

$$\begin{pmatrix} \beta_1 & -\beta_{12} \\ -\beta_{21} & \beta_2 \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = \begin{pmatrix} t_1 \\ t_2 \end{pmatrix},$$

we have a relation

$$(\beta_1 - \beta_{21})s_1 + (\beta_2 - \beta_{12})s_2 = t_1 + t_2.$$

If the numbers β_{21} and β_{12} were both non-zero, the coefficient $\mathbf{u}^{t_1+t_2}$ would become the Apéry monomial $\mathbf{u}^{(\beta_1-\beta_{21})s_1}\mathbf{u}^{(\beta_2-\beta_{12})s_2}$. Therefore the matrix $\log_{\mathbf{Y}} \mathbf{Z}$ is triangular for $n = 2$. The main result of this section is the case $n = 3$.

Theorem 5.1. *For a flat rectangular algebra $R[\![\mathbf{u}^{s_1}, \mathbf{u}^{s_2}, \mathbf{u}^{s_3}]\!]$, the matrix $\log_{\mathbf{Y}} \mathbf{Z}$ is triangular after a suitable permutation of indices. In particular it is non-singular.*

Proof. Our proof consists of three steps: (1) $\beta_{ij}\beta_{ji} = 0$ for all $i \neq j$. (2) $t_i > 0$ for some i. (3) $\beta_{ij} = \beta_{ik} = 0$ for some $\{i, j, k\} = \{1, 2, 3\}$. Then we can change indices to that $\beta_{31} = \beta_{32} = 0$. With $\beta_{12}\beta_{21} = 0$, we may change indices again so that furthermore $\beta_{21} = 0$. After these changes of indices, the matrix $\log_{\mathbf{Y}} \mathbf{Z}$ becomes upper triangular.

Step 1. We show first that $\beta_{ij}\beta_{ji} = 0$ for all $i \neq j$. If not, say $\beta_{12}\beta_{21} \neq 0$, we claim that the conditions $\beta_i = \beta_{ji} + \beta_{ki}$ on columns would hold for all $\{i, j, k\} = \{1, 2, 3\}$. From

$$\begin{pmatrix} \beta_1 & -\beta_{12} & -\beta_{13} \\ -\beta_{21} & \beta_2 & -\beta_{23} \\ -\beta_{31} & -\beta_{32} & \beta_3 \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} = \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix},$$

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we have a relation

$$(\beta_1 - \beta_{21})s_1 + (\beta_2 - \beta_{12})s_2 = (\beta_{23} + \beta_{13})s_3 + (t_1 + t_2).$$

The Apéry monomial $\mathbf{u}^{(\beta_1-\beta_{21})s_1}\mathbf{u}^{(\beta_2-\beta_{12})s_2}$ is divisible by the coefficient $\mathbf{u}^{t_1+t_2}$. Hence $t_1 = t_2 = 0$. If $\beta_{23} + \beta_{13} < \beta_3$, we would have two different factorizations

$$\mathbf{u}^{(\beta_1-\beta_{21})s_1}\mathbf{u}^{(\beta_2-\beta_{12})s_2} = \mathbf{u}^{(\beta_{23}+\beta_{13})s_3}$$

of a monomial in the rectangle. Hence $0 \le \beta_{23} + \beta_{13} - \beta_3 < \beta_3$ and we have another relation

$$(\beta_1 - \beta_{21})s_1 + (\beta_2 - \beta_{12})s_2 = \beta_{31}s_1 + \beta_{32}s_2 + (\beta_{23} + \beta_{13} - \beta_3)s_3 + t_3.$$

The Apéry monomial $\mathbf{u}^{(\beta_1-\beta_{21})s_1}\mathbf{u}^{(\beta_2-\beta_{12})s_2}$ is divisible by the coefficient $\mathbf{u}^{t_3}$. Hence $t_3 = 0$ and

$$\mathbf{u}^{(\beta_1-\beta_{21})s_1}\mathbf{u}^{(\beta_2-\beta_{12})s_2} = \mathbf{u}^{\beta_{31}s_1}\mathbf{u}^{\beta_{32}s_2}\mathbf{u}^{(\beta_{23}+\beta_{13}-\beta_3)s_3}.$$

Monomials in the rectangle are distinct. Hence the column conditions $\beta_i = \beta_{ji} + \beta_{ki}$ hold for all $\{i, j, k\} = \{1, 2, 3\}$.

To get a contradiction from $\beta_{12}\beta_{21} \neq 0$, we work on elements of the form $\beta_i s_i + \beta_{jk}s_k$, where $\{i, j, k\} = \{1, 2, 3\}$. By the column conditions, we have

$$\beta_i s_i + \beta_{jk}s_k = (\beta_{ji} + \beta_{ki})s_i + \beta_{jk}s_k = \beta_j s_j + \beta_{ki}s_i.$$

Therefore $\beta_i s_i + \beta_{jk}s_k$ represents the same number for any $\{i, j, k\} = \{1, 2, 3\}$. Write

$$\mathbf{u}^{\beta_i s_i}\mathbf{u}^{\beta_{jk}s_k} = \mathbf{u}^t\mathbf{u}^{\alpha_i s_i}\mathbf{u}^{\alpha_j s_j}\mathbf{u}^{\alpha_k s_k},$$

where $0 \le \alpha_i < \beta_i$, $0 \le \alpha_j < \beta_j$, $0 \le \alpha_k < \beta_k$ and $\mathbf{u}^t \in R$. If $\alpha_i > 0$, the Apéry monomial $\mathbf{u}^{(\beta_i-\alpha_i)s_i}\mathbf{u}^{\beta_{jk}s_k}$ is divisible by the coefficient $\mathbf{u}^t$. Then $t = 0$ and we would have two different factorizations

$$\mathbf{u}^{(\beta_i-\alpha_i)s_i}\mathbf{u}^{\beta_{jk}s_k} = \mathbf{u}^{\alpha_j s_j}\mathbf{u}^{\alpha_k s_k}$$

of a monomial in the rectangle. Hence α_i vanishes, and so do α_j and α_k. Now $\beta_i s_i + \beta_{jk}s_k = t$. By a similar argument, we have $\beta_i s_i + \beta_{kj}s_j = t' \in \log_{\mathbf{u}} R$. For $i = 3$, we obtain a contradiction by two different representations

$$\mathbf{u}^{t'}\mathbf{u}^{\beta_{12}s_2} = \mathbf{u}^t\mathbf{u}^{\beta_{21}s_1}.$$

Step 2. Since $\beta_{12}\beta_{21} = 0$, we may assume $\beta_{12} = 0$ by changing indices. If $\beta_{13} = 0$, then $\det \log_{\mathbf{Y}} \mathbf{Z} = \beta_1\beta_2\beta_3 - \beta_1\beta_{32}\beta_{23} > 0$. If $\beta_{13} \neq 0$, then $\beta_{31} = 0$ and $\det \log_{\mathbf{Y}} \mathbf{Z}$ can be computed according to vanishing of β_{32}: If furthermore $\beta_{32} = 0$, then $\det \log_{\mathbf{Y}} \mathbf{Z} = \beta_1\beta_2\beta_3 - \beta_{21}\beta_{12}\beta_3 > 0$. Otherwise, $\beta_{32} \neq 0$ implies that $\beta_{23} = 0$ and $\det \log_{\mathbf{Y}} \mathbf{Z} = \beta_1\beta_2\beta_3 - \beta_{21}\beta_{32}\beta_{13} > 0$. In any cases, $\det \log_{\mathbf{Y}} \mathbf{Z} > 0$. Therefore

$$\begin{vmatrix} \beta_1 s_1 - \beta_{12}s_2 - \beta_{13}s_3 & -\beta_{12} & -\beta_{13} \\ -\beta_{21}s_1 + \beta_2 s_2 - \beta_{23}s_3 & \beta_2 & -\beta_{23} \\ -\beta_{31}s_1 - \beta_{32}s_2 + \beta_3 s_3 & -\beta_{32} & \beta_3 \end{vmatrix} = s_1 \det \log_{\mathbf{Y}} \mathbf{Z} > 0.$$

Recall that the entries of the adjoint of $\log_{\mathbf{Y}} \mathbf{Z}$ are all non-negative. Hence $t_i = \beta_i s_i - \beta_{ij}s_j - \beta_{ik}s_k > 0$ for some $\{i, j, k\} = \{1, 2, 3\}$.

Step 3. Provided that $\beta_{ij}s_j + \beta_{ik}s_k = \beta_i s_i - t_i > 0$ for all $\{i,j,k\} = \{1,2,3\}$, we want to get a contradiction. After changing indices, we may assume that $t_3 > 0$ and $\beta_{21} = 0$. Let α_1 be a positive integer in $\log_{\mathbf{u}} R$. Then $\alpha := \alpha_1 s_1 \in \log_{\mathbf{u}} R$. We choose the largest element $s_0 \in \log_{\mathbf{u}} R$ such that there exists a factorization

$$\mathbf{u}^\alpha = \mathbf{u}^{s_0}(\mathbf{u}^{s_1})^{a_1}(\mathbf{u}^{s_2})^{a_2}(\mathbf{u}^{s_3})^{a_3}$$

satisfying the condition $s_0 < \alpha$. Write $a_1 = n_1\beta_1 + a_1'$, where $n_1 \in \mathbb{N}$ and $0 \leq a_1' < \beta_1$. Let $a_2' := a_2 + n_1\beta_{12}$ and $a_3' := a_3 + n_1\beta_{13}$. Then we have another factorization

$$\mathbf{u}^\alpha = \mathbf{u}^{s_0+n_1t_1}(\mathbf{u}^{s_1})^{a_1'}(\mathbf{u}^{s_2})^{a_2'}(\mathbf{u}^{s_3})^{a_3'}.$$

Since $\beta_{12}s_2 + \beta_{13}s_3 > 0$, the condition $s_0 + n_1t_1 < \alpha$ still holds. By the maximality of s_0, the number n_1t_1 has to vanish. Write $a_2' = n_2\beta_2 + a_2''$, where $n_2 \in \mathbb{N}$ and $0 \leq a_2'' < \beta_2$. Let $a_3''' := a_3'' + n_2\beta_{23}$. Since we assume $\beta_{21} = 0$, we have one more factorization

$$\mathbf{u}^\alpha = \mathbf{u}^{s_0+n_2t_2}(\mathbf{u}^{s_1})^{a_1'}(\mathbf{u}^{s_2})^{a_2''}(\mathbf{u}^{s_3})^{a_3'''}.$$

The condition $s_0 + n_2t_2 < \alpha$ still holds from the assumption $\beta_{21}s_1 + \beta_{23}s_3 > 0$. By the maximality of s_0 again, the number n_2t_2 has to vanish. We claim that $a_3''' < \beta_3$. Otherwise, we would have a factorization

$$\mathbf{u}^\alpha = \mathbf{u}^{s_0+t_3}(\mathbf{u}^{s_1})^{a_1'+\beta_{31}}(\mathbf{u}^{s_2})^{a_2''+\beta_{32}}(\mathbf{u}^{s_3})^{a_3'''-\beta_3}$$

contradicting the maximality of s_0, since we assume $\beta_{31}s_1 + \beta_{32}s_2 > 0$ and $t_3 > 0$. As an element in the rectangle, the monomial $(\mathbf{u}^{s_1})^{a_1'}(\mathbf{u}^{s_2})^{a_2''}(\mathbf{u}^{s_3})^{a_3'''}$ is Apéry. Now we have different representations $\mathbf{u}^\alpha\mathbf{u}^0$ and $\mathbf{u}^{s_0}\mathbf{u}^{a_1's_1+a_2''s_2+a_3'''s_3}$ of $\mathbf{u}^\alpha$. This can not happen in a flat algebra. □

Corollary 5.2. *A flat rectangular algebra $R[\![\mathbf{u}^{s_1}, \mathbf{u}^{s_2}, \mathbf{u}^{s_3}]\!]$ is complete intersection.*

Example 5.3. In the flat algebra $\kappa[\![\mathbf{u}^{16}, \mathbf{u}^{24}, \mathbf{u}^{31}, \mathbf{u}^{46}, \mathbf{u}^{44}]\!]/\kappa[\![\mathbf{u}^{16}, \mathbf{u}^{24}]\!]$, the set of the Apéry monomials is a rectangle of size $2 \times 2 \times 2$ with the relation

$$\begin{pmatrix} 2 & -1 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} 31 \\ 46 \\ 44 \end{pmatrix} = \begin{pmatrix} 16 \\ 2 \times 24 \\ 4 \times 16 + 24 \end{pmatrix}.$$

The matrix $\log_{\mathbf{Y}} \mathbf{Z}$ is triangular.

We define $\log_{\mathbf{Y}} \mathbf{Z}$ only for flat rectangular algebras. The following algebra is not flat. The corresponding 3×3 matrix is singular.

Example 5.4. The Apéry monomials of the algebra $\kappa[\![\mathbf{u}^3, \mathbf{u}^5, \mathbf{u}^7]\!]/\kappa[\![\mathbf{u}^{17}, \mathbf{u}^{19}]\!]$ are 1, $\mathbf{u}^3$, $\mathbf{u}^{18}$, $\mathbf{u}^{21}$ and $\mathbf{u}^s$ for $5 \leq s \leq 16$. They form a rectangle of size $4 \times 2 \times 2$ with the relation

$$\begin{pmatrix} 4 & -1 & -1 \\ -1 & 2 & -1 \\ -3 & -1 & 2 \end{pmatrix}\begin{pmatrix} 3 \\ 5 \\ 7 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Using a result of Bresinsky [3], we have the following result for an algebra with 4 minimal monomials.

Theorem 5.5. *Let $\mathbf{u}^{s_1}, \mathbf{u}^{s_2}, \mathbf{u}^{s_3}, \mathbf{u}^{s_4}$ be the minimal monomials of a rectangular algebra R'/R. If $R = \kappa[\![\mathbf{u}^s]\!]$ for some s in the semigroup generated by s_1, s_2, s_3, s_4, then R'/R is complete intersection.*

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Proof. We may assume that s_1, s_2, s_3, s_4 are integers with the greatest common divisor 1. In the rectangle of size $\beta_1 \times \beta_2 \times \beta_3 \times \beta_4$, we have relations

$$\beta_i s_i = \beta_{ij} s_j + \beta_{ik} s_k + \beta_{il} s_l + \lambda_i s$$

for $\{i, j, k, l\} = \{1, 2, 3, 4\}$, where $\lambda_i \in \mathbb{N}$. Since flat rectangular algebras are Gorenstein, the semigroup $\langle s_1, s_2, s_3, s_4 \rangle$ is symmetric. If the algebra is not complete intersection, then [3, Theorem 3] says

$$(\mathbf{u}^{s_1})^{\alpha_1}(\mathbf{u}^{s_3})^{\alpha_3} = (\mathbf{u}^{s_2})^{\alpha_2}(\mathbf{u}^{s_4})^{\alpha_4}$$

for some $0 < \alpha_i < c_i$, where $c_i = \min\{n \,|\, 0 < ns_i \in \langle s_j; j \neq i \rangle\}$. Since monomials in the rectangle are distinct, $\alpha_i \geq \beta_i$ for some i. Say $\alpha_1 \geq \beta_1$. Then $\beta_1 < c_1$. Write $s = n_1 s_1 + n_2 s_2 + n_3 s_3 + n_4 s_4$, where $n_1, n_2, n_3, n_4 \in \mathbb{N}$. Then

$$(\beta_1 - n_1\lambda_1)s_1 = (\beta_{12} + n_2\lambda_1)s_2 + (\beta_{13} + n_3\lambda_1)s_3 + (\beta_{14} + n_4\lambda_1)s_4.$$

By the minimality of c_1, the non-negative number $\beta_1 - n_1\lambda_1$ has to vanish. Therefore $\beta_{12} = \beta_{13} = \beta_{14} = n_2\lambda_1 = n_3\lambda_1 = n_4\lambda_1 = 0$ and $\lambda_1 s = \beta_1 s_1 = n_1\lambda_1 s_1$. Consequently, $\lambda_1 > 0$ and $s = n_1 s_1$. The monomial $(\mathbf{u}^{s_1})^{\alpha_1} = \mathbf{u}^{\lambda_1 s}(\mathbf{u}^{s_1})^{\alpha_1 - \beta_1}$ is not Apéry, nor is $(\mathbf{u}^{s_2})^{\alpha_2}(\mathbf{u}^{s_4})^{\alpha_4} = (\mathbf{u}^{s_1})^{\alpha_1}(\mathbf{u}^{s_3})^{\alpha_3}$. Therefore $\alpha_i \geq \beta_i$ for $i = 2$ or 4. Say $\alpha_2 \geq \beta_2$. Then $\beta_2 < c_2$ and $\lambda_2 s = \beta_2 s_2$ by the same argument as above. Now the relation $\beta_2 s_2 = \lambda_2 n_1 s_1$ contradicts the minimality of c_2. □

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