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Completeness of the induced cotorsion pairs in categories of quiver representations ☆

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ABSTRACT

Given a complete hereditary cotorsion pair $(\mathcal{A}, \mathcal{B})$ in an abelian category $\mathcal{C}$ satisfying certain conditions, we study the completeness of the induced cotorsion pairs $(\Phi(\mathcal{A}), \Phi(\mathcal{A})^{\perp})$ and $({}^{\perp}\Psi(\mathcal{B}), \Psi(\mathcal{B}))$ in the category $\mathrm{Rep}(Q, \mathcal{C})$ of $\mathcal{C}$-valued representations of a given quiver Q. We show that if Q is left rooted, then the cotorsion pair $(\Phi(\mathcal{A}), \Phi(\mathcal{A})^{\perp})$ is complete, and if Q is right rooted, then the cotorsion pair $({}^{\perp}\Psi(\mathcal{B}), \Psi(\mathcal{B}))$ is complete. Besides, we work on the infinite line quiver A_{∞}^{∞}, which is neither left rooted nor right rooted. We prove that these cotorsion pairs in $\mathrm{Rep}(A_{\infty}^{\infty}, R)$ are complete, as well.

1. Introduction

Let $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair in an abelian category $\mathcal{C}$, which has enough $\mathcal{A}$-objects and enough $\mathcal{B}$-objects. We now know that there are various ways of lifting such cotorsion pair $\mathcal{C}$ to a cotorsion pair in, $\mathbf{C}(\mathcal{C})$, the category of chain complexes over $\mathcal{C}$; see for example [7]. Some of these lifts are:

(i) $({}^{\perp}\mathbf{C}(\mathcal{B}), \mathbf{C}(\mathcal{B}))$, generated by all disks $D^i(A)$ for every object $A \in \mathcal{A}$.
(ii) $(\widetilde{\mathcal{A}}, dg\ \widetilde{\mathcal{B}})$, cogenerated by all spheres $S^i(B)$ for every object $B \in \mathcal{B}$.
(iii) $(\mathbf{C}(\mathcal{A}), \mathbf{C}(\mathcal{A})^{\perp})$, cogenerated by all disks $D^i(B)$ for every object $B \in \mathcal{B}$.
(iv) $(dg\ \widetilde{\mathcal{A}}, \widetilde{\mathcal{B}})$, generated by all spheres $S^i(A)$ for every object $A \in \mathcal{A}$.

The most important application of these induced cotorsion pairs in $\mathbf{C}(\mathcal{C})$ emerges in finding abelian model structures on $\mathbf{C}(\mathcal{C})$, whose homotopy category is the derived category $\mathbf{D}(\mathcal{C})$; see [9] for a detailed treatment about the correspondence between cotorsion pairs and abelian model structures. In [15], the authors proved

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that whenever the category $\mathcal{C}$ is (co)complete and the cotorsion pair $(\mathcal{A}, \mathcal{B})$ is complete hereditary, the so-called *induced cotorsion pairs* given in (ii) and (iv) are complete and compatible, and therefore, yield to such a model structure on $\mathbf{C}(\mathcal{C})$.

Thinking of chain complexes over $\mathcal{C}$ in terms of $\mathcal{C}$-valued representations of the infinite line quiver with relations, it is natural to ask how to obtain 'quiver representation version' of these cotorsion pairs, that is, how to extend a given cotorsion pair in the ground category $\mathcal{C}$ to a cotorsion pair in the category $\mathrm{Rep}(Q, \mathcal{C})$ of $\mathcal{C}$-valued representations of any quiver Q. Several related results on this problem can be found in the literature for certain cotorsion pairs in the category R-Mod of (left) R-modules, and for certain quivers, see [4], [2], [5]. In [8], the authors handle the subject in a general framework, working with any abelian category $\mathcal{C}$ and any quiver Q, and unify these aforementioned works.

For every vertex i in a quiver Q, under certain conditions, the ith evaluation functor $\mathrm{ev}_i : \mathrm{Rep}(Q, \mathcal{C}) \to \mathcal{C}$ has both the left adjoint $f_i(-)$ and the right adjoint $g_i(-)$, see (2.10) and (2.11). Together with the ith stalk functor $s_i : \mathcal{C} \to \mathrm{Rep}(Q, \mathcal{C})$, which is analogous to the sphere functor $S^i(-)$, it is proved in [8, Proposition 7.3] that a cotorsion pair $(\mathcal{A}, \mathcal{B})$ in an abelian category $\mathcal{C}$, satisfying certain mild conditions, leads to the following cotorsion pairs in $\mathrm{Rep}(Q, \mathcal{C})$:

(i') $({}^{\perp}\mathrm{Rep}(Q, \mathcal{B}), \mathrm{Rep}(Q, \mathcal{B}))$, generated by the class $f_*(\mathcal{A})$.
(ii') $(\Phi(\mathcal{A}), \Phi(\mathcal{A})^{\perp})$, cogenerated by the class $s_*(\mathcal{B})$.
(iii') $(\mathrm{Rep}(Q, \mathcal{A}), \mathrm{Rep}(Q, \mathcal{A})^{\perp})$, cogenerated by the class $g_*(\mathcal{B})$.
(iv') $({}^{\perp}\Psi(\mathcal{B}), \Psi(\mathcal{B}))$, generated by the class $s_*(\mathcal{A})$.

For the full description of the aforementioned classes, we refer to (3.8) and Proposition 3.11. Besides, they prove that the first two ones (i') and (ii') coincide if the quiver Q is left rooted. Dually, the last two ones (iii') and (iv') coincide if the quiver Q is right rooted

In order to understand better the analogy with the case $\mathbf{C}(\mathcal{C})$, one should be aware that the ith disk functor $D^i(-)$ is the left adjoint of the ith evaluation functor $\mathrm{ev}_i : \mathbf{C}(\mathcal{C}) \to \mathcal{C}$, $\mathrm{ev}_i(X) = X_i$, and besides, the $(i+1)$th disk functor $D^{i+1}(-)$ is the right adjoint of ev_i. Furthermore, the intersections of the cotorsion pairs in (i) and (ii) with the subcategory, $\mathbf{C}_-(\mathcal{C})$, of chain complexes with zeros in non-negative degrees, are the same, that is, $({}^{\perp}\mathbf{C}_-(\mathcal{B}) \cap \mathbf{C}_-(\mathcal{C}), \mathbf{C}(\mathcal{B}) \cap \mathbf{C}_-(\mathcal{C})) = (\widetilde{\mathcal{A}} \cap \mathbf{C}_-(\mathcal{C}), dg\ \widetilde{\mathcal{B}} \cap \mathbf{C}_-(\mathcal{C}))$. Dually, the intersections of each of the cotorsion pairs in (iii) and (iv) with the subcategory, $\mathbf{C}_+(\mathcal{C})$, of chain complexes with zeros in non-positive degrees, are the same.

From this perspective, we naturally are interested in answering the question proposed in [8, Question 7.7] on extending the results given in [15] to the cotorsion pairs given in the category $\mathrm{Rep}(Q, \mathcal{C})$ of $\mathcal{C}$-valued representations of a quiver Q. Namely, in this paper, we study the completeness of the induced cotorsion pairs $(\Phi(\mathcal{A}), \Phi(\mathcal{A})^{\perp})$ and $({}^{\perp}\Psi(\mathcal{B}), \Psi(\mathcal{B}))$ in $\mathrm{Rep}(Q, \mathcal{C})$ whenever the cotorsion pair $(\mathcal{A}, \mathcal{B})$ in $\mathcal{C}$ is complete and hereditary.

We now give a summary of the layout of the paper. In Sections 2, we summarize necessary results on quivers and quiver representations. For that, we mostly follow the paper [8]. In Section 3, provide basic notions and tools about cotorsion pairs. In Section 4, we prove our main result:

Theorem 4.6. *Let $\mathcal{C}$ be an abelian category, and $(\mathcal{A}, \mathcal{B})$ be a complete hereditary cotorsion pair in $\mathcal{C}$.*

(1) *If Q is a left rooted quiver, and if $\mathcal{C}$ has exact $|Q_1^{*\to i}|$-indexed coproducts for every vertex i in Q, then the cotorsion pair $(\Phi(\mathcal{A}), \Phi(\mathcal{A})^{\perp})$ is complete in* $\mathrm{Rep}(Q, \mathcal{C})$.
(2) *If Q is a right rooted quiver, and if $\mathcal{C}$ has exact $|Q_1^{i\to *}|$-indexed products for every vertex i in Q, then the cotorsion pair $({}^{\perp}\Psi(\mathcal{B}), \Psi(\mathcal{B}))$ is complete in* $\mathrm{Rep}(Q, \mathcal{C})$.

At this point, we note a subtle detail in the previous theorem. As clarified in [8, Remark 4.2], the cotorsion pairs $(\Phi(\mathcal{A}), \Phi(\mathcal{A})^{\perp})$ and $({}^{\perp}\Psi(\mathcal{B}), \Psi(\mathcal{B}))$ to exist, it isn't necessary to assume the category $\mathcal{C}$ to be (co)complete and $\mathcal{C}$ to have enough projectives and injectives as stated in [8, Theorem 7.4]. That's why we just assume $\mathcal{C}$ to have $|Q_1^{*\to i}|$-indexed coproducts or $|Q_1^{i\to *}|$-indexed products for every vertex i in Q.

One should note that if the cotorsion pair $(\mathcal{A}, \mathcal{B})$ is generated by a set, and the category $\mathcal{C}$ is a Grothendieck category with enough $\mathcal{A}$-objects, then the category $\operatorname{Rep}(Q, \mathcal{C})$ is a Grothendieck category, and the cotorsion pairs given in (i') and (iv') are also generated by a set, therefore, complete. Even though most of the cotorsion pairs in the theory are known to be generated by a set, there are examples of complete hereditary cotorsion pairs which are not known to be generated by a set. For instance, let $\mathrm{GP}(R)$ denote the class of Gorenstein projective R-modules, then under conditions given in [6, Corollary 1], the cotorsion pair $(\mathrm{GP}(R), \mathrm{GP}(R)^{\perp})$ is complete hereditary but not known to be generated by a set. Having these examples in hand, our result provides a more categorical tool in a (co)complete category without depending on generation arguments, and extends the class of cotorsion pairs for which the induced cotorsion pairs (ii') and (iv') are complete.

It seems hard to give an answer in a full generality on the completeness of the cotorsion pairs $(\Phi(\mathcal{A}), \Phi(\mathcal{A})^{\perp})$ and $({}^{\perp}\Psi(\mathcal{B}), \Psi(\mathcal{B}))$ for any quiver Q. However, in Section 5, we focus on the infinite line quiver A_{∞}^{∞}, which is neither left rooted nor right rooted, and we are able to prove the following:

Theorem 5.6. *Let Q be the infinite line quiver A_{∞}^{∞} and $(\mathcal{A}, \mathcal{B})$ be a complete hereditary cotorsion pair in the category R-*Mod *of left R-modules. Then the cotorsion pairs $(\Phi(\mathcal{A}), \Phi(\mathcal{A})^{\perp})$ and $({}^{\perp}\Psi(\mathcal{B}), \Psi(\mathcal{B}))$ in* $\operatorname{Rep}(Q, R)$ *are complete and hereditary, as well.*

For example, if $\mathcal{F}lat$ and $\mathcal{C}ot$ denote the classes of flat and cotorsion R-modules, respectively, then the pair $(\mathcal{F}lat, \mathcal{C}ot)$ is known to be a hereditary cotorsion pair generated by a set, hence, complete. Then, from the construction the cotorsion pair $({}^{\perp}\Psi(\mathcal{C}ot), \Psi(\mathcal{C}ot)))$ in $\operatorname{Rep}(A_{\infty}^{\infty}, R)$ is generated by a set, thus, it is complete. On the other hand, it is not immediate if the cotorsion pair $(\Phi(\mathcal{F}lat), \Phi(\mathcal{F}lat)^{\perp})$ is generated by a set. However, by our result, we can say that it is complete, as well.

2. Quiver representations

Throughout this paper, we fix the notation $\mathcal{C}$ for an abelian category.

2.1. A quiver, denoted by $Q = (Q_0, Q_1)$, is a directed graph where Q_0 and Q_1 are the set of vertices and arrows, respectively. For $i, j \in Q_0$, $Q(i, j)$ denotes the set of paths in Q from i to j. For every $i \in Q_0$, e_i is *the trivial path*. If $p \in Q(i, j)$, we write $s(p) = i$ and $t(p) = j$, called its *source* and *target*, respectively. Thus, one can think of Q as a category with the object class Q_0 and $\operatorname{Hom}_Q(i, j) := Q(i, j)$.

For a given vertex i in Q, we let

$$Q_1^{i\to *} := \{a \in Q_1 |\ s(a) = i\} \quad \text{and} \quad Q_1^{*\to i} := \{a \in Q_1 |\ t(a) = i\}. \tag{2.1}$$

From now on, the letter Q will denote a quiver with the sets of vertices and arrows Q_0, Q_1, respectively.

2.2. For a given quiver Q, the *opposite quiver* Q^{op} is the quiver obtained by reversing arrows of Q.

2.3. For a given set S, $|S|$ denotes the cardinality of the set S. We will say that a category $\mathcal{C}$ has $|Q_1^{*\to i}|$-*indexed coproducts for every vertex i in a quiver Q* if every family $\{C_u\}_{u\in U}$ of objects in $\mathcal{C}$ has the coproduct in $\mathcal{C}$ whenever $|U| \leq |Q_1^{*\to i}|$ for some $i \in Q_0$. Furthermore, if for every $i \in Q_0$ any $|Q_1^{*\to i}|$-indexed coproduct of monomorphisms is a monomorphism, we say that $\mathcal{C}$ has *exact* $|Q_1^{*\to i}|$-*indexed coproducts for every vertex i in Q*.

Note that since in this work we work with abelian categories, if the quiver Q is *target-finite*, that is, the set $Q_1^{*\to i}$ is finite for every $i \in Q_0$, then any abelian category satisfies the aforementioned definitions. So these definitions make sense if there is a vertex i in the quiver Q such that $|Q_1^{*\to i}|$ ($|Q_1^{i\to *}|$) is an infinite cardinal.

Dually, we will say that $\mathcal{C}$ has (exact) $|Q_1^{i\to *}|$-*indexed products for every vertex i in Q* if $\mathcal{C}^{op}$ has (*exact*) $|(Q^{op})_1^{*\to i}|$-*indexed coproducts for every $i \in Q_0$*.

2.4. Left rooted quiver. [4], [8, 2.5] For any quiver $Q = (Q_0, Q_1)$, there is a transfinite sequence $\{V_\alpha\}_\alpha$ of subsets of Q_0, defined as follows:

- $V_0 := \emptyset$, $V_1 := \{i \in Q_0 |$ there is no arrow a with $t(a) = i\}$.
- If V_α is defined for some ordinal α,

$$V_{\alpha+1} := \{i \in Q_0 |\ i \text{ is not target of any arrow a with } s(a) \in Q_0 \setminus \bigcup_{\gamma \leq \alpha} V_\gamma\}.$$

- For a limit ordinal β, $V_\beta := \bigcup_{\gamma<\beta} V_\gamma$.

We recall the following results, which are relevant to our purposes and proved explicitly in [8].

2.5. Proposition. [8, Lemma 2.7] Consider the family $\{V_\alpha\}_\alpha$ of vertices in a quiver Q, defined in 2.4. There is a chain of the form

$$V_1 \subseteq V_2 \subseteq \ldots \subseteq V_\alpha \subseteq V_{\alpha+1} \subseteq \ldots \subseteq Q_0.$$

2.6. Proposition. [8, Corollary 2.8] If $a : i \to j$ is an arrow in a quiver Q with $j \in V_{\alpha+1}$ for some ordinal α, then $i \in V_\alpha$.

2.7. Example. Let Q be the simplest loop quiver $\begin{tikzcd} \cdot 1 \arrow[loop left, "a"] \end{tikzcd}$. Then $V_\alpha = \emptyset$ for any ordinal α.

One can show that if a quiver has an oriented cycle (a nontrivial path with the same source and target), any vertex on it will not belong to V_α for any ordinal α. Following the terminology given in [8], this leads to the so-called *left rooted* quiver, introduced in [4], in which every vertex belongs to V_α for some ordinal α: A quiver Q is called *left rooted* if there exists an ordinal λ such that $V_\lambda = Q_0$. Equivalently, it has no infinite sequence of arrows of the form $\ldots \bullet \to \bullet \to \bullet$, see [4, Proposition 3.6]. In a dual manner, a quiver Q is said to be *right rooted* if the opposite quiver Q^{op} is left rooted.

2.8. We let $\mathrm{Rep}(Q, \mathcal{C})$ denote the category of $\mathcal{C}$-valued functors from Q. Equivalently, $X \in \mathrm{Rep}(Q, \mathcal{C})$ if and only if X assigns to $i \in Q_0$ an object $X(i) \in \mathcal{C}$, and to every arrow $a : i \to j$ a morphism $X(a) : X(i) \to X(j)$ in $\mathcal{C}$ with $X(e_i)$ the identity map on $X(i)$. If $p \in Q(i, j)$, then $X(p) : X(i) \to X(j)$ is obtained by composing. An object $X \in \mathrm{Rep}(Q, \mathcal{C})$ is called a $\mathcal{C}$-*valued representation* of the quiver Q.

A morphism $\eta : X \to X'$ between two $\mathcal{C}$-valued representations of Q is just a natural transformation of functors. In fact, η is a family $\{\eta(i) :\ X(i) \to X'(i)\}_{i\in Q_0}$ of morphisms in $\mathcal{C}$ such that the following diagram commutes

$$\begin{tikzcd} X(i) \arrow[r, "X(a)"] \arrow[d, "\eta(i)"'] & X(j) \arrow[d, "\eta(j)"] \\ X'(i) \arrow[r, "X'(a)"'] & X'(j) \end{tikzcd}$$

for every arrow $a : i \to j$ in Q.

2.9. The evaluation functor. Given a vertex i in a quiver Q, the *ith evaluation functor*

$$\mathrm{ev}_i : \quad \mathrm{Rep}(Q, \mathcal{C}) \longrightarrow \mathcal{C}$$

assigns to a representation $X \in \mathrm{Rep}(Q, \mathcal{C})$ the object $X(i)$.

2.10. Given a quiver Q, assume that $\mathcal{C}$ has the coproduct of any family $\{C_u\}_{u \in U}$ where $|U| \leq |Q(i,j)|$ for some $i, j \in Q_0$. Following the notation given in [8], for a given vertex i in Q, we consider the functor $f_i^{Q,\mathcal{C}} : \mathcal{C} \to \mathrm{Rep}(Q, \mathcal{C})$ defined as follows: For an object $C \in \mathcal{C}$, the $\mathcal{C}$-valued representation $f_i^{Q,\mathcal{C}}(C)$ has

$$f_i^{Q,\mathcal{C}}(C)(j) := \bigoplus_{p \in Q(i,j)} C_p, \qquad \text{for all } j \in Q_0 \quad (C_p = C).$$

If $a : j \to k \in Q_1$ and $p \in Q(i,j)$, then pa is a path from i to k. Therefore, there is a canonical inclusion

$$\iota_{pa} : \ C = C_p = C_{pa} \hookrightarrow f_i(C)(k).$$

By the universal property of coproducts, there is a unique morphism $f_i^{Q,\mathcal{C}}(C)(a)$, which fits in the following commutative diagram

$$\begin{tikzcd}
C_p \arrow[rr, equal] \arrow[d, hook, "\iota_p"'] & & C_{pa} \arrow[d, hook, "\iota_{pa}"] \\
f_i^{Q,\mathcal{C}}(C)(j) \arrow[rr, "f_i^{Q,\mathcal{C}}(C)(a)"'] & & f_i^{Q,\mathcal{C}}(C)(k).
\end{tikzcd}$$

Mostly we omit the letters Q and $\mathcal{C}$, and we write f_i

2.11. Given a quiver Q, assume that $\mathcal{C}$ has the product of any family $\{C_u\}_{u \in U}$ where $|U| \leq |Q(i,j)|$ for some $i, j \in Q_0$. Then the opposite category $\mathcal{C}^{op}$ satisfies the condition given in (2.10) for the opposite quiver Q^{op}. We let g_i denote the functor $(f_i^{Q^{op},\mathcal{C}^{op}})^{op}$.

2.12. Proposition. [8, Theorem 3.7] Given a quiver Q, suppose that $\mathcal{C}$ has the (co)product of any family $\{C_u\}_{u \in U}$ where $|U| \leq |Q(i,j)|$ for some $i, j \in Q_0$. For any vertex i in Q, the pairs (f_i, ev_i) and (ev_i, g_i) are adjoint pairs.

2.13. Note that all (co)limits in $\mathrm{Rep}(Q, \mathcal{C})$ are calculated componentwise, hence, the category $\mathrm{Rep}(Q, \mathcal{C})$ of $\mathcal{C}$-valued representations inherits homological properties of $\mathcal{C}$.

Besides, if $\mathcal{C}$ is a (co)complete abelian category, and if G is a generator (or cogenerator) of $\mathcal{C}$, then the set $\{f_i(G)\}_{i \in Q_0}$ ($\{g_i(G)\}_{i \in Q_0}$) is a (co)generating set for the category $\mathrm{Rep}(Q, \mathcal{C})$.

2.14. The stalk functor. For a given vertex i in Q, the *ith stalk functor*

$$s_i : \quad \mathcal{C} \longrightarrow \mathrm{Rep}(Q, \mathcal{C})$$

is defined as

$$s_i(C) := \begin{cases} C, & i = j; \\ 0, & i \neq j, \end{cases}$$

with $s_i(a) = 0$ for any arrow a in Q.

2.15. Assume that $\mathcal{C}$ has $|Q_1^{*\to i}|$-indexed coproducts for every $i \in Q_0$. Let i be a vertex in Q and $X \in \mathrm{Rep}(Q, \mathcal{C})$. By the universal property of coproducts, there is a unique morphism

$$\varphi_i^X : \quad \bigoplus_{a \in Q_1^{*\to i}} X(s(a)) \longrightarrow X(i) \ .$$

We let $c_i(X) := \mathrm{Coker}\ \varphi_i^X$. Then it yields a functor

$$c_i : \quad \mathrm{Rep}(Q, \mathcal{C}) \longrightarrow \mathcal{C} \ .$$

Dually, if the category $\mathcal{C}$ has $|Q_1^{i\to *}|$-indexed products for every $i \in Q_0$, the functor $k_i : \ \mathrm{Rep}(Q, \mathcal{C}) \to \mathcal{C}$ is defined as

$$\mathrm{Ker}\ \left(\ X(i) \xrightarrow{\psi_i^X} \prod_{a \in Q_1^{i\to *}} X(t(a))\ \right),$$

where ψ_i^X is the canonical morphism induced from the universal property of products.

2.16. Proposition. [8, Theorem 4.5]

(1) If $\mathcal{C}$ is a category with $|Q_1^{*\to i}|$-indexed coproducts for every $i \in Q_0$, then (c_i, s_i) is an adjoint pair.
(2) If $\mathcal{C}$ is a category with $|Q_1^{i\to *}|$-indexed products for every $i \in Q_0$, then (s_i, k_i) is an adjoint pair.

3. Cotorsion pairs

In order to state our object of interest, we continue recalling necessary terminologies and results on cotorsion pairs. For a given pair of objects X, Y in a category $\mathcal{C}$ and a positive integer n, $\mathrm{Ext}^n_{\mathcal{C}}(X, Y)$ denotes the nth Yoneda extension class, see [10, Section III-5]. Even though, it carries an abelian group structure, it may fail to be a set unless $\mathcal{C}$ is an efficient abelian category, see [14, Corollary 5.5]. For convenience, we will omit the letter $\mathcal{C}$ in the notation.

For a given class $\mathcal{A}$ of objects of $\mathcal{C}$, we let

$$\begin{aligned} \mathcal{A}^{\perp} &= \{X \in \mathcal{C} |\ \mathrm{Ext}^1(A, X) = 0 \text{ for all } A \in \mathcal{A}\}, \\ {}^{\perp}\mathcal{A} &= \{X \in \mathcal{C} |\ \mathrm{Ext}^1(X, A) = 0 \text{ for all } A \in \mathcal{A}\}. \end{aligned}$$

3.1. A pair $(\mathcal{A}, \mathcal{B})$ of classes of objects in $\mathcal{C}$ is said to be a *cotorsion pair* provided that

$$\mathcal{A}^{\perp} = \mathcal{B} \quad \text{and} \quad {}^{\perp}\mathcal{B} = \mathcal{A}.$$

The category $\mathcal{C}$ is said to *have enough $\mathcal{A}$-objects* (*$\mathcal{B}$-objects*) if for every object $C \in \mathcal{C}$ there exists an epimorphism $A \twoheadrightarrow C$ (a monomorphism $C \hookrightarrow B$) with $A \in \mathcal{A}$ ($B \in \mathcal{B}$).

Given a class $\mathcal{F}$ of objects in $\mathcal{C}$, there are two associated cotorsion pairs

$$\begin{aligned} &({}^{\perp}(\mathcal{F}^{\perp}), \mathcal{F}^{\perp}), \quad \text{called } \textit{cotorsion pair generated by } \mathcal{F}, \\ &({}^{\perp}\mathcal{F}, ({}^{\perp}\mathcal{F})^{\perp}), \quad \text{called } \textit{cotorsion pair cogenerated by } \mathcal{F}. \end{aligned}$$

3.2. Let $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair in $\mathcal{C}$ and C be an object in $\mathcal{C}$. C is said to *have a special $\mathcal{A}$-precover* if there is a short exact sequence

$$0 \longrightarrow B \longrightarrow A \longrightarrow C \longrightarrow 0\ , \qquad \text{with } A \in \mathcal{A},\ B \in \mathcal{B},$$

and to *have a special* $\mathcal{B}$*-preenvelope* if there is a short exact sequence

$$0 \longrightarrow C \longrightarrow B' \longrightarrow A' \longrightarrow 0\ , \qquad \text{with } A' \in \mathcal{A},\ B' \in \mathcal{B}.$$

A cotorsion pair $(\mathcal{A}, \mathcal{B})$ is said to *have enough projectives* (*injectives*) if every object of $\mathcal{C}$ has a special $\mathcal{A}$-precover (a special $\mathcal{B}$-preenvelope). It is called *complete* if it has both enough projectives and enough injectives.

The following is a general form of the so-called Salce's Lemma (see [12]), which can be easily proved by using pull-back and push-out arguments.

3.3. Lemma. Let $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair in a category $\mathcal{C}$.

1. If $(\mathcal{A}, \mathcal{B})$ has enough projectives, and if $\mathcal{C}$ has enough $\mathcal{B}$-objects, then the cotorsion pair $(\mathcal{A}, \mathcal{B})$ is complete.
2. If $(\mathcal{A}, \mathcal{B})$ has enough injectives, and if $\mathcal{C}$ has enough $\mathcal{A}$-objects, then the cotorsion pair $(\mathcal{A}, \mathcal{B})$ is complete.

Proof. Because of duality of arguments, we just provide a sketch of proof for the first statement (1). Let C be an object in $\mathcal{C}$. By assumption, there is a short exact sequence $\mathbb{E}: \quad 0 \to C \to B \to K \to 0$ with $B \in \mathcal{B}$. Again, by hypothesis, there is an epimorphism $f : A \to K$ with kernel in $\mathcal{B}$ and $A \in \mathcal{A}$. Then the short exact sequence $\mathbb{E}f$ obtained by taking pullback of $\mathbb{E}$ over f is the desired special $\mathcal{B}$-preenvelope of C. $\square$

3.4. A cotorsion pair $(\mathcal{A}, \mathcal{B})$ in $\mathcal{C}$ is said to be *hereditary* if $\mathrm{Ext}^n(A, B) = 0$ for every positive integer n, $A \in \mathcal{A}$ and $B \in \mathcal{B}$. It is a well-known fact that if the category $\mathcal{C}$ has enough $\mathcal{A}$-objects and $\mathcal{B}$-objects, the following are equivalent:

(1) $(\mathcal{A}, \mathcal{B})$ is hereditary.
(2) The class $\mathcal{A}$ is closed under kernels of epimorphisms.
(3) The class $\mathcal{B}$ is closed under cokernels of monomorphisms.

The following lemma plays a crucial role in our main results. Even though in [1, Theorem 3.1] it is stated for $\mathcal{C} = R$-Mod, its proof is categorical, which permits us to state it in a more general form for any abelian category.

3.5. Lemma. [1, Theorem 3.1] Let $(\mathcal{A}, \mathcal{B})$ be a hereditary cotorsion pair in $\mathcal{C}$ and

$$0 \longrightarrow C_1 \longrightarrow C \longrightarrow C_2 \longrightarrow 0$$

be an exact sequence in $\mathcal{C}$. If there are short exact sequences $0 \longrightarrow B_1 \longrightarrow T_1 \longrightarrow C_1 \longrightarrow 0$ and $0 \longrightarrow B_2 \longrightarrow A_2 \longrightarrow C_2 \longrightarrow 0$ where $B_1, B_2 \in \mathcal{B}$ and $A_2 \in \mathcal{A}$, then there is a commutative diagram with exact rows and columns

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & B_1 \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & B_2 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & T_1 \arrow[r] \arrow[d] & T_2 \arrow[r] \arrow[d] & A_2 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_1 \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & C_2 \arrow[r] \arrow[d] & 0. \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

where $B \in \mathcal{B}$. Furthermore, if the left column is a special precover of C_1, then the middle column is a special $\mathcal{A}$-precover of C, as well. The dual statement holds for special $\mathcal{B}$-preenvelopes.

Proof. We just provide a sketch of the proof, which will be needed in the proof of Theorem 5.3. Taking the pullback of the morphisms $A_2 \twoheadrightarrow C_2$ and $C \twoheadrightarrow C_2$, one obtains a short exact sequence $\mathbb{E}' \in \operatorname{Ext}^1(A_2, C_1)$. Since the cotorsion pair is hereditary and $B_1 \in \mathcal{B}$, $\operatorname{Ext}^1(A_1, T_1) \cong \operatorname{Ext}^1(A_1, C_1)$, therefore, there exists a unique short exact sequence $\mathbb{E} : 0 \to T_1 \to T_2 \to A_2 \to 0$ whose pushout with the morphism $T_1 \twoheadrightarrow C_1$ is $\mathbb{E}'$. $\square$

3.6. A *continuous chain* of objects in $\mathcal{C}$ is a directed system $\{C_\alpha, g_{\alpha\alpha'}\}_{\alpha \leq \alpha' \leq \lambda}$ (indexed by an ordinal λ) of objects of $\mathcal{C}$ such that for every ordinal $\alpha < \lambda$, the morphism $g_{\alpha,\alpha+1}$ is a monomorphism, and for every limit ordinal $\beta \leq \lambda$, $C_\beta = \varinjlim_{\alpha<\beta} C_\alpha$.

Given a class $\mathcal{F}$ of objects of a category $\mathcal{C}$, an object $C \in \mathcal{C}$ is said to *have an $\mathcal{F}$-filtration* if there is a continuous chain $\{C_\alpha\}_{\alpha \leq \lambda}$ of subobjects of C with $C_0 = 0$, $C_\lambda = C$ and $C_{\alpha+1}/C_\alpha$ is isomorphic to an object in $\mathcal{F}$ for every ordinal $\alpha < \lambda$.

Dually, a *cocontinuous chain* of objects in $\mathcal{C}$ is a directed system $\{C_\alpha, g_{\alpha'\alpha}\}_{\alpha \leq \alpha' \leq \lambda}$ (indexed by an ordinal λ) of objects of $\mathcal{C}$ such that for every ordinal $\alpha < \lambda$, the morphism $g_{\alpha+1,\alpha}$ is an epimorphism, and for every limit ordinal $\beta \leq \lambda$, $C_\beta = \varprojlim_{\alpha<\beta} C_\alpha$.

An object $C \in \mathcal{C}$ is said to *have an $\mathcal{F}$-cofiltration* if there is a cocontinuous chain $\{C_\alpha, g_{\alpha'\alpha}\}_{\alpha \leq \alpha' \leq \lambda}$ with $C_0 = 0$, $C_\lambda = C$ and $\operatorname{Ker} g_{\alpha+1,\alpha}$ is isomorphic to an object in $\mathcal{F}$ for every ordinal $\alpha < \lambda$.

The following Lemma, known as Eklof Lemma, plays a crucial role in our main result, and shows that the left part of a cotorsion pair is closed under filtrations while the right part is closed under cofiltrations.

3.7. Eklof Lemma. Let $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair in an abelian category $\mathcal{C}$. Then we have:

(1) If an object $C \in \mathcal{C}$ has an $\mathcal{A}$-filtration, then C belongs to $\mathcal{A}$, as well.
(2) If an object $C \in \mathcal{C}$ has an $\mathcal{B}$-cofiltration, then C belongs to $\mathcal{B}$, as well.

Proof. Let $\{C_\alpha\}_{\alpha \leq \lambda}$ be an $\mathcal{A}$-filtration of C. If $\beta \leq \lambda$ is a limit ordinal, and if $\mathbb{E}$ is a short exact sequence in $\operatorname{Ext}^1(C_\beta, B)$, where $B \in \mathcal{B}$, then consider the short exact sequence $\mathbb{E}g_{\alpha\beta}$ obtained by taking pullback along $C_\alpha \to C_\beta$. Apply the arguments given in the proof of [3, Theorem 7.3.4]. The arguments for (2) are dual to that of (1). $\square$

As claimed, next we recall the classes in $\mathrm{Rep}(Q,\mathcal{C})$ associated to a class in $\mathcal{C}$, which are defined in [8, Section 7], and which will be needed for the construction of cotorsion pairs in $\mathrm{Rep}(Q,\mathcal{C})$ arising from a cotorsion pair in the ground category $\mathcal{C}$.

3.8. Given a class $\mathcal{F}$ of objects in $\mathcal{C}$, we let

$$\begin{aligned} s_*(\mathcal{F}) &= \{s_i(F)|\ F \in \mathcal{F} \text{ and } i \in Q_0\}, \\ \mathrm{Rep}(Q,\mathcal{F}) &= \{X \in \mathrm{Rep}(Q,\mathcal{C})|\ X(i) \in \mathcal{F} \text{ for all } i \in Q_0\}, \\ f_*(\mathcal{F}) &= \{f_i(F)|\ F \in \mathcal{F} \text{ and } i \in Q_0, \text{ where } f_i \text{ is defined }\}, \\ g_*(\mathcal{F}) &= \{g_i(F)|\ F \in \mathcal{F} \text{ and } i \in Q_0, \text{ where } g_i \text{ is defined }\}, \\ \Psi(\mathcal{F}) &= \{X \in \mathrm{Rep}(Q,\mathcal{C})|\ \psi_i^X \text{ is epic and Ker } \psi_i^X \in \mathcal{F} \text{ for all } i \in Q_0\}, \\ \Phi(\mathcal{F}) &= \{X \in \mathrm{Rep}(Q,\mathcal{C})|\ \varphi_i^X \text{ is monic and Coker } \varphi_i^X \in \mathcal{F} \text{ for all } i \in Q_0\}. \end{aligned}$$

3.9. Proposition. [8, Proposition 5.2] Let $C \in \mathcal{C}$ and $X \in \mathrm{Rep}(Q,\mathcal{C})$. Let i be a vertex in a quiver Q.

(1) If the functor f_i exists, for all non-negative integers n there is an isomorphism

$$\mathrm{Ext}^n(f_i(C), X) \cong \mathrm{Ext}^n(C, \mathrm{ev}_i(X)).$$

(2) If the functor g_i exists, for all non-negative integers n there is an isomorphism

$$\mathrm{Ext}^n(X, g_i(\mathcal{C})) \cong \mathrm{Ext}^n(\mathrm{ev}_i(X), C).$$

The following lemma is a general form of the result given in [8, Proposition 5.4].

3.10. Proposition. Let i be a vertex in a quiver Q. Let $X \in \mathrm{Rep}(Q,\mathcal{C})$ and $C \in \mathcal{C}$.

(1) Assume that $\mathcal{C}$ has $|Q_1^{*\to i}|$-indexed coproducts. Then there is an inclusion

$$\mathrm{Ext}^1(c_i(X), C) \hookrightarrow \mathrm{Ext}^1(X, s_i(C)).$$

If φ_i^X is a monomorphism, then it is an isomorphism.

(2) Assume that $\mathcal{C}$ has $|Q_1^{i\to *}|$-indexed products. Then there is an inclusion

$$\mathrm{Ext}^1(C, k_i(X)) \hookrightarrow \mathrm{Ext}^1(s_i(C), X).$$

If ψ_i^X is an epimorphism, then it is an isomorphism.

Proof. The two statements are dual. So we will prove the first statement (1). Let

$$\mathbb{E}:\ 0 \longrightarrow C \xrightarrow{\alpha} T \xrightarrow{t} c_i(X) \longrightarrow 0$$

be a short exact sequence in $\mathcal{C}$. Taking the pullback of t and $\sigma:\ X(i) \to c_i(X)$, we get the following commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & \\
 & C \arrow[r, equal] \arrow[d, "\alpha'"'] & C \arrow[d, "\alpha"] & \\
\bigoplus_{a:*\to i} X(s(a)) \arrow[r, "\beta"] \arrow[d, equal] & T' \arrow[r, "\sigma'"] \arrow[d, "t'"'] & T \arrow[r] \arrow[d, "t"] & 0 \\
\bigoplus_{a:*\to i} X(s(a)) \arrow[r, "\varphi_i^X"'] & X(i) \arrow[r, "\sigma"'] \arrow[d] & c_i(X) \arrow[r] \arrow[d] & 0. \\
 & 0 & 0 &
\end{tikzcd}$$

We let Y be a $\mathcal{C}$-valued representation of Q defined by

$$Y(j) := \begin{cases} T', & \text{if } j = i \\ X(j), & \text{if } j \neq i \end{cases} \qquad \text{for } j \in Q_0.$$

For an arrow $a : j \to k$ in Q, we define the morphism $Y(a) : \ Y(j) \to Y(k)$ as follows:

Case 1: If $j \neq i$ and $k \neq i$, then $Y(a) = X(a)$.
Case 2: If $j \neq i$ and $k = i$, then $Y(a) := \iota_a \circ \beta$, where $\iota_a : X(s(a)) \hookrightarrow \bigoplus_{a \in Q_1^{*\to i}} X(s(a))$ is the canonical inclusion.
Case 3: If $j = i$ and $k \neq i$, then $Y(a) = X(a) \circ t'$.
Case 4: If $j = k = i$, then $\sigma \circ X(a) = 0$, and by using the pullback square, there exists a unique morphism $Y(a) : T' \to T'$ which fits in the following commutative diagram

$$\begin{tikzcd}
T' \arrow[dr, "Y(a)"] \arrow[drr, bend left, "0"] \arrow[dd, "t'"'] & & \\
 & T' \arrow[r, "\sigma'"'] \arrow[dd, "t'"] & T \arrow[dd, "t"] \\
X(i) \arrow[dr, "X(a)"'] & & \\
 & X(i) \arrow[r, "\sigma"'] & c_i(X).
\end{tikzcd} \tag{3.1}$$

Claim: There is a short exact sequence of the form

$$\widetilde{\mathbb{E}} : \ 0 \longrightarrow s_i(C) \xrightarrow{\widetilde{\alpha}} Y \xrightarrow{\widetilde{t}} X \longrightarrow 0$$

in $\operatorname{Rep}(Q, \mathcal{C})$ where for a vertex j in Q

$$\widetilde{\alpha}(j) := \begin{cases} \alpha', & \text{if } j = i \\ 0, & \text{if } j \neq i \end{cases} \quad \text{and} \quad \widetilde{t}(j) := \begin{cases} t', & \text{if } j = i \\ \mathrm{id}, & \text{if } j \neq i. \end{cases}$$

It is easy to verify that $\widetilde{t}: Y \to X$ is a morphism of $\mathcal{C}$-valued representations. As for $\widetilde{\alpha}$, we just need to prove the Case 4, that is, for an arrow $a: i \to i$, $Y(a) \circ h = 0$. From the diagram (3.1), we have $t' \circ Y(a) \circ \alpha' = X(a) \circ t' \circ \alpha' = 0$, by uniqueness, $Y(a) \circ \alpha' = 0$. Finally, it is obvious that $\widetilde{\mathbb{E}}$ is a short exact sequence in $\mathrm{Rep}(Q, \mathcal{C})$.

The assignment to a short exact sequence $\mathbb{E} \in \mathrm{Ext}^1(c_i(X), C)$ a short exact sequence $\widetilde{\mathbb{E}}$ in $\mathrm{Ext}^1(X, s_i(C))$ is well-defined, therefore, it yields to a homomorphism $\mathrm{Ext}^1(X, s_i(C)) \to \mathrm{Ext}^1(c_i(X), C)$ of abelian groups. Now we need to show that it is injective. Suppose that the short exact sequence $\widetilde{\mathbb{E}}$ splits and $h = \{h_i\}_{i \in Q_0}$ is a coretraction of $\widetilde{\alpha}$, that is, $h \circ \widetilde{\alpha} = \mathrm{id}$. If $Q_1^{* \to i} = \emptyset$, then $\bigoplus_{a: * \to i} X(s(a)) = 0$, hence, $\widetilde{\alpha}(i) = \alpha' = \alpha$ and $h_i \circ \alpha = \mathrm{id}$. If $Q_1^{* \to i} \neq \emptyset$, then for every $a \in Q_1^{* \to i}$, $h_i \circ Y(a) = 0$, and therefore $h_i \circ \beta = 0$. Since $T = \mathrm{Coker}\, \beta$, there exists a unique morphism $z: T \to C$ such that $z \circ \sigma' = h_i$. Now, $z \circ \alpha = z \circ \sigma' \circ \alpha' = h_i \circ \alpha' = \mathrm{id}$. So the short exact sequence $\mathbb{E}$ splits, as well. □

The following result shows that under suitable conditions, a cotorsion pair $(\mathcal{A}, \mathcal{B})$ in a category $\mathcal{C}$ gives rises to four canonical cotorsion pairs in $\mathrm{Rep}(Q, \mathcal{C})$, which appear in the literature when $\mathcal{C} := R$-Mod and Q is a left or right rooted quiver.

3.11. Proposition. [8, Theorem 7.4, Theorem 7.9] Let $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair in a (co)complete abelian category $\mathcal{C}$, which has enough $\mathcal{A}$-objects and $\mathcal{B}$-objects. Assume that the cotorsion pair $(\mathcal{A}, \mathcal{B})$ is generated by a class $\mathcal{A}_0$ and cogenerated by a class $\mathcal{B}_0$. Then the following are cotorsion pairs in the category $\mathrm{Rep}(Q, \mathcal{C})$ of $\mathcal{C}$-valued representations of a quiver Q:

(1) $({}^{\perp}\mathrm{Rep}(Q, \mathcal{B}), \mathrm{Rep}(Q, \mathcal{B}))$, generated by $f_*(\mathcal{A}_0)$,
(2) $(\Phi(\mathcal{A}), \Phi(\mathcal{A})^{\perp})$, cogenerated by $s_*(\mathcal{B}_0)$,
(3) $(\mathrm{Rep}(Q, \mathcal{A}), \mathrm{Rep}(Q, \mathcal{A})^{\perp})$, generated by $g_*(\mathcal{B}_0)$,
(4) $({}^{\perp}\Psi(\mathcal{B}), \Psi(\mathcal{B}))$, cogenerated by $s_*(\mathcal{A}_0)$.

If Q is left rooted, then the cotorsion pairs given in (1-2) coincide. If Q is right rooted, then the cotorsion pairs given in (3-4) coincide.

3.12. Remark. Note that the pairs in Proposition 3.11 are proved to be a cotorsion pair in [8, Theorem 7.4] under the condition that $\mathcal{C}$ is a category with enough injectives and projectives, $\mathrm{Proj} \subseteq \mathcal{A}_0$ and $\mathrm{Inj} \subseteq \mathcal{B}_0$. As a matter of fact, it depends on [8, Proposition 5.6], which still holds if $\mathcal{C}$ has enough $\mathcal{A}$-objects and enough $\mathcal{B}$-objects.

3.13. Remark. The existence of the cotorsion pairs $(2 - 4)$ in Proposition 3.11 doesn't depend on the (co)completeness of the category $\mathcal{C}$ so that it can be replaced with the condition that $\mathcal{C}$ has $|Q_1^{* \to i}|$-indexed coproducts and $|Q_1^{i \to *}|$-indexed products for every $i \in Q_0$ just as in Proposition 3.10. Besides, by using Eklof Lemma, if the quiver Q is left rooted, the transfinite induction done in [8, Theorem 7.9] would still work, and therefore, we have $\mathrm{Rep}(Q, \mathcal{B}) \subseteq \Phi(\mathcal{A})^{\perp}$. Dually, if Q is right rooted, then $\mathrm{Rep}(Q, \mathcal{A}) \subseteq^{\perp} \Psi(\mathcal{B})$.

4. Completeness of cotorsion pairs in $\mathrm{Rep}(Q, \mathcal{C})$

From now on, we fix the notation $(\mathcal{A}, \mathcal{B})$ for a cotorsion pair in an abelian category $\mathcal{C}$. In this section, we aim at proving the completeness of the induced cotorsion pairs $(\Phi(\mathcal{A}), \Phi(\mathcal{A})^{\perp})$ and $({}^{\perp}\Psi(\mathcal{B}), \Psi(\mathcal{B}))$ for certain quivers. For that, we need some preparative lemmas.

4.1. Lemma.

(1) If $(\mathcal{A}, \mathcal{B})$ in $\mathcal{C}$ has enough injectives, then every $\mathcal{C}$-valued representation of a quiver Q can be embedded in an object of $\operatorname{Rep}(Q, \mathcal{B})$.
(2) If $(\mathcal{A}, \mathcal{B})$ in $\mathcal{C}$ has enough projectives, then every $\mathcal{C}$-valued representation of a quiver Q is an epimorphic image of an object of $\operatorname{Rep}(Q, \mathcal{A})$.

Proof. Let X be a $\mathcal{C}$-valued representation of a quiver Q. For every $i \in Q_0$, we fix a special B-preenvelope

$$0 \longrightarrow X(i) \xrightarrow{\eta_i} B_i \longrightarrow A_i \longrightarrow 0\ .$$

Since $(\mathcal{A}, \mathcal{B})$ is a cotorsion pair, for every arrow $a:\ i \to j$ in Q, there is a morphism $B(a)$ which fits in the following commutative diagram

$$\begin{tikzcd}
X(i) \arrow[r, "\eta_i"] \arrow[d, "X(a)"'] & B_i \arrow[d, "B(a)"] \\
X(j) \arrow[r, "\eta_j"'] & B_j
\end{tikzcd} .$$

We let B be the $\mathcal{C}$-valued representation of Q with $B(i) := B_i$ for every vertex i in Q, and morphisms $B(a)$ for every arrow a in Q. By construction, $B \in \operatorname{Rep}(Q, \mathcal{B})$ and the family $\eta := \{\eta_i\}_{i \in Q_0}$ is a monomorphism from X to B. $\square$

4.2. The following lemma is essentially based on the construction given in [8, 5.5]. Just to comprehend the idea of the proof, we show how it works on a simple case. Let Q be the quiver

$$\begin{tikzcd}
\overset{1}{\bullet} \arrow[dr, "a"] & & \\
& \overset{3}{\bullet} \arrow[r, "c"'] & \overset{4}{\bullet}. \\
\overset{2}{\bullet} \arrow[ur, "b"'] & &
\end{tikzcd}$$

Then it is a left rooted quiver with $V_0 = \emptyset$, $V_1 = \{1, 2\}$, $V_2 = \{1, 2, 3\}$ and $V_3 = \{1, 2, 3, 4\}$.

Now we assume that $\mathcal{C}$ has enough $\mathcal{B}$-objects. Let X be any $\mathcal{C}$-valued representation of Q. Obviously, the morphisms φ_1^X and φ_2^X are monomorphisms because the vertices 1 and 2 are in V_1. So we let $E^1 := X$. But, it may fail the morhphisms φ_3^X and φ_4^X to be monomorphisms. In order to repair it, we firstly consider a monomorphism $\varepsilon_3 :\ X(1) \oplus X(2) \hookrightarrow B_3$ with $B_3 \in \mathcal{B}$. Then we let E^2 be the $\mathcal{C}$-valued representation

$$\begin{tikzcd}
X(1) \arrow[dr, "E^2(a)"] & & \\
& X(3) \oplus B_3 \arrow[r, "E^2(c)"'] & X(4) \\
X(2) \arrow[ur, "E^2(b)"'] & &
\end{tikzcd}$$

where $E^2(a) = \begin{pmatrix} X(a) \\ \\ \varepsilon_3 \circ \iota \end{pmatrix}$, $E^2(b) = \begin{pmatrix} X(b) \\ \\ \varepsilon_3 \circ \iota \end{pmatrix}$ and $E^2(c) = X(c) \circ \pi$, and ι and π are the canonical inclusion and projection morphisms, respectively. We observe that $\varphi_3^{E^2} = \begin{pmatrix} \varphi_3^X & \varepsilon_3 \end{pmatrix}$ is now a monomorphism, and there is a vertex-wise split epimorphism $0 \to s_3(B_3) \to E^2 \to E^1 \to 0$. Now we proceed the same argument for the representation E^2 at the vertex 4. Then we obtain the $\mathcal{C}$-valued representation E^3

$$\begin{tikzcd}
X(1) \arrow[dr, "E^2(a)"] & & \\
& X(3) \oplus B_3 \arrow[r, "E^3(c)"'] & X(4) \oplus B_4. \\
X(2) \arrow[ur, "E^2(b)"'] & &
\end{tikzcd}$$

One can easily verify that $E := E^3 \in \Phi(\mathcal{C})$, and there is a vertex-wise split epimorphism $E \to X$ with kernel $B := s_3(B_3) \oplus s_4(B_4) \in \mathrm{Rep}(Q, \mathcal{B})$.

4.3. Lemma. Let X be a $\mathcal{C}$-valued representation of a quiver Q.

(1) If Q is a left rooted quiver, and if $\mathcal{C}$ has enough $\mathcal{B}$-objects and $|Q_1^{*\to i}|$-indexed coproducts for every $i \in Q_0$, then there is a short exact sequence $0 \longrightarrow B \longrightarrow E \longrightarrow X \longrightarrow 0$ with $E \in \Phi(\mathcal{C})$ and $B \in \mathrm{Rep}(Q, \mathcal{B})$.
(2) If Q is a right rooted quiver, and if $\mathcal{C}$ has enough $\mathcal{A}$-objects and $|Q_1^{i\to *}|$-indexed products for every $i \in Q_0$, then there is a short exact sequence $0 \longrightarrow X \longrightarrow E' \longrightarrow A \longrightarrow 0$ with $E' \in \Psi(\mathcal{C})$ and $C \in \mathrm{Rep}(Q, \mathcal{A})$.

Proof. We will prove the first statement (1), the proof of (2) is dual.

Let $\{V_\alpha\}_{\alpha \leq \lambda}$ be the λ-transfinite sequence of vertices given in (2.4) with $V_\lambda = Q_0$. By transfinite induction, we will construct a cocontinuous inverse λ-sequence $\{E^\alpha\}_{\alpha \leq \lambda}$ in $\mathrm{Rep}(Q, \mathcal{C})$ which satisfies:

(i) For every ordinal $0 < \alpha < \lambda$ and $i \in Q_0 \backslash V_\alpha$, $E^\alpha(i) = X(i)$ and $\mathrm{Ker}(E^{\alpha+1} \twoheadrightarrow E^\alpha) \in \mathrm{Rep}(Q, \mathcal{B})$.
(ii) For every ordinal $\alpha \leq \lambda$ and $i \in V_\alpha$, $\varphi_i^{E^\alpha}$ is a monomorphism.
(iii) For every $0 < \alpha < \alpha' \leq \lambda$, the morphism $E^{\alpha'} \twoheadrightarrow E^\alpha$ is a vertex-wise split epimorphism, and $E^\alpha(i) = E^{\alpha'}(i)$ whenever $i \notin V_{\alpha'} \backslash V_\alpha$.

We set $E^0 = 0$ and $E^1 = X$. Suppose that we have E^α for some ordinal $0 < \alpha < \lambda$. We now construct $E^{\alpha+1}$ as follows:

For every $i \in V_{\alpha+1}/V_\alpha$, we fix a monomorphism

$$\varepsilon_i^\alpha : \quad \bigoplus_{a \in Q_1^{*\to i}} E^\alpha(s(a)) \hookrightarrow B_i^{\alpha+1} ,$$

where $B_i^{\alpha+1} \in \mathcal{B}$. From (2.6), any arrow $a \in Q_1^{*\to i}$ has the source $s(a)$ in V_α, and therefore, $s(a) \neq i$ and there exists the canonical inclusion

$$\varepsilon_i^\alpha \circ \iota^{s(a),i} : \quad E^\alpha(s(a)) \xhookrightarrow{\iota^{s(a),i}} \bigoplus_{a \in Q_1^{*\to i}} E^\alpha(s(a)) \hookrightarrow B_i^{\alpha+1}. \tag{4.1}$$

By assumption, $i \in Q_0 \backslash V_\alpha$, hence, $E^\alpha(a): E^\alpha(s(a)) \to E^\alpha(i) = X(i)$. Using the universal property of products, there is a canonical morphism

$$\begin{pmatrix} E^\alpha(a) \\ \\ \varepsilon_i^\alpha \circ \iota^{s(a),i} \end{pmatrix} : \quad E^\alpha(s(a)) \longrightarrow X(i) \oplus B_i^{\alpha+1} , \tag{4.2}$$

which induces a monomorphism

$$\bigoplus_{a \in Q_1^{* \to i}} E^\alpha(s(a)) \hookrightarrow X(i) \oplus B_i^{\alpha+1}.$$

We define $E^{\alpha+1}$ as follows

$$E^{\alpha+1}(i) := \begin{cases} E^\alpha(i), & i \notin V_{\alpha+1} \backslash V_\alpha \\ \\ X(i) \oplus B_i^{\alpha+1}, & \text{otherwise.} \end{cases}$$

Let $a: j \to k$ be an arrow in Q.

- If $k \in V_{\alpha+1} \backslash V_\alpha$, then $j \in V_\alpha$, and therefore, $E^{\alpha+1}(a)$ is the morphism given in (4.2).
- If $j \in V_{\alpha+1} \backslash V_\alpha$, then $k \notin V_{\alpha+1}$, and $E^{\alpha+1}(k) = E^\alpha(k) = X(k)$, hence, $E^{\alpha+1}(a): E^{\alpha+1}(j) \to X(k)$ is the composition of the projection map $E^{\alpha+1}(j) \to E^\alpha(j) = X(j)$ followed by $E^\alpha(a) = X(a)$.
- For the other cases, $E^{\alpha+1}(a) = E^\alpha(a)$.

One can easily check that the canonical projection map at each vertex gives rise to a vertex-wise split epimorphism $E^{\alpha+1} \twoheadrightarrow E^\alpha$ with

$$\text{Ker } (E^{\alpha+1} \twoheadrightarrow E^\alpha) = \prod_{i \in V_{\alpha+1} \backslash V_\alpha} s_i(B_i^{\alpha+1}) \in \text{Rep}(Q, \mathcal{B}) \subseteq \Phi(\mathcal{A})^\perp.$$

As a consequence, $E^{\alpha+1}$ satisfies the desired conditions.

If $\beta \leq \lambda$ is a limit ordinal and E^α is constructed for every ordinal $\alpha < \beta$, then the inverse limit $\varprojlim_{\alpha<\beta} E^\alpha$ exists because at each vertex i of Q, the inverse sequence $\{E^\alpha(i)\}_{\alpha<\beta}$ is eventually constant. Indeed, if $i \in V_\beta = \bigcup_{\alpha<\beta} V_\alpha$, then $i \in V_\alpha$ for some ordinal $\alpha < \beta$. From the condition (iii), for any ordinal $\alpha \leq \alpha' < \beta$, $E^\alpha(i) = E^{\alpha'}(i)$. If $i \notin V_\beta$, then, by hypothesis, $E^\alpha(i) = X(i)$ for any ordinal $\alpha < \beta$. In other words, $\varprojlim_{\alpha<\beta} E^\alpha$ is of the form

$$\varprojlim_{\alpha<\beta} E^\alpha(i) = \begin{cases} E^\alpha(i), & i \in V_\beta \\ \\ X(i), & i \notin V_\beta. \end{cases}$$

We let $E^\beta := \varprojlim_{\alpha<\beta} E^\alpha$. E^β satisfies the desired conditions.

As a consequence, we have an inverse limit in $\text{Rep}(Q, \mathcal{C})$

$$E^\lambda \twoheadrightarrow \ldots \twoheadrightarrow E^2 \twoheadrightarrow E^1 = X.$$

By the condition (iii), the morphism $E^\lambda \to E^1$ is a vertex-wise split epimorphism. Hence, letting $E := E^\lambda$, we have a short exact sequence

$$0 \longrightarrow B \longrightarrow E \longrightarrow X \longrightarrow 0\ .$$

To show that $B \in \mathrm{Rep}(Q, \mathcal{B})$, we let $B^\alpha := \mathrm{Ker}(E^\alpha \twoheadrightarrow E^1 = X)$. For any ordinals $\alpha \leq \beta \leq \lambda$ there exists a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & B^\beta \arrow[r] \arrow[d, "g_{\alpha\beta}"'] & E^\beta \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & B^\alpha \arrow[r] & E^\alpha \arrow[r] & X \arrow[r] & 0.
\end{tikzcd}$$

Using pullback arguments, the morphism $g_{\alpha\beta}:\ B^\beta \to B^\alpha$ is a vertex-wise split epimorphism, as well. In other words, $\{g_{\alpha\beta}\}_{\alpha \leq \beta \leq \lambda}$ is a continuous inverse λ-sequence with $B^\lambda = B$. Note that for an ordinal $\alpha < \lambda$

$$\mathrm{Ker}\ g_{\alpha,\alpha+1} = \mathrm{Ker}\ (E^{\alpha+1} \to E^\alpha) \in \mathrm{Rep}(Q, \mathcal{B}).$$

It implies that for every $i \in Q_0$, $B(i)$ has a $\mathcal{B}$-cofiltration, therefore, $B \in \mathrm{Rep}(Q, \mathcal{B})$. $\square$

4.4. Consider the quiver Q given in (4.2). Let $E \in \Phi(\mathcal{C})$ be a $\mathcal{C}$-valued representation of Q. We consider the following associated representations E^1, E^2 and E^3, respectively,

$$\begin{tikzcd}
E(1) \arrow[dr] & & \\
& 0 \arrow[r] & 0, \\
E(2) \arrow[ur] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
E(1) \arrow[dr, hook, "E(a)"] & & \\
& E(3) \arrow[r] & 0, \\
E(2) \arrow[ur, hook, "E(b)"'] & &
\end{tikzcd}
\qquad
E$$

Assume that $(\mathcal{A}, \mathcal{B})$ is a complete hereditary cotorsion pair in $\mathcal{C}$. So we fix the following special $\mathcal{A}$-precovers

$$0 \longrightarrow B_1 \longrightarrow A_1 \longrightarrow E(1) \longrightarrow 0$$

$$0 \longrightarrow B_2 \longrightarrow A_2 \longrightarrow E(2) \longrightarrow 0$$

$$0 \longrightarrow B_3' \longrightarrow A_3' \longrightarrow \mathrm{Coker}\ \varphi_3^E \longrightarrow 0$$

$$0 \longrightarrow B_4' \longrightarrow A_4' \longrightarrow \mathrm{Coker}\ \varphi_4^E \longrightarrow 0.$$

We let $A^1 := s_1(A_1) \oplus s_2(A_2)$ and $B^1 := s_1(B_1) \oplus s_2(B_2)$. So there is a canonical short exact sequence $\mathbb{E}^1:\ 0 \to B^1 \to A^1 \to E^1 \to 0$. Using Lemma 3.5, we have the following commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & B_1 \oplus B_2 \arrow[r, "g"] \arrow[d] & B_3 \arrow[r] \arrow[d] & B'_3 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A_1 \oplus A_2 \arrow[r, "f"] \arrow[d] & A_3 \arrow[r] \arrow[d] & A'_3 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & E(1) \oplus E(2) \arrow[r, "\varphi_3^E"] \arrow[d] & E(3) \arrow[r] \arrow[d] & \text{Coker } \varphi_3^E \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

whose middle row is a special $\mathcal{A}$-precover of $E(3)$. Again, we let A^2 and B^2 be the following representations

$$\begin{tikzcd}
A_1 \arrow[dr, "f \circ \iota"] & & \\
 & A_3 \arrow[r] & 0, \\
A_2 \arrow[ur, "f \circ \iota"'] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
B_1 \arrow[dr, "g \circ \iota"] & & \\
 & B_3 \arrow[r] & 0. \\
B_2 \arrow[ur, "g \circ \iota"'] & &
\end{tikzcd}$$

Clearly, $\varphi_3^{A^2} = f$ is a monomorphism with cokernel $A'_3 \in \mathcal{A}$, and $B^2 \in \operatorname{Rep}(Q, \mathcal{B}) \subseteq \Phi(\mathcal{B})$. Besides, there exists the canonical short exact sequence $\mathbb{E}^2 : \ 0 \to B^2 \to A^2 \to E^2 \to 0$ in $\operatorname{Rep}(Q, \mathcal{C})$. Repeating the previous argument at the vertex 4 by using short exact sequences $0 \to B_3 \to A_3 \to E(3) \to 0$ and $0 \to B'_4 \to A'_4 \to \text{Coker } \varphi_4^E \to 0$, we obtain a short exact sequence

$$0 \longrightarrow B^3 \longrightarrow A^3 \longrightarrow E^3 = E \longrightarrow 0$$

where $A := A^3$ and $B := B^3$ are of the form

$$\begin{tikzcd}
A_1 \arrow[dr, "f \circ \iota"] & & \\
 & A_3 \arrow[r, hook] & A_4, \\
A_2 \arrow[ur, "f \circ \iota"'] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
B_1 \arrow[dr, "g \circ \iota"] & & \\
 & B_3 \arrow[r, hook] & B_4, \\
B_2 \arrow[ur, "g \circ \iota"'] & &
\end{tikzcd}$$

respectively, and $A \in \Phi(\mathcal{A})$ and $B \in \operatorname{Rep}(Q, \mathcal{B}) \subseteq \Phi(\mathcal{B})$. This method and its dual can be generalized to any (right) left rooted quiver as follows.

4.5. Lemma. Assume that $(\mathcal{A}, \mathcal{B})$ is a complete hereditary cotorsion pair in $\mathcal{C}$.

(1) If Q is left rooted and $\mathcal{C}$ has exact $|Q_1^{* \to i}|$-indexed coproducts for every $i \in Q_0$, then any $E \in \Phi(\mathcal{C})$ has a special $\Phi(\mathcal{A})$-precover.

(2) If Q is right rooted and $\mathcal{C}$ has exact $|Q_1^{i\to *}|$-indexed products for every $i \in Q_0$, then any $E' \in \Psi(\mathcal{C})$ has a special $\Phi(\mathcal{B})$-preenvelope.

Proof. We just prove the first statement because of the duality of arguments.

Let $\{V_\alpha\}_{\alpha\leq\lambda}$ be the λ-transfinite sequence of vertices given in (2.4) with $V_\lambda = Q_0$. By transfinite induction on ordinals $\alpha \leq \lambda$, we will define a cocontinuous inverse λ-sequence of short exact sequences $\mathbb{E}^\alpha : 0 \to B^\alpha \to A^\alpha \to E^\alpha \to 0$ in $\mathrm{Rep}(Q, \mathcal{C})$, which satisfies

(i) $E^\lambda = E$
(ii) If $\alpha < \alpha' \leq \lambda$ and $i \notin V_{\alpha'} \backslash V_\alpha$, then $\mathbb{E}^\alpha(i) = \mathbb{E}^{\alpha'}(i)$.
(iii) For every ordinal $\alpha \leq \lambda$, $B^\alpha \in \Phi(\mathcal{B})$.
(iv) For every ordinal $\alpha \leq \lambda$ and $i \in V_\alpha$, $\varphi_i^{A^\alpha}$ is a monomorphism with Coker $\varphi_i^{A^\alpha} \in \mathcal{A}$.

Firstly, for every $i \in Q_0$ we fix a special $\mathcal{A}$-precover $0 \to B_i' \to A_i' \to \mathrm{Coker}\ \varphi_i^E \to 0$.

We let E^α be the representation

$$E^\alpha(i) := \begin{cases} E(i), & i \in V_\alpha; \\ 0, & \text{otherwise.} \end{cases} \quad , \quad E^\alpha(a) := \begin{cases} E(a), & i, j \in V_\alpha; \\ 0, & \text{otherwise,} \end{cases}$$

where $a : i \to j \in Q_1$. Clearly $E^0 = 0$ and $E^\lambda = E$. So we let $\mathbb{E}^0$ be a short exact sequence of zero representations. Suppose that we have $\mathbb{E}^\alpha$ for some ordinal $\alpha < \lambda$. We now construct $\mathbb{E}^{\alpha+1}$ as follows:

Let i be a vertex in $V_{\alpha+1}\backslash V_\alpha$. Essentially, we want to use Lemma 3.5, but the set $Q_1^{*\to i}$ may be infinite, so the coproduct $\bigoplus_{Q_1^{*\to i}} B^\alpha(s(a))$ may fail to belong to $\mathcal{B}$. Therefore we consider a special $\mathcal{B}$-preenvelope of $\bigoplus_{Q_1^{*\to i}} B^\alpha(s(a))$

$$0 \longrightarrow \bigoplus_{Q_1^{*\to i}} B^\alpha(s(a)) \longrightarrow B_i^\alpha \longrightarrow A_i^\alpha \longrightarrow 0.$$

Applying pushout arguments, we have the following commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \bigoplus_{Q_1^{*\to i}} B^\alpha(s(a)) \arrow[r] \arrow[d] & \bigoplus_{Q_1^{*\to i}} A^\alpha(s(a)) \arrow[r] \arrow[d] & \bigoplus_{Q_1^{*\to i}} E^\alpha(s(a)) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & B_i^\alpha \arrow[r] \arrow[d] & \overline{A}_i^\alpha \arrow[r] \arrow[d] & \bigoplus_{Q_1^{*\to i}} E^\alpha(s(a)) \arrow[r] & 0 \\
0 \arrow[r] & A_i^\alpha \arrow[r, equal] \arrow[d] & A_i^\alpha \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd} \tag{4.3}$$

Since the class $\mathcal{A}$ is closed under coproducts and extensions, $\overline{A}_i^\alpha \in \mathcal{A}$. Note that the first row is exact because $\mathcal{C}$ has exact $|Q_1^{*\to i}|$-indexed coproducts for every $i \in Q_0$. Applying Lemma 3.5, we have the following commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d, dashed] & 0 \arrow[d] & \\
0 \arrow[r, dashed] & B_i^{\alpha} \arrow[r, dashed] \arrow[d] & B_i \arrow[r, dashed] \arrow[d, dashed] & B_i' \arrow[r, dashed] \arrow[d] & 0 \\
0 \arrow[r, dashed] & \overline{A}_i^{\alpha} \arrow[r, dashed] \arrow[d] & A_i \arrow[r, dashed] \arrow[d, dashed] & A_i' \arrow[r, dashed] \arrow[d] & 0 \\
0 \arrow[r] & \oplus_{a\in Q_1^{*\to i}} E^{\alpha}(s(a)) \arrow[r, "\varphi_i^E"] \arrow[d] & E(i) \arrow[r] \arrow[d, dashed] & \text{Coker } \varphi_i^E \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{4.4}$$

in which all three columns are special $\mathcal{A}$-precovers. We set $\mathbb{E}^{\alpha+1}(i) := 0 \to B_i \to A_i \to E(i) \to 0$ if $i \in V_{\alpha+1}\backslash V_{\alpha}$, and $\mathbb{E}^{\alpha+1}(i) = \mathbb{E}^{\alpha}(i)$ if $i \notin V_{\alpha+1}\backslash V_{\alpha}$.

Let $a: \ j \to k \in Q_1$.

- If $k \in V_{\alpha+1}\backslash V_{\alpha}$, then $j \in V_{\alpha}$ and $B^{\alpha+1}(a)$ and $A^{\alpha+1}(a)$ are the canonical morphisms

$$\begin{aligned}
&B^{\alpha}(j) \hookrightarrow \bigoplus_{a\in Q_1^{*\to k}} B^{\alpha}(s(a)) \hookrightarrow B_k^{\alpha} \hookrightarrow B_k, \\
&A^{\alpha}(j) \hookrightarrow \bigoplus_{a\in Q_1^{*\to i}} A^{\alpha}(s(a)) \hookrightarrow \overline{A}_k^{\alpha} \hookrightarrow A_k.
\end{aligned} \tag{4.5}$$

- If $k \in V_{\alpha}$, then $j \in V_{\alpha}$, and $B^{\alpha+1}(a) = B^{\alpha}(a)$ and $A^{\alpha+1}(a) = A^{\alpha}(a)$.
- If $k \in Q_0\backslash V_{\alpha+1}$, then $B^{\alpha+1}(k) = B^{\alpha}(k) = 0$ and $A^{\alpha+1}(k) = A^{\alpha}(k) = 0$. So $B^{\alpha+1}(a) = A^{\alpha+1}(a) = 0$.

Clearly, $B^{\alpha+1} \in \mathrm{Rep}(Q,\mathcal{B}) \subseteq \Phi(\mathcal{B})$, the morphism $\varphi_i^{A^{\alpha+1}}$ is a monomorphism for every $i \in V_{\alpha+1}$. Just left to show that for every $i \in V_{\alpha+1}$, Coker $(\varphi_i^{A^{\alpha+1}}) \in \mathcal{A}$. Due to the construction, it suffices to show it for $i \in V_{\alpha+1}\backslash V_{\alpha}$. Using the diagrams given in (4.3) and (4.5), we have the following pullback

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \bigoplus_{Q_1^{*\to i}} A^{\alpha+1}(s(a)) \arrow[r] \arrow[d, equal] & \overline{A}_i^{\alpha} \arrow[r] \arrow[d] & A_i^{\alpha} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \bigoplus_{Q_1^{*\to i}} A^{\alpha+1}(s(a)) \arrow[r, "\varphi_i^{A^{\alpha+1}}"] & A_i \arrow[r] \arrow[d] & \text{Coker } (\varphi_i^{A^{\alpha+1}}) \arrow[r] \arrow[d] & 0 \\
 & & A_i' \arrow[r, equal] \arrow[d] & A_i' \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

So Coker $(\varphi_i^{A^{\alpha+1}}) \in \mathcal{A}$.

If $\beta \leq \lambda$ is a limit ordinal, then the inverse limit $\varprojlim_{\alpha<\beta} \mathbb{E}^\alpha$ of short exact sequences in $\mathrm{Rep}(Q, \mathcal{C})$ exists because at each vertex i in Q, the sequence remains constant after some steps. Let $\mathbb{E}^\beta := \varprojlim_{\alpha<\beta} \mathbb{E}^\alpha$. As argumented in the proof of Lemma 4.3, for every vertex $i \in V_\beta$, there is an ordinal $\alpha < \beta$ such that $i \in V_\alpha$ and $\mathbb{E}^\beta(i) := \mathbb{E}^\alpha(i)$, and if $i \notin V_\beta$, $\mathbb{E}^\beta(i) = 0$. So $\mathbb{E}^\beta$ is a short exact sequence which satisfies the desired conditions.

By transfinite induction, we have a short exact sequence

$$\mathbb{E}: \quad 0 \longrightarrow B \longrightarrow A \longrightarrow E \longrightarrow 0$$

where $A := A^\lambda \in \Phi(\mathcal{A})$ and $B := B^\lambda \in \mathrm{Rep}(Q, \mathcal{B}) \subseteq \Phi(\mathcal{A})^\perp$. $\square$

4.6. Theorem. Suppose that $(\mathcal{A}, \mathcal{B})$ is a complete hereditary cotorsion pair in $\mathcal{C}$.

(1) If Q is left rooted, and if $\mathcal{C}$ has exact $|Q_1^{*\to i}|$-coproducts for every vertex i in Q, then the cotorsion pair $(\Phi(\mathcal{A}), \Phi(\mathcal{A})^\perp)$ is complete in $\mathrm{Rep}(Q, \mathcal{C})$.
(2) If Q is right rooted, and if $\mathcal{C}$ has exact $|Q_1^{i\to *}|$-products for every vertex i in Q, then the cotorsion pair $({}^\perp\Psi(\mathcal{B}), \Psi(\mathcal{B}))$ is complete in $\mathrm{Rep}(Q, \mathcal{C})$.

Proof. Using Lemma 4.3, Lemma 4.5 and classical pullback-pushout arguments, one may easily see that the cotorsion pair $(\Phi(\mathcal{A}), \Phi(\mathcal{A})^\perp)$ has enough projectives, and the cotorsion pair $({}^\perp\Psi(\mathcal{B}), \Psi(\mathcal{B}))$ has enough injectives. By Lemma 3.3, Lemma 4.1 and Remark 3.12, they are complete, as well. $\square$

5. A not left nor right rooted quiver

In this section, we are interested in enlarging the class of quivers for which the induced cotorsion pairs $(\Phi(\mathcal{A}), \Phi(\mathcal{A})^\perp)$ and $({}^\perp\Psi(\mathcal{B}), \Psi(\mathcal{B}))$ are complete whenever the cotorsion pair $(\mathcal{A}, \mathcal{B})$ is so. It seems hard to give a complete answer. However, we are able to prove it for the infinite line quiver A_∞^∞.

5.1. Lemma. Let $\mathcal{C}$ be a (co)complete category. Consider a family $\{F^i : J \to \mathcal{C}\}_{i\in I}$ of functors over a small category J.

(1) If the category $\mathcal{C}$ has exact coproducts, then the canonical morphism

$$\bigoplus_{i\in I} \varprojlim_{j\in J} F^i \longrightarrow \varprojlim_{j\in J} \bigoplus_{i\in I} F^i \tag{5.1}$$

is a monomorphism.
(2) If the category $\mathcal{C}$ has exact products, then the canonical morphism

$$\varinjlim_{j\in J} \bigoplus_{i\in I} F^i \longrightarrow \bigoplus_{i\in I} \varprojlim_{j\in J} F^i \tag{5.2}$$

is an epimorphism.

Proof. By duality of arguments, we just prove the first statement (1). Note that for every $i \in I$, the limit $\varprojlim_{j\in J} F^i = \mathrm{Ker}\ \big(\prod_{j\in J} F^i(j) \xrightarrow{\eta^i} \prod_{\lambda:\ j\to j'} F^i(j') \big)$, see [13, Chapter IV, Proposition 8.2]. For every finite subset $S \subset I$, there exists the following commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \varprojlim_{j\in J} \bigoplus_{i\in S} F^i(j) \arrow[r] \arrow[d] & \prod_{j\in J} \bigoplus_{i\in S} F^i(j) \arrow[r] \arrow[d, hook] & \prod_{\lambda:\ j\to j'} \bigoplus_{i\in S} F^i(j') \arrow[d, hook] \\
0 \arrow[r] & \varprojlim_{j\in J} \bigoplus_{i\in I} F^i(j) \arrow[r] & \prod_{j\in J} \bigoplus_{i\in I} F^i(j) \arrow[r] & \prod_{\lambda:\ j\to j'} \bigoplus_{i\in I} F^i(j')
\end{tikzcd}$$

whose last two columns are monic because products preserve monomorphisms, and thus, the first column is monic, as well. Since the set S is finite, the coproduct in the first row can be taken out of the products and hence, it would be of the form

$$0 \longrightarrow \bigoplus_{i\in S} \varprojlim_{j\in J} F^i(j) \longrightarrow \bigoplus_{i\in S} \prod_{j\in J} F^i(j) \xrightarrow{\oplus\eta^i} \bigoplus_{i\in S} \prod_{j\in J} F^i(j').$$

Since coproducts preserve monomorphisms by assumption, the morphism given in (5.1) is a monomorphism. $\square$

The following lemma is a kind of generalization of Lemma 4.3, but one needs to impose certain conditions on the category $\mathcal{C}$, instead.

5.2. Lemma. Let Q be a quiver without loop. Let $\mathcal{C}$ be a (co)complete category, and $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair in $\mathcal{C}$. Consider a $\mathcal{C}$-valued representation of X.

(1) If $\mathcal{C}$ has enough $\mathcal{B}$-objects and exact coproducts, then there is a short exact sequence $0 \longrightarrow B \longrightarrow E \longrightarrow X \longrightarrow 0$ of $\mathcal{C}$-valued representations of Q with $E \in \Phi(\mathcal{C})$ and $B \in \Phi(\mathcal{A})^{\perp}$.
(2) If $\mathcal{C}$ has enough $\mathcal{A}$-objects and exact products, then there is a short exact sequence $0 \longrightarrow X \longrightarrow E' \longrightarrow A \longrightarrow 0$ of $\mathcal{C}$-valued representations of Q with $E' \in \Psi(\mathcal{C})$ and $A \in^{\perp} \Psi(\mathcal{B})$.

Proof. We can well order the set Q_0. Then there is a unique ordinal λ which is order isomorphic to Q_0. We rename vertices of Q by ordinals $\alpha < \lambda$. Notice that the set $\mathbb{N} \times (\lambda \cup \{\lambda\})$ with the lexicographic order is well-ordered, as well. By transfinite induction on $\mathbb{N} \times (\lambda \cup \{\lambda\})$, we will construct an inverse system $\{E^{(n,\alpha)}\}_{\substack{n\in\mathbb{Z} \\ \alpha\leq\lambda}}$ of C-valued representations of the quiver Q, which satisfy:

(i) For every $(n, \alpha) \leq (n, \alpha')$, the morphism $E^{(n,\alpha')} \to E^{(n,\alpha)}$ is a vertex-wise split epimorphism with $E^{(n,\alpha)}(\beta) = E^{(n,\alpha')}(\beta)$ for every ordinal $\beta \leq \alpha$ or $\alpha' < \beta < \lambda$, if possible.
(ii) For every (n, α) with $n > 0$ and $\alpha < \lambda$, $\varphi_\alpha^{E^{(n,\alpha)}}$ is a monomorphism.
(iii) If an ordinal $\alpha' < \lambda$ is the successor ordinal of α, then the morphism $E^{(n,\alpha')} \to E^{(n,\alpha)}$ has the kernel in $\Phi(\mathcal{A})^{\perp}$ for every $n \in \mathbb{N}$.

For every ordinal $\alpha \leq \lambda$, we let $E^{(0,\alpha)} := X$, and so $E^0 := \varprojlim_{\alpha<\beta} E^{(0,\alpha)} = X$.

Now we construct $E^{(1,0)}$ as follows:

Consider a monomorphism $\bigoplus_{a\in Q^{*\to 0}} X(s(a)) \hookrightarrow B^{(1,0)}$, where $B^{(1,0)} \in \mathcal{B}$. We define $E^{(1,0)}$ by

$$E^{(1,0)}(\alpha) := \begin{cases} X(\alpha), & \alpha \neq 0 \\ X(0) \oplus B^{(1,0)}, & \alpha = 0 \end{cases}$$

for every vertex $\alpha < \lambda$, and with the canonical morphisms between vertices as indicated in the proof of Lemma 4.3. By assumption, $0 \notin Q_1^{*\to 0}$, so the morphism $\varphi_0^{E^{(1,0)}}$ is a monomorphism. The morphism

$E^{(1,0)} \to X = E^{(0,\lambda)}$, which is defined by the canonical projection at each vertex, is a vertex-wise split epimorphism, and has kernel $s_0(B^{(1,0)})$. In the same manner, if $\alpha' = \alpha + 1 < \lambda$, the $\mathcal{C}$-valued representation $E^{(1,\alpha')}$ of Q is constructed through $E^{(1,\alpha)}$ and the vertex α'.

Now let $\beta < \lambda$ be a limit ordinal. Suppose that $E^{(1,\alpha)}$ is constructed for every ordinal $\alpha < \beta$. Then the inverse limit $\varprojlim_{\alpha<\beta} E^{(1,\alpha)}$ exists because, by the condition given in (i), for every vertex $\alpha' < \lambda$ the sequence $\{E^{(1,\alpha)}(\alpha')\}_{\alpha<\beta}$ is eventually constant. In fact, letting $T^{(1,\beta)} := \varprojlim_{\alpha<\beta} E^{(1,\alpha)}$, $T^{(1,\beta)}$ is of the form

$$T^{(1,\beta)}(\alpha') := \begin{cases} E^{(1,\alpha')}(\alpha'), & \alpha' < \beta \\ X(\alpha'), & \beta \leq \alpha' < \lambda. \end{cases} \tag{5.3}$$

Hence, there is a splitting epimorphism $T^{(1,\beta)} \to E^0$. In the same way as done above, we proceed the argument in order to define $E^{(1,\beta)}$ through $T^{(1,\beta)}$ and the vertex β.

Finally we let $E^1 := E^{(1,\lambda)} := \varprojlim_{\alpha<\lambda} E^{(1,\alpha)}$, which has a similar form given in (5.3), and the canonical vertex-wise split epimorphism $E^1 \to E^0$.

As an aside, $B^1 := \mathrm{Ker}(E^1 \to E^0) \in \Phi(\mathcal{A})^\perp$. Indeed, letting $B^{(1,\alpha)} := \mathrm{Ker}\ (E^{(1,\alpha)} \to E^0)$, for every $\alpha \leq \beta \leq \lambda$, there is a commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & B^{(1,\beta)} \arrow[r] \arrow[d, "g^1_{(\alpha\beta)}"'] & E^{(1,\beta)} \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & B^{(1,\alpha)} \arrow[r] & E^{(1,\alpha)} \arrow[r] & X \arrow[r] & 0. \end{tikzcd}$$

Using pullback arguments, one can easily verify that the morphism $g^1_{(\alpha\beta)} : B^{(1,\beta)} \to B^{(1,\alpha)}$ is a vertex-wise split epimorphism, as well. Therefore, $\{g^1_{(\alpha\beta)}\}_{\alpha\leq\beta\leq\lambda}$ is a cocontinuous inverse λ-sequence with $\varprojlim_{\alpha<\lambda} B^{(1,\alpha)} = B^{(1,\lambda)} = B^1$. Note that for a successor ordinal $\alpha' = \alpha + 1 < \lambda$

$$\mathrm{Ker}\ g^1_{(\alpha,\alpha')} = \mathrm{Ker}\ (E^{(1,\alpha')} \to E^{(1,\alpha)}) = s_{\alpha'}(B^{(1,\alpha')}) \in \Phi(\mathcal{A})^\perp.$$

In other words, $\{g^1_{(\alpha\beta)}\}_{\alpha\leq\beta\leq\lambda}$ is a $\Phi(\mathcal{A})^\perp$-cofiltration of B^1. By Eklof Lemma, $B^1 \in \Phi(\mathcal{A})^\perp$.

For a given integer $n > 0$, using $\mathcal{C}$-valued representation $E^{n-1} := E^{(n-1,\lambda)} = \varprojlim_{\alpha<\lambda} E^{(n-1,\alpha)}$, we repeat the previous argument to obtain the family $\{E^{(n,\alpha)}\}_{\alpha\leq\lambda}$, and finally a vertex-wise split epimorphism $E^n := E^{(n,\lambda)} = \varprojlim_{\alpha<\lambda} E^{(n,\alpha)} \to E^{n-1}$ with the kernel in $\Phi(\mathcal{A})^\perp$.

Finally, we have a cocontinuous inverse system

$$\ldots \to E^2 \to E^1 \to E^0 = X$$

whose morphisms between consecutive objects are vertex-wise splitting epimorphisms and $\mathrm{Ker}\ (E^{n+1} \to E^n) = B^{n+1} \in \Phi(\mathcal{A})^\perp$. We denote $E := \varprojlim_{n\in\mathbb{N}} E^n$. By [11, Lemma 66], there is a short exact sequence

$$0 \longrightarrow B \longrightarrow E \longrightarrow X \longrightarrow 0$$

where $B := \varprojlim_{n\in\mathbb{Z}} B^n$. As done before, B has a $\Phi(\mathcal{A})^\perp$-cofiltration $\{B^n\}_{n\in\mathbb{N}}$, therefore, $B \in \Phi(\mathcal{A})^\perp$.

Now we need to prove that $E \in \Phi(\mathcal{C})$. Let α be a vertex in Q, that is, an ordinal $\alpha < \lambda$. Since the set $\{(n,\alpha)\}_{n>0}$ is cofinal in $\mathbb{N} \times (\lambda \cup \{\lambda\})$,

$$E(\alpha) = \varprojlim_{n>0} E^{(n,\alpha)}(\alpha).$$

However, by the condition (ii), $\varphi_\alpha^{E^{(n,\alpha)}}$ is monic for every $n > 0$, and inverse limit preserves monomorphisms, so the morphism

$$\varprojlim_{n>0} \varphi_\alpha^{E^{(n,\alpha)}} : \varprojlim_{n>0} \bigoplus_{a \in Q_1^{*\to\alpha}} E^{(n,\alpha)}(s(a)) \to E(\alpha)$$

is a monomorphism, as well. Using the universal property of limits, the morphism φ_α^E can be factorized as

$$\bigoplus_{a \in Q_1^{*\to\alpha}} E(s(\alpha)) = \bigoplus_{a \in Q_1^{*\to\alpha}} \varprojlim_{n>0} E^{(n,\alpha)}(s(a)) \to \varprojlim_{n>0} \bigoplus_{a \in Q_1^{*\to\alpha}} E^{(n,\alpha)}(s(a)) \hookrightarrow E(\alpha)$$

with the canonical morphisms. By Lemma 5.1, the first morphism is monic, so is φ_α^E. □

5.3. Lemma. Let Q be the infinite line quiver

$$A_\infty^\infty : \quad \ldots \longrightarrow \bullet_1 \longrightarrow \bullet_0 \longrightarrow \bullet_{-1} \longrightarrow \bullet_{-2} \longrightarrow \ldots,$$

and $(\mathcal{A}, \mathcal{B})$ be a complete hereditary cotorsion pair in R-Mod.

(1) For any $E \in \Phi(R\text{-Mod}) \subseteq \mathrm{Rep}(Q, R)$, there is a short exact sequence

$$0 \longrightarrow B \longrightarrow A \longrightarrow E \longrightarrow 0$$

with $A \in \Phi(\mathcal{A})$ and $B \in \Phi(A)^\perp$.

(2) For any $E' \in \Psi(R\text{-Mod}) \subseteq \mathrm{Rep}(Q, R)$, there is a short exact sequence

$$0 \longrightarrow E' \longrightarrow B' \longrightarrow A' \longrightarrow 0$$

with $B' \in \Psi(\mathcal{B})$ and $A' \in {}^\perp \Psi(A)$.

Proof. We just prove (1), the second one is dual. Consider an object $E \in \Phi(\mathcal{C})$

$$\ldots \longrightarrow E_1 \hookrightarrow E_0 \hookrightarrow E_{-1} \hookrightarrow E_{-2} \hookrightarrow \ldots.$$

For every vertex i in Q, we fix a special $\mathcal{A}$-precover $\mathbb{E}_i : \ 0 \to B_i \to A_i \to c_i(X) \to 0$. Taking the pullback of the morphisms $E_0 \twoheadrightarrow c_0(E)$ and $A_0 \twoheadrightarrow c_0(E)$, and using Lemma 3.5, we have the following short exact sequence $0 \to B^0 \to A^0 \to E \to 0$ of R-module valued representations of Q

$$\begin{tikzcd}
B^0 : \ \cdots \arrow[r] & 0 \arrow[r] \arrow[dd] & 0 \arrow[r] \arrow[dd] & B^0_0 \arrow[rr, hook] \arrow[dd, hook] \arrow[dr, equal] & & B^0_{-1} \arrow[rr, hook] \arrow[dd, hook] \arrow[dr, two heads] & & B^0_{-2} \arrow[r, hook] \arrow[dd, hook] \arrow[dr, two heads] & & \cdots \\
& & & & B^0 \arrow[dd, hook] & & B_{-1} \arrow[dd, hook] & & B_{-2} \arrow[dd, hook] & \\
A^0 : \ \cdots \arrow[r] & E_2 \arrow[r, hook] \arrow[dd, equal] & E_1 \arrow[r, hook] \arrow[dd, equal] & T^0_0 \arrow[rr, hook] \arrow[dd, two heads] \arrow[dr, two heads] & & T^0_{-1} \arrow[rr, hook] \arrow[dd, two heads] \arrow[dr, two heads] & & T^0_{-2} \arrow[r, hook] \arrow[dd, two heads] \arrow[dr, two heads] & & \cdots \\
& & & & A_0 \arrow[dd, two heads] & & A_{-1} \arrow[dd, two heads] & & A_{-2} \arrow[dd, two heads] & \\
E : \ \cdots \arrow[r] & E_2 \arrow[r, hook] & E_1 \arrow[r, hook] & E_0 \arrow[rr, hook] \arrow[dr, two heads] & & E_{-1} \arrow[rr, hook] \arrow[dr, two heads] & & E_{-2} \arrow[r, hook] \arrow[dr, two heads] & & \cdots \\
& & & & c_0(E) & & c_{-1}(E) & & c_{-2}(E) &
\end{tikzcd}$$

As seen easily above, applying the functor c_j, $j \leq 0$, given in (2.15), to the previous short exact sequence, one gets exactly $\mathbb{E}_j$.

In the same manner, for any integer $i \geq 0$, we obtain a short exact sequence in $\mathrm{Rep}(Q, R)$

$$0 \to B^i \to A^i \to E \to 0$$

with $B^i, A^i \in \Phi(R\text{-Mod})$, and the induced short exact sequence $0 \to c_j(B^i) \to c_j(A^i) \to c_j(E) \to 0$ is the same as $\mathbb{E}_j$, for $j \leq i$, and $B^i(j) = 0$, $j > i$. One can easily show that B^j has a $s_*(\mathcal{B})$-cofiltration, so it belongs to $\Phi(\mathcal{B})^\perp$.

We claim that for every $i \geq 0$, the morphism $A^{i+1} \twoheadrightarrow A^i$ has a factorization of the form

$$\begin{tikzcd} A^{i+1} \arrow[r, two heads] \arrow[dr, two heads] & A^i \arrow[d, two heads] \\ & E \end{tikzcd}$$

such that $\mathrm{Ker}(A^{i+1} \twoheadrightarrow A^i) = f_{i+1}(B_{i+1}) \in \Phi(\mathcal{B})^\perp$. We will show it for $i = 0$. Firstly, if $j \geq 2$, $A^1(j) = A^0(j) = E_j$.

If $j = 1$, then, by the construction, there is an epimorphism $A^1(1) = T_1^1 \twoheadrightarrow E_1 = A^0(0)$ with the kernel B_1.

Now suppose that there is a vertex $j \geq 0$ such that this factorization exists for the left part $\geq j$. Since $(\mathcal{A}, \mathcal{B})$ is hereditary, we have a commutative diagram

$$\begin{tikzcd} \mathrm{Ext}^1(A_{j-1}, T_j^1) \arrow[r, "\cong"] \arrow[d, equal] & \mathrm{Ext}^1(A_{j-1}, T_j^0) \arrow[d, "\cong"] \\ \mathrm{Ext}^1(A_{j-1}, T_j^1) \arrow[r, "\cong"'] & \mathrm{Ext}^1(A_{j-1}, E_j). \end{tikzcd}$$

Taking the pullback of the morphism $A_{j-1} \twoheadrightarrow c_{j-1}(E)$ and $E_{j-1} \twoheadrightarrow c_{j-1}(E)$, we obtain a short exact sequence $\mathbb{F} \in \mathrm{Ext}^1(A_{j-1}, E_j)$. Using arguments in the proof of Lemma 3.5 and from the previous commutative diagram, the short exact sequence $0 \to T_j^0 \to T_{j-1}^0 \to A_{j-1} \to 0$ is the image of the short exact sequence $0 \to T_j^1 \to T_{j-1}^1 \to A_{j-1} \to 0$, that is, pushout along the morphism $T_j^1 \twoheadrightarrow T_j^0$. As a result, there is a morphism $T_{j-1}^1 \twoheadrightarrow T_{j-1}^0$ which fits in the commutative diagram and has the kernel B_1. It proves our claim.

Finally, we have an inverse system of short exact sequences of R-module valued representations of the quiver Q

$$\begin{tikzcd} 0 \arrow[r] & B^{i+1} \arrow[r] \arrow[d] & A^{i+1} \arrow[r] \arrow[d] & E \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & B^i \arrow[r] & A^i \arrow[r] & E \arrow[r] & 0, \end{tikzcd}$$

with $i \geq 0$. Taking the limit of this system, finally we have a short exact sequence $0 \to B \to A \to E \to 0$. Since $\mathrm{Ker}\ (B^{i+1} \to B^i) = f_{i+1}(B_{i+1}) \in \Phi(\mathcal{B})^\perp$, by Eklof Lemma, $B \in \Phi(\mathcal{B})^\perp$. Besides $A \in \Phi(R\text{-Mod})$ because inverse limits preserve monomorphisms. Note that $c_j(A^i) = A_j$ for every $j \leq i$, therefore, $c_j(A) = A_j$ for every integer j. As a consequence $A \in \Phi(\mathcal{A})$. $\square$

5.4. Remark. The previous lemma can be stated over a (co)complete category $\mathcal{C}$ in which for every cocontinuous inverse system over$\{C_n\}_{n \in \mathbb{N}}$, the morphism $\varprojlim_{n \in \mathbb{N}} C_n \to C_1$ is an epimorphism.

5.5. The author thinks that the proof of the previous lemma can be generalized to any quiver without loops and finite arrows between any two vertices by proceeding carefully a transfinite induction.

5.6. Theorem. Under the setup as in Lemma 5.3, the cotorsion pairs $(\Phi(\mathcal{A}), \Phi(\mathcal{A})^{\perp})$ and $({}^{\perp}\Psi(\mathcal{B}), \Psi(\mathcal{B}))$ are complete.

Proof. By Lemma 5.2, Lemma 5.3 and pullback-pushout arguments, the cotorsion pairs $(\Phi(\mathcal{A}), \Phi(\mathcal{A})^{\perp})$ and $({}^{\perp}\Psi(\mathcal{B}), \Psi(\mathcal{B}))$ have enough projectives and injectives, respectively. Note also that the category $\mathrm{Rep}(Q, R)$ has enough $\Phi(\mathcal{A})$-objects and $\Psi(\mathcal{B})$-objects. In fact, for every $X \in \mathrm{Rep}(Q, R)$ we have

$$X \hookrightarrow \prod_{i \in Q_0} f_i(B_i) \ , \quad \text{and} \quad \bigoplus_{i \in Q_0} g_i(A_i) \twoheadrightarrow X,$$

where for every $i \in Q$, $X(i) \hookrightarrow B_i$ is a monomorphism with $B_i \in \mathcal{B}$, and $A_i \twoheadrightarrow X(i)$ is an epimorphism with $A_i \in \mathcal{A}$. Then Lemma 3.3 can be applied. $\square$

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