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Classical limit of quantum Borcherds-Bozec algebras [☆]

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A R T I C L E I N F O

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A B S T R A C T

Let $\mathfrak{g}$ be a Borcherds-Bozec algebra, $U(\mathfrak{g})$ be its universal enveloping algebra and $U_q(\mathfrak{g})$ be the corresponding quantum Borcherds-Bozec algebra. We show that the classical limit of $U_q(\mathfrak{g})$ is isomorphic to $U(\mathfrak{g})$ as Hopf algebras. Thus, $U_q(\mathfrak{g})$ can be regarded as a quantum deformation of $U(\mathfrak{g})$. We also provide explicit formulas for the commutation relations among the generators of $U_q(\mathfrak{g})$.

0. Introduction

The *quantum Borcherds-Bozec algebras* were introduced by Bozec in his research of perverse sheaves theory for quivers with loops [1–3]. They can be treated as a further generalization of quantum generalized Kac-Moody algebras. Even though they use the same Borcherds-Cartan data, the constructions of the quantum groups are quite different.

More precisely, the quantum Borcherds-Bozec algebras have more generators and defining relations than quantum generalized Kac-Moody algebras. For each simple root α_i with imaginary index, there are infinitely many generators e_{il}, f_{il} $(l \in \mathbb{Z}_{>0})$ whose degrees are $l\alpha_i$ and $-l\alpha_i$, respectively. Bozec deals with these generators by treating them as similar positions as divided powers $\theta_i^{(l)}$ in Lusztig algebras.

Geometric approach to the half parts of quantum groups can be traced back to Lusztig's work [15]. In [16], Lusztig constructed the canonical bases for the half parts of the quantized enveloping algebras of Kac-

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Moody algebras. The canonical bases correspond to certain simple perverse sheaves on representation space of quivers without loops. By allowing quivers to have edge loops, in [9], Kang and Schiffmann generalized Lusztig's work to quantum generalized Kac-Moody algebras, while in this case, the canonical bases correspond to certain semisimple perverse sheaves rather than simple perverse sheaves. (See also [12].) Motivated by the results for quivers with one vertex and multiple loops in [17], Bozec considered the general definition of Lusztig sheaves for arbitrary quivers, possibly carrying loops, and constructed the canonical bases for the half parts of quantum Borcherds-Bozec algebras in terms of simple perverse sheaves [1,2].

In [2], Bozec studied the crystal basis theory for quantum Borcherds-Bozec algebras. He defined the notion of Kashiwara operators and generalized crystals. He also proved several critical results, which provides an important framework for Kashiwara's grand-loop argument (cf. [10]). He also provided a geometric construction of the crystal for the negative half of quantum Borcherds-Bozec algebras based on the theory of perverse sheaves associated to quivers with loops (cf. [7,11]), and a geometric realization of generalized crystals for the integrable highest weight representations *via* Nakajima's quiver varieties (cf. [8,18]).

For a Kac-Moody algebra $\mathfrak{g}$, Lusztig showed that the integrable highest weight module $\overline{L}$ over $U(\mathfrak{g})$ can be deformed to those integrable highest weight module L over $U_q(\mathfrak{g})$ in such a way that the dimensions of weight spaces are invariant under the deformation (cf. [14]). Let $\mathscr{A} = \mathbb{Q}[q, q^{-1}]$ be the Laurent polynomial rings. Lusztig constructed a $\mathscr{A}$-subalgebra $U_{\mathscr{A}}$ of $U_q(\mathfrak{g})$ generated by divided powers and $k_i^{\pm}$, and defined a $U_{\mathscr{A}}$-submodule $L_{\mathscr{A}}$ of L. He proved that $F_0 \otimes_{\mathscr{A}} L_{\mathscr{A}}$ is isomorphic to $\overline{L}$ as $U(\mathfrak{g})$-modules, where $F_0 = \mathscr{A}/I$ and I is the ideal of $\mathscr{A}$ generated by $(q-1)$.

In [4, Chapter 3], Hong and Kang modified Lusztig's approach, and showed that the $U_q(\mathfrak{g})$ is a deformation of $U(\mathfrak{g})$ as a Hopf algebra. Moreover, a highest weight $U(\mathfrak{g})$-module admits a deformation to a highest weight $U_q(\mathfrak{g})$-module. They used the $\mathbb{A}_1$-form of $U_q(\mathfrak{g})$ and highest weight $U_q(\mathfrak{g})$-module, where $\mathbb{A}_1$ is the localization of $\mathbb{Q}[q]$ at the ideal $(q-1)$. We can see that $\mathscr{A} = \mathbb{Q}[q, q^{-1}] \subseteq \mathbb{A}_1$.

In this paper, we study the classical limit theory of quantum Borcherds-Bozec algebras. We first review some basic notions of Borcherds-Bozec algebras and quantum Borcherds-Bozec algebras. For their representation theory, the readers may refer to [5,6]. As we show in Appendix, the commutation relations between e_{il} and f_{jk} are rather complicated. For the aim of classical limit, we need another set of generators. Thanks to Bozec, there exists an alternative set of primitive generators in $U_q(\mathfrak{g})$, denoted by s_{il} and t_{il}, which satisfy a simpler set of commutation relations

$$s_{il}t_{jk} - t_{jk}s_{il} = \delta_{ij}\delta_{lk}\tau_{il}(K_i^l - K_i^{-l})$$

for some constants $\tau_{il} \in \mathbb{Q}(q)$. Using Lusztig's approach, we prove that these generators also satisfy the Serre-type relations (cf. [13, Chapter 1]).

In Section 3, we define the $\mathbb{A}_1$-form of quantum Borcherds-Bozec algebras and their highest weight representations. We show that the triangular decomposition of $U_q(\mathfrak{g})$ carries over to $\mathbb{A}_1$-form. In Section 4, we study the process of taking the limit $q \to 1$. Let $U_1 = \mathbb{Q} \otimes_{\mathbb{A}_1} U_{\mathbb{A}_1}$ be a $\mathbb{Q}$-algebra, where $U_{\mathbb{A}_1}$ is the $\mathbb{A}_1$-form of $U_q(\mathfrak{g})$. We prove that the classical limit U_1 of $U_q(\mathfrak{g})$ is isomorphic to the universal enveloping algebra $U(\mathfrak{g})$ as Hopf algebras, and when we take the classical limit, the Verma module and highest weight modules of $U_q(\mathfrak{g})$ *tend to* those Verma module and highest weight modules of $U(\mathfrak{g})$, respectively. Finally, we provide the concrete commutation relations between the generators e_{il} and f_{jk} of $U_q(\mathfrak{g})$ in Appendix. They have an interesting combinatorial structure.

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1. Borcherds-Bozec algebras

Let I be an index set possibly countably infinite. An integer-valued matrix $A = (a_{ij})_{i,j \in I}$ is called an *even symmetrizable Borcherds-Cartan matrix* if it satisfies the following conditions:

(i) $a_{ii} = 2, 0, -2, -4, \ldots$,
(ii) $a_{ij} \in \mathbb{Z}_{\leq 0}$ for $i \neq j$,
(iii) there is a diagonal matrix $D = \mathrm{diag}(r_i \in \mathbb{Z}_{>0} \mid i \in I)$ such that DA is symmetric.

Set $I^{\mathrm{re}} := \{i \in I \mid a_{ii} = 2\}$, the set of *real indices* and $I^{\mathrm{im}} := \{i \in I \mid a_{ii} \leq 0\}$, the set of *imaginary indices*. We denote by $I^{\mathrm{iso}} := \{i \in I \mid a_{ii} = 0\}$ the set of *isotropic indices*.

A *Borcherds-Cartan datum* consists of

(a) an even symmetrizable Borcherds-Cartan matrix $A = (a_{ij})_{i,j \in I}$,
(b) a free abelian group $P^\vee = \left(\bigoplus_{i \in I} \mathbb{Z} h_i\right) \oplus \left(\bigoplus_{i \in I} \mathbb{Z} d_i\right)$, the *dual weight lattice*,
(c) $\mathfrak{h} = \mathbb{Q} \otimes_{\mathbb{Z}} P^\vee$, the *Cartan subalgebra*,
(d) $P = \{\lambda \in \mathfrak{h}^* \mid \lambda(P^\vee) \subseteq \mathbb{Z}\}$, the *weight lattice*,
(e) $\Pi^\vee = \{h_i \in P^\vee \mid i \in I\}$, the set of *simple coroots*,
(f) $\Pi = \{\alpha_i \in P \mid i \in I\}$, the set of *simple roots*, which is linearly independent over $\mathbb{Q}$ and satisfies

$$\alpha_j(h_i) = a_{ij}, \ \alpha_j(d_i) = \delta_{ij} \ \text{ for all } i, j \in I,$$

(g) for each $i \in I$, there is an element $\Lambda_i \in P$ such that

$$\Lambda_i(h_j) = \delta_{ij}, \ \Lambda_i(d_j) = 0 \ \text{ for all } i, j \in I.$$

The $\Lambda_i (i \in I)$ are called the *fundamental weights*.

We denote by

$$P^+ := \{\lambda \in P \mid \lambda(h_i) \geq 0 \ \text{ for all } i \in I\}$$

the set of *dominant integral weights*. The free abelian group $Q := \bigoplus_{i \in I} \mathbb{Z}\alpha_i$ is called the *root lattice*. Set $Q_+ = \sum_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i$ and $Q_- = -Q_+$. For $\beta = \sum k_i \alpha_i \in Q_+$, we define its *hight* to be $\mathrm{ht}(\beta) := \sum k_i$.

There is a non-degenerate symmetric bilinear form $(\ , \ )$ on $\mathfrak{h}^*$ satisfying

$$(\alpha_i, \lambda) = r_i \lambda(h_i) \ \text{ for all } \lambda \in \mathfrak{h}^*,$$

and therefore we have

$$(\alpha_i, \alpha_j) = r_i a_{ij} = r_j a_{ji} \ \text{ for all } i, j \in I.$$

For $i \in I^{\mathrm{re}}$, we define the *simple reflection* $\omega_i \in GL(\mathfrak{h}^*)$ by

$$\omega_i(\lambda) = \lambda - \lambda(h_i)\alpha_i \ \text{ for } \lambda \in \mathfrak{h}^*.$$

The subgroup W of $GL(\mathfrak{h}^*)$ generated by ω_i $(i \in I^{\mathrm{re}})$ is called the *Weyl group* of $\mathfrak{g}$. One can easily verify that the symmetric bilinear form $(\ , \ )$ is W-invariant.

Let $I^\infty := (I^{\mathrm{re}} \times \{1\}) \cup (I^{\mathrm{im}} \times \mathbb{Z}_{>0})$. For simplicity, we will often write i for $(i, 1)$ if $i \in I^{\mathrm{re}}$.

Definition 1.1. The *Borcherds-Bozec algebra* $\mathfrak{g}$ associated with a Borcherds-Cartan datum $(A, P, \Pi, P^\vee, \Pi^\vee)$ is the Lie algebra over $\mathbb{Q}$ generated by the elements e_{il}, f_{il} $((i,l) \in I^\infty)$ and $\mathfrak{h}$ with defining relations

$$\begin{aligned} &[h, h'] = 0 \ \text{ for } h, h' \in \mathfrak{h}, \\ &[e_{ik}, f_{jl}] = k\, \delta_{ij}\, \delta_{kl}\, h_i \ \text{ for } i, j \in I, k, l \in \mathbb{Z}_{>0}, \\ &[h, e_{jl}] = l\alpha_j(h) e_{jl}, \quad [h, f_{jl}] = -l\alpha_j(h) f_{jl}, \\ &(\mathrm{ad} e_i)^{1-la_{ij}}(e_{jl}) = 0 \ \text{ for } i \in I^{\mathrm{re}},\ i \neq (j,l), \\ &(\mathrm{ad} f_i)^{1-la_{ij}}(f_{jl}) = 0 \ \text{ for } i \in I^{\mathrm{re}},\ i \neq (j,l), \\ &[e_{ik}, e_{jl}] = [f_{ik}, f_{jl}] = 0 \ \text{ for } a_{ij} = 0. \end{aligned} \tag{1.1}$$

Let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$. Since we have the following equations in $U(\mathfrak{g})$

$$(\mathrm{ad} x)^m(y) = \sum_{k=0}^{m} (-1)^k \binom{m}{k} x^{m-k} y x^k \ \text{ for } x, y \in U(\mathfrak{g}), m \in \mathbb{Z}_{\geq 0},$$

we obtain the presentation of $U(\mathfrak{g})$ with generators and relations given below.

Proposition 1.2. *The universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ is an associative algebra over $\mathbb{Q}$ with unity generated by e_{il}, f_{il} $((i,l) \in I^\infty)$ and $\mathfrak{h}$ subject to the following defining relations*

$$\begin{aligned} &hh' = h'h \ \textit{ for } h, h' \in \mathfrak{h}, \\ &e_{ik} f_{jl} - f_{jl} e_{ik} = k\, \delta_{ij}\, \delta_{kl}\, h_i \ \textit{ for } i, j \in I, k, l \in \mathbb{Z}_{>0}, \\ &h e_{jl} - e_{jl} h = l\alpha_j(h) e_{jl}, \quad h f_{jl} - f_{jl} h = -l\alpha_j(h) f_{jl}, \\ &\sum_{k=0}^{1-la_{ij}} (-1)^k \binom{1-la_{ij}}{k} e_i^{1-la_{ij}-k} e_{jl}\, e_i^k = 0 \ \textit{ for } i \in I^{re},\ i \neq (j,l), \\ &\sum_{k=0}^{1-la_{ij}} (-1)^k \binom{1-la_{ij}}{k} f_i^{1-la_{ij}-k} f_{jl}\, f_i^k = 0 \ \textit{ for } i \in I^{re},\ i \neq (j,l), \\ &e_{ik} e_{jl} - e_{jl} e_{ik} = f_{ik} f_{jl} - f_{jl} f_{ik} = 0 \ \textit{ for } a_{ij} = 0. \end{aligned} \tag{1.2}$$

The universal enveloping algebra $U(\mathfrak{g})$ has a Hopf algebra structure given by

$$\begin{aligned} &\Delta(x) = x \otimes 1 + 1 \otimes x, \\ &\varepsilon(x) = 0, \\ &S(x) = -x \ \text{ for } x \in \mathfrak{g}, \end{aligned} \tag{1.3}$$

where $\Delta : U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$ is the comultiplication, $\varepsilon : U(\mathfrak{g}) \to \mathbb{Q}$ is the counit, and $S : U(\mathfrak{g}) \to U(\mathfrak{g})$ is the antipode.

Furthermore, by the Poincaré-Brikhoff-Witt Theorem, the universal enveloping algebra also has the *triangular decomposition*

$$U(\mathfrak{g}) \cong U^-(\mathfrak{g}) \otimes U^0(\mathfrak{g}) \otimes U^+(\mathfrak{g}), \tag{1.4}$$

where $U^+(\mathfrak{g})$ (resp. $U^0(\mathfrak{g})$ and $U^-(\mathfrak{g})$) is the subalgebra of $U(\mathfrak{g})$ generated by the elements e_{il} (resp. $\mathfrak{h}$ and f_{il}) for $(i,l) \in I^\infty$.

In [5], Kang studied the representation theory of the Borcherds-Bozec algebras. We quote some results that we will use later.

Proposition 1.3. *[5]*

(a) *Let $\lambda \in P^+$ and $V(\lambda) = U(\mathfrak{g})v_\lambda$ be the irreducible highest weight $\mathfrak{g}$-module. Then we have*

$$\begin{aligned} &f_i^{\lambda(h_i)+1} v_\lambda = 0 \quad \text{for } i \in I^{re}, \\ &f_{il}\, v_\lambda = 0 \quad \text{for } (i,l) \in I^\infty \text{ with } \lambda(h_i) = 0. \end{aligned} \tag{1.5}$$

(b) *Every highest weight $\mathfrak{g}$-module with highest weight $\lambda \in P^+$ satisfying (1.5) is isomorphic to $V(\lambda)$.*

2. Quantum Borcherds-Bozec algebras

Let q be an indeterminate and set

$$q_i = q^{r_i}, \quad q_{(i)} = q^{\frac{(\alpha_i,\alpha_i)}{2}}.$$

Note that $q_i = q_{(i)}$ if $i \in I^{\mathrm{re}}$. For each $i \in I^{\mathrm{re}}$ and $n \in \mathbb{Z}_{\geq 0}$, we define

$$[n]_i = \frac{q_i^n - q_i^{-n}}{q_i - q_i^{-1}}, \quad [n]_i! = \prod_{k=1}^{n} [k]_i, \quad \begin{bmatrix} n \\ k \end{bmatrix}_i = \frac{[n]_i!}{[k]_i![n-k]_i!}$$

Let $\mathscr{F} = \mathbb{Q}(q)\langle f_{il} \mid (i,l) \in I^\infty \rangle$ be the free associative algebra over $\mathbb{Q}(q)$ generated by the symbols f_{il} for $(i,l) \in I^\infty$. By setting $\deg f_{il} = -l\alpha_i$, $\mathscr{F}$ becomes a Q_--graded algebra. For a homogeneous element u in $\mathscr{F}$, we denote by $|u|$ the degree of u, and for any $A \subseteq Q_-$, set $\mathscr{F}_A = \{x \in \mathscr{F} \mid |x| \in A\}$.

We define a *twisted* multiplication on $\mathscr{F} \otimes \mathscr{F}$ by

$$(x_1 \otimes x_2)(y_1 \otimes y_2) = q^{-(|x_2|,|y_1|)} x_1 y_1 \otimes x_2 y_2,$$

and equip $\mathscr{F}$ with a co-multiplication δ defined by

$$\delta(f_{il}) = \sum_{m+n=l} q_{(i)}^{-mn} f_{im} \otimes f_{in} \quad \text{for } (i,l) \in I^\infty.$$

Here, we understand $f_{i0} = 1$ and $f_{il} = 0$ if $l < 0$.

Proposition 2.1. *[1,2] For any family $\nu = (\nu_{il})_{(i,l)\in I^\infty}$ of non-zero elements in $\mathbb{Q}(q)$, there exists a symmetric bilinear form $(\ ,\ )_L : \mathscr{F} \times \mathscr{F} \to \mathbb{Q}(q)$ such that*

(a) $(x,y)_L = 0$ *if* $|x| \neq |y|$,
(b) $(1,1)_L = 1$,
(c) $(f_{il}, f_{il})_L = \nu_{il}$ *for all* $(i,l) \in I^\infty$,
(d) $(x, yz)_L = (\delta(x), y \otimes z)_L$ *for all* $x, y, z \in \mathscr{F}$.

Here, $(x_1 \otimes x_2, y_1 \otimes y_2)_L = (x_1,y_1)_L(x_2,y_2)_L$ *for any* $x_1, x_2, y_1, y_2 \in \mathscr{F}$.

From now on, we assume that

$$\nu_{il} \in 1 + q\mathbb{Z}_{\geq 0}[[q]] \quad \text{for all } (i,l) \in I^\infty. \tag{2.1}$$

Then, the bilinear form $(\, , \,)_L$ is non-degenerate on $\mathscr{F}(i) = \bigoplus_{l \geq 1} \mathscr{F}_{-l\alpha_i}$ for $i \in I^{\mathrm{im}} \backslash I^{\mathrm{iso}}$.

Let $\widehat{U}$ be the associative algebra over $\mathbb{Q}(q)$ with $\mathbf{1}$ generated by the elements q^h ($h \in P^\vee$) and e_{il}, f_{il} ($(i,l) \in I^\infty$) with defining relations

$$\begin{aligned}
&q^0 = \mathbf{1}, \quad q^h q^{h'} = q^{h+h'} \ \text{ for } h, h' \in P^\vee \\
&q^h e_{jl} q^{-h} = q^{l\alpha_j(h)} e_{jl}, \ \ q^h f_{jl} q^{-h} = q^{-l\alpha_j(h)} f_{jl} \ \text{ for } h \in P^\vee, (j,l) \in I^\infty, \\
&\sum_{k=0}^{1-la_{ij}} (-1)^k \begin{bmatrix} 1 - la_{ij} \\ k \end{bmatrix}_i e_i{}^{1-la_{ij}-k} e_{jl} e_i^k = 0 \ \text{ for } i \in I^{\mathrm{re}}, \, i \neq (j,l), \\
&\sum_{k=0}^{1-la_{ij}} (-1)^k \begin{bmatrix} 1 - la_{ij} \\ k \end{bmatrix}_i f_i{}^{1-la_{ij}-k} f_{jl} f_i^k = 0 \ \text{ for } i \in I^{\mathrm{re}}, \, i \neq (j,l), \\
&e_{ik} e_{jl} - e_{jl} e_{ik} = f_{ik} f_{jl} - f_{jl} f_{ik} = 0 \ \text{ for } a_{ij} = 0.
\end{aligned} \tag{2.2}$$

We extend the grading by setting $|q^h| = 0$ and $|e_{il}| = l\alpha_i$.

The algebra $\widehat{U}$ is endowed with the co-multiplication $\Delta : \widehat{U} \to \widehat{U} \otimes \widehat{U}$ given by

$$\begin{aligned}
&\Delta(q^h) = q^h \otimes q^h, \\
&\Delta(e_{il}) = \sum_{m+n=l} q_{(i)}^{mn} e_{im} \otimes K_i^{-m} e_{in}, \\
&\Delta(f_{il}) = \sum_{m+n=l} q_{(i)}^{-mn} f_{im} K_i^n \otimes f_{in},
\end{aligned} \tag{2.3}$$

where $K_i = q_i^{h_i}, i \in I$.

Let $\widehat{U}^{\leq 0}$ be the subalgebra of $\widehat{U}$ generated by f_{il} and q^h, for all $(i,l) \in I^\infty$ and $h \in P^\vee$, and $\widehat{U}^+$ be the subalgebra generated by e_{il} for all $(i,l) \in I^\infty$. In [1], Bozec showed that one can extended $(\, , \,)_L$ to a symmetric bilinear form $(\, , \,)_L$ on $\widehat{U}$ satisfying

$$\begin{aligned}
&(q^h, 1)_L = 1, \ (q^h, f_{il})_L = 0, \\
&(q^h, K_j)_L = q^{-\alpha_j(h)}, \\
&(x, y)_L = (\omega(x), \omega(y))_L \ \text{ for all } x, y \in \widehat{U}^+,
\end{aligned} \tag{2.4}$$

where $\omega : \widehat{U} \to \widehat{U}$ is the involution defined by

$$\omega(q^h) = q^{-h}, \ \omega(e_{il}) = f_{il}, \ \omega(f_{il}) = e_{il} \ \text{ for } h \in P^\vee, \ (i,l) \in I^\infty.$$

For any $x \in \widehat{U}$, we shall use the Sweedler's notation, and write

$$\Delta(x) = \sum x_{(1)} \otimes x_{(2)}.$$

Following the Drinfeld double process, we define $\tilde{U}$ as the quotient of $\widehat{U}$ by the relations

$$\sum (a_{(1)}, b_{(2)})_L \omega(b_{(1)}) a_{(2)} = \sum (a_{(2)}, b_{(1)})_L a_{(1)} \omega(b_{(2)}) \ \text{ for all } a, b \in \widehat{U}^{\leq 0}. \tag{2.5}$$

Definition 2.2. Given a Borcherds-Cartan datum $(A, P, \Pi, P^\vee, \Pi^\vee)$, the *quantum Borcherds-Bozec algebra* $U_q(\mathfrak{g})$ is defined to be the quotient algebra of $\tilde{U}$ by the radical of $(\, , \,)_L$ restricted to $\tilde{U}^- \times \tilde{U}^+$.

Let U^+ (resp. U^-) be the subalgebra of $U_q(\mathfrak{g})$ generated by e_{il} (resp. f_{il}) for all $(i,l) \in I^\infty$. We shall denote by U^0 the subalgebra of $U_q(\mathfrak{g})$ generated by q^h for all $h \in P^\vee$. It is easy to see that q^h $(h \in P^\vee)$ is a $\mathbb{Q}(q)$-basis of U^0.

In [6], Kang and Kim showed that the co-multiplication $\Delta : \widehat{U} \to \widehat{U} \otimes \widehat{U}$ passes down to $U_q(\mathfrak{g})$ and therefore, $U_q(\mathfrak{g})$ becomes a Hopf algebra. They also proved the quantum Borcherds-Bozec algebra has a *triangular decomposition.*

Theorem 2.3. *[6] The quantum Borcherds-Bozec algebra $U_q(\mathfrak{g})$ has the following triangular decomposition:*

$$U_q(\mathfrak{g}) \cong U^- \otimes U^0 \otimes U^+. \tag{2.6}$$

By the defining relation (2.5), we obtain complicated commutation relations between e_{il} and f_{jk} for $(i,l),(j,k) \in I^\infty$. We shall derive explicit formulas for these complicated commutation relations in Appendix A. But, as we already see in (1.2), the commutation relations in the universal enveloping algebra $U(\mathfrak{g})$ of Borcherds-Bozec algebra $\mathfrak{g}$ are rather simple

$$e_{ik}f_{jl} - f_{jl}e_{ik} = k\,\delta_{ij}\,\delta_{kl}\,h_i \quad \text{for } i,j \in I, k,l \in \mathbb{Z}_{>0}. \tag{2.7}$$

Thanks to Bozec, there exists another set of generators in $U_q(\mathfrak{g})$ called *primitive generators*. They satisfy a simpler set of commutation relations, and we shall prove that these generators also satisfy all the defining relations of $U_q(\mathfrak{g})$ described in (2.2).

We denote by $\mathcal{C}_l$ (resp. $\mathcal{P}_l$) the set of compositions (resp. partitions) of l, and denote by $\eta : U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$ the $\mathbb{Q}$-algebra homomorphism defined by

$$\eta(e_{il}) = e_{il},\ \eta(f_{il}) = f_{il},\ \eta(q^h) = q^{-h},\ \eta(q) = q^{-1} \quad \text{for } h \in P^\vee,\ (i,l) \in I^\infty. \tag{2.8}$$

As usual, let $S : U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$ and $\epsilon : U_q(\mathfrak{g}) \to \mathbb{Q}(q)$ be the *antipode* and the *counit* of $U_q(\mathfrak{g})$, respectively. Then, we have the following proposition.

Proposition 2.4. *[1,2] For any $i \in I^{im}$ and $l \geq 1$, there exist unique elements $t_{il} \in U^-_{-l\alpha_i}$ and $s_{il} = \omega(t_{il})$ such that*

(1) $\mathbb{Q}(q)\langle f_{il} \mid l \geq 1\rangle = \mathbb{Q}(q)\langle t_{il} \mid l \geq 1\rangle$ *and* $\mathbb{Q}(q)\langle e_{il} \mid l \geq 1\rangle = \mathbb{Q}(q)\langle s_{il} \mid l \geq 1\rangle$,
(2) $(t_{il}, z)_L = 0$ *for all* $z \in \mathbb{Q}(q)\langle f_{i1}, \cdots, f_{il-1}\rangle$, $(s_{il}, z)_L = 0$ *for all* $z \in \mathbb{Q}(q)\langle e_{i1}, \cdots, e_{il-1}\rangle$,
(3) $t_{il} - f_{il} \in \mathbb{Q}(q)\langle f_{ik} \mid k < l\rangle$ *and* $s_{il} - e_{il} \in \mathbb{Q}(q)\langle e_{ik} \mid k < l\rangle$,
(4) $\eta(t_{il}) = t_{il},\ \eta(s_{il}) = s_{il}$,
(5) $\delta(t_{il}) = t_{il} \otimes 1 + 1 \otimes t_{il},\ \delta(s_{il}) = s_{il} \otimes 1 + 1 \otimes s_{il}$,
(6) $\Delta(t_{il}) = t_{il} \otimes 1 + K_i^l \otimes t_{il},\ \Delta(s_{il}) = s_{il} \otimes K_i^{-l} + 1 \otimes s_{il}$,
(7) $S(t_{il}) = -K_i^{-l}t_{il},\ S(s_{il}) = -s_{il}K_i^l$.

If we set $\tau_{il} = (t_{il}, t_{il})_L = (s_{il}, s_{il})_L$, we have the following commutation relations in $U_q(\mathfrak{g})$

$$s_{il}t_{jk} - t_{jk}s_{il} = \delta_{ij}\delta_{lk}\tau_{il}(K_i^l - K_i^{-l}). \tag{2.9}$$

Assume that $i \in I^{\text{im}}$ and let $\mathbf{c} = (c_1, \cdots, c_m)$ be an element in $\mathcal{C}_l$ or in $\mathcal{P}_l$. We set

$$t_{i,\mathbf{c}} = \prod_{j=1}^{m} t_{ic_j} \text{ and } s_{i,\mathbf{c}} = \prod_{j=1}^{m} s_{ic_j}.$$

Notice that $\{t_{i,\mathbf{c}} \mid \mathbf{c} \in \mathcal{C}_l\}$ is a basis of $U^-_{-l\alpha_i}$.

For any $i \in I^{\mathrm{iso}}$ and $\mathbf{c}, \mathbf{c}' \in \mathcal{P}_l$, if $\mathbf{c} \neq \mathbf{c}'$, then by induction, we have

$$(t_{i,\mathbf{c}}, t_{i,\mathbf{c}'})_L = (s_{i,\mathbf{c}}, s_{i,\mathbf{c}'})_L = 0.$$

For any $i \in I^{\mathrm{im}} \backslash I^{\mathrm{iso}}$ and $\mathbf{c}, \mathbf{c}' \in \mathcal{C}_l$, if the partitions obtained by rearranging $\mathbf{c}$ and $\mathbf{c}'$ are not equal, then we have

$$(t_{i,\mathbf{c}}, t_{i,\mathbf{c}'})_L = (s_{i,\mathbf{c}}, s_{i,\mathbf{c}'})_L = 0.$$

For each $i \in I^{\mathrm{re}}$, we also use the notation t_{i1} and s_{i1}. Here we set

$$t_{i1} = f_{i1}, \quad s_{i1} = e_{i1}.$$

Sometimes, we simply write t_i (resp. s_i) instead of t_{i1} (resp. s_{i1}) in this case. By mimicking Definition 1.2.13 in [13], we have the following definition.

Definition 2.5. For every $(i,l) \in I^\infty$, we define linear maps $e'_{i,l}, e''_{i,l} : U^- \to U^-$ by

$$e'_{i,l}(1) = 0, \; e'_{i,l}(t_{jk}) = \delta_{ij}\delta_{lk} \text{ and } e'_{i,l}(xy) = e'_{i,l}(x)y + q^{l(|x|,\alpha_i)} x e'_{i,l}(y) \tag{2.10}$$

$$e''_{i,l}(1) = 0, \; e''_{i,l}(t_{jk}) = \delta_{ij}\delta_{lk} \text{ and } e''_{i,l}(xy) = q^{l(|y|,\alpha_i)} e''_{i,l}(x)y + x e''_{i,l}(y) \tag{2.11}$$

for any homogeneous elements x, y in U^-.

Proposition 2.6.

(a) *For any $x, y \in U^-$, we have*

$$(t_{il}y, x)_L = \tau_{il}(y, e'_{i,l}(x))_L \quad \textit{and} \quad (yt_{il}, x)_L = \tau_{il}(y, e''_{i,l}(x))_L$$

(b) *The maps $e'_{i,l}$ and $e''_{i,l}$ preserve the radical of $(\ ,\ )_L$.*

(c) *Let $x \in U^-$, we have*

  (i) *If $e'_{i,l}(x) = 0$ for all $(i,l) \in I^\infty$, then $x = 0$.*

  (ii) *If $e''_{i,l}(x) = 0$ for all $(i,l) \in I^\infty$, then $x = 0$.*

Proof. (a) For any homogeneous element $x \in U^-$. We first show that

$$\delta(x) = t_{il} \otimes e'_{i,l}(x) + \sum_{w \neq (i,l)} t_w \otimes y_w, \tag{2.12}$$

where $t_w = t_{(j_1,l_1)} \cdots t_{(j_r,l_r)}$ if $w = (j_1,l_1)...(j_r,l_r)$ is a word in I^∞, and y_w are elements in U^-.

Since $e'_{i,l}$ is a linear map, it is enough to check (2.12) by assuming that x is a monomial in t_{jk}. Fix $(i,l) \in I^\infty$. We show it by induction on the number of t_{il} that appears in x. If x doesn't contain t_{il}, then $e'_{i,l}(x) = 0$ and there is no term of the form $t_{il} \otimes -$. Now assume that x contains t_{il}, then we can write $x = x_1 t_{il} x_2$ for some monomials x_1, x_2 such that x_1 doesn't contain t_{il}. So we have

$$e'_{i,l}(x) = e'_{i,l}(x_1 t_{il} x_2) = q^{l(|x_1|,\alpha_i)} x_1 e'_{i,l}(t_{il} x_2) = q^{l(|x_1|,\alpha_i)} x_1 [x_2 + q^{l(-l\alpha_i,\alpha_i)} t_{il} e'_{i,l}(x_2)]. \tag{2.13}$$

On the other hand

$$\delta(x) = \delta(x_1)(t_{il} \otimes 1 + 1 \otimes t_{il})\delta(x_2). \tag{2.14}$$

By induction hypothesis, the term $t_{il} \otimes -$ only appears in

$$(1 \otimes x_1)(t_{il} \otimes 1)(1 \otimes x_2) + (1 \otimes x_1)(1 \otimes t_{il})(t_{il} \otimes e'_{i,l}(x_2)), \tag{2.15}$$

which is equal to

$$\begin{aligned} &t_{il} \otimes q^{(|x_1|, l\alpha_i)} x_1 x_2 + t_{il} \otimes q^{-(|x_1|-l\alpha_i, -l\alpha_i)} x_1 t_{il} e'_{i,l}(x_2) \\ &= t_{il} \otimes q^{l(|x_1|, \alpha_i)} x_1 [x_2 + q^{-l(l\alpha_i, \alpha_i)} t_{il} e'_{i,l}(x_2)]. \end{aligned} \tag{2.16}$$

This shows (2.12).

Similarly, we can show that

$$\delta(x) = e''_{i,l}(x) \otimes t_{il} + \sum_{w \neq (i,l)} z_w \otimes t_w. \tag{2.17}$$

Since $e'_{i,l}$ and $e''_{i,l}$ are linear maps, Equations (2.12) and (2.17) hold for any $x, y \in U^-$.

For any $\mathbf{c} \in \mathcal{C}_{il}$, we have $(t_{il}, t_{i\mathbf{c}})_L = \delta_{(l),\mathbf{c}} \tau_{il}$. Thus

$$(t_{il} y, x)_L = \tau_{il}(y, e'_{i,l}(x))_L \quad \text{and} \quad (y t_{il}, x)_L = \tau_{il}(y, e''_{i,l}(x))_L \tag{2.18}$$

for any $x, y \in U^-$.

(b) Since $\tau_{il} = (t_{il}, t_{il})_L \neq 0$, our assertion follows.

(c) Note that each monomial ends with some t_{jk}'s. By (a), if $e''_{i,l}(x) = 0$ for all $(i, l) \in I^\infty$, then x belongs to the radial of $(\ ,\ )_L$, which is equal to 0 in U^-. $\square$

For any $i \in I^{\text{re}}$ and $n \in \mathbb{N}$, we set

$$t_i^{(n)} = \frac{t_i^n}{[n]_i!}.$$

By a similar argument as [13, 1.4.2], we have the following Lemma.

Lemma 2.7. *We have*

$$\delta(t_i^{(n)}) = \sum_{p+p'=n} q_i^{-pp'} t_i^{(p)} \otimes t_i^{(p')} \tag{2.19}$$

for any $i \in I^{\text{re}}$ and $n \in \mathbb{N}$.

Proposition 2.8. *For any $i \in I^{\text{re}}$, $(j, l) \in I^\infty$, and $i \neq (j, l)$, we have*

$$\sum_{p+p'=1-la_{ij}} (-1)^p t_i^{(p)} t_{jl} t_i^{(p')} = 0$$

in $U_q(\mathfrak{g})$.

Proof. If $i \in I^{\text{re}}$, we have $a_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}$. Set

$$R_{i,(j,l)} = \sum_{p+p'=1-la_{ij}} (-1)^p t_i^{(p)} t_{jl} t_i^{(p')}.$$

By (2.6), we only need to show that $e''_\mu(R_{i,(j,l)}) = 0$ for all $\mu \in I^\infty$. It is clear that

$$e''_\mu(R_{i,(j,l)}) = 0 \quad \text{if } \mu \neq i, (j,l).$$

By the definition of e''_i, we have

$$\begin{aligned} e''_i(t_i^{(p)} t_{jl} t_i^{(p')}) &= q^{(\alpha_i, -p'\alpha_i)} e''_i(t_i^{(p)} t_{jl}) t_i^{(p')} + t_i^{(p)} t_{jl} e''_i(t_i^{(p')}) \\ &= q^{-p'(\alpha_i,\alpha_i)} q^{-(\alpha_i, l\alpha_j)} q_i^{(1-p)} t_i^{(p-1)} t_{jl} t_i^{(p')} + q_i^{(1-p')} t_i^{(p)} t_{jl} t_i^{(p'-1)}. \end{aligned} \tag{2.20}$$

Thus

$$\begin{aligned} e''_i(R_{i,(j,l)}) = &\sum_{p+p'=1-la_{ij}} (-1)^p q^{-p'(\alpha_i,\alpha_i)} q^{-(\alpha_i, l\alpha_j)} q_i^{(1-p)} t_i^{(p-1)} t_{jl} t_i^{(p')} \\ &+ \sum_{p+p'=1-la_{ij}} (-1)^p q_i^{(1-p')} t_i^{(p)} t_{jl} t_i^{(p'-1)} \\ = &\sum_{0 \le p \le 1-la_{ij}} (-1)^p q^{-(1-la_{ij}-p)(\alpha_i,\alpha_i)} q^{-(\alpha_i, l\alpha_j)} q_i^{(1-p)} t_i^{(p-1)} t_{jl} t_i^{(1-la_{ij}-p)} \\ &+ \sum_{0 \le p \le 1-la_{ij}} (-1)^p q_i^{(la_{ij}+p)} t_i^{(p)} t_{jl} t_i^{(-la_{ij}-p)}. \end{aligned} \tag{2.21}$$

The coefficient of $t_i^{(p)} t_{jl} t_i^{(-la_{ij}-p)}$ in the first sum of (2.21) is

$$\begin{aligned} &(-1)^{p+1} q^{-(-la_{ij}-p)(\alpha_i,\alpha_i)} q^{-(\alpha_i, l\alpha_j)} q_i^{-p} \\ &= (-1)^{p+1} q^{(l\frac{2(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)}+p)(\alpha_i,\alpha_i) - l(\alpha_i,\alpha_j) + (-p)\frac{(\alpha_i,\alpha_i)}{2}} \\ &= (-1)^{p+1} q^{l(\alpha_i,\alpha_j) + p\frac{(\alpha_i,\alpha_i)}{2}} \\ &= (-1)^{p+1} q_i^{(la_{ij}+p)}. \end{aligned} \tag{2.22}$$

Hence, we have $e''_i(R_{i,(j,l)}) = 0$.

By the definition of e''_{jl}, we have

$$e''_{jl}(t_i^{(p)} t_{jl} t_i^{(p')}) = q^{-l(\alpha_j, p'\alpha_i)} e''_{jl}(t_i^{(p)} t_{jl}) t_i^{(p')} = q^{-l(\alpha_j, p'\alpha_i)} t_i^{(p)} t_i^{(p')}. \tag{2.23}$$

So

$$e''_{jl}(R_{i,(j,l)}) = \sum_{0 \le p' \le 1-la_{ij}} (-1)^{(1-la_{ij}-p')} q^{-l(\alpha_j, p'\alpha_i)} t_i^{(1-la_{ij}-p')} t_i^{(p')}. \tag{2.24}$$

By [13, 1.3.4], we obtain

$$\sum_{0 \le p' \le 1-l\frac{2(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)}} (-1)^{(1-l\frac{2(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)}-p')} q^{-l(\alpha_j, p'\alpha_i)} \begin{bmatrix} 1 - l\frac{2(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)} \\ p' \end{bmatrix}_i = 0.$$

Hence, we get $e''_{jl}(R_{i,(j,l)}) = 0$. This finishes the proof. $\square$

By the above arguments, we have primitive generators t_{il} ($(i,l) \in I^\infty$) in U^- of degree $-l\alpha_i$ and s_{il} ($(i,l) \in I^\infty$) in U^+ of degree $l\alpha_i$ satisfying

$$\begin{aligned} &s_{il}t_{jk} - t_{jk}s_{il} = \delta_{ij}\delta_{lk}\tau_{il}(K_i^l - K_i^{-l}), \\ &\sum_{k=0}^{1-la_{ij}} (-1)^k \begin{bmatrix} 1-la_{ij} \\ k \end{bmatrix}_i {t_i}^{1-la_{ij}-k} t_{jl} t_i^k = 0 \ \text{ for } i \in I^{\mathrm{re}},\ i \neq (j,l). \end{aligned} \tag{2.25}$$

By using the involution ω, we get

$$\sum_{k=0}^{1-la_{ij}} (-1)^k \begin{bmatrix} 1-la_{ij} \\ k \end{bmatrix}_i {s_i}^{1-la_{ij}-k} s_{jl} s_i^k = 0 \ \text{ for } i \in I^{\mathrm{re}},\ i \neq (j,l). \tag{2.26}$$

Since t_{il} (resp. s_{il}) can be written as a homogeneous polynomial of f_{ik} (resp. e_{ik}) for $k \leq l$, we have

$$q^h t_{jl} q^{-h} = q^{-l\alpha_j(h)} t_{jl}, \quad q^h s_{jl} q^{-h} = q^{l\alpha_j(h)} s_{jl} \ \text{ for } h \in P^\vee, (j,l) \in I^\infty, \tag{2.27}$$

and

$$[t_{ik}, t_{jl}] = [s_{ik}, s_{jl}] = 0 \ \text{ for } a_{ij} = 0. \tag{2.28}$$

3. $\mathbb{A}_1$-form of the quantum Borcherds-Bozec algebras

We consider the localization of $\mathbb{Q}[q]$ at the ideal $(q-1)$:

$$\begin{aligned} \mathbb{A}_1 &= \{f(q) \in \mathbb{Q}(q) \mid f \text{ is regular at } q = 1\} \\ &= \{g/h \mid g, h \in \mathbb{Q}[q],\ h(1) \neq 0\} \end{aligned} \tag{3.1}$$

Let $\mathbb{J}_1$ be the unique maximal ideal of the local ring $\mathbb{A}_1$, which is generated by $(q-1)$. Then we have an isomorphism of fields

$$\mathbb{A}_1/\mathbb{J}_1 \xrightarrow{\sim} \mathbb{Q}, \quad f(q) + \mathbb{J}_1 \mapsto f(1).$$

Note that, for $i \in I^{\mathrm{re}}$, $[n]_i$ and $\begin{bmatrix} n \\ k \end{bmatrix}_i$ are elements of $\mathbb{Z}[q, q^{-1}] \subseteq \mathbb{A}_1$. For any $h \in P^\vee$, $n \in \mathbb{Z}$, we formally define

$$(q^h; n)_q = \frac{q^h q^n - 1}{q - 1} \in U^0.$$

Definition 3.1. We define the $\mathbb{A}_1$-*form*, denote by $U_{\mathbb{A}_1}$ of the quantum Borcherds-Bozec algebra $U_q(\mathfrak{g})$ to be the $\mathbb{A}_1$-subalgebra generated by the elements s_{il}, T_{il}, q^h and $(q^h;0)_q$, for all $(i,l) \in I^\infty$ and $h \in P^\vee$, where

$$T_{il} = \frac{1}{\tau_{il}} \frac{1}{q_i^2 - 1} t_{il} \ \text{ for } (i,l) \in I^\infty. \tag{3.2}$$

Let $U_{\mathbb{A}_1}^+$ (resp. $U_{\mathbb{A}_1}^-$) be the $\mathbb{A}_1$-subalgebra of $U_{\mathbb{A}_1}$ generated by the elements s_{il} (resp. T_{il}) for $(i,l) \in I^\infty$, and $U_{\mathbb{A}_1}^0$ be the subalgebra of $U_{\mathbb{A}_1}$ generated by q^h and $(q^h;0)_q$ for ($h \in P^\vee$).

Lemma 3.2.

(a) $(q^h;n)_q \in U^0_{\mathbb{A}_1}$ *for all* $n \in \mathbb{Z}$ *and* $h \in P^\vee$.
(b) $\dfrac{K_i^l - K_i^{-l}}{q_i^2 - 1} \in U^0_{\mathbb{A}_1}$.

Proof. It is straightforward to check that

$$\begin{aligned} (q^h;n)_q &= q^n (q^h;0)_q + \frac{q^n - 1}{q-1}, \\ \frac{K_i^l - K_i^{-l}}{q_i^2 - 1} &= \frac{q-1}{q_i^2-1}(1 + K_i^{-l})\frac{K_i^l - 1}{q-1}. \end{aligned} \tag{3.3}$$

The lemma follows. $\square$

The next proposition shows that the triangular decomposition (2.6) of $U_q(\mathfrak{g})$ carries over to its $\mathbb{A}_1$-form.

Proposition 3.3. *We have a natural isomorphism of* $\mathbb{A}_1$*-modules*

$$U_{\mathbb{A}_1} \cong U^-_{\mathbb{A}_1} \otimes U^0_{\mathbb{A}_1} \otimes U^+_{\mathbb{A}_1} \tag{3.4}$$

induced from the triangular decomposition of $U_q(\mathfrak{g})$.

Proof. Consider the canonical isomorphism $\varphi : U_q(\mathfrak{g}) \xrightarrow{\sim} U^- \otimes U^0 \otimes U^+$ given by multiplication. By (2.25) and (2.27), we have the following commutation relations

$$\begin{aligned} s_{il}(q^h;0)_q &= (q^h; -l\alpha_i(h))_q s_{il}, \\ (q^h;0)_q T_{il} &= T_{il}(q^h; -l\alpha_i(h))_q, \\ s_{il}T_{jk} - T_{jk}s_{il} &= \delta_{ij}\delta_{lk}\frac{K_i^l - K_i^{-l}}{q_i^2 - 1}. \end{aligned} \tag{3.5}$$

Combining with (3.2), the image of φ lies inside $U^-_{\mathbb{A}_1} \otimes U^0_{\mathbb{A}_1} \otimes U^+_{\mathbb{A}_1}$. $\square$

The representation theory of quantum Borcherds-Bozec algebras has been studied by Kang and Kim in [6]. In the following sections, we shall use some notions defined in [6], which are similar to those in classical representation theory of quantum groups.

Fix $\lambda \in P$, let V^q be a highest weight $U_q(\mathfrak{g})$-module with highest weight λ and highest weight vector v_λ. Then we have the $\mathbb{A}_1$-form for the highest weight modules.

Definition 3.4. The $\mathbb{A}_1$-*form* of V^q is defined to be the $U_{\mathbb{A}_1}$-module $V_{\mathbb{A}_1} = U_{\mathbb{A}_1} v_\lambda$.

By the definition of highest weight module and $V_{\mathbb{A}_1}$, it is easy to see that $V_{\mathbb{A}_1} = U^-_{\mathbb{A}_1} v_\lambda$. The highest weight $U_q(\mathfrak{g})$-module V^q has the weight space decomposition

$$V^q = \bigoplus_{\mu \le \lambda} V^q_\mu, \tag{3.6}$$

where $V^q_\mu = \{v \in V^q \mid q^h v = q^{\mu(h)} v \ \text{ for all } h \in P^\vee\}$. For each $\mu \in P$, we define the *weight space* $(V_{\mathbb{A}_1})_\mu = V_{\mathbb{A}_1} \cap V^q_\mu$. The following proposition shows that $V_{\mathbb{A}_1}$ also has the weight space decomposition.

Proposition 3.5. $V_{\mathbb{A}_1} = \bigoplus_{\mu\le\lambda}(V_{\mathbb{A}_1})_\mu$.

Proof. The proof is the same as [4, Proposition 3.3.6]. $\square$

Proposition 3.6. *For each $\mu \in P$, the weight space $(V_{\mathbb{A}_1})_\mu$ is a free $\mathbb{A}_1$-module with $rank_{\mathbb{A}_1}(V_{\mathbb{A}_1})_\mu = dim_{\mathbb{Q}(q)}V^q_\mu$.*

Proof. We first show that $(V_{\mathbb{A}_1})_\mu$ is finite generated as an $\mathbb{A}_1$-module. Note that $V_{\mathbb{A}_1} = U^-_{\mathbb{A}_1}v_\lambda$ and every element in $U^-_{\mathbb{A}_1}$ is a polynomial of T_{il} with coefficients in $\mathbb{A}_1$. Assume that $\lambda = \mu+\alpha$ for some $\alpha \in Q_+$. Then for each $v \in \mathbb{A}_1$ with weight μ, v must be a $V_{\mathbb{A}_1}$-linear combination of $\{T_{i_1l_1}\cdots T_{i_pl_p}v_\lambda \mid l_1\alpha_{l_1}+\cdots l_p\alpha_{l_p} = \alpha\}$, which is a finite set.

Let $\{T_\zeta v_\lambda\}$ be a $\mathbb{Q}(q)$-basis of V^q_μ, where T_ζ are monomials in T_{il}. The set $\{T_\zeta v_\lambda\}$ is certainly contained in $(V_{\mathbb{A}_1})_\mu$ and is also $\mathbb{A}_1$-linearly independent. So we have $\mathrm{rank}_{\mathbb{A}_1}(V_{\mathbb{A}_1})_\mu \ge \dim_{\mathbb{Q}(q)}V^q_\mu$. Let $\{u_1,\cdots,u_p\}$ be an $\mathbb{A}_1$-linearly independent subset of $(V_{\mathbb{A}_1})_\mu$. Consider a $\mathbb{Q}(q)$-linear dependence relation

$$c_1(q)u_1+\cdots+c_p(q)u_p = 0,\ c_k(q)\in\mathbb{Q}(q) \quad \text{for } k = 1,\cdots,p.$$

Multiplying some powers of $(q-1)$ if needed, we may assume that all $c_k(q) \in \mathbb{A}_1$, which implies that $c_k(q) = 0$ for all $k = 1,\cdots,p$. Hence $u_1,\cdots,u_p$ are linearly independent over $\mathbb{Q}(q)$ and $\mathrm{rank}_{\mathbb{A}_1}(V_{\mathbb{A}_1})_\mu \le \dim_{\mathbb{Q}(q)}V^q_\mu$. This finishes the proof. $\square$

Corollary 3.7. *The $\mathbb{Q}(q)$-linear map $\varphi : \mathbb{Q}(q)\otimes_{\mathbb{A}_1} V_{\mathbb{A}_1} \to V^q$ given by $c\otimes v \mapsto cv$ is an isomorphism.*

4. Classical limit of quantum Borcherds-Bozec algebras

Define the $\mathbb{Q}$-linear vector spaces

$$\begin{aligned} U_1 &= (\mathbb{A}_1/\mathbb{J}_1)\otimes_{\mathbb{A}_1} U_{\mathbb{A}_1} \cong U_{\mathbb{A}_1}/\mathbb{J}_1U_{\mathbb{A}_1},\\ V^1 &= (\mathbb{A}_1/\mathbb{J}_1)\otimes_{\mathbb{A}_1} V_{\mathbb{A}_1} \cong V_{\mathbb{A}_1}/\mathbb{J}_1V_{\mathbb{A}_1}. \end{aligned} \tag{4.1}$$

Then V^1 is naturally a U^1-module. Consider the natural maps

$$\begin{aligned} U_{\mathbb{A}_1} &\to U_1 = U_{\mathbb{A}_1}/\mathbb{J}_1U_{\mathbb{A}_1},\\ V_{\mathbb{A}_1} &\to V^1 = V_{\mathbb{A}_1}/\mathbb{J}_1V_{\mathbb{A}_1}. \end{aligned} \tag{4.2}$$

The passage under these maps is referred to as taking the classical limit. We will denote by $\overline{x}$ the image of x under the classical limit. Notice that q is mapped to 1 under these maps.

For each $\mu \in P$, set $V^1_\mu = (\mathbb{A}_1/\mathbb{J}_1)\otimes_{\mathbb{A}_1}(V_{\mathbb{A}_1})_\mu$. Then we have

Proposition 4.1.

(a) $V^1 = \bigoplus_{\mu\le\lambda} V^1_\mu$.
(b) *For each $\mu \in P$, $dim_{\mathbb{Q}}V^1_\mu = rank_{\mathbb{A}_1}(V_{\mathbb{A}_1})_\mu = dim_{\mathbb{Q}(q)}V^q_\mu$.*

Denote by $\overline{h} \in U_1$ the classical limit of the element $(q^h;0)_q \in U_{\mathbb{A}_1}$. As in [4], we have the following lemma.

Lemma 4.2.

(i) *For all $h \in P^\vee$, we have $\overline{q^h} = 1$.*
(ii) *For any $h, h' \in P^\vee$, $\overline{h+h'} = \overline{h}+\overline{h'}$. Hence, we have $\overline{nh} = n\overline{h}$ for $n \in \mathbb{Z}$.*

Define the subalgebras $U_1^0 = \mathbb{Q} \otimes U_{\mathbb{A}_1}^0$ and $U_1^{\pm} = \mathbb{Q} \otimes U_{\mathbb{A}_1}^{\pm}$. The next theorem shows that we can define a surjective homomorphism from the universal enveloping algebra $U(\mathfrak{g})$ to U_1, and as a $U(\mathfrak{g})$-module, V^1 is a highest weight module with highest weight $\lambda \in P$ and highest weight vector $\overline{v}_\lambda$.

Theorem 4.3.

(a) *The elements $\overline{s}_{il}$, $\overline{T}_{il}$ $((i,l) \in I^\infty)$ and $\overline{h}$ $(h \in P^\vee)$ satisfy the defining relations of $U(g)$. Hence there exists a surjective $\mathbb{Q}$-algebra homomorphism $\psi : U(\mathfrak{g}) \to U_1$ sending e_{il} to $\overline{s}_{il}$, f_{il} to $\overline{T}_{il}$, and h to $\overline{h}$. In particular, the U_1-module V^1 has a $U(\mathfrak{g})$-module structure.*

(b) *For each $\mu \in P$, $h \in P^\vee$, the element $\overline{h}$ acts on V_μ^1 as scalar multiplication by $\mu(h)$. So V_μ^1 is the μ-weight space of the $U(\mathfrak{g})$-module V^1.*

(c) *As a $U(\mathfrak{g})$-module, V^1 is a highest weight module with highest weight $\lambda \in P$ and highest weight vector $\overline{v}_\lambda$.*

Proof. (a) Since $\dfrac{K_i^l - K_i^{-l}}{q_i^2 - 1} = \dfrac{q-1}{q_i^2-1}(1 + K_i^{-l})\dfrac{K_i^l - 1}{q-1}$, when we take classical limit, we get

$$\overline{\frac{K_i^l - K_i^{-l}}{q_i^2 - 1}} = \frac{1}{2r_i} \cdot 2 \cdot l r_i \overline{h}_i = l\overline{h}_i.$$

By (2.25), we have the following equation in U_1

$$\overline{s}_{il}\overline{T}_{jk} - \overline{T}_{jk}\overline{s}_{il} = \delta_{ij}\delta_{lk} l\overline{h}_i,$$

and it is the same as the commutation relations in $U(\mathfrak{g})$.

Since

$$q^h s_{jl} = q^{l\alpha_j(h)} s_{jl} q^h, \quad q^h T_{jl} = q^{-l\alpha_j(h)} T_{jl} q^h \quad \text{for } h \in P^\vee, (j,l) \in I^\infty,$$

we have $\dfrac{q^h - 1}{q-1} s_{il} = s_{il} \dfrac{q^{l\alpha_i(h)} q^h - 1}{q-1}$ and

$$\frac{q^h - 1}{q-1} s_{il} - s_{il}\frac{q^h - 1}{q-1} = s_{il}\frac{q^{l\alpha_i(h)} - 1}{q-1} q^h. \tag{4.3}$$

Thus $\overline{h}\overline{s}_{il} - \overline{s}_{il}\overline{h} = l\alpha_i(h)\overline{s}_{il}$. Similarly, we have

$$\overline{h}\, \overline{T}_{il} - \overline{T}_{il}\overline{h} = -l\alpha_i(h)\overline{T}_{il}.$$

It is easy to check the commutation relations

$$[\overline{T}_{ik}, \overline{T}_{jl}] = [\overline{s}_{ik}, \overline{s}_{jl}] = 0 \quad \text{for } a_{ij} = 0. \tag{4.4}$$

For $i \in I^{\mathrm{re}}$, we have

$$\overline{[n]}_i = n \quad \text{and} \quad \overline{\begin{bmatrix} n \\ k \end{bmatrix}_i} = \binom{n}{k}.$$

Hence the remaining Serre relations follow.

(b) For $v \in (V_{\mathbb{A}_1})_\mu$ and $h \in P^\vee$, we have $(q^h;0)_q v = \dfrac{q^{\mu(h)-1}}{q-1} v$. Hence when we take the classical limit, we obtain $\overline{h}v = \mu(h)v$.

(c) As a $U(\mathfrak{g})$-module, by (2), we have $h\overline{v}_\lambda = \overline{h}\overline{v}_\lambda = \lambda(h)\overline{v}_\lambda$ in V^1 for all $h \in P^\vee$. For each $(i,l) \in I^\infty$, $s_{il}\overline{v}_\lambda$ is zero. Therefore, $V^1 = U_1^- \overline{v}_\lambda = U^-(\mathfrak{g})\overline{v}_\lambda$ and hence V^1 is a highest weight module with highest weight $\lambda \in P$ and highest weight vector $\overline{v}_\lambda$. $\square$

Combining Proposition 4.1 (b) and Theorem 4.3 (b), we have $\mathrm{ch}V^1 = \mathrm{ch}V^q$. For a dominant integral weight $\lambda \in P^+$, the irreducible highest weight $U_q(\mathfrak{g})$-module $V^q(\lambda)$ has the following property.

Proposition 4.4. *[6] Let $\lambda \in P^+$ and $V^q(\lambda)$ be the irreducible highest weight module with highest weight λ and highest weight vector v_λ. Then the following statements hold.*

(a) *If $i \in I^{re}$, then $f_i^{\lambda(h_i)+1} v_\lambda = 0$.*
(b) *If $i \in I^{im}$ and $\lambda(h_i) = 0$, then $f_{ik} v_\lambda = 0$ for all $k > 0$.*

We now conclude that the classical limit of the irreducible highest weight $U_q(\mathfrak{g})$-module $V^q(\lambda)$ is isomorphic to the irreducible highest $U(\mathfrak{g})$-module $V(\lambda)$.

Theorem 4.5. *If $\lambda \in P^+$ and V^q is the irreducible highest weight $U_q(\mathfrak{g})$-module $V^q(\lambda)$ with highest weight λ, then V^1 is isomorphic to the irreducible highest weight module $V(\lambda)$ over $U(\mathfrak{g})$ with highest weight λ.*

Proof. By Proposition 4.4, if $i \in I^{\mathrm{re}}$, then $T_i^{\lambda(h_i)+1} v_\lambda = 0$; if $i \in I^{\mathrm{im}}$ and $\lambda(h_i) = 0$, then $T_{ik} v_\lambda = 0$ for all $k > 0$. Therefore, V^1 is a highest weight $U_q(\mathfrak{g})$-module with highest weight λ and highest weight vector $\overline{v}_\lambda$ satisfying:

(a) If $i \in I^{\mathrm{re}}$, then $f_i^{\lambda(h_i)+1} \overline{v}_\lambda = \overline{T}_i^{\lambda(h_i)+1} \overline{v}_\lambda = 0$.
(b) If $i \in I^{\mathrm{im}}$ and $\lambda(h_i) = 0$, then $f_{ik}\overline{v}_\lambda = \overline{T}_{ik}\overline{v}_\lambda = 0$ for all $k > 0$.

Hence $V^1 \cong V(\lambda)$ by Proposition 1.3. $\square$

By Proposition 4.1 (b), the character of $V^q(\lambda)$ is the same as the character of $V(\lambda)$, which is given by (see, [5,3])

$$\begin{aligned} \mathrm{ch}V(\lambda) &= \frac{\sum_{w \in W} \epsilon(w) e^{w(\lambda+\rho)-\rho} w(S_\lambda)}{\prod_{\alpha \in \Delta_+} (1-e^{-\alpha})^{\dim \mathfrak{g}_\alpha}} \\ &= \frac{\sum_{w \in W} \sum_{s \in F_\lambda} \epsilon(w)\epsilon(s) e^{w(\lambda+\rho-s)-\rho}}{\prod_{\alpha \in \Delta_+} (1-e^{-\alpha})^{\dim \mathfrak{g}_\alpha}}. \end{aligned} \tag{4.5}$$

Theorem 4.6. *The classical limit U_1 of $U_q(\mathfrak{g})$ is isomorphic to the universal enveloping algebra $U(\mathfrak{g})$ as $\mathbb{Q}$-algebras.*

Proof. By Theorem 4.3 (a), we already have an epimorphism $\psi : U(\mathfrak{g}) \twoheadrightarrow U_1$ sending e_{il}, f_{il}, h to $\overline{s}_{il}$, $\overline{T}_{il}$, $\overline{h}$, respectively. So it is sufficient to show that ψ is injective.

We first show that the restriction ψ_0 of ψ to $U^0(\mathfrak{g})$ is an isomorphism of $U^0(\mathfrak{g})$ onto U_1^0. Note that ψ_0 is certainly surjective. Since $\chi = \{h_i \mid i \in I\} \cup \{d_i \mid i \in I\}$ is a $\mathbb{Z}$-basis of the free $\mathbb{Z}$-lattice $P^\vee$, it is also a $\mathbb{Q}$-basis of the Cartan subalgebra $\mathfrak{h}$. Thus any element of $U^0(\mathfrak{g})$ may be written as a polynomial in χ. Suppose $g \in \mathrm{Ker}\psi_0$. Then, for each $\lambda \in P$, we have

$$0 = \psi_0(g) \cdot \overline{v}_\lambda = \lambda(g)\overline{v}_\lambda,$$

where v_λ is a highest weight vector of a highest weight $U_q(\mathfrak{g})$-module of highest weight λ and $\lambda(g)$ denotes the polynomial in $\{\lambda(x) \mid x \in \chi\}$ corresponding to g. Hence, we have $\lambda(g) = 0$ for every $\lambda \in P$. Since we may take any integer value for $\lambda(x)$ $(x \in \chi)$, g must be zero, which implies that ψ_0 is injective.

Next, we show that the restriction of ψ to $U^-(\mathfrak{g})$, denoted by ψ_-, is an isomorphism of $U^-(\mathfrak{g})$ onto U_1^-. Suppose $\mathrm{Ker}\psi_- \neq 0$, and take a non-zero element $u = \sum a_\zeta f_\zeta \in \mathrm{Ker}\psi_-$, where $a_\zeta \in \mathbb{Q}$ and f_ζ are monomials in f_{il}'s $(i,l) \in I^\infty$. Let N be the maximal length of the monomials f_ζ in the expression of u and choose a dominant integral weight $\lambda \in P^+$ such that $\lambda(h_i) > N$ for all $i \in I$. The kernel of the $U^-(\mathfrak{g})$-module homomorphism $\varphi : U^-(\mathfrak{g}) \to V^1$ given by $x \mapsto \psi(x) \cdot \overline{v}_\lambda$ is the left ideal of $U^-(\mathfrak{g})$ generated by $f_i^{\lambda(h_i)+1}$ for $i \in I^{\mathrm{re}}$ and f_{il} for $i \in I^{\mathrm{im}}$ with $\lambda(h_i) = 0$. Because of the choice of λ, it is generated by $f_i^{\lambda(h_i)+1}$ for all $i \in I^{\mathrm{re}}$.

Therefore, $u = \sum a_\zeta f_\zeta \notin \mathrm{Ker}\varphi$. That is, $\psi_-(u) \cdot \overline{v}_\lambda = \psi(u) \cdot \overline{v}_\lambda \neq 0$, which is a contradiction. Hence, $\mathrm{Ker}\psi_- = 0$ and $U^-(\mathfrak{g})$ is isomorphic to U_1^-.

Similarly, we have $U^+(\mathfrak{g}) \cong U_1^+$. Hence, by the triangular decomposition, we have the linear isomorphisms

$$U(\mathfrak{g}) \cong U^-(\mathfrak{g}) \otimes U^0(\mathfrak{g}) \otimes U^+(\mathfrak{g}) \cong U_1^- \otimes U_1^0 \otimes U_1^+ \cong U_1,$$

where the last isomorphism follows from Proposition 3.3. It is easy to see that this isomorphism is actually an algebra isomorphism. □

We now show that U_1 inherits a Hopf algebra structure from that of $U_q(\mathfrak{g})$. It suffices to show that $U_{\mathbb{A}_1}$ inherits the Hopf algebra structure from that of $U_q(\mathfrak{g})$. Since

$$\begin{aligned}
&\Delta(T_{il}) = T_{il} \otimes 1 + K_i^l \otimes T_{il},\ \Delta(s_{il}) = s_{il} \otimes K_i^{-l} + 1 \otimes s_{il},\\
&\Delta(q^h) = q^h \otimes q^h,\\
&S(T_{il}) = -K_i^{-l} T_{il},\ S(s_{il}) = -s_{il} K_i^l,\ S(q^h) = q^{-h},\\
&\epsilon(T_{il}) = \epsilon(s_{il}) = 0,\ \epsilon(q^h) = 1,
\end{aligned} \tag{4.6}$$

we have

$$\begin{aligned}
&\Delta((q^h;0)_q) = \frac{q^h \otimes q^h - 1 \otimes 1}{q-1} = (q^h;0)_q \otimes 1 + q^h \otimes (q^h;0)_q,\\
&S((q^h;0)_q) = (q^{-h};0)_q,\\
&\epsilon((q^h;0)_q) = 0.
\end{aligned} \tag{4.7}$$

Hence the maps $\Delta \colon U_{\mathbb{A}_1} \to U_{\mathbb{A}_1} \otimes U_{\mathbb{A}_1}$, $\epsilon \colon U_{\mathbb{A}_1} \to \mathbb{A}_1$, and $S \colon U_{\mathbb{A}_1} \to U_{\mathbb{A}_1}$ are all well-defined and $U_{\mathbb{A}_1}$ inherits a Hopf algebra structure from that of $U_q(\mathfrak{g})$.

Let us show that the Hopf algebra structure of U_1 coincides with that of $U(\mathfrak{g})$ under the isomorphism we have been considering. Taking the classical limit of the equations in (4.6) and in (4.7), we have

$$\begin{aligned}
&\Delta(\overline{T}_{il}) = \overline{T}_{il} \otimes 1 + 1 \otimes \overline{T}_{il},\ \Delta(\overline{s}_{il}) = \overline{s}_{il} \otimes 1 + 1 \otimes \overline{s}_{il},\ \Delta(\overline{h}) = \overline{h} \otimes 1 + 1 \otimes \overline{h},\\
&S(\overline{T}_{il}) = -\overline{T}_{il},\ S(\overline{s}_{il}) = -\overline{s}_{il},\ S(\overline{h}) = -\overline{h},\\
&\epsilon(\overline{T}_{il}) = \epsilon(\overline{s}_{il}) = \epsilon(\overline{h}) = 0.
\end{aligned} \tag{4.8}$$

This coincides with (1.3). Therefore, we have the following corollary.

Corollary 4.7. *The classical limit U_1 of $U_q(\mathfrak{g})$ inherits a Hopf algebra structure from that of $U_q(\mathfrak{g})$ so that U_1 and $U(\mathfrak{g})$ are isomorphic as Hopf algebras over $\mathbb{Q}$.*

Since $U^-(\mathfrak{g}) \cong U_1^-$, by the same argument in [4, Theorem 3.4.10], we have the following theorem when we take the classical limit on the Verma module over $U_q(\mathfrak{g})$.

Theorem 4.8. *[4] If $\lambda \in P$ and V^q is the Verma module $M^q(\lambda)$ over $U_q(\mathfrak{g})$ with highest weight λ, then its classical limit V^1 is isomorphic to the Verma module $M(\lambda)$ over $U(\mathfrak{g})$ with highest weight λ.*

Appendix A

We shall provide an explicit commutation relations for e_{ik} and f_{jl}, for $(i,k),(j,l) \in I^\infty$ in $U_q(\mathfrak{g})$. Recall that, we have the co-multiplication formulas

$$\Delta(f_{il}) = \sum_{m+n=l} q_{(i)}^{-mn} f_{im} K_i^n \otimes f_{in}.$$

Then, the defining relation (2.5) yields the following lemma.

Lemma A.1. *[6] For any $i, j \in I$ and $k, l \in \mathbb{Z}_{>0}$, we have*

(a) *If $i \neq j$, then e_{ik} and f_{jl} are commutative.*
(b) *If $i = j$, we have the following relations in $U_q(\mathfrak{g})$ for all $k, l > 0$*

$$\sum_{\substack{m+n=k \\ n+s=l}} q_{(i)}^{n(m-s)} \nu_{in} e_{is} f_{im} K_i^{-n} = \sum_{\substack{m+n=k \\ n+s=l}} q_{(i)}^{-n(m-s)} \nu_{in} f_{im} e_{is} K_i^n. \tag{A.1}$$

Since

$$K_i^n e_{im} K_i^{-n} = q_{(i)}^{2nm} e_{im},$$
$$K_i^n f_{im} K_i^{-n} = q_{(i)}^{-2nm} f_{im},$$

we can modify the equations (A.1) as the following form

$$\sum_{\substack{m+n=k \\ n+s=l}} q_{(i)}^{n(s-m)} \nu_{in} K_i^{-n} e_{is} f_{im} = \sum_{\substack{m+n=k \\ n+s=l}} q_{(i)}^{n(m-s)} \nu_{in} K_i^n f_{im} e_{is}. \tag{A.2}$$

If $i \in I^{\mathrm{re}}$, then $k = l = 1$ and $m = s$, so there are only one commutation relation in this case

$$e_i f_i + \nu_{i1} K_i^{-1} = f_i e_i + \nu_{i1} K_i. \tag{A.3}$$

If $i \in I^{\mathrm{im}}$ (we omit the notation "i" in this case for simplicity), we first assume that $k = l$. By (A.2), we have

$$\begin{aligned}
k = l = 1, \quad & e_1 f_1 + \nu_1 K^{-1} = f_1 e_1 + \nu_1 K, \\
k = l = 2, \quad & e_2 f_2 + \nu_1 K^{-1} e_1 f_1 + \nu_2 K^{-2} = f_2 e_2 + \nu_1 K f_1 e_1 + \nu_2 K^2, \\
& \qquad \cdots \\
k = l = n, \quad & e_n f_n + \nu_1 K^{-1} e_{n-1} f_{n-1} + \cdots + \nu_{n-1} K^{1-n} e_1 f_1 + \nu_n K^{-n} \\
& = f_n e_n + \nu_1 K f_{n-1} e_{n-1} + \cdots + \nu_{n-1} K^{n-1} f_1 e_1 + \nu_n K^n.
\end{aligned} \tag{A.4}$$

By direct calculation, we can write $e_n f_n - f_n e_n$ in the following way

$$e_n f_n - f_n e_n = \alpha_1 f_{n-1} e_{n-1} + \alpha_2 f_{n-2} e_{n-2} + \cdots + \alpha_{n-1} f_1 e_1 + \alpha_n,$$

where

$$\begin{aligned}
\alpha_1 &= \nu_1(K - K^{-1}),\\
\alpha_2 &= \nu_2(K^2 - K^{-2}) - \nu_1 K^{-1}\alpha_1 = \nu_2(K^2 - K^{-2}) - \nu_1^2 K^{-1}(K - K^{-1}),\\
\alpha_3 &= \nu_3(K^3 - K^{-3}) - \nu_1 K^{-1}\alpha_2 - \nu_2 K^{-2}\alpha_1\\
&= \nu_3(K^3 - K^{-3}) - \nu_1\nu_2 K^{-1}(K^2 - K^{-2}) + (\nu_1^3 - \nu_1\nu_2)K^{-2}(K - K^{-1}),\\
&\cdots\\
\alpha_n &= \nu_n(K^n - K^{-n}) - \nu_1 K^{-1}\alpha_{n-1} - \nu_2 K^{-2}\alpha_{n-2} - \cdots - \nu_{n-1}K^{-(n-1)}\alpha_1.
\end{aligned} \tag{A.5}$$

If $m \in \mathbb{N}$ and $\mathbf{c} = (c_1, \cdots, c_d)$ is a composition of m (i.e. $\mathbf{c} \in \mathcal{C}_m$), then we set $\nu_{\mathbf{c}} = \prod_{k=1}^{d} \nu_k$ and $\|\mathbf{c}\| = d$. By induction, we have

$$e_n f_n = \sum_{p=1}^{n} \left\{ \sum_{r=1}^{p} \left[\nu_r \vartheta_{p-r} K^{r-p}(K^r - K^{-r})\right] \right\} f_{n-p} e_{n-p} + f_n e_n, \tag{A.6}$$

where $\vartheta_m = \sum_{\mathbf{c}\in\mathcal{C}_m} (-1)^{\|\mathbf{c}\|}\nu_{\mathbf{c}}$. For example, $\vartheta_4 = \nu_1^4 - 3\nu_1^2\nu_2 + 2\nu_1\nu_3 + \nu_2^2 - \nu_4$.

Next, we assume that $k - l = t$, then $m - s = t$. By (A.2), we have

$$\sum_{n=0}^{l} q_{(i)}^{-nt}\nu_n K^{-n} e_{l-n} f_{k-n} = \sum_{n=0}^{l} q_{(i)}^{nt}\nu_n K^n f_{k-n} e_{l-n}.$$

Hence, we have

$$\begin{aligned}
&e_l f_k + q_{(i)}^{-t}\nu_1 K^{-1} e_{l-1} f_{k-1} + \cdots + q_{(i)}^{-(l-1)t}\nu_{l-1}K^{-(l-1)} e_1 f_{t+1} + q_{(i)}^{-lt}\nu_l K^{-l} f_t\\
&= f_k e_l + q_{(i)}^{t}\nu_1 K f_{k-1} e_{l-1} + \cdots + q_{(i)}^{(l-1)t}\nu_{l-1}K^{(l-1)} f_{t+1} e_1 + q_{(i)}^{lt}\nu_l K^{l} f_t.
\end{aligned}$$

We substitute K by $q_{(i)}^t K$ in formula (A.6) and obtain

$$e_l f_k = \sum_{p=1}^{l} \left\{ \sum_{r=1}^{p} \left[\nu_r \vartheta_{p-r} (q_{(i)}^t K)^{r-p}((q_{(i)}^t K)^r - (q_{(i)}^t K)^{-r})\right] \right\} f_{k-p} e_{l-p} + f_k e_l. \tag{A.7}$$

Finally, we assume that $l - k = t$, then $s - m = t$. By (A.2), we get

$$\sum_{n=0}^{k} q_{(i)}^{nt}\nu_n K^{-n} e_{l-n} f_{k-n} = \sum_{n=0}^{k} q_{(i)}^{-nt}\nu_n K^n f_{k-n} e_{l-n}.$$

Hence, we have

$$\begin{aligned}
&e_l f_k + q_{(i)}^{t}\nu_1 K^{-1} e_{l-1} f_{k-1} + \cdots + q_{(i)}^{(l-1)t}\nu_{l-1}K^{-(l-1)} e_{t+1} f_1 + q_{(i)}^{lt}\nu_l K^{-l} e_t\\
&= f_k e_l + q_{(i)}^{-t}\nu_1 K f_{k-1} e_{l-1} + \cdots + q_{(i)}^{-(l-1)t}\nu_{l-1}K^{(l-1)} f_1 e_{t+1} + q_{(i)}^{-lt}\nu_l K^{l} e_t.
\end{aligned}$$

We substitute K by $q_{(i)}^{-t} K$ in formula (A.6) and obtain

$$e_l f_k = \sum_{p=1}^{k} \left\{ \sum_{r=1}^{p} \left[\nu_r \vartheta_{p-r} (q_{(i)}^{-t} K)^{r-p} ((q_{(i)}^{-t} K)^{r} - (q_{(i)}^{-t} K)^{-r}) \right] \right\} f_{k-p} e_{l-p} + f_k e_l. \tag{A.8}$$

Combining Formulas (A.6), (A.7), and (A.8), we have the following statement.

Proposition A.2. *For $i \in I^{im}$, we have the following commutation relations for all $k, l > 0$*

$$e_{il} f_{ik} - f_{ik} e_{il} = \sum_{p=1}^{min\{k,l\}} \left\{ \sum_{r=1}^{p} \left[\nu_{ir} \vartheta_{i,p-r} (q_{(i)}^{k-l} K_i)^{r-p} ((q_{(i)}^{k-l} K_i)^{r} - (q_{(i)}^{k-l} K_i)^{-r}) \right] \right\} f_{i,k-p} e_{i,l-p},$$

where $\vartheta_{i,p-r} = \sum_{\mathbf{c} \in \mathcal{C}_{p-r}} (-1)^{\|\mathbf{c}\|} \nu_{i,\mathbf{c}}$.

References

[1] T. Bozec, Quivers with loops and perverse sheaves, Math. Ann. 362 (2015) 773–797.
[2] T. Bozec, Quivers with loops and generalized crystals, Compos. Math. 152 (2016) 1999–2040.
[3] T. Bozec, O. Schiffmann, E. Vasserot, On the number of points of nilpotent quiver varieties over finite fields, arXiv: 1701.01797.
[4] J. Hong, S.-J. Kang, Introduction to Quantum Groups and Crystal Bases, Graduate Studies in Mathematics, vol. 42, Amer. Math. Soc., 2002.
[5] S.-J. Kang, Borcherds-Bozec algebras, root multiplicities and the Schofield construction, Commun. Contemp. Math. 21 (3) (2019).
[6] S.-J. Kang, Y.R. Kim, Quantum Borcherds-Bozec algebras and their integrable representations, J. Pure Appl. Algebra 224 (10) (2020) 106388, arXiv:1912.06115.
[7] S.-J. Kang, M. Kashiwara, O. Schiffmann, Geometric construction of crystal bases for quantum generalized Kac-Moody algebras, Adv. Math. 222 (2009) 996–1015.
[8] S.-J. Kang, M. Kashiwara, O. Schiffmann, Geometric construction of highest weight crystals for quantum generalized Kac-Moody algebras, Math. Ann. 354 (2012) 193–208.
[9] S.-J. Kang, O. Schiffmann, Canonical bases for quantum generalized Kac-Moody algebras, Adv. Math. 200 (2006) 455–478.
[10] M. Kashiwara, On crystal bases of the q-analogue of universal enveloping algebras, Duke Math. J. 63 (1991) 465–516.
[11] M. Kashiwara, Y. Saito, Geometric construction of crystal bases, Duke Math. J. 89 (1997) 9–36.
[12] Y. Li, Z. Lin, Canonical bases of Borcherds-Cartan type, Nagoya Math. J. 194 (2009) 169–193.
[13] G. Lusztig, Introduction to Quantum Groups, Modern Birkhäuser Classics, Birkhäuser/Springer, New York, 2010.
[14] G. Lusztig, Quantum deformation of certain simple modules over enveloping algebras, Adv. Math. 70 (1988) 237–249.
[15] G. Lusztig, Canonical bases arising from quantized enveloping algebras, J. Am. Math. Soc. 3 (1990) 447–498.
[16] G. Lusztig, Quivers, perverse sheaves, and quantized enveloping algebras, J. Am. Math. Soc. 4 (1991) 365–421.
[17] G. Lusztig, Tight monomials in quantized enveloping algebras, in: Quantum Deformations of Algebras and Their Representations, in: Isr. Math. Conf. Proc., vol. 7, Bar-Ilan Univ., Ramat Gan, 1993, pp. 117–132.
[18] Y. Saito, Crystal bases and quiver varieties, Math. Ann. 324 (2002) 675–688.