



On the existence of birational maximal Cohen-Macaulay modules over biradical extensions in mixed characteristic



Prashanth Sridhar

University of Kansas, 405 Snow Hall, 1460 Jayhawk Blvd, Lawrence, KS, 66045, USA

ARTICLE INFO

Article history:

Received 17 November 2020
Received in revised form 9 April 2021
Available online 13 May 2021
Communicated by S. Iyengar

MSC:

13B22; 13C10; 13C14; 13C15;
13D22; 13E05; 13H05

Keywords:

Birational maximal Cohen-Macaulay module
Biradical extension
Mixed characteristic
Unramified regular local ring
Integral closure

ABSTRACT

Let S be an unramified regular local ring of mixed characteristic $p \geq 3$ and S^p the subring of S obtained by lifting to S the image of the Frobenius map on S/pS . Let R be the integral closure of S in a biradical extension of degree p^2 of its quotient field obtained by adjoining p -th roots of sufficiently general square free elements $f, g \in S^p$. We show that R admits a birational maximal Cohen-Macaulay module. It is noted that R is not automatically Cohen-Macaulay.

© 2021 Elsevier B.V. All rights reserved.

1. Introduction

The existence of maximal Cohen-Macaulay (MCM) modules (2.1) over catenary rings is largely an open problem. Hochster conjectured that every complete local domain admits an MCM module (see for example [3] and [11]), but this is known to be true only in very few cases. The primary goal of this article is to show the existence of an MCM module for a new class of rings.

Hochster's conjecture reduces to the integral closure of a complete regular local ring in a normal extension of its fraction field, so that one may approach the problem from this viewpoint. Let S denote an unramified regular local ring with quotient field L and R the integral closure of S in a finite field extension K/L . In [10] it is shown that if K/L is Abelian and the degree of the extension is not divisible by the characteristic of the residue field, then R is Cohen-Macaulay. Note that this applies when S contains the rational numbers. The proof uses the fact that the group algebra $k[G]$ is a product of fields when the residue field k of S is

E-mail address: prashanth@ku.edu.

algebraically closed and $G = \text{Gal}(K/L)$. There is no direct analog of the argument when $\text{char}(k)$ divides the order of G . In fact, the conclusion fails in mixed characteristic as shown in [7] and [5].

Motivated by the above phenomenon and the studies in [5], [6] and [12], we consider extensions K/L with the property that R is S -free when S contains a field but not necessarily so otherwise. Kummer theory tells us that Abelian extensions of a field of characteristic zero containing “suitable” roots of unity are repeated radical extensions. Thus it is natural to study repeated n -th root extensions of an unramified regular local ring of mixed characteristic $p > 0$ with the property that $p|n$. If S were to contain the rational numbers or if S were of mixed characteristic $p > 0$ and $p \nmid n$, then it follows that the integral closure of S in an arbitrary repeated radical extension is Cohen-Macaulay, see [12] and [2].

Now assume additionally that S has mixed characteristic $p \geq 3$. If K is the extension by a p -th root of a square free element of S , then R is Cohen-Macaulay as shown in [5]. In contrast, we shall see that the integral closure R fails to be Cohen-Macaulay in a finite square free tower of p -th roots. In this paper, we consider the existence of maximal Cohen-Macaulay modules over the integral closure of S in an extension of degree p^2 , obtained by adjoining p -th roots of sufficiently general square free elements. The generality and square free conditions on the elements are natural, since any given multi-radical extension can be embedded in a sufficiently large, general, square free tower.

Let S^p denote the subring of S obtained by lifting the image of the Frobenius map on S/pS to S . Towards the above goal, the immediate obstruction is when $f, g \in S^p$, see [8]. We now outline our principal findings. Fix $f, g \in S^p$ square free, non-units that form a regular sequence in S or units that are not p -th powers in S . Let $\omega^p = f$ and $\mu^p = g$. If f, g are units, assume further that $[L(\omega, \mu) : L] = p^2$. Given integers $n, k \geq 1$, let $S^{p^k \wedge p^n} \subset S$ be the multiplicative subset of S consisting of elements expressible in the form $x^{p^k} + y \cdot p^n$ for some $x, y \in S$. For a discussion of the case $p = 2$, see [12], where the results are sharper since in that case such extensions are automatically Abelian. When $p \geq 3$, the presence of a p -th root of unity in S necessarily ramifies p , so we do not quite have the same leg room. The main result of this paper is

Theorem 4.10. *Let (S, \mathfrak{m}) be an unramified regular local ring of mixed characteristic $p \geq 3$. Then*

1. *R is Cohen-Macaulay if*
 - (a) *At least one of $S[\omega], S[\mu]$ is not integrally closed.*
 - (b) *$S[\omega], S[\mu]$ are integrally closed and $fg^i \notin S^{p \wedge p^2}$ for all $1 \leq i \leq p-1$.*
2. *Let $S[\omega], S[\mu]$ be integrally closed such that $fg \in S^{p \wedge p^2}$. Then R is Cohen-Macaulay if and only if $Q := (p, f, g) \subset S$ is a two generated ideal or all of S . Moreover, $p.d._S(R) \leq 1$ and $\nu_S(R) \leq p^2 + 1$.*
3. *If $Q := (p, f, g) \subseteq S$ has grade three, R admits a birational maximal Cohen-Macaulay module.*

In section 2, we set up convention and make some preliminary remarks that will be used subsequently. In section 3, we provide some sufficient conditions for R to be Cohen-Macaulay by showing a more general version of 4.10(1), see 3.1 and 3.3. We also identify the conductor of R to the complete intersection ring $A := S[\omega, \mu]$ in a crucial case. In section 4, we prove parts (2) and (3) of 4.10. We see that R is not “too far” from being Cohen-Macaulay, in the sense that it can be generated by $p^2 + 1$ elements over the base ring S and $p.d._S(R) \leq 1$. However, it is not as close as it appears, since if $\dim(S) \geq 3$ it could be that R does not even satisfy Serre’s condition S_3 . We then show that R admits a birational maximal Cohen-Macaulay module when $Q := (p, h_1, h_2)$ has grade three. The condition on Q can be viewed as a further generality condition on the chosen elements. For an A -module M , let $M^* = \text{Hom}_A(M, A)$ be the dual module. The strategy here is to realize $R = I^*$ for a suitable ideal $I \subseteq A$ and identifying an ideal $J \subseteq A$ such that J^* is an I^* module and J^* is S -free.

2. Preliminaries

All rings considered are commutative and Noetherian and all modules finitely generated.

Convention 2.1.

1. Let R be a Noetherian ring and M an R -module. For $G \subseteq M$ a subset, the notation $M = \langle G \rangle_R$ means that M is generated as a R -module by G .
2. Let (R, \mathfrak{m}) be a local ring of dimension d . A nonzero R -module M is a **maximal Cohen-Macaulay module** (MCM) over R if it is finitely generated and every (some) system of parameters of R is a regular sequence on M . If R is an arbitrary Noetherian ring, then an R -module M is a **maximal Cohen-Macaulay module** if for all maximal ideals $\mathfrak{m} \subseteq R$, $M_{\mathfrak{m}}$ is an MCM module over $R_{\mathfrak{m}}$.
3. Let R be a domain and M an R -module. Denote by M_R^* , the dual module $\text{Hom}_R(M, R)$. If R is clear from the context, denote it by M^* . Suppose that $M \subseteq K$, where K denotes the field of fractions of R . Then via the identification $\text{Hom}_R(M, R) \simeq (R :_K M)$, we use M^* to denote $(R :_K M)$ as well.
4. If R is a Noetherian ring of dimension at least one, denote

$$NNL_1(R) := \{P \in \text{Spec}(R) \mid \text{height}(P) = 1, R_P \text{ is not a DVR}\}$$

5. Suppose S is a ring and $p \in \mathbb{Z}$ is such that $p \in S$ is a non-unit. Let $F : S/pS \rightarrow S/pS$ be the Frobenius map. Let S^p denote the subring of S obtained by lifting the image of F to S . Let $S^{p^k \wedge p^n}$ for $k, n \geq 1$ denote the multiplicative subset of S of elements expressible in the form $x^{p^k} + yp^n$ for some $x, y \in S$. In particular, $S^{p \wedge p} = S^p$.
6. For a local ring R and an R -module M , denote $\nu_R(M)$ for the minimal number of generators of M over R .

Remark 2.2. Let $S \subseteq C \subseteq D$ be an extension of Noetherian domains such that S is integrally closed, D is module finite over S and D is birational to C . Then if C is regular in codimension one, so is D . (see [13][Theorem 2.4] for example)

Remark 2.3. Let $S \subseteq D$ be an extension of Noetherian domains such that going down holds. Let \overline{D} denote the integral closure of D in its field of fractions K and assume \overline{D} is finite over D . If $c \cdot u, c \cdot v \in D :_K \overline{D}$ with $c \in D$ and $u, v \in S$ such that there exists no height one prime of S containing both of them, then $NNL_1(D) \subseteq V(c)$.

Remark 2.4. Let A be a Noetherian Gorenstein local domain and R the integral closure of A . Assume that R is module finite over A . Then for every height one unmixed ideal $I \subseteq A$, I^* is Cohen-Macaulay if and only if A/I is Cohen-Macaulay (see [12][Proposition 2.11] for example). In particular, R is Cohen-Macaulay if and only if the conductor of R to A is Cohen-Macaulay.

Remark 2.5. Let $S \subseteq R$ be a finite extension of local rings such that S is Gorenstein and R is Cohen-Macaulay. Then for an R -module M , we have $\text{Hom}_R(M, \omega_R) \simeq \text{Hom}_S(M, S)$ as R -modules (and S -modules), where ω_R is the canonical module of R .

Convention 2.6. We will assume the following notation for the rest of the paper unless otherwise specified. Let S denote a Noetherian, integrally closed domain of dimension d and L its field of fractions. Assume $\text{Char}(L) = 0$. Fix $3 \leq p \in \mathbb{Z}$ a prime and assume that $p \in S$ is a principal prime such that S/pS is integrally closed. Note here that an unramified regular local ring of mixed characteristic p satisfies the above hypothesis, though not all results here require this specific setting.

Say that a subset $E \subset S$ satisfies \mathcal{A}_1 , if for all distinct $x, y \in E$, there exists no height one prime $Q \subset S$ such that $(x, y) \subset Q$. An element $c \in S$ is square free if for all height one primes $P \subset S$ containing c , $PS_P = (c)S_P$.

Fix $f, g \in S^p$ square free, non-units, satisfying \mathcal{A}_1 or $f, g \in S^p$ units such that they are not p -th powers in S . Write $f = h_1^p + a \cdot p$ and $g = h_2^p + b \cdot p$ with $h_1, h_2, a, b \in S$.

Let W, U be indeterminates over S . We have the monic irreducible polynomials $F(W) := W^p - f \in S[W]$ and $G(U) := U^p - g \in S[U]$. Let ω, μ be roots of $F(W)$ and $G(U)$ respectively and set $K := L(\omega, \mu)$. Assume that $G(U)$ is irreducible over $L(\omega)$, so that $[K : L] = p^2$. If S is a unique factorization domain and f, g are non units, it can be shown that $[K : L] = p^2$ is automatic. Let R be the integral closure of S in K , that is R is the integral closure of $A := S[\omega, \mu]$. Note that, $A \simeq S[W, U]/(F(W), G(U))$, $S[\omega] \simeq S[W]/(F(W))$ and $S[\mu] \simeq S[U]/(G(U))$.

Remark 2.7. From [12][Proposition 2.10], if exactly one of f, g lies in S^p then R is S -free. Moreover [12][Example 2.12] shows that R need not be S -free when $f, g \notin S^p$. However, to construct an MCM module over R it suffices to consider the case $f, g \in S^p$ when S is a complete unramified regular local ring with perfect residue field, see [8]. This motivates us to understand the case $f, g \in S^p$.

Lemma 2.8 ([5]). *Let $p = 2k + 1$ and $h \in S \setminus pS$. Let W be an indeterminate over S . If*

$$C := (W^p - h^p) - (W - h)^p = \sum_{j=1}^k (-1)^{j+1} \binom{p}{j} (W \cdot h)^j [W^{p-2j} - h^{p-2j}] \quad (1)$$

$C' := (p(W - h))^{-1} \cdot C$ and $\tilde{P} := (p, W - h)S[W]$, then $C' \notin \tilde{P}$.

Lemma 2.9. *Let $p = 2k + 1$ and $h \in S \setminus pS$. Let W be an indeterminate over S . Suppose C' is as defined in 2.8. Then $C' \equiv h^{p-1} \pmod{(p, W - h)S[W]}$.*

Proof. We have in $S[W]$

$$\begin{aligned} C' &= \sum_{j=1}^k (-1)^{j+1} j^{-1} \binom{p-1}{j-1} (W \cdot h)^j [W^{p-2j-1} + \dots + h^{p-2j-1}] \\ &\equiv \sum_{j=1}^k (-1)^{j+1} j^{-1} \binom{p-1}{j-1} h^{2j} \cdot (p-2j) \cdot h^{p-2j-1} \pmod{(p, W-h)} \\ &\equiv -2h^{p-1} \sum_{j=1}^k (-1)^{j+1} \binom{p-1}{j-1} \pmod{(p, W-h)} \\ &\equiv -h^{p-1} (-1)^{k+1} \binom{2k}{k} \pmod{(p, W-h)} \\ &\equiv h^{p-1} \pmod{(p, W-h)} \quad \square \end{aligned}$$

Convention 2.10. Suppose $h_1, h_2 \in S \setminus pS$. In this case, let C'_1 and C'_2 denote respectively the elements in the rings $S[W]$ and $S[U]$ obtained by setting $h = h_1$ and $h = h_2$ in 2.8. Denote by c'_1 and c'_2 their respective images in the rings $S[\omega]$ and $S[\mu]$ respectively. If $h_1 = 0$ ($h_2 = 0$), simply set $c'_1 = 0$ ($c'_2 = 0$). Denote by d_i the corresponding element in $S[\omega\mu^i]$ for $1 \leq i \leq p-1$.

Proposition 2.11 ([5]). *With notation as specified above, $S[\omega]$ is integrally closed if and only if $f \notin S^{p \wedge p^2}$. Further, if $S[\omega]$ is not integrally closed, write $f = h_1^p + a' \cdot p^2$ for some $a' \in S$. Then*

- (a) $\overline{S[\omega]} = (P^*)_{S[\omega]} = S[\omega, \tau_1]$ where $\tau_1 = p^{-1} \cdot (\omega^{p-1} + h_1\omega^{p-2} + \cdots + h_1^{p-1})$ and $P := (p, \omega - h_1)S[\omega]$.
- (b) There are exactly two height one primes in $\overline{S[\omega]}$ containing p , namely $P := (p, \omega - h_1, \tau_1)\overline{S[\omega]}$ and $Q := (p, \omega - h_1, \tau - c'_1)\overline{S[\omega]}$. Further, $P_P = (\omega - h_1)_P$ and $Q_Q = (p)_Q$.
- (c) The element τ_1 satisfies $l(T) := T^2 - c'_1T - a' \cdot (\omega - h_1)^{p-2} \in S[\omega][T]$.
- (d) $\overline{S[\omega]} = \langle 1, \omega, \omega^2, \dots, \omega^{p-2}, \tau_1 \rangle_S$ is S -free.

3. Cohen-Macaulay integral closures

In this section we identify scenarios where R is S -free by showing more general versions of 1(a) and 1(b) in Theorem 4.10. We maintain notation as set up in 2.6.

Proposition 3.1. R is S -free if at least one of the rings $S[\omega], S[\mu]$ is not integrally closed.

Proof. We organize the proof as follows:

1. Assume $S[\omega]$ and $S[\mu]$ are both not integrally closed. We then
 - (a) Identify a finite birational overring $A \hookrightarrow \mathcal{R}A$ such that $\mathcal{R}A$ satisfies R_1 .
 - (b) Identify a “natural” finite birational overring $\mathcal{R}A \hookrightarrow Z$ such that Z is S -free, so that $R = Z$ is S -free.
2. Assume exactly one of the rings $S[\omega], S[\mu]$ is integrally closed. We then take an identical path as indicated in (1) above.

1. (a) From 2.11, $f, g \in S^{p \wedge p^2}$. Write $f = h_1^p + a' \cdot p^2$ and $g = h_2^p + b' \cdot p^2$ for some $a', b' \in S$. Note that $h_1, h_2 \neq 0$ since f, g are square free. We have from 2.11 that $S[\omega, \tau_1], S[\mu, \tau_2]$ are the respective normalizations of $S[\omega]$ and $S[\mu]$ where

$$\begin{aligned}\tau_1 &= p^{-1} \cdot (\omega^{p-1} + h_1\omega^{p-2} + \cdots + h_1^{p-1}) \\ \tau_2 &= p^{-1} \cdot (\mu^{p-1} + h_2\mu^{p-2} + \cdots + h_2^{p-1})\end{aligned}$$

Set $E := A[\tau_1, \tau_2]$. Let X, Y be indeterminates over A and let $\phi : A[X, Y] \rightarrow E$ be the projection map sending $X \rightarrow \tau_1$ and $Y \rightarrow \tau_2$. From 2.11:

$$X^2 - c'_1X - a'(\omega - h_1)^{p-2}, Y^2 - c'_2Y - b'(\mu - h_2)^{p-2} \in \text{Ker}(\phi)$$

Height one primes in E containing p correspond to height three primes in $A[X, Y]$ containing $\text{Ker}(\phi)$ and p . Since $P := (p, \omega - h_1, \mu - h_2)$ is the unique height one prime in A containing p , any such height three prime in $A[X, Y]$ has to contain either X or $X - c'_1$. Likewise it contains either Y or $Y - c'_2$. Therefore if $Q \subseteq A[X, Y]$ is a height three prime containing p and $\text{Ker}(\phi)$, it must be that $(p, \omega - h_1, \mu - h_2, X - m, Y - n) \subseteq Q$ for some $m, n \in A$ and hence the containment must be an equality. Moreover, there is at least one height one prime in E containing p , since p is not a unit in S . Therefore, the only possibilities for height one primes in E containing p are

$$\begin{aligned}P_1 &:= (p, \omega - h_1, \mu - h_2, \tau_1, \tau_2) \\ P_2 &:= (p, \omega - h_1, \mu - h_2, \tau_1, \tau_2 - c'_2) \\ P_3 &:= (p, \omega - h_1, \mu - h_2, \tau_1 - c'_1, \tau_2) \\ P_4 &:= (p, \omega - h_1, \mu - h_2, \tau_1 - c'_1, \tau_2 - c'_2)\end{aligned}$$

We have $\omega \cdot F'(\omega) = p \cdot f \in (S[\mu, \tau_2, \omega] :_K R)$ and identically $p \cdot g \in (S[\omega, \tau_1, \mu] :_K R)$ (see for example [4][Theorem 12.1.1]) and hence $p \cdot f, p \cdot g \in (E :_K R)$. From 2.3, $NNL_1(E) \subseteq V(p)$. But the localizations of E at P_2, P_3 and P_4 are regular with uniformizing parameters being the images of $\omega - h_1, \mu - h_2$ and p respectively. For example, consider P_{2P_2} . Let $Q_1 := (p, \omega - h_1, \tau_1)S[\omega, \tau_1]$ and $Q_2 := (p, \mu - h_2, \tau_2 - c'_2)S[\mu, \tau_2]$. From 2.11, $Q_{1Q_1} = (\omega - h_1)_{Q_1}$ and $Q_{2Q_2} = (p)_{Q_2}$. Thus $P_{2P_2} = (\omega - h_1)_{P_2}$. The P_3 and P_4 cases are similar. Note that however $P_{1P_1} = (\omega - h_1, \mu - h_2)_{P_1}$.

Set $\eta_1 = p^{-1}(\omega - h_1)(\mu - h_2)^{p-2} \in K$. Let X be an indeterminate over E . Then η_1 satisfies $l(X) \in E[X]$ where $l(X) := X^{p-1} - (\tau_1 - c'_1)(\tau_2 - c'_2)^{p-2}$ since $p \cdot \tau_1 = (\omega - h_1)^{p-1} + p \cdot c'_1$ (similarly for $p \cdot \tau_2$). We claim that $\mathcal{R}A := E[\eta_1]$ is regular in codimension one. From 2.2, $NNL_1(\mathcal{R}A) \subseteq V(P_1\mathcal{R}A)$. Denote by $\overline{l(X)}$ the image of $l(X)$ in $(E/P_1E)[X]$. From 2.9 we have $c'_1 \equiv h_1^{p-1}$ and $c'_2 \equiv h_2^{p-1}$ in the ring E/P_1E , and thus¹

$$\overline{l(X)} = X^{p-1} - (h_1 h_2^{p-2})^{p-1} = \prod_{k=1}^{p-1} (X + k h_1 h_2^{p-2}) \in (E/P_1E)[X]$$

Thus, the only possibilities for height one primes in $\mathcal{R}A$ lying over P_1 are

$$Q_k = (p, \omega - h_1, \mu - h_2, \tau_1, \tau_2, \eta_1 + k h_1 h_2^{p-2})\mathcal{R}A$$

for $1 \leq k \leq p-1$ and we have

$$Q_k Q_k = (\omega - h_1, \mu - h_2, \eta_1 + k h_1 h_2^{p-2})_{Q_k}$$

Since $(\mu - h_2)\eta_1 = (\tau_2 - c'_2)(\omega - h_1)$ and

$$\eta_1, \tau_2 - c'_2, \prod_{j=1, j \neq k}^{p-1} (\eta_1 + j h_1 h_2^{p-2}) \notin Q_k$$

we have $Q_k Q_k = (\mu - h_2)_{Q_k} = (\omega - h_1)_{Q_k}$. Therefore $\mathcal{R}A$ is regular in codimension one.

(b) Set $Z := \langle T \rangle_S$ where $T := T_1 \cup T_2 \cup T_3$ and

$$\begin{aligned} T_1 &:= \{(\omega - h_1)^i (\mu - h_2)^j \mid 0 \leq i, j \leq p-1, i+j < p-1\} \\ T_2 &:= \{p^{-1}(\omega - h_1)^i (\mu - h_2)^j \mid 0 \leq i, j \leq p-1, i+j \geq p-1, i+j \neq 2p-2\} \\ T_3 &:= \{p^{-2}(\omega - h_1)^{p-1} (\mu - h_2)^{p-1}\} \end{aligned}$$

For every choice of $0 \leq i, j \leq p-1$, there is a unique element in T with “leading coefficient” $\omega^i \mu^j$. Therefore the order of T is p^2 . Moreover, since A is S -free with a basis given by $D := \{(\omega - h_1)^i (\mu - h_2)^j \mid 0 \leq i, j \leq p-1\}$, the elements of T are linearly independent over S . From the relations

$$\begin{aligned} (\omega - h_1)^p &= a' p^2 - p c'_1 (\omega - h_1) \\ (\mu - h_2)^p &= b' p^2 - p c'_2 (\mu - h_2) \end{aligned}$$

we get that Z is a ring. Since Z satisfies S_2 , if we show $\mathcal{R}A \subseteq Z$, then from 2.2 $Z = R$. Since $D \subseteq T$, we have $A \subseteq Z$. Since $\eta_1 = p^{-1}(\omega - h_1)(\mu - h_2)^{p-2} \in T$, it only remains to be seen that $\tau_1, \tau_2 \in Z$. But this is clear from the relation $\tau_1 = p^{-1}(\omega - h_1)^{p-1} + c'_1$ (analogously for τ_2). Thus $Z = R$ and R is S -free.

¹ If R is a ring of characteristic p and X, Y indeterminates over R , then for $X^{p-1} - Y^{p-1} \in R[X, Y]$, $X^{p-1} - Y^{p-1} = \prod_{i=1}^{p-1} (X + iY)$.

2. (a) Assume without loss of generality $\overline{S[\omega]} = S[\omega, \tau]$ where $\tau = p^{-1}(\omega^{p-1} + \cdots + h_1^{p-1})$ and that $S[\mu]$ is integrally closed. Notice that if $P := (p, \mu - h_2) \subseteq S[\mu]$ is the unique height one prime in $S[\mu]$ containing p , then $P_P = (\mu - h_2)_P$ since $(\mu - h_2)(\mu^{p-1} + \cdots + h_2^{p-1}) = bp$ and $b \notin pS$. From 2.11, $f \in S^{p \wedge p^2}$, so write $f = h_1^p + a'p^2$.

Set $E := S[\omega, \mu, \tau]$. From 2.11, it follows that there are precisely two height one primes in E containing p , namely $P_1 := (p, \omega - h_1, \tau, \mu - h_2)$ and $P_2 := (p, \omega - h_1, \tau - c'_1, \mu - h_2)$. From 2.3, E is regular in codimension one outside of P_1 and P_2 . It follows from 2.11(b) that $P_1 P_1 = (\omega - h_1, \mu - h_2)_{P_1}$ and $P_2 P_2 = (p, \mu - h_2)_{P_2} = (\mu - h_2)_{P_2}$.

Set $\eta_1 := p^{-1}(\omega - h_1)(\mu - h_2)^{p-1} \in K$. $\eta_1 \in R$ since it satisfies

$$l(X) := X^{p-1} - (\tau - c'_1)k_2^{p-2}(\mu - h_2) \in E[X] \quad (2)$$

where $k_2 = p^{-1}(\mu - h_2)^p \in A \setminus P$. Set $\mathcal{R}A := E[\eta_1]$. From 2.2, $\mathcal{R}A$ is regular in codimension one if height one primes in $\mathcal{R}A$ lying over $P_1 E$ are regular. From the above integral equation for η_1 over E , it is clear that the only such height one prime in $\mathcal{R}A$ is $Q_1 := (p, \omega - h_1, \tau, \mu - h_2, \eta_1)$. Now $Q_1 Q_1 = (\omega - h_1, \mu - h_2, \eta_1)_{Q_1}$. But $\eta_1(\mu - h_2) = k_2(\omega - h_1)$ and $k_2 \notin Q_1$. Further, since $\tau - c'_1 \notin Q_1$, $Q_1 Q_1 = (\eta_1)_{Q_1}$. Therefore $\mathcal{R}A$ is regular in codimension one.

(b) Set $Z := \langle T \rangle_S$ where $T = T_1 \cup T_2 \cup T_3$ and

$$\begin{aligned} T_1 &:= \{(\omega - h_1)^i(\mu - h_2)^j \mid i + j < p, 0 \leq i \leq p - 2, 0 \leq j \leq p - 1\} \\ T_2 &:= \{p^{-1}(\omega - h_1)^i(\mu - h_2)^j \mid i + j \geq p, 1 \leq i \leq p - 1, 1 \leq j \leq p - 1\} \\ T_3 &:= \{\tau - c'_1 = p^{-1}(\omega - h_1)^{p-1}\} \end{aligned}$$

For every $0 \leq i, j \leq p - 1$, there is a unique element in T with “leading coefficient” $\omega^i \mu^j$. Therefore the order of T is p^2 . Moreover since A is S -free with a basis given by $D := \{(\omega - h_1)^i(\mu - h_2)^j \mid 0 \leq i, j \leq p - 1\}$, the elements of T are linearly independent over S . From the relations

$$\begin{aligned} (\omega - h_1)^p &= a'p^2 - pc'_1(\omega - h_1) \\ (\mu - h_2)^p &= bp - pc'_2(\mu - h_2) \end{aligned}$$

we see that Z is a ring. Since Z satisfies S_2 , if we show that $\mathcal{R}A \subseteq Z$, then from 2.2 $Z = R$. It now suffices to note that $D \subseteq Z$, $\eta_1 \in T$ and $\tau := p^{-1}(\omega - h_1)^{p-1} + c'_1 \in Z$, so that $\mathcal{R}A \subseteq Z$. Thus $R = Z$ is S -free. \square

Lemma 3.2. *With established notation, assume that $S[\omega]$ and $S[\mu]$ are integrally closed. The following hold*

1. $(P^{(p-1)})_A^* = \langle T \rangle_S$, where $P^{(p-1)}$ denotes the $(p - 1)$ -th symbolic power of the unique height one prime $P \subseteq A$ containing p and $T := T_1 \cup T_2$, with

$$\begin{aligned} T_1 &:= \{(\omega - h_1)^i(\mu - h_2)^j \mid 0 \leq i, j \leq p - 1, i + j < p\} \\ T_2 &:= \{p^{-1}(\omega - h_1)^i(\mu - h_2)^j \mid 0 \leq i, j \leq p - 1, i + j \geq p\} \end{aligned}$$

2. The ring $A/P^{(p-1)}$ is Cohen-Macaulay.

Proof. For every $0 \leq i, j \leq p - 1$, there is a unique element in T with “leading coefficient” $\omega^i \mu^j$. Therefore the order of T is p^2 . Moreover since A is S -free with a basis given by $D := \{(\omega - h_1)^i(\mu - h_2)^j \mid 0 \leq i, j \leq p - 1\}$, the elements of T are linearly independent over S . From the relations

$$(\omega - h_1)^p = ap - pc'_1(\omega - h_1)$$

$$(\mu - h_2)^p = bp - pc'_2(\mu - h_2)$$

we see that $\langle T \rangle_S$ is a ring (in particular it is an A -module). Moreover, since it is Cohen-Macaulay, (2) immediately follows from (1) by using 2.4. Therefore, only (1) remains to be shown. Since $(P^{(p-1)})^*$ and $\langle T \rangle_S$ are birational, S_2 A -modules, it suffices to show their equality in codimension one. If $Q \subseteq A$, $Q \neq P$ is a height one prime, the equality is clear. So localize A at P and assume (A, P) local for the rest of the proof. Then $\langle T \rangle_S = A[\eta]$ where $\eta := (\omega - h_1)^{-1}(\mu - h_2)$. Note that $A = B'/(G(U))$, where $B' := S[\omega][U]$. Set $\tilde{I} := (\omega - h_1, U - h_2)^{p-1} \subseteq B'$. We have $G(U) \in \tilde{I}$:

$$\begin{aligned} U^p - g &= (U - h_2)^p + p(C'_2(U - h_2) - b) \\ &= -k_1^{-1}(\omega - h_1)(C'_2(U - h_2) - b) \cdot \Delta_1 + 0 \cdot \Delta_2 + \cdots + 0 \cdot \Delta_{p-1} - (U - h_2) \cdot \Delta_p \end{aligned}$$

where $\Delta_i = (-1)^i e_i$ with $e_i = (\omega - h_1)^{p-i}(U - h_2)^{i-1}$ and $k_1 = p^{-1}(\omega - h_1)^p$. That is \tilde{I} is the lift to B' of P^{p-1} . Further, \tilde{I} is grade two perfect since it arises as the ideal of maximal minors of the $p \times (p-1)$ matrix M :

$$M = \begin{bmatrix} U - h_2 & 0 & \cdots & 0 & 0 \\ \omega - h_1 & U - h_2 & 0 & \cdots & 0 \\ 0 & \omega - h_1 & U - h_2 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \omega - h_1 & U - h_2 \\ 0 & \cdots & 0 & 0 & \omega - h_1 \end{bmatrix}$$

Let M' be the $p \times p$ matrix obtained by adjoining M with the column of coefficients of $G(U)$:

$$M' = \begin{bmatrix} U - h_2 & 0 & \cdots & 0 & 0 & -k_1^{-1}(\omega - h_1)(C'_2(U - h_2) - b) \\ \omega - h_1 & U - h_2 & 0 & \cdots & 0 & 0 \\ 0 & \omega - h_1 & U - h_2 & \cdots & 0 & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & \omega - h_1 & U - h_2 & 0 \\ 0 & \cdots & 0 & 0 & \omega - h_1 & -(U - h_2) \end{bmatrix}$$

By [9][Lemma 2.5] (or [5][Prop 2.1]), $(P^{p-1})^*$ is generated as an A -module by the set $\{\delta_i^{-1}M'_{i,i} \mid 1 \leq i \leq p\}$, where δ_i denotes the image of Δ_i in A and $M'_{i,i}$ the image in A of the (i, i) -th cofactor of M' . This is exactly the set $\{\eta^{p-1}, \eta^{p-2}, \dots, \eta, 1\}$. Since η satisfies the integral equation $X^p - k_1^{-1}k_2 \in A[X]$ (with $k_2 = p^{-1}(\mu - h_2)^p$), this implies $(P^{p-1})^* = A[\eta] = \langle T \rangle_S$. Thus the proof is complete. \square

Proposition 3.3. *With established notation, R is S -free if $S[\omega]$ and $S[\mu]$ are integrally closed and $fg^i \notin S^{p \wedge p^2}$ for $1 \leq i \leq p-1$. Further in this case, $P^{(p-1)}$ is the conductor of R to A where P is the unique height one prime in A containing p and $P^{(p-1)}$ denotes the $(p-1)$ -th symbolic power of P .*

Proof. Since $S[\omega]$ and $S[\mu]$ are integrally closed, we have from 2.11 that $f, g \notin S^{p \wedge p^2}$. Write $f = h_1^p + ap$ and $g = h_2^p + bp$ with $a, b \notin pS$. We first note the following: The condition $fg^i \notin S^{p \wedge p^2}$ for all $1 \leq i \leq p-1$ is equivalent to the condition $\prod_{i=1}^{p-1}(ah_2^p + ibh_1^p) \notin pS$. This follows since for $1 \leq i \leq p-1$

$$\begin{aligned} fg^i &= (h_1^p + ap)(h_2^p + bp)^i \\ &= (h_1h_2^i)^p + h_2^{p(i-1)}(ah_2^p + ibh_1^p) \cdot p + q \cdot p^2 \end{aligned}$$

for some $q \in S$. We organize the proof as follows:

1. We first construct the normalization of A locally at $NNL_1(A)$. Since $NNL_1(A) = \{P\}$, we only need to construct R_P .
2. Using (1), identify a finite birational overring $A \hookrightarrow \mathcal{R}A$ such that $\mathcal{R}A$ satisfies R_1 . Then choose a suitable finite birational overring $\mathcal{R}A \hookrightarrow Z$ such that Z is S -free. This would show $R = Z$ is S -free.

1. From 2.3, A is regular in codimension one outside of $V(p)$. Moreover $P := (p, \omega - h_1, \mu - h_2)$ is the only height one prime in A containing p . Localize at P and assume (A, P) local for part (1). Now $(\omega - h_1)^p = pk_1$ and $(\mu - h_2)^p = pk_2$ where $k_1 = a - c'_1(\omega - h_1)$ and $k_2 = b - c'_2(\mu - h_2)$. Since $k_1, k_2 \notin P$, we have $P = (\omega - h_1, \mu - h_2)$. The element $\eta = (\omega - h_1)^{-1}(\mu - h_2) \in K$ satisfies the integral equation $l(X) := X^p - k_1^{-1}k_2 \in A[X]$. If $\eta \in A$, then $P_P = (\omega - h_1)_P$ so A is integrally closed. But from [12][Proposition 2.7] this is impossible. Therefore $A[\eta]$ is a proper birational extension of A . We claim that $E := A[\eta]$ is regular when $\prod_{i=1}^{p-1}(ah_2^p + ibh_1^p) \notin pS$. We will observe that E is local with maximal ideal $Q \subseteq E$ either of the form $Q = PE$ or $Q = (P, \eta - r)E$ for some suitable $r \in S \setminus pS$. To see this, let $\phi : \tilde{E} := S[W, U]_{\tilde{P}}[X] \rightarrow E$ be the natural projection map sending $W \mapsto \omega, U \mapsto \mu, X \mapsto \eta$, where W, U, X are indeterminates over S and $\tilde{P} := (p, W - h_1, U - h_2)$. Let $Q \subseteq E$ be any maximal ideal and let \tilde{Q} be the preimage of Q under ϕ . Now

$$l(X) \equiv X^p - ba^{-1} \in (A/P)[X] \simeq (S/pS)[X] \quad (3)$$

Since S/pS is a field, if $l(X)$ is an irreducible polynomial over $(S/pS)[X]$ then

$$(p, W - h_1, U - h_2, l(X)) \subseteq \tilde{Q}$$

is a height four prime containing p . Therefore the above inclusion must be an equality. If on the other hand $l(X)$ is reducible over S/pS then $l(X) \equiv (X - r)^p \in (S/pS)[X]$ for some $r \in S \setminus pS$. So in this case

$$(p, W - h_1, U - h_2, X - r) \subset \tilde{Q}$$

is a height four prime. Again, the above inclusion must then be an equality. Therefore in either case E is local and the maximal ideal is either of the form $PE = (\omega - h_1, \mu - h_2)E$ or $Q := (\omega - h_1, \mu - h_2, \eta - r)E$. In the first case, since $\eta \cdot (\omega - h_1) = \mu - h_2$, E is a DVR. In the second case, we have $Q_Q = (\omega - h_1, \eta - r)_Q$. We now show that $Q_Q = (\eta - r)_Q$. We have for some $m \in E$:

$$\begin{aligned} (\eta - r)^p &= \eta^p - ba^{-1} + pm \\ &= k_1^{-1}k_2 - ba^{-1} + pm \\ &= k_1^{-1}[-c'_2(\mu - h_2) + ba^{-1}c'_1(\omega - h_1) + pmk_1] \\ &= k_1^{-1}(\omega - h_1)[-c'_2\eta + ba^{-1}c'_1 + (\omega - h_1)^{p-1}m] \end{aligned}$$

So Q is principal if $\alpha := c'_2\eta - ba^{-1}c'_1$ is invertible in E . To show $\alpha \in E$ is a unit, it suffices to show $(ac'_2\eta - bc'_1)^p \in E$ is invertible. From 2.9, $c'_1 \equiv h_1^{p-1} \pmod{Q}$ and $c'_2 \equiv h_2^{p-1} \pmod{Q}$. We then have

$$\begin{aligned} (ac'_2\eta - bc'_1)^p &\equiv a^p(c'_2)^p\eta^p - b^p(c'_1)^p \pmod{Q} \\ &\equiv b(a^{p-1}(c'_2)^p - b^{p-1}(c'_1)^p) \pmod{Q} \\ &\equiv b[a^{p-1}(h_2^{p-1})^p - b^{p-1}(h_1^{p-1})^p] \pmod{Q} \\ &\equiv b \prod_{i=1}^{p-1} (ah_2^p + ibh_1^p) \pmod{Q} \end{aligned}$$

Thus $\alpha \in E$ is a unit. Hence $E = A[\eta]$ is regular, that is $E = R$.

2. Set $\mathcal{R}A := A[k_1\eta]$ for $\eta = (\omega - h_1)^{-1}(\mu - h_2)$ and $k_1 = p^{-1}(\omega - h_1)^p$. Note that $k_1\eta = p^{-1}(\omega - h_1)^{p-1}(\mu - h_2)$ and that it satisfies the integral equation $X^p - k_1^{p-1}k_2 \in A[X]$ for $k_2 := p^{-1}(\mu - h_2)^p$. Since $k_1 \notin P$, by 2.2 and part (1) of the proof, $\mathcal{R}A$ is regular in codimension one.

Set $Z := \langle T \rangle_S$ where T is as in the statement of 3.2. We see that Z is a ring from the relations

$$\begin{aligned}(\omega - h_1)^p &= ap - pc'_1(\omega - h_1) \\ (\mu - h_2)^p &= bp - pc'_2(\mu - h_2)\end{aligned}$$

Moreover it is a free S -module of rank p^2 . Clearly $\mathcal{R}A \subseteq Z$, so Z inherits R_1 from $\mathcal{R}A$. Thus $Z = R$ and R is S -free.

Finally, from 3.2(1) $P^{(p-1)}$ is contained in the conductor J of R to A . Since A_P is a one dimensional Gorenstein local ring, $J_P = (P^{p-1})_P$ and thus $J \subseteq P^{(p-1)}$. Thus $P^{(p-1)}$ is the conductor of R to A . \square

Remark 3.4. The condition $fg^i \notin S^{p \wedge p^2}$ for $1 \leq i \leq p-1$ in 3.3 is saying that some suitable subrings of A are integrally closed. As noted in the proof of 3.3, the condition is equivalent to $\prod_{i=1}^{p-1}(ah_2^p + ibh_1^p) \notin pS$. Let $1 \leq k, i \leq p-1$ and $1 \leq i(k) \leq p-1$ be such that $i(k) - ik \in p\mathbb{Z}$. The condition $\prod_{i=1}^{p-1}(ah_2^p + ibh_1^p) \notin pS$ is saying that for all $1 \leq i \leq p-1$, $A_i := S[\omega\mu^{i(1)}, \dots, \omega^j\mu^{i(j)}, \dots, \omega^{p-1}\mu^{i(p-1)}]$ is integrally closed. Indeed

$$(\omega\mu^i)^p = fg^i = (h_1h_2^i)^p + (ah_2^{ip} + ibh_1^p h_2^{ip-p})p + p^2q$$

for some $q \in S$. If $i = 1$, we have that fg is squarefree in S and by 2.11 $S[\omega\mu]$ is integrally closed. If $i \neq 1$, the given condition is equivalent to saying that

$$NNL_1(S[\omega\mu^i]) \cap V(p) = \emptyset$$

Moreover $NNL_1(S[\omega\mu^i]) \subseteq V(g)$. Choose k such that $i(k) = 1$, so that $S[\omega\mu^i, \omega^k\mu]$ is a finite birational extension of both $S[\omega\mu^i]$ and $S[\omega^k\mu]$. Now $NNL_1(S[\omega^k\mu]) \subseteq V(f)$. So by 2.2, $S[\omega\mu^i, \omega^k\mu]$ is regular in codimension one since $f, g \in S$ satisfy \mathcal{A}_1 . Since A_i is a finite birational extension of $S[\omega\mu^i, \omega^k\mu]$, it is regular in codimension one for the same reason. The remark follows since it is easily checked that A_i is S -free.

Remark 3.5. The powers of the prime $P^{p-1} \subseteq A$ in 3.3 are not P -primary in general. For example if $p = 3$, observe that $3a, 3b \in P^2$. However, it holds that $P^{(p-1)} = (p) + P^{p-1}$.

4. Existence of birational MCM modules

In this section we look at cases where R is not S -free. That is in the primary case of interest, when S is an unramified regular local ring of mixed characteristic p , we look at non-Cohen-Macaulay integral closures R . More specifically, we will show that $p.d._S(R) = 1$ under some natural conditions and show that in this case R admits a birational MCM module.

From 3.1 and 3.3, if we are looking for a non S -free R , we must have that $S[\omega]$ and $S[\mu]$ are integrally closed such that there exists an $1 \leq i \leq p-1$ satisfying $fg^i \in S^{p \wedge p^2}$. The reader can easily see that if it exists, such an “ i ” is unique. We start by identifying an ideal $I \subseteq A$, such that $I^* = R$ under this circumstance.

Convention 4.1. We maintain notation as set up in 2.6 and make the additional assumption that $f, g \notin S^{p \wedge p^2}$. Write $f = h_1^p + ap$, $g = h_2^p + bp$ with $a, b \notin pS$. Assume further that $h_1, h_2 \neq 0$. Note here that if $h_1 = 0$ (or $h_2 = 0$), we have by 3.3 that R is S -free. Let $P := (p, \omega - h_1, \mu - h_2) \subseteq A$ denote the unique height one prime in A containing p .

Lemma 4.2. For $H := (p, \omega\mu^i - h_1h_2^i) \subseteq A$, $H_P^* = \langle 1, \tau_i \rangle_{A_P}$ where

$$\tau_i = p^{-1}[(\omega\mu^i)^{p-1} + h_1h_2^i(\omega\mu^i)^{p-2} + \cdots + (h_1h_2^i)^{p-1}] \in K$$

Proof. Localize A at P and assume (A, P) local. Consider the ideal

$$\tilde{H} := (p, W\mu^i - h_1h_2^i) \subseteq S[\mu][W]$$

We have $F(W) \in \tilde{H}$:

$$F(W) - h_2^{-ip}[(W\mu^i)^{p-1} + h_1h_2^i(W\mu^i)^{p-2} + \cdots + (h_1h_2^i)^{p-1}] \cdot (W\mu^i - h_1h_2^i) \in pS[\mu][W] \quad (4)$$

Clearly \tilde{H} is a grade two perfect ideal in $S[\mu][W]$ and is the ideal of maximal minors of the matrix E :

$$E = \begin{bmatrix} W\mu^i - h_1h_2^i \\ p \end{bmatrix}$$

Adjoining the column of coefficients from (4) appropriately, we have for some $\alpha \in S[\mu][W]$ the matrix E' :

$$E' = \begin{bmatrix} W\mu^i - h_1h_2^i & \alpha \\ p & h_2^{-ip}[(W\mu^i)^{p-1} + h_1h_2^i(W\mu^i)^{p-2} + \cdots + (h_1h_2^i)^{p-1}] \end{bmatrix}$$

From [5][Proposition 2.1], $H^* = \langle E'_{11}/\delta_1, E'_{22}/\delta_2 \rangle_A$ where E'_{ii} and δ_i denote the image in A of the (i, i) -th cofactor of E' and the i -th (signed) minor of E respectively. Thus $H^* = \langle 1, \tau_i \rangle_A$ \square

Lemma 4.3. With established notation, let $fg^i \in S^{p \wedge p^2}$. Then for

$$I := pA + P^{p-2} \cdot (\omega\mu^i - h_1h_2^i)A$$

we have $I_A^* = R$.

Proof. Since I_A^* and R are birational S_2 A -modules, it suffices to show the desired equality in codimension one. If $Q \neq P$ is a height one prime in A , $I_Q^* = R_Q = A_Q$. Therefore localize A at P and assume (A, P) and (S, pS) are one dimensional local rings for the remainder of the proof.

We have $A = S[\mu, \omega\mu^i]$. Note that $g, fg^i \in S$ are units and therefore trivially are square free and satisfy \mathcal{A}_1 . Moreover, $S[\mu]$ is integrally closed and $S[\omega\mu^i]$ is not. Thus we are in the setting of 3.1(2). From the proof of 3.1(2)(a), we get that $R = A[\tau, \eta]$ where $\tau = p^{-1}[(\omega\mu^i)^{p-1} + h_1h_2^i(\omega\mu^i)^{p-2} + \cdots + (h_1h_2^i)^{p-1}]$ and $\eta = p^{-1}(\omega\mu^i - h_1h_2^i)(\mu - h_2)^{p-1}$. Since $\tau\eta \in A$, we see from 2.11(c) and equation (2) that $R = \langle 1, \eta, \dots, \eta^{p-2}, \tau \rangle_A$.

Since $p \in P^{p-1}$, a straightforward calculation gives

$$P^{p-1} \cap (p, \omega\mu^i - h_1h_2^i) = (p) + P^{p-1} \cap (\omega\mu^i - h_1h_2^i) = (p) + P^{p-2} \cdot (\omega\mu^i - h_1h_2^i) = I$$

From 3.2, $(P^{p-1})^* = A[\eta] \subseteq R$. Since A is Gorenstein, $A :_K R \subseteq P^{p-1}$. Let H be as in 4.2. Combining 4.2 and 2.11(c), we get that $H^* = A[\tau] \subseteq R$. Again since H is reflexive, $A :_K R \subseteq H$. Therefore, $A :_K R \subseteq P^{p-1} \cap H = I$.

To show $IR \subseteq A$, note that $I\eta^i \subseteq A$ for $1 \leq i \leq p-2$, since $I \subseteq P^{p-1} = A[\eta]^*$. Similarly, $I\tau \subseteq A$ since $I \subseteq H = A[\tau]^*$. Thus we have shown $I = A :_K R$ and the proof is complete. \square

We now set out to show R need not be Cohen-Macaulay.

Lemma 4.4. Let $S[\omega], S[\mu]$ be integrally closed such that $fg \in S^{p \wedge p^2}$. Then

1. $R \subseteq \langle \{1\} \cup p^{-1} \cdot (\omega - h_1, \mu - h_2)^{p-1} \rangle_A$.
2. Consider $y = p^{-1}(\sum_{i=1}^p a_i(\mu - h_2)^{p-i}(\omega - h_1)^{i-1}) \in K$ with the $a_i \in A$. Then $y \in R$ if and only if for all $2 \leq i \leq p$, $a_{i-1}h_2 + a_ih_1 \in P$.

Proof. From 4.3, $p \cdot R \subseteq A$, so consider an arbitrary element $y := p^{-1} \cdot x \in R$ with $x \in A$. From 4.3, $x \cdot (\omega - h_1)^{p-2}(\omega\mu - h_1h_2) \in pA$. Lifting to $B := S[W, U]$ and denoting lifts by \sim

$$\tilde{x}(W - h_1)^{p-2}(WU - h_1h_2) \in (p, F(W), G(U)) \quad (5)$$

Write

$$\omega\mu - h_1h_2 = (\omega - h_1)(\mu - h_2) + h_2(\omega - h_1) + h_1(\mu - h_2) \quad (6)$$

Lifting the identity in (6) to B we see that $\tilde{x} \in (p, W - h_1, (U - h_2)^{p-1})$. By symmetry $\tilde{x} \in (p, (W - h_1)^{p-1}, U - h_2)$ and hence

$$\tilde{x} \in (p, (U - h_2)^{p-1}, (W - h_1)^{p-1}, (W - h_1)(U - h_2))$$

This is because for a regular sequence $(q, y, z) \subseteq B$

$$(q, y, z^n) \cap (q, y^n, z) = (q, y^n, z^n, yz) \quad (7)$$

Since $1 \in R$, towards describing A -module generators for R we may assume that $y = p^{-1}x$ with

$$x = a_1 \cdot (\mu - h_2)^{p-1} + a_2 \cdot (\omega - h_1)^{p-1} + a_3 \cdot (\omega - h_1)(\mu - h_2) \quad (8)$$

for some $a_1, a_2, a_3 \in A$. Suppose we can write

$$y = p^{-1}[a_1(\mu - h_2)^{p-1} + a_2(\omega - h_1)(\mu - h_2)^{p-2} + \cdots + a_p(\omega - h_1)^{p-1} + b \cdot (\omega - h_1)^i(\mu - h_2)^i] \quad (9)$$

with $1 \leq i < (p-1)/2$ and $a_i, b \in A$. By (8), we can do this for $i = 1$, where $a_j = 0$ for $2 \leq j \leq p-1$. Now using (6) we get that $y \cdot (\omega - h_1)^{p-1-i}(\mu - h_2)^{i-1}(\omega\mu - h_1h_2) \in pA$ if and only if

$$a_{i+1}h_1(\omega - h_1)^{p-1}(\mu - h_2)^{p-1} + a_ih_2(\omega - h_1)^{p-1}(\mu - h_2)^{p-1} + bh_1(\omega - h_1)^{p-1}(\mu - h_2)^{2i} \in pA$$

Pulling back to B

$$\tilde{b}h_1(U - h_2)^{2i} + h_2\tilde{a}_i(U - h_2)^{p-1} + h_1\tilde{a}_{i+1}(U - h_2)^{p-1} \in (p, (U - h_2)^p, W - h_1) \quad (10)$$

and thus $\tilde{b} \in (p, W - h_1, (U - h_2)^{p-2i-1})$. By symmetry, $\tilde{b} \in (p, U - h_2, (W - h_1)^{p-2i-1})$ and hence by (7)

$$\tilde{b} \in (p, (U - h_2)^{p-2i-1}, (W - h_1)^{p-2i-1}, (W - h_1)(U - h_2))$$

Therefore

$$py \in (p, (\omega - h_1)^{i+1}(\mu - h_2)^{i+1}) + (\omega - h_1, \mu - h_2)^{p-1}$$

Starting from (8) and iterating the argument from (9) to this point sufficiently many times, we see that

$$R \subseteq \langle \{1\} \cup p^{-1} \cdot (\omega - h_1, \mu - h_2)^{p-1} \rangle_A$$

Consider

$$y = p^{-1} \left(\sum_{i=1}^p a_i (\mu - h_2)^{p-i} (\omega - h_1)^{i-1} \right) \in K \quad (11)$$

with the $a_i \in A$. From 4.3, $y \in R$ if and only if for all $2 \leq i \leq p$

$$y \cdot (\omega - h_1)^{p-i} (\mu - h_2)^{i-2} (\omega \mu - h_1 h_2) \in A \quad (12)$$

From (6) the above statements are equivalent to

$$(a_{i-1} h_2 + a_i h_1) (\mu - h_2)^{p-1} (\omega - h_1)^{p-1} \in pA \quad (13)$$

for each $2 \leq i \leq p$. Lifting to B , we see that (13) is equivalent to

$$a_{i-1} h_2 + a_i h_1 \in P \quad (14)$$

Thus the proof is complete. \square

Lemma 4.5. Assume S is an unramified regular local ring of mixed characteristic $p \geq 3$. Let $S[\omega], S[\mu]$ be integrally closed such that $fg \in S^{p \wedge p^2}$. Then $\nu_S(R) \leq p^2 + 1$. More explicitly, set $\eta_i := p^{-1} (\omega - h_1)^i (\mu - h_2)^{p-i}$ for $1 \leq i \leq p-1$. We have $R = \langle A[\eta_1, \dots, \eta_{p-1}] \cup \{\epsilon\} \rangle_S$ for:

$$\epsilon := p^{-1} \sum_{i=1}^p (-1)^i c^{p-i} e^{i-1} (\mu - h_2)^{p-i} (\omega - h_1)^{i-1}$$

where $h_1 \equiv zc \pmod{pS}$, $h_2 \equiv ze \pmod{pS}$ for some $z \in S \setminus pS$ and $c, e \in S$ relatively prime.

Proof. Suppose $\langle T \rangle_S$ is as in 3.2. Note that it is just the ring $A[\eta_1, \dots, \eta_{p-1}]$. Since $\langle T \rangle_S$ is S -free of rank p^2 , the assertion $\nu_S(R) \leq p^2 + 1$ follows from the second assertion.

Since S/pS is regular local (a UFD), $h_1 \equiv (zc) \pmod{pS}$, $h_2 \equiv (ze) \pmod{pS}$ for some $z \in S \setminus pS$ and $c, e \in S$ relatively prime. First suppose c or e is a unit in S . Then it follows from 4.4(1) and (2) that $R = \langle A[\eta_1, \dots, \eta_{p-1}] \cup \{\epsilon\} \rangle_A$. Notice that if $i + j > 0$, then $(\omega - h_1)^i (\mu - h_2)^j \in (A[\eta_1, \dots, \eta_{p-1}] :_K \epsilon)$. Thus $R = \langle A[\eta_1, \dots, \eta_{p-1}] \cup \{\epsilon\} \rangle_S$ in this case.

Next assume that neither c or e is a unit, so that $(p, c, e) \subseteq S$ forms a regular sequence. Now $\epsilon \in R$ from 4.4(2). From 4.4(1), it suffices to look at elements of the form

$$y = p^{-1} \left(\sum_{i=1}^p a_i (\mu - h_2)^{p-i} (\omega - h_1)^{i-1} \right) \in R$$

In view of 4.4(2), the condition $a_1 h_2 + a_2 h_1 \in P$ upon lifting to $B := S[W, U]$ (denoting lifts by \sim) tells us that \tilde{a}_1, \tilde{a}_2 arise from the first syzygy of the grade five complete intersection B -ideal, $\tilde{Q} := (p, c, e, W - h_1, U - h_2)$. In particular

$$\tilde{a}_2 \in (p, c, W - h_1, U - h_2) \cap (p, e, W - h_1, U - h_2) = (p, ce, W - h_1, U - h_2)$$

since $a_2 h_2 + a_3 h_1 \in P$ as well. Since $A[\eta_1, \dots, \eta_{p-1}] \subseteq R$, towards describing A -module generators for R we may assume $\tilde{a}_2 = \alpha ce$ for some $\alpha \in B$ and consequently that $\tilde{a}_1 = -\alpha c^2$ and $\tilde{a}_3 = -\alpha e^2$.

Now let $3 \leq i < p$ be such that for all $1 \leq k \leq i$, $a_k = (-1)^k \alpha c^{i-k} e^{k-1}$ for some $\alpha \in A$. Lifting $a_i h_2 + a_{i+1} h_1 \in P$ to B , we have $(-1)^i \alpha e^i + a_{i+1} c \in \tilde{P}$. Since $A[\eta_1, \dots, \eta_{p-1}] \subseteq R$, we may assume that

$\alpha = \alpha'c$ for some $\alpha' \in B$ and hence that $a_{i+1} = (-1)^{i+1}\alpha' \cdot e^i$. Iterating this argument, we get that $R = \langle A[\eta_1, \dots, \eta_{p-1}] \cup \{\epsilon\} \rangle_A$. Finally, if $i + j > 0$ then $(\omega - h_1)^i(\mu - h_2)^j \in (A[\eta_1, \dots, \eta_{p-1}] :_K \epsilon)$ and the conclusion follows. \square

Proposition 4.6. *Let (S, \mathfrak{m}) be an unramified regular local ring of mixed characteristic $p \geq 3$ such that $S[\omega]$ and $S[\mu]$ are integrally closed and $fg \in S^{p \wedge p^2}$. Then R is Cohen-Macaulay if and only if $Q := (p, h_1, h_2) \subset S$ is a two generated ideal or all of S . Moreover, $p.d_S(R) \leq 1$.*

Proof. Since S/pS is a UFD, $h_1 \equiv z \pmod{pS}$, $h_2 \equiv ze \pmod{pS}$ for some $z \in S \setminus pS$ and $c, e \in S$ relatively prime. Then Q is a two generated ideal or all of R if and only if c or e is a unit. From 4.5, $R = \langle A[\eta_1, \dots, \eta_{p-1}] \cup \{\epsilon\} \rangle_S$. Suppose that c is a unit. Then

$$(\mu - h_2)^{p-1} \in \langle \epsilon, (\omega - h_1)(\mu - h_2)^{p-2}, \dots, (\omega - h_1)^{p-1} \rangle_S.$$

Thus R is S -free of rank p^2 and hence is Cohen-Macaulay.

Now assume neither c nor e is a unit, that is Q is either grade three perfect or grade two and not perfect. We know from 4.5 that $R = \langle A[\eta_1, \dots, \eta_{p-1}] \cup \{\epsilon\} \rangle_S$. With T as in 3.2, define $\Gamma : T \rightarrow \mathbb{Z}$, $\Gamma' : T \rightarrow \mathbb{Z}$ by

$$\begin{aligned} \Gamma(p^{-k}(\omega - h_1)^i(\mu - h_2)^j) &= i + j \\ \Gamma'(p^{-k}(\omega - h_1)^i(\mu - h_2)^j) &= i \end{aligned}$$

Define a total ordering on T as follows: for $x, y \in T$, if $\Gamma(x) \geq \Gamma(y)$ then $x \geq y$ and if $\Gamma(x) = \Gamma(y)$, then $x \geq y$ if $\Gamma'(x) \geq \Gamma'(y)$. Let $\alpha : S^{p^2+1} \rightarrow R$ be the S -projection map defined by the generating set $T \cup \{\epsilon\}$ such that the basis element e_{p^2+1} maps to ϵ and the image of the basis elements e_i , $i \neq p^2 + 1$ is defined by the ordered set T . Consider $U = [u_i] \in \text{Ker}(\alpha)$. Since A is S -free with a basis given by $\{(\omega - h_1)^i(\mu - h_2)^j \mid 0 \leq i, j \leq p-1\}$, we get that $u_i = 0$ for $m \leq i \leq p^2$, where $m = p^2 - 2^{-1}(p-1)p + 1$. Let $p\epsilon = \sum_{i=1}^{m-1} v_i x_i$ with the $v_i \in S$ and x_i from the ordered set T . Since $p \in S$ is prime we get the following free resolution of R over S :

$$0 \longrightarrow S \xrightarrow{\psi^T} S^{p^2+1} \xrightarrow{\alpha} R \longrightarrow 0 \quad (15)$$

where $\psi = [v_1 \dots v_{m-1} \ 0 \dots 0 \ -p]$. The above resolution is minimal since $\psi^T(S) \subseteq \mathfrak{m}S^{p^2+1}$, so that $p.d_S(R) = 1$. The proof is now complete. \square

Remark 4.7. Let S be an unramified regular local ring of mixed characteristic $p \geq 3$. Note that the ideal $(p, h_1, h_2) \subseteq S$ is a two generated ideal or all of R if and only if the same property holds for the ideal $(p, f, g) \subseteq S$. Similarly, the ideal (p, h_1, h_2) has grade three if and only if the ideal (p, f, g) has the same property.

Example 4.8. The conditions in 4.6 give a non-empty class of non Cohen-Macaulay integral closures R . For an example where $Q = (p, h_1, h_2)$ has grade three, consider $S = \mathbb{Z}[X, Y]_{(3, X, Y)}$ where X, Y are indeterminates over $\mathbb{Z}_{(3)}$. Let

$$\begin{aligned} f &= -5X^3 + 9 = X^3 + 3(3 - 2X^3) \\ g &= -2Y^3 + 9 = Y^3 + 3(3 - Y^3) \end{aligned}$$

and $\omega^3 = f, \mu^3 = g$. Then f, g are square free elements that form a regular sequence in S . It is easily checked that $[K : L] = 9$ and that this choice satisfies the hypothesis of 4.6, so that $p.d_S(R) = 1$.

For an example where Q has grade two but $p.d_S(S/Q) = 3$, let $S = \mathbb{Z}[X, Y]_{(p, X, Y)}$ for some prime number $p \geq 3$. Set

$$\begin{aligned} f &= (1-p)X^{2p} + p^2 = (X^2)^p + p(p - X^{2p}) \\ g &= (1+p)(XY)^p + p^2 = (XY)^p + p(p + (XY)^p) \end{aligned}$$

Then $f, g \in S$ are square free and form a regular sequence in S . It is easily verified that $[K : L] = p^2$ and that the choice satisfies the hypothesis of 4.6, so that $p.d_S(R) = 1$. \square

Lemma 4.9. *With established notation, the following holds*

$$P_A^* = \langle 1, p^{-1}(\omega - h_1)^{p-1}(\mu - h_2)^{p-1} \rangle_A$$

Proof. Set $P_1 := (p, \omega - h_1)$ and $P_2 := (p, \mu - h_2)$, so that $P^* = P_1^* \cap P_2^*$. Let $\tilde{P}_1 := (p, W - h_1) \subseteq S[\mu][W]$. It is the maximal minors of

$$E = \begin{bmatrix} W - h_1 \\ p \end{bmatrix}$$

We have $F(W) \in \tilde{P}_1$, $F(W) = a \cdot (-p) + (W^{p-1} + h_1 W^{p-2} + \dots + h_1^{p-1})(W - h_1)$. Adjoining the appropriate column of coefficients

$$E' = \begin{bmatrix} W - h_1 & a \\ p & W^{p-1} + \dots + h_1^{p-1} \end{bmatrix}$$

From [9][Lemma 2.5] $P_1^* = \langle E'_{11}/\delta_1, E'_{22}/\delta_2 \rangle_A$ where E'_{ii} and δ_i denote the image in A of the (i, i) -th cofactor of E' and the i -th (signed) minor of E . Therefore

$$P_1^* = \langle 1, p^{-1}(\omega^{p-1} + \dots + h_1^{p-1}) \rangle_A$$

Identically

$$P_2^* = \langle 1, p^{-1}(\mu^{p-1} + \dots + h_2^{p-1}) \rangle_A$$

Now consider $y \in P^* = P_1^* \cap P_2^*$. Write for some $\alpha_1, \alpha_2, \beta_1, \beta_2 \in A$

$$py = p\alpha_1 + \beta_1(\omega^{p-1} + \dots + h_1^{p-1}) = p\alpha_2 + \beta_2(\mu^{p-1} + \dots + h_2^{p-1})$$

Lifting to $B := S[W, U]$ and denoting lifts by \sim

$$p(\tilde{\alpha}_1 - \tilde{\alpha}_2) + \tilde{\beta}_1(W^{p-1} + \dots + h_1^{p-1}) - \tilde{\beta}_2(U^{p-1} + \dots + h_2^{p-1}) \in (F(W), G(U))$$

Writing $W^{p-1} + \dots + h_1^{p-1} = (W - h_1)^{p-1} + p \cdot C'_1$ (respectively for $U^{p-1} + \dots + h_2^{p-1}$),

$$\tilde{\beta}_1(W - h_1)^{p-1} - \tilde{\beta}_2(U - h_2)^{p-1} \in (p, F(W), G(U))$$

This gives $\tilde{\beta}_1 \in (p, W - h_1, (U - h_2)^{p-1})$. Since $1 \in P^*$ and $(\omega - h_1)(\omega^{p-1} + \dots + h_1^{p-1}) \in pA$, we get $P^* \subseteq \langle 1, p^{-1}(\omega - h_1)^{p-1}(\mu - h_2)^{p-1} \rangle_A$. Since the reverse inclusion is obvious, the proof is complete. \square

Theorem 4.10. *Let (S, \mathfrak{m}) be an unramified regular local ring of mixed characteristic $p \geq 3$. Then*

1. R is Cohen-Macaulay if

(a) At least one of $S[\omega], S[\mu]$ is not integrally closed.

(b) $S[\omega], S[\mu]$ are integrally closed and $fg^i \notin S^{p \wedge p^2}$ for all $1 \leq i \leq p-1$.

2. Let $S[\omega], S[\mu]$ be integrally closed and $fg \in S^{p \wedge p^2}$. Then R is Cohen-Macaulay if and only if $Q := (p, f, g) \subseteq S$ is a two generated ideal or all of S . Moreover, $p.d_S(R) \leq 1$ and $\nu_S(R) \leq p^2 + 1$.

3. If $Q := (p, f, g) \subseteq S$ has grade three, R admits a birational maximal Cohen-Macaulay module.

Proof. We have shown 1(a) in 3.1 and 1(b) in 3.3. The proof of (2) follows from 4.6, 4.5 and 4.7.

Now assume Q has grade three. From part (1), we may assume that $S[\omega], S[\mu]$ are integrally closed and $fg^i \in S^{p \wedge p^2}$ for some (unique) $1 \leq i \leq p-1$. From 4.3, $I^* = R$ for $I := pA + (\omega\mu^i - h_1h_2^i) \cdot P^{p-2}$. Set $M := (IP)^*$. Then M is an R -module since $(A :_K IP) = ((A :_K I) :_K P) = (R :_K P)$. We will show $\text{depth}_S(M) = d$, so that M is an MCM module over R . By definition

$$M = (IP)^* = (p \cdot P + (\omega\mu^i - h_1h_2^i) \cdot P^{p-1})^* = F_1 \cap F_2,$$

where $F_1 = p^{-1}P^*$ and $F_2 = (\omega\mu^i - h_1h_2^i)^{-1}(P^{p-1})^*$. This is because for ideals $H, N \subseteq A$, $(A :_K H + N) = (A :_K H) \cap (A :_K N)$ as A -modules. Now $A/P \simeq S/pS$ as S -modules, therefore by the depth lemma P is S -free. By 2.5, $\text{Hom}_A(P, A) \simeq \text{Hom}_S(P, S)$ as S -modules and hence P^* is Cohen-Macaulay. On the other hand, since $(P^{p-1})^*$ and $(P^{(p-1)})^*$ are birational S_2 modules that agree in codimension one, we have $(P^{p-1})^* = (P^{(p-1)})^*$. From 3.2(2) and 2.4 we then have that $(P^{p-1})^*$ is Cohen-Macaulay. Therefore F_1 and F_2 are Cohen-Macaulay since $F_1 \simeq P^*$ and $F_2 \simeq (P^{p-1})^*$ as A -modules and S -modules. We have the natural short exact sequence of S -modules

$$0 \longrightarrow F_1 \cap F_2 \longrightarrow F_1 \oplus F_2 \longrightarrow F_1 + F_2 \longrightarrow 0 \quad (16)$$

To complete the proof it suffices to show that $\text{depth}_S(F_1 + F_2) \geq d-1$. Set

$$\mathcal{F} := p(\omega\mu^i - h_1h_2^i) \cdot (F_1 + F_2) = \mathcal{F}_1 + \mathcal{F}_2,$$

where $\mathcal{F}_1 := (\omega\mu^i - h_1h_2^i)P^*$ and $\mathcal{F}_2 := p(P^{p-1})^*$. Clearly $F_1 + F_2 \simeq \mathcal{F}$ as A -modules and hence as S -modules. From 3.2(1), $\mathcal{F}_2 = (p) + P^p$ and from 4.9

$$\mathcal{F}_1 = (\omega\mu^i - h_1h_2^i, p^{-1}(\omega\mu^i - h_1h_2^i)(\omega - h_1)^{p-1}(\mu - h_2)^{p-1})A. \quad (17)$$

Set $m := \omega\mu^i - h_1h_2^i$. We make the following two claims:

1. $\mathcal{F} = \mathcal{F}_2 + (m)$.
2. $(\mathcal{F}_2 :_A m) = (p) + P^{p-1}$.

Assume both claims hold. Since $(\mathcal{F}_2 :_A m) \simeq \mathcal{F}_2 \cap (m)$ as A -modules and hence S -modules, we have a natural short exact sequence of S -modules

$$0 \longrightarrow (p) + P^{p-1} \longrightarrow \mathcal{F}_2 \oplus (m) \longrightarrow \mathcal{F} \longrightarrow 0 \quad (18)$$

If $\text{depth}_S((p) + P^{p-1}) = d$, then $\text{depth}_S(\mathcal{F}) \geq d-1$ and we are done. But $\text{depth}_S((p) + P^{p-1}) = d$ if and only if $A/((p) + P^{p-1})$ is Cohen-Macaulay. For $B := S[W, U]_{(\mathfrak{m}, W-h_1, U-h_2)}$ we have as B -modules

$$A/((p) + P^{p-1}) \simeq B/((p) + (W - h_1, U - h_2)^{p-1}).$$

Since B/pB is regular local and any power of a complete intersection B -ideal is perfect, we are through. Therefore only the claims remain to be proved.

Set $\mathcal{Q} := \mathcal{F}_2 + (m)$. For claim (1), from (17) we only need to show

$$s := p^{-1}m(\omega - h_1)^{p-1}(\mu - h_2)^{p-1} \in \mathcal{Q}.$$

Since $fg^i \in S^{p \wedge p^2}$, we get $ah_2^p + ibh_1^p \in pS$. Moreover, $\text{grade}(Q) = 3$ implies $a - qh_1^p \in pS$ and $b + i^{-1}qh_2^p \in pS$ for some $q \in S$. Write

$$m = (\omega - h_1)(\mu^i - h_2^i) + h_2^i(\omega - h_1) + h_1(\mu^i - h_2^i), \quad (19)$$

and recall that $(\omega - h_1)^p = p(a - c'_1(\omega - h_1))$ and $(\mu - h_2)^p = p(b - c'_2(\mu - h_2))$. Then

$$\begin{aligned} s &\equiv p^{-1}[h_2^i(\omega - h_1) + h_1(\mu^i - h_2^i)](\omega - h_1)^{p-1}(\mu - h_2)^{p-1} \pmod{\mathcal{Q}} \\ &\equiv ah_2^i(\mu - h_2)^{p-1} + bh_1(\mu^{i-1} + \cdots + h_2^{i-1})(\omega - h_1)^{p-1} \pmod{\mathcal{Q}} \\ &\equiv ah_2^i(\mu - h_2)^{p-1} + ibh_1h_2^{i-1}(\omega - h_1)^{p-1} \pmod{\mathcal{Q}} \\ &\equiv qh_1h_2^i[h_1^{p-1}(\mu - h_2)^{p-1} - h_2^{p-1}(\omega - h_1)^{p-1}] \pmod{\mathcal{Q}}. \end{aligned} \quad (20)$$

Now $(\omega\mu^i - h_1h_2^i) \cdot P^{p-2} \subseteq \mathcal{Q}$, (19) and $P^p \subseteq \mathcal{Q}$ imply

$$h_2^i(\omega - h_1) + h_1(\mu^i - h_2^i) \in (\mathcal{Q} :_A P^{p-2}).$$

Therefore for all $0 \leq j \leq p-2$

$$\begin{aligned} &(h_2^i(\omega - h_1) + h_1(\mu^i - h_2^i)) \cdot h_2^{p-2-j}h_1^j(\omega - h_1)^{p-2-j}(\mu - h_2)^j \\ &= h_2^{i-1}[(h_2(\omega - h_1))^{p-j-1}(h_1(\mu - h_2))^j] + \\ &\quad (h_1(\mu - h_2))^{j+1}(\mu^{i-1} + \cdots + h_2^{i-1})(h_2(\omega - h_1))^{p-2-j} \in \mathcal{Q}. \end{aligned}$$

Thus

$$h_2^{i-1}[(h_2(\omega - h_1))^{p-j-1}(h_1(\mu - h_2))^j] \equiv -ih_2^{i-1}(h_1(\mu - h_2))^{j+1}(h_2(\omega - h_1))^{p-2-j} \pmod{\mathcal{Q}}.$$

It then follows that for any $1 \leq k \leq p-j-1$

$$h_2^{i-1}[(h_2(\omega - h_1))^{p-j-1}(h_1(\mu - h_2))^j] \equiv (-i)^k h_2^{i-1}(h_1(\mu - h_2))^{j+k}(h_2(\omega - h_1))^{p-j-1-k} \pmod{\mathcal{Q}}.$$

In particular for $j = 0$ and $k = p-1$ we get

$$h_2^{i-1}(h_2(\omega - h_1))^{p-1} \equiv h_2^{i-1}(h_1(\mu - h_2))^{p-1} \pmod{\mathcal{Q}}. \quad (21)$$

Combining (21) and (20), we see that $s \in \mathcal{Q}$ and thus claim (1) holds.

To show one containment in claim (2), note that $(p) + P^{p-1} \subseteq (\mathcal{F}_2 :_A m)$ since $p + P^p = \mathcal{F}_2$. For the reverse inclusion, consider $y \in (\mathcal{F}_2 :_A m)$. Lifting to B and denoting lifts by \sim

$$\tilde{y}(WU^i - h_1h_2^i) \in (p, F(W), G(U)) + (W - h_1, U - h_2)^p = (p) + (W - h_1, U - h_2)^p. \quad (22)$$

Using (19) we have $\tilde{y} \in (p, (W - h_1)^{p-1}, U - h_2)$. Similarly $\tilde{y} \in (p, W - h_1, (U - h_2)^{p-1})$ and from (7)

$$\tilde{y} \in (p, (W - h_1)^{p-1}, (U - h_2)^{p-1}, (W - h_1)(U - h_2)). \quad (23)$$

Now assume for some $1 \leq i \leq 2^{-1}(p-1)-1$,

$$\tilde{y} \in (p, (W-h_1)^i(U-h_2)^i) + (W-h_1, U-h_2)^{p-1}. \quad (24)$$

Write $\tilde{y} = \alpha \cdot (W-h_1)^i(U-h_2)^i + \beta$ for some $\alpha \in B$ and $\beta \in (p) + (W-h_1, U-h_2)^{p-1}$. From (22):

$$\alpha \cdot (W-h_1)^i(U-h_2)^i(WU^i - h_1h_2^i) \in (p) + (W-h_1, U-h_2)^p. \quad (25)$$

Using the regular sequence $(p, (U-h_2)^{i+1}, (W-h_1)^{i+1}, h_2^i) \subseteq B$, we get

$$\alpha \in (p, (W-h_1)^{p-2i-1}, U-h_2).$$

Similarly, using the regular sequence $(p, (W-h_1)^{i+1}, (U-h_2)^{i+1}, h_1(U^{i-1} + \dots + h_2^{i-1})) \subseteq B$ we get $\alpha \in (p, W-h_1, (U-h_2)^{p-2i-1})$. Thus by (7):

$$\alpha \in (p, (W-h_1)^{p-2i-1}, (U-h_2)^{p-2i-1}, (W-h_1)(U-h_2))$$

and hence

$$\tilde{y} \in (p, (W-h_1)^{i+1}(U-h_2)^{i+1}) + (W-h_1, U-h_2)^{p-1}.$$

Thus starting from (23) we may induct on i to get

$$\tilde{y} \in (p, (W-h_1)^{2^{-1}(p-1)}(U-h_2)^{2^{-1}(p-1)}) + (W-h_1, U-h_2)^{p-1} = (p) + (W-h_1, U-h_2)^{p-1}.$$

This shows $(\mathcal{F}_2 :_A m) = (p) + P^{p-1}$ and all claims have been proved. Thus R admits a birational MCM module. \square

Remark 4.11. If $\text{grade}(Q) = 2$ and $p.d_S(S/Q) = 3$ in the context of 4.10(3), we are not able to construct a birational MCM module over R at present. However, if we allow an extension of the quotient field, then constructing an MCM module over R may be possible in this case, see [8].

Remark 4.12. By a vector bundle on the punctured spectrum of a regular local ring (S, \mathfrak{m}) or simply a bundle on S we mean a finitely generated reflexive S -module M such that M_P is S_P -free for all non maximal ideals $P \subseteq S$. One could use 4.10(2) to generate examples of non-trivial bundles M on localizations of polynomial rings or power series rings over $\mathbb{Z}_{(p)}$ of dimension d at least three such that $\text{rank}_S(M) = p^2 + d - 3$. Moreover, these bundles would satisfy $p.d_S(M) = 1$.

Let $d \geq 3$ and (T, \mathfrak{n}) be a d -dimensional unramified regular local ring of mixed characteristic p . Choose $(S, \mathfrak{m}) \subseteq T$ a three dimensional subring of T that is an unramified regular local ring of mixed characteristic p and a quotient of T by a regular sequence (such a choice is possible for example when T is a localization of a polynomial ring over $\mathbb{Z}_{(p)}$). Let $S \subseteq (E, \mathfrak{n}') \subseteq T$ be such that E is regular local and $S = E/(t)$ for some $0 \neq t \in E$. Using 4.10(2), with the base ring as S , construct R such that it is not S -free. Choose a minimal S -free resolution

$$0 \longrightarrow S \xrightarrow{\psi^T} S^{p^2+1} \longrightarrow R \longrightarrow 0 \quad (26)$$

Let M' be the cokernel of the E -matrix $\phi := \begin{bmatrix} \psi^T & | & t \end{bmatrix}$

$$0 \longrightarrow E \xrightarrow{\phi} E^{p^2+2} \longrightarrow M' \longrightarrow 0 \quad (27)$$

so that $p.d._E(M) = 1$. The ideal of maximal minors of ψ^T is \mathfrak{m} -primary since R is a bundle over S . Therefore the ideal of maximal minors of ϕ is \mathfrak{n}' -primary and hence it is free on the punctured spectrum of E . Proceeding this way, we can construct a finite module M over T that is free on the punctured spectrum of T and $p.d._T(M) = 1$. Moreover, since M is an S_2 T -module, it is T -reflexive (see [1][Proposition 1.4.1] for example).

Acknowledgement

I would like to thank my Ph.D. advisor Prof. Daniel Katz for suggesting this problem and for his support through the course of this work. I would also like to thank the referee for a careful reading of the paper and suggestions for improvement.

References

- [1] Winfried Bruns, H. Jürgen Herzog, Cohen-Macaulay Rings, 2 edition, Cambridge Studies in Advanced Mathematics, Cambridge University Press, 1998.
- [2] Craig Huneke, Daniel Katz, Uniform symbolic topologies in Abelian extensions, Trans. Am. Math. Soc. 372 (3) (May 2019) 1735–1750.
- [3] Melvin Hochster, Cohen-Macaulay modules, in: James W. Brewer, Edgar A. Rutter (Eds.), Conference on Commutative Algebra, Springer Berlin Heidelberg, Berlin, Heidelberg, 1973, pp. 120–152.
- [4] Craig Huneke, Irena Swanson, Integral Closure of Ideals, Rings, and Modules, vol. 13, Cambridge University Press, 2006.
- [5] Daniel Katz, On the existence of maximal Cohen-Macaulay modules over p -th root extensions, Proc. Am. Math. Soc. 127 (9) (1999) 2601–2609.
- [6] Daniel Katz, Conductors in mixed characteristic, J. Algebra 571 (2021) 350–375, Commutative Algebra and its Interactions with Algebraic Geometry: a volume in honor of Craig Huneke on the occasion of his 65th birthday.
- [7] Jee Koh, Degree p extensions of an unramified regular local ring of mixed characteristic p , J. Algebra 99 (2) (1986) 310–323.
- [8] Daniel Katz, Prashanth Sridhar, Small CM modules over repeated radical extensions in mixed characteristic, in preparation.
- [9] Steven Kleiman, Bernd Ulrich, Gorenstein algebras, symmetric matrices, self-linked ideals, and symbolic powers, Trans. Am. Math. Soc. 349 (12) (1997) 4973–5000.
- [10] Paul Roberts, Abelian extensions of regular local rings, Proc. Am. Math. Soc. 78 (3) (1980) 307–310.
- [11] Paul C. Roberts, The homological conjectures, in: Progress in Commutative Algebra 1: Combinatorics and Homology, 2012, p. 199.
- [12] Prashanth Sridhar, Existence of birational small Cohen-Macaulay modules over biquadratic extensions in mixed characteristic, J. Algebra 582 (2021) 100–116.
- [13] Wolmer V. Vasconcelos, Computing the integral closure of an affine domain, Proc. Am. Math. Soc. 113 (3) (1991) 633–638.