Metrics on triangulated categories ☆

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A R T I C L E   I N F O

A B S T R A C T

In a 1973 article Lawvere defined (among many other things) *metrics* on categories—the article has been enormously influential over the years, spawning a huge literature. In recent work, which is surveyed in the current note, we pursue a largely-unexplored angle: we *complete* categories with respect to their Lawvere metrics.

This turns out to be particularly interesting when the category is triangulated and the Lawvere metric is *good;* a metric is good if it is translation invariant and the balls of radius $\varepsilon > 0$ shrink rapidly enough as ε decreases. The definitions are all made precise at the beginning of the note. And the main theorem is that a certain natural subcategory $\mathfrak{S}(\mathcal{S})$, of the completion of $\mathcal{S}$ with respect to a good metric, is triangulated.

There is also a theorem which, under restrictive conditions, gives a procedure for computing $\mathfrak{S}(\mathcal{S})$. As examples we discuss the special cases (1) where $\mathcal{S}$ is the homotopy category of finite spectra, and (2) where $\mathcal{S} = \mathbf{D}^b(R\text{–mod})$, the derived category of bounded complexes of finitely generated R–modules over a noetherian ring R.

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Reminder 1. Following a 1973 article of Lawvere [11,12], more precisely the discussion on pages 139-140 of [11],[1] a *metric* on a category is a function that assigns a positive real number (length) to every morphism, in such a way that for every identity map id : $x \longrightarrow x$ we have Length(id) = 0 and the triangle inequality is satisfied. The triangle inequality means: if $x \xrightarrow{f} y \xrightarrow{g} z$ are composable morphisms then

$$\text{Length}(gf) \quad \leq \quad \text{Length}(f) + \text{Length}(g) \ .$$

Lawvere's article does many other things and has had an enormous influence over the years—when I last checked on *Google Scholar* it had 732 citations. To the best of my knowledge the myriad applications have

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[1] Lawvere calls these *normed categories;* if we wanted to be faithful to Lawvere's terminology we would use the word "norm" rather than "metric". But to the author the term "norm" suggests that the metric is compatible with the action of some ring with a multiplicative absolute value.

essentially all gone in directions totally different from the one we will be pursuing in this note. There is only a handful of exceptions, we will cite these when they become relevant.

We begin with a string of definitions.

Definition 2. Suppose we are given a category $\mathcal{C}$. Two Lawvere metrics Length_1 and Length_2 are declared *equivalent* if, for any real number $\varepsilon > 0$, there exists a number $\delta > 0$ such that

$$\{\mathrm{Length}_1(a \to b) < \delta\} \quad \Longrightarrow \quad \{\mathrm{Length}_2(a \to b) < \varepsilon\} \,,$$
$$\{\mathrm{Length}_2(a \to b) < \delta\} \quad \Longrightarrow \quad \{\mathrm{Length}_1(a \to b) < \varepsilon\} \,.$$

Definition 3. Let $\mathcal{C}$ be a category with a Lawvere metric. A *Cauchy sequence* in $\mathcal{C}$ is a sequence $E_1 \longrightarrow E_2 \longrightarrow E_3 \longrightarrow \cdots$ of composable morphisms in which the maps $E_i \longrightarrow E_j$ eventually become very short. More precisely: for any $\varepsilon > 0$ there exists an $M > 0$ such that the morphisms $E_i \longrightarrow E_j$ satisfy

$$\mathrm{Length}(E_i \to E_j) \quad < \quad \varepsilon$$

whenever $i, j > M$ and $i \leq j$.

We are accustomed from analysis to the idea of completing a metric space with respect to its metric, and now we want to do the same for Lawvere metrics on categories. And the idea is simple enough: the Yoneda embedding takes any category $\mathcal{C}$ to a subcategory of a cocomplete category, with the traditional definition that a category is cocomplete if all small colimits exist. Hence the completion of $\mathcal{C}$ should just be the closure of the image under Yoneda of $\mathcal{C}$. We make this precise in:

Definition 4. Let $\mathcal{C}$ be a category with a metric. Let $Y : \mathcal{C} \longrightarrow \mathrm{Hom}[\mathcal{C}^{\mathrm{op}}, \mathrm{Set}]$ be the Yoneda functor, that is the functor sending an object $c \in \mathcal{C}$ to the representable functor $Y(c) = \mathrm{Hom}(-, c)$.

 (i) Let $\mathcal{L}'(\mathcal{C})$ be the *completion* of $\mathcal{C}$, meaning the full subcategory of $\mathrm{Hom}[\mathcal{C}^{\mathrm{op}}, \mathrm{Set}]$ whose objects are the colimits in $\mathrm{Hom}[\mathcal{C}^{\mathrm{op}}, \mathrm{Set}]$ of Cauchy sequences in $\mathcal{C}$.
 (ii) Let $\mathfrak{C}'(\mathcal{C})$ be the full subcategory of $\mathrm{Hom}[\mathcal{C}^{\mathrm{op}}, \mathrm{Set}]$ whose objects we will call *compactly supported*. An object $F \in \mathrm{Hom}[\mathcal{C}^{\mathrm{op}}, \mathrm{Set}]$, that is a functor $F : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Set}$, is declared to be compactly supported if it takes sufficiently short morphisms to isomorphisms. That is: F belongs to $\mathfrak{C}'(\mathcal{C})$ if there exists an $\varepsilon > 0$ such that

$$\{\mathrm{Length}(a \to b) < \varepsilon\} \quad \Longrightarrow \quad \{F(b) \longrightarrow F(a) \text{ is an isomorphism}\}.$$

(iii) Now let $\mathfrak{S}'(\mathcal{C}) = \mathfrak{C}'(\mathcal{C}) \cap \mathcal{L}'(\mathcal{C})$.

Next assume the category $\mathcal{C}$ is *pre-additive.* This means that $\mathrm{Hom}(a, b)$ is an abelian group for every pair of objects $a, b \in \mathcal{C}$, and the composition is bilinear.[2] In this situation the Yoneda map factors as a composite

$$\mathcal{C} \xrightarrow{\;\;\widetilde{Y}\;\;} \mathrm{Mod}\text{–}\mathcal{C} \xrightarrow{\;\;\Phi\;\;} \mathrm{Hom}[\mathcal{C}^{\mathrm{op}}, \mathrm{Set}]$$

where $\mathrm{Mod}\text{–}\mathcal{C}$ is the category whose objects are the *additive functors* of the form $\mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Ab}$. And since $\mathrm{Mod}\text{–}\mathcal{C}$ is cocomplete we now have the option of taking the closure of the image of $\widetilde{Y}$ instead of the closure of the image of Y. This leads us to

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[2] In the more recent literature what used to be called pre-additive categories goes by the name $\mathbb{Z}$–linear categories, or even just $\mathbb{Z}$–categories.

Definition 5. Let $\mathcal{C}$ be a *pre-additive* category with a metric. Then

(i) Let $\mathfrak{L}(\mathcal{C})$ be the completion of $\widetilde{Y}(\mathcal{C})$ in Mod–$\mathcal{C}$; it is the full subcategory of Mod–$\mathcal{C}$ whose objects are the colimits in Mod–$\mathcal{C}$ of Cauchy sequences in $\mathcal{C}$.

(ii) Let $\mathfrak{C}(\mathcal{C}) = \Phi^{-1}\mathfrak{C}'(\mathcal{C})$.

(iii) Finally let $\mathfrak{S}(\mathcal{C}) = \mathfrak{C}(\mathcal{C}) \cap \mathfrak{L}(\mathcal{C})$.

Remark 6. We leave it to the reader to compare our description of the completion with what can be found in Lawvere [11, Proposition, bottom of p. 163, and its proof which goes on to p. 164].

Categories with metrics—that is what Lawvere [11] calls normed categories—may be viewed as categories enriched over a certain closed monoidal category, see Betti and Galuzzi [2] for a detailed exposition. And there is a notion of completing an enriched category with respect to a class of colimits, the reader can find it already in Kelly's book [5], but for a more direct approach see Kelly and Schmitt [9]. It doesn't seem automatic that the specialization of the general theory to the case at hand agrees with what we've done in this article. That is: using Kelly's construction we may complete a normed category $\mathcal{C}$, enriched as in [2], with respect to Cauchy sequences—and what's obtained doesn't in general seem to agree with our $\mathfrak{L}'(\mathcal{C})$. There are conditions that suffice to guarantee agreement: in Kubiś [10] the reader can see that adding the extra axiom that $\mathrm{Length}(f) \leq \mathrm{Length}(gf) + \mathrm{Length}(g)$ is sufficient. In this article we work mostly with "good metrics", which will be spelled out in Definition 10. Our good metrics happen *not* to satisfy the Kubiś axiom. The interested reader can nevertheless check that restricting to good metrics also suffices to guarantee the agreement of $\mathfrak{L}'(\mathcal{C})$ with the Cauchy completion due to Kelly.

For yet another construction of the category $\mathfrak{L}'(\mathcal{C})$ see Krause [8]. Krause only looks at one particular metric but the method generalizes. In Krause's approach the category $\mathfrak{L}'(\mathcal{C})$ is presented as the Gabriel-Zisman localization (see [4]) of the category of Cauchy sequences, where one formally inverts the Ind-isomorphisms.

The idea of studying the categories $\mathfrak{C}'(\mathcal{C})$ and $\mathfrak{S}'(\mathcal{C})$ seems to have arisen only in [16].

Remark 7. When I wrote [16] I was unaware of the earlier work by Lawvere, Betti, Galuzzi, Kelly, Schmitt and Kubiś; one of the aims of this survey is to present the results with the notation as close as possible to the older papers, but nevertheless compatible enough with [16] so that the interested reader can easily read further. Since Lawvere introduces metrics on arbitrary categories, the right notion of the completion in his generality is $\mathfrak{L}'(\mathcal{C})$ or Kelly's more sophisticated enriched completion. In [16] we assume at the outset that $\mathcal{C}$ is a triangulated category, hence only mention $\mathfrak{L}(\mathcal{C})$. It is easy to check that, when $\mathcal{C}$ is pre-additive, the functor $\Phi : \mathrm{Mod}–\mathcal{C} \longrightarrow \mathrm{Hom}[\mathcal{C}^{\mathrm{op}}, \mathrm{Set}]$ restricts to an equivalence $\mathfrak{L}(\mathcal{C}) \longrightarrow \mathfrak{L}'(\mathcal{C})$, and hence also to an equivalence $\mathfrak{S}(\mathcal{S}) \longrightarrow \mathfrak{S}'(\mathcal{S})$.

Remark 8. All we have shown so far is that there is no law barring a mathematician from making a string of ridiculous definitions. To persuade the reader that this formalism has some value we need to use it to prove a theorem.

In the interest of full disclosure: in the generality of the paragraphs above I can't prove anything worthwhile. The only obvious observation is that the constructions are robust under replacing one metric by an equivalent other. The Cauchy sequences depend only on the equivalence class of the metric, hence so do the categories $\mathfrak{L}'(\mathcal{C})$ and $\mathfrak{L}(\mathcal{C})$. The definitions of the categories $\mathfrak{C}'(\mathcal{C})$ and $\mathfrak{C}(\mathcal{C})$ make it clear that these two categories are also unperturbed by replacing a metric by an equivalent. Hence the same is true for $\mathfrak{S}'(\mathcal{C}) = \mathfrak{C}'(\mathcal{C}) \cap \mathfrak{L}'(\mathcal{C})$ and $\mathfrak{S}(\mathcal{C}) = \mathfrak{C}(\mathcal{C}) \cap \mathfrak{L}(\mathcal{C})$.

So much for triviality. To get anywhere we need to narrow our attention considerably.

Heuristic 9. Let $\mathcal{S}$ be a *triangulated category.* We will only consider "translation invariant"[3] metrics on $\mathcal{S}$, meaning for any homotopy cartesian square

$$
\begin{array}{ccc}
a & \xrightarrow{\ f\ } & b \\
\downarrow & & \downarrow \\
c & \xrightarrow{\ g\ } & d
\end{array}
$$

we postulate that

$$
\mathrm{Length}(f) \quad = \quad \mathrm{Length}(g) \ .
$$

Given any morphism $f : a \longrightarrow b$ we may form the homotopy cartesian square

$$
\begin{array}{ccc}
a & \xrightarrow{\ f\ } & b \\
\downarrow & & \downarrow \\
0 & \xrightarrow{\ g\ } & x
\end{array}
$$

and our assumption tells us that

$$
\mathrm{Length}(f) \quad = \quad \mathrm{Length}(g) \ .
$$

Hence it suffices to know the lengths of the morphisms $0 \longrightarrow x$. Replacing the metric by an equivalent, if necessary, we may assume our metric takes values in the set of rational numbers of the form $\{\frac{1}{n} \mid n \in \mathbb{N}\}$. To know everything about the metric it therefore suffices to specify the balls

$$
B_n \quad = \quad \left\{ x \in \mathcal{S} \ \middle| \ \text{the morphism } 0 \longrightarrow x \text{ has length } \leq \frac{1}{n} \right\} \ .
$$

To paraphrase the discussion above: if $f : x \longrightarrow y$ is a morphism, to compute its length you complete to a triangle $x \xrightarrow{\ f\ } y \longrightarrow z$, and then

$$
\mathrm{Length}(f) \quad = \quad \inf \left\{ \frac{1}{n} \ \middle| \ z \in B_n \right\} \ .
$$

Furthermore we will restrict our attention to non-archimedian metrics, that is metrics that satisfy the strong triangle inequality. This means: if $x \xrightarrow{\ f\ } y \xrightarrow{\ g\ } z$ are composable morphisms, then $\mathrm{Length}(gf) \leq \max\big(\mathrm{Length}(f), \mathrm{Length}(g)\big)$. By the translation-invariance it suffices to consider the case $x = 0$; that is it suffices to show that the composable morphisms $0 \xrightarrow{\ f\ } y \xrightarrow{\ g\ } z$ satisfy $\mathrm{Length}(gf) \leq \max\big(\mathrm{Length}(f), \mathrm{Length}(g)\big)$. Completing g to a triangle $y \xrightarrow{\ g\ } z \longrightarrow w$ this comes down to $\{y, w \in B_n\} \Longrightarrow \{z \in B_n\}$.

The discussion above motivates

Definition 10. Let $\mathcal{S}$ be a triangulated category. A *good metric* on $\mathcal{S}$ is a sequence of full subcategories $\{B_n, n \in \mathbb{N}\}$, containing 0 and with $B_1 = \mathcal{S}$, and furthermore satisfying

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[3] The word "translation" is sometimes used for the shift functor $\Sigma : \mathcal{S} \longrightarrow \mathcal{S}$; our translation invariance has nothing to do with this translation Σ.

(i) $B_n * B_n = B_n$, which means that if there exists a triangle $b \longrightarrow x \longrightarrow b'$ with $b, b' \in B_n$, then $x \in B_n$.

(ii) $\Sigma^{-1} B_{n+1} \cup B_{n+1} \cup \Sigma B_{n+1} \subset B_n$.

Remark 11. In Heuristic 9 we explained where part (i) of Definition 10 comes from, it guarantees that the translation-invariant metric given by the balls B_n is non-archimedian. The hypothesis (ii) of Definition 10 has not yet been motivated. We clearly must have $B_{n+1} \subset B_n$, the ball of radius $\frac{1}{n+1}$ must be contained in the ball of radius $\frac{1}{n}$. But it turns out to be convenient to assume the balls decrease rapidly enough for the stronger hypothesis (ii) to hold; it guarantees that the automorphism Σ is a "homeomorphism" with respect to the metric—in other words the metric $\{\Sigma B_n, n \in \mathbb{N}\}$ is equivalent to the metric $\{B_n, n \in \mathbb{N}\}$.

Note that we are *not* assuming that the metric is compatible with any other automorphism of $\mathcal{S}$.

Example 12. Suppose $\mathcal{S}$ is a triangulated category, $\mathcal{A}$ is an abelian category and $H : \mathcal{S} \longrightarrow \mathcal{A}$ is a homological functor. Put $B_1 = \mathcal{S}$. If for $n > 1$ we set B_n as given in the formulas below, we obtain three (inequivalent) good metrics on $\mathcal{S}$.

(i) $B_n = \{s \in \mathcal{S} \mid H^i(s) = 0 \text{ for all } i \text{ in the range } i > -n\}$.

(ii) $B_n = \{s \in \mathcal{S} \mid H^i(s) = 0 \text{ for all } i \text{ in the range } i < n\}$.

(iii) $B_n = \{s \in \mathcal{S} \mid H^i(s) = 0 \text{ for all } i \text{ in the range } -n < i < n\}$.

Note that if $\{B_n, n \in \mathbb{N}\}$ define a good metric on $\mathcal{S}$ then $\{B_n^{\mathrm{op}}, n \in \mathbb{N}\}$ define a good metric on $\mathcal{S}^{\mathrm{op}}$, which we will call the dual metric. Now a homological functor $H : \mathcal{S} \longrightarrow \mathcal{A}$ has a dual $H^{\mathrm{op}} : \mathcal{S}^{\mathrm{op}} \longrightarrow \mathcal{A}^{\mathrm{op}}$, and the reader can check that (i), applied to $H^{\mathrm{op}} : \mathcal{S}^{\mathrm{op}} \longrightarrow \mathcal{A}^{\mathrm{op}}$, gives a good metric equal to the dual of that obtained from (ii) applied to $H : \mathcal{S} \longrightarrow \mathcal{A}$.

The metric of (iii) is self-dual.

One more definition before the first theorem.

Definition 13. Let $\mathcal{S}$ be a triangulated category with a good metric. With the category $\mathfrak{S}(\mathcal{S})$ as in Definition 5(iii), we define the distinguished triangles in $\mathfrak{S}(\mathcal{S})$ to be the colimits in $\mathfrak{S}(\mathcal{S}) \subset \mathrm{Mod\text{-}}\mathcal{S}$ of Cauchy sequences of distinguished triangles in $\mathcal{S}$.

Explanation 14. What this means is the following. If we are given a Cauchy sequence of distinguished triangles in $\mathcal{S}$, we can always form the colimit in the cocomplete category $\mathrm{Mod\text{-}}\mathcal{S}$, and by the definition of $\mathfrak{L}(\mathcal{S})$ this colimit must lie in $\mathfrak{L}(\mathcal{S})$. In general there is no guarantee that the colimit will lie in the subcategory $\mathfrak{S}(\mathcal{S}) \subset \mathfrak{L}(\mathcal{S})$. What Definition 13 does is declare that those colimits which happen to lie in $\mathfrak{S}(\mathcal{S})$ are distinguished triangles in $\mathfrak{S}(\mathcal{S})$.

And now we come to

Theorem 15. *With the distinguished triangles as in Definition 13, the category $\mathfrak{S}(\mathcal{S})$ is triangulated.*

The proof of a slightly stronger theorem [the hypotheses on the metric are slightly less restrictive] may be found in [16, Theorem 2.11].

Remark 16. Up to Theorem 15 all we saw was a string of increasingly bizarre definitions. We've said it before: in this free world of ours there is no law prohibiting a mathematician from making up a long sequence of absurd-looking definitions.

Then, out of all the seemingly pointless formalism, we magically pulled out Theorem 15. Perhaps it takes an expert to appreciate how surprising the result is. Triangulated categories have been around since the early

1960s—meaning for about 55 years. And the conventional wisdom has always been that they don't reproduce. Until very recently there were no interesting recipes that began with a triangulated category $\mathcal{S}$, and out of it cooked up another triangulated category $\mathcal{T}$—in this context we view (full) triangulated subcategories and Verdier quotients as dull, trivial constructions. In more detail: the first whiff of such a recipe came in 2005 in Keller [7]. Keller proved that, given a triangulated category $\mathcal{S}$ and an automorphism $\sigma : \mathcal{S} \longrightarrow \mathcal{S}$, then the category $\mathcal{T} = \mathcal{S}/\sigma$ sometimes [rarely] has a triangulated structure so that the quotient map is triangulated—but the conditions are very stringent. And the only other known recipe was found in 2011 by Balmer [1]: given a separable monoid R in a tensor triangulated category $\mathcal{S}$, the category $\mathcal{T}$ whose objects are the R–modules in $\mathcal{S}$, and whose morphisms are those morphisms in $\mathcal{S}$ which respect the R–module structure, is triangulated. The distinguished triangles in $\mathcal{T}$ are precisely those sequences $T \longrightarrow T' \longrightarrow T'' \longrightarrow \Sigma T$ whose image in $\mathcal{S}$ is a distinguished triangle. OK: these two relatively recent exceptions aside, the accepted wisdom has long been that you need some enhancement to produce triangulated categories in a nontrivial way.

Theorem 15 gives a third recipe, and the natural question is whether the end product is of any value. Given an input triangulated category $\mathcal{S}$, together with its good metric, is the output category $\mathfrak{S}(\mathcal{S})$ a worthwhile object of study? And to answer this we need examples. In Example 12 we saw three ways to produce good metrics, out of any homological functor $H : \mathcal{S} \longrightarrow \mathcal{A}$. For each of these the question arises: what is the triangulated category $\mathfrak{S}(\mathcal{S})$? Is it of any interest?

In general I don't know how to compute $\mathfrak{S}(\mathcal{S})$. The only procedure I know so far assumes that $\mathcal{S}$ has an embedding into a larger triangulated category $\mathcal{T}$, and this embedding satisfies a strong condition. In the presence of such an embedding $\mathfrak{S}(\mathcal{S})$ may be computed as a triangulated subcategory of $\mathcal{T}$. Below we will spell out carefully the exact statements.

In order to make the above precise we need some more definitions—my apologies to the reader, we will get to the point in Theorem 20.

Definition 17. Let $\mathcal{S}$ be a triangulated category with a good metric. Suppose we are given a fully faithful triangulated functor $F : \mathcal{S} \longrightarrow \mathcal{T}$; we consider also the functor $\mathcal{Y} : \mathcal{T} \longrightarrow \text{Mod–}\mathcal{S}$, which takes an object $A \in \mathcal{T}$ to the functor $\text{Hom}\big(F(-), A\big)$. The functor F is called a *good extension with respect to the metric* if $\mathcal{T}$ has countable coproducts, and for every Cauchy sequence E_* in $\mathcal{S}$ the natural map $\underrightarrow{\text{colim}}\, \widetilde{Y}(E_*) \longrightarrow \mathcal{Y}\big(\underrightarrow{\text{Hocolim}}\, F(E_*)\big)$ is an isomorphism.

Explanation 18. The functors F, $\mathcal{Y}$ and $\widetilde{Y}$ are related by a canonical natural isomorphism $\widetilde{Y} \cong \mathcal{Y} \circ F$. And we remind the reader: given a sequence $T_1 \longrightarrow T_2 \longrightarrow T_3 \longrightarrow \cdots$ of composable morphisms in $\mathcal{T}$, the homotopy colimit is defined to be the third edge of the triangle

$$\coprod_{i=1}^{\infty} T_i \xrightarrow{\;\;\text{id} - \text{shift}\;\;} \coprod_{i=1}^{\infty} T_i \xrightarrow{\;\;\varphi\;\;} \underrightarrow{\text{Hocolim}}\; T_*$$

where $(\text{shift}) : \coprod_{i=1}^{\infty} T_i \longrightarrow \coprod_{i=1}^{\infty} T_i$ is the unique map rendering commutative, for any integer $n \geq 1$, the square below

$$
\begin{array}{ccc}
T_n & \longrightarrow & T_{n+1} \\
{\scriptstyle(\text{inc})_n}\big\downarrow & & \big\downarrow{\scriptstyle(\text{inc})_{n+1}} \\
\coprod_{i=1}^{\infty} T_i & \xdashrightarrow{\;\;\text{shift}\;\;} & \coprod_{i=1}^{\infty} T_i
\end{array}
$$

with (inc) being the canonical inclusion into the coproduct. The object $\underrightarrow{\mathrm{Hocolim}} \ T_*$ is only defined up to (non-canonical) isomorphism in $\mathcal{T}$, but the isomorphism can be assumed to respect the map φ. Hence any such isomorphism, between two candidates for $\underrightarrow{\mathrm{Hocolim}} \ T_*$, will respect all the composites $T_n \xrightarrow{\mathrm{inc}} \coprod_{i=1}^{\infty} T_i \xrightarrow{\varphi} \underrightarrow{\mathrm{Hocolim}} \ T_*$; we write these as $\varphi_n : T_n \longrightarrow \underrightarrow{\mathrm{Hocolim}} \ T_*$. The vanishing of the composite $\varphi \circ (\mathrm{id} - \mathrm{shift})$, in the displayed maps of the triangle above, guarantees that, for each integer $n \geq 1$, the composite $T_n \longrightarrow T_{n+1} \xrightarrow{\varphi_{n+1}} \underrightarrow{\mathrm{Hocolim}} \ T_*$ must be equal to $\varphi_n : T_n \longrightarrow \underrightarrow{\mathrm{Hocolim}} \ T_*$.

If we are given a sequence $E_1 \longrightarrow E_2 \longrightarrow E_3 \longrightarrow \cdots$ in the category $\mathcal{S}$, then the functor F takes it to a sequence $F(E_1) \longrightarrow F(E_2) \longrightarrow F(E_3) \longrightarrow \cdots$ in the category $\mathcal{T}$. The paragraph above gives, for each n, a map $\varphi_n : F(E_n) \longrightarrow \underrightarrow{\mathrm{Hocolim}} \ F(E_*)$. Applying to this the functor $\mathcal{Y}$ we deduce the second morphism in the composable pair below

$$\widetilde{Y}(E_n) \xrightarrow{\quad \sim \quad} \mathcal{Y}F(E_n) \xrightarrow{\ \mathcal{Y}(\varphi_n)\ } \mathcal{Y}\big(\underrightarrow{\mathrm{Hocolim}} \ F(E_i) \big)$$

where the first morphism comes from the canonical isomorphism $\widetilde{Y} \cong \mathcal{Y}F$. And these maps assemble to a single morphism $\underrightarrow{\mathrm{colim}}\, \widetilde{Y}(E_*) \longrightarrow \mathcal{Y}\big(\underrightarrow{\mathrm{Hocolim}} \ F(E_*) \big)$, unique up to (non-canonical) isomorphism. Hence postulating that this map is an isomorphism, as in Definition 17, makes sense independent of choices.

Note that this is a strong restriction. If $\mathcal{S}$ has countable coproducts we might be tempted to let F be the identity $\mathrm{id} : \mathcal{S} \longrightarrow \mathcal{S}$. But then it becomes a strong hypothesis to assume that the Cauchy sequences all satisfy the condition that $\underrightarrow{\mathrm{colim}}\, \widetilde{Y}(E_*) \longrightarrow \widetilde{Y}\big(\underrightarrow{\mathrm{Hocolim}} \ E_* \big)$ is an isomorphism.

Definition 19. Suppose $\mathcal{S}$ is a triangulated category with a good metric, and let $F : \mathcal{S} \longrightarrow \mathcal{T}$ be a good extension. We define

(i) The full subcategory $\widehat{\mathfrak{L}}(\mathcal{S}) \subset \mathcal{T}$ has for objects all the homotopy colimits of Cauchy sequences in $\mathcal{S}$.
(ii) The full subcategory $\widehat{\mathfrak{S}}(\mathcal{S}) \subset \mathcal{T}$ is given by the formula $\widehat{\mathfrak{S}}(\mathcal{S}) = \widehat{\mathfrak{L}}(\mathcal{S}) \cap \mathcal{Y}^{-1}\big(\mathfrak{C}(\mathcal{S}) \big)$, with $\mathfrak{C}(\mathcal{S}) \subset \mathrm{Mod}\text{–}\mathcal{S}$ as in Definition 5(ii).

Theorem 20. *The category $\widehat{\mathfrak{S}}(\mathcal{S})$ is a triangulated subcategory of $\mathcal{T}$, and the functor $\mathcal{Y} : \mathcal{T} \longrightarrow \mathrm{Mod}\text{–}\mathcal{S}$ restricts to a triangulated equivalence $\mathcal{Y} : \widehat{\mathfrak{S}}(\mathcal{S}) \longrightarrow \mathfrak{S}(\mathcal{S})$.*

The proof of a slightly stronger theorem [once again, the hypotheses on the metric are slightly less restrictive] may be found in [16, Theorem 3.15].

Remark 21. We have a fully faithful functor $\widetilde{Y} : \mathcal{S} \longrightarrow \mathrm{Mod}\text{–}\mathcal{S}$ and, in the presence of a good extension, another fully faithful functor $F : \mathcal{S} \longrightarrow \mathcal{T}$. If we confuse $\mathcal{S}$ with its essential images we can view it as a subcategory in each of $\mathrm{Mod}\text{–}\mathcal{S}$ and $\mathcal{T}$. And then we have subcategories $\widehat{\mathfrak{S}}(\mathcal{S}) \subset \mathcal{T}$ and $\mathfrak{S}(\mathcal{S}) \subset \mathrm{Mod}\text{–}\mathcal{S}$, and it's natural to wonder what one can say about the subcategories $\mathcal{S} \cap \widehat{\mathfrak{S}}(\mathcal{S}) \subset \mathcal{T}$ and $\mathcal{S} \cap \mathfrak{S}(\mathcal{S}) \subset \mathrm{Mod}\text{–}\mathcal{S}$. To avoid getting too confused, between the incarnation of $\mathcal{S}$ as a subcategory of $\mathcal{T}$ and as a subcategory of $\mathrm{Mod}\text{–}\mathcal{S}$, for most of this remark our notation will be careful; we will not confound $\mathcal{S}$ with either of its images.

The functor $F : \mathcal{S} \longrightarrow \mathcal{T}$ is a fully faithful, triangulated functor, while the subcategory $\widehat{\mathfrak{S}}(\mathcal{S}) \subset \mathcal{T}$ is triangulated. Hence $F^{-1}\big[\widehat{\mathfrak{S}}(\mathcal{S}) \big]$ is a triangulated subcategory of $\mathcal{S}$, and the functor F restricts to a fully faithful, triangulated functor $F^{-1}\big[\widehat{\mathfrak{S}}(\mathcal{S}) \big] \longrightarrow \widehat{\mathfrak{S}}(\mathcal{S})$.

Now $\widehat{\mathfrak{S}}(\mathcal{S})$ lies in $\widehat{\mathfrak{L}}(\mathcal{S}) \subset \mathcal{T}$, and the functor $\mathcal{Y} : \mathcal{T} \longrightarrow \mathrm{Mod}\text{–}\mathcal{S}$ obviously takes $\widehat{\mathfrak{L}}(\mathcal{S}) \subset \mathcal{T}$ to $\mathfrak{L}(\mathcal{S}) \subset \mathrm{Mod}\text{–}\mathcal{S}$; hence $\mathcal{Y}$ restricts to a functor $\mathcal{Y}|_{\widehat{\mathfrak{L}}(\mathcal{S})} : \widehat{\mathfrak{L}}(\mathcal{S}) \longrightarrow \mathfrak{L}(\mathcal{S})$. And it turns out to be easy to show that the functor $\mathcal{Y}|_{\widehat{\mathfrak{L}}(\mathcal{S})}$ is essentially surjective, full and conservative. This means: every object in $\mathfrak{L}(\mathcal{S})$ is isomorphic to an object in the image of $\mathcal{Y}|_{\widehat{\mathfrak{L}}(\mathcal{S})}$, any morphism in $\mathfrak{L}(\mathcal{S})$ between objects in the image of $\mathcal{Y}|_{\widehat{\mathfrak{L}}(\mathcal{S})}$ is in the image

of $\mathcal{Y}|_{\widehat{\mathfrak{L}}(\mathcal{S})}$, and a morphism in $\widehat{\mathfrak{L}}(\mathcal{S})$ is an isomorphism if and only if $\mathcal{Y}|_{\widehat{\mathfrak{L}}(\mathcal{S})}$ takes it to an isomorphism. And the relevance of this for us is that the commutative square

$$
\begin{array}{ccc}
\widehat{\mathfrak{S}}(\mathcal{S}) & \longhookrightarrow & \widehat{\mathfrak{L}}(\mathcal{S}) \\
\downarrow{\scriptstyle \mathcal{Y}|_{\widehat{\mathfrak{S}}(\mathcal{S})}} & & \downarrow{\scriptstyle \mathcal{Y}|_{\widehat{\mathfrak{L}}(\mathcal{S})}} \\
\mathfrak{S}(\mathcal{S}) & \longhookrightarrow & \mathfrak{L}(\mathcal{S})
\end{array}
$$

is a strict pullback square. The point is that, from their definitions, the categories $\widehat{\mathfrak{S}}(\mathcal{S})$ and $\mathfrak{S}(\mathcal{S})$ are replete subcategories in, respectively, $\mathcal{T}$ and Mod–$\mathcal{S}$; this means they contain all isomorphs of any of their objects. Theorem 20 tells us that the vertical map on the left is an equivalence—hence any object $x \in \mathfrak{S}(\mathcal{S})$ is isomorphic to $\mathcal{Y}(z)$ with z an object of $\widehat{\mathfrak{S}}(\mathcal{S})$. But if we have an object $t \in \widehat{\mathfrak{L}}(\mathcal{S})$ with $\mathcal{Y}(t) = x \cong \mathcal{Y}(z)$, then the isomorphism must lift to $\widehat{\mathfrak{L}}(\mathcal{S})$, and hence $t \cong z$ must belong to the replete subcategory $\widehat{\mathfrak{S}}(\mathcal{S}) \subset \widehat{\mathfrak{L}}(\mathcal{S})$.

Now recall the Yoneda embedding $\widetilde{Y} : \mathcal{S} \longrightarrow \mathfrak{L}(\mathcal{S}) \subset \text{Mod–}\mathcal{S}$. We have the triangle of functors

$$
\begin{array}{ccc}
 & \xrightarrow{\ F\ } & \widehat{\mathfrak{L}}(\mathcal{S}) \\
\mathcal{S} & & \downarrow{\scriptstyle \mathcal{Y}|_{\widehat{\mathfrak{L}}(\mathcal{S})}} \\
 & \xrightarrow[\widetilde{Y}]{} & \mathfrak{L}(\mathcal{S})
\end{array}
$$

which commutes up to natural isomorphism. It immediately follows that

$$
\widetilde{Y}^{-1}\big[\mathfrak{S}(\mathcal{S})\big] \quad = \quad F^{-1}\mathcal{Y}_{\widehat{\mathfrak{L}}(\mathcal{S})}^{-1}\big[\mathfrak{S}(\mathcal{S})\big] \quad = \quad F^{-1}\big[\widehat{\mathfrak{S}}(\mathcal{S})\big]
$$

The first equality is because the inverse images under the isomorphic functors $\widetilde{Y} \simeq F \circ \big[\mathcal{Y}|_{\widehat{\mathfrak{L}}(\mathcal{S})}\big]$, of the replete subcategory $\mathfrak{S}(\mathcal{S})$, must be equal. And the second equality comes from the paragraph above, which informs us that $\mathcal{Y}_{\widehat{\mathfrak{L}}(\mathcal{S})}^{-1}\big[\mathfrak{S}(\mathcal{S})\big] = \widehat{\mathfrak{S}}(\mathcal{S})$.

Rewriting the second paragraph of the current Remark, by appealing to the equality $\widetilde{Y}^{-1}\big[\mathfrak{S}(\mathcal{S})\big] = F^{-1}\big[\widehat{\mathfrak{S}}(\mathcal{S})\big]$, we deduce first that $\widetilde{Y}^{-1}\big[\mathfrak{S}(\mathcal{S})\big]$ is a triangulated subcategory of $\mathcal{S}$, and then that the functor $\widetilde{Y} : \mathcal{S} \longrightarrow \text{Mod–}\mathcal{S}$ restricts to a fully faithful, triangulated functor $\widetilde{Y}^{-1}\big[\mathfrak{S}(\mathcal{S})\big] \longrightarrow \mathfrak{S}(\mathcal{S})$.

All of the discussion above assumed we were in the presence of a good extension $F : \mathcal{S} \longrightarrow \mathcal{T}$. But the assertion of the last paragraph turns out to be robust. Even though the category $\mathfrak{L}(\mathcal{S})$ is rarely triangulated it contains both $\mathcal{S}$ and $\mathfrak{S}(\mathcal{S})$ as subcategories—in the case of $\mathfrak{S}(\mathcal{S})$ this is by definition, while for $\mathcal{S}$ we commit the notational crime of confusing $\mathcal{S}$ with its essential image under $\widetilde{Y} : \mathcal{S} \longrightarrow \mathfrak{L}(\mathcal{S})$. Each of $\mathcal{S}$ and $\mathfrak{S}(\mathcal{S})$ has its own triangulated structure. And it is always true that $\mathcal{S} \cap \mathfrak{S}(\mathcal{S})$ has a (unique) triangulated structure so that each of the two embeddings, into $\mathcal{S}$ and into $\mathfrak{S}(\mathcal{S})$, is triangulated.

The import of Theorem 20 and Remark 21 is that any good extension of $\mathcal{S}$ contains both $\mathcal{S}$ and $\mathfrak{S}(\mathcal{S})$ as triangulated subcategories, and the embedding of $\mathfrak{S}(\mathcal{S})$ into $\mathcal{T}$ is explicit enough to facilitate computations, both of $\mathfrak{S}(\mathcal{S})$ and of $\mathcal{S} \cap \mathfrak{S}(\mathcal{S})$. The author will be the first to admit that better computational tools would be wonderful—this is all we have right now.

Notwithstanding the current limitations on what we know, Theorem 20 does produce interesting examples. We give a few.

Example 22. Let $\mathcal{S}$ be the homotopy category of finite spectra. Let us remind the reader: the objects in this category may be taken to be pairs (X, n), where X is a pointed, finite CW-complex and $n \in \mathbb{Z}$ is an integer, positive or negative—the way to think of this is that the object (X, n) is the nth suspension

of X. And $\Sigma : \mathcal{S} \longrightarrow \mathcal{S}$ is the functor taking a pair (X, n) to the pair $(\Sigma X, n)$, where ΣX is the ordinary suspension of the pointed CW-complex X. The morphisms are precisely what one would expect, given that we want to force the functor $\Sigma : \mathcal{S} \longrightarrow \mathcal{S}$ to be invertible—for any two objects (X, m) and (Y, n) in $\mathcal{S}$, the abelian group $\text{Hom}_{\mathcal{S}}\big[(X, m), (Y, n)\big]$ is defined to be the colimit as $k \longrightarrow \infty$ of the (eventual) abelian groups $\text{Hom}_{\text{CW-complexes}}(\Sigma^{m+k} X, \Sigma^{n+k} Y)$. This means: if k is large enough, so that both $m + k$ and $n + k$ are ≥ 2, then the Hom-set above is the abelian group of homotopy equivalence classes of pointed continuous maps $\Sigma^{m+k} X \longrightarrow \Sigma^{n+k} Y$.

Let $H : \mathcal{S} \longrightarrow \text{Ab}$ be the homological functor which takes a spectrum (X, n) to its zeroth stable homotopy group; in the notation above this means

$$H(X, n) \quad = \quad \underrightarrow{\text{colim}} \, \text{Hom}_{\text{CW-complexes}}(\mathbb{S}^k, \Sigma^{n+k} X) \, ,$$

where $\mathbb{S}^k$ is the k–dimensional sphere. In the standard notation of homotopy theorists $H(X, n) = \pi_0(\Sigma^n X)$ and $H^i(X) = \pi_{-i}(\Sigma^n X)$, where π_{-i} is the $(-i)^{\text{th}}$ stable homotopy group. Now let the good metric be as in Example 12(i).

Let $F : \mathcal{S} \longrightarrow \mathcal{T}$ be the embedding of the homotopy category of finite spectra into the homotopy category of all spectra—the homotopy category of all spectra is not quite so easy to describe simply, hence let us leave this out. For us what's important is that the functor F can be shown to be a good extension. And the computation of $\widehat{\mathfrak{S}}(\mathcal{S})$, which by Theorem 20 is canonically triangle equivalent to $\mathfrak{S}(\mathcal{S})$, can be carried out. It shows that $\widehat{\mathfrak{S}}(\mathcal{S}) \subset \mathcal{T}$ is given by the formula

$$\widehat{\mathfrak{S}}(\mathcal{S}) \quad = \quad \left\{ x \in \mathcal{T} \,\middle|\, \begin{array}{l} H^i(x) = 0 \text{ for all but finitely many } i \in \mathbb{Z}, \text{ and} \\ H^i(x) \text{ is a finitely generated } \mathbb{Z}\text{–module for all } i \in \mathbb{Z} \end{array} \right\} .$$

The assertions in the paragraphs above follow from the far more general [16, Example 4.2].

With $\mathcal{S}$ still as above, it's known that $\mathcal{S} \cap \widehat{\mathfrak{S}}(\mathcal{S}) \cong \mathcal{S} \cap \mathfrak{S}(\mathcal{S}) = \{0\}$ and that $\mathcal{S}$ and $\mathfrak{S}(\mathcal{S})$ are not triangle equivalent. Let us recall.

Let $\mathbb{S} \in \mathcal{S}$ be the zero-sphere; in the notation of the first paragraph of the current Example this means $\mathbb{S} = (\mathbb{S}^n, -n)$ where $n > 0$ is an integer and $\mathbb{S}^n$ is the n–dimensional sphere. And let $K(\mathbb{Z}, 0)$ be the Eilenberg-MacLane spectrum defined by

$$\pi_{-i}\big[K(\mathbb{Z}, 0)\big] \quad = \quad H^i\big[K(\mathbb{Z}, 0)\big] \quad = \quad \begin{cases} \mathbb{Z} & \text{if } i = 0 \\ 0 & \text{otherwise} \end{cases}$$

Next we adopt the terminology of Bondal and Van den Bergh [3, 2.1]: an object G in a triangulated category $\mathcal{R}$ is a *classical generator* if the smallest thick subcategory containing G is all of $\mathcal{R}$. It's not difficult to see that $\mathbb{S}$ is a classical generator for $\mathcal{S}$ while $K(\mathbb{Z}, 0)$ is a classical generator for $\widehat{\mathfrak{S}}(\mathcal{S})$. But it's also known that

$$\begin{array}{ll} \text{Hom}(\mathbb{S}, \Sigma^n \mathbb{S}) = 0 & \text{if } 0 < n \\ \text{Hom}\big(K(\mathbb{Z}, 0), \Sigma^n K(\mathbb{Z}, 0)\big) = 0 & \text{if } n < 0 \end{array}$$

and that, for all $n \in \mathbb{Z}$, the modules above are finitely generated $\mathbb{Z}$–modules. The vanishing assertions are ancient, and the finite generation was proved in Serre's 1951 PhD thesis [18]. Somewhat more recent is the computation that

$$\text{Hom}\big(K(\mathbb{Z}, 0), \Sigma^n \mathbb{S}\big) = 0 \quad \text{for all } n \neq 1$$

while $\text{Hom}\big(K(\mathbb{Z}, 0), \Sigma \mathbb{S}\big)$ is a $\mathbb{Q}$–vector space: the reader can find this in Lin's 1976 article [13, Theorem 3.6]. The interested reader can look at Margolis 1974 article [14] for an (independent) approach to results

similar to Lin's, and at Ravenel [17, Section 4] for later developments and extensions. Anyway: because $\mathbb{S}$ and $K(\mathbb{Z}, 0)$ classically generate the respective subcategories, we immediately deduce

$$
\begin{array}{llll}
\text{For any pair } G, H \in \mathcal{S} & \mathrm{Hom}(G, \Sigma^n H) = 0 & \text{if } 0 \ll n \\
\text{For any pair } G, H \in \widehat{\mathfrak{S}}(\mathcal{S}) & \mathrm{Hom}\left(G, \Sigma^n H\right) = 0 & \text{if } n \ll 0 \\
\text{For any pair } G \in \widehat{\mathfrak{S}}(\mathcal{S}), H \in \mathcal{S} & \mathrm{Hom}\left(G, \Sigma^n H\right) = 0 & \text{if } |n| \gg 0
\end{array}
$$

while for all n we have

$$
\begin{array}{lll}
\text{For any pair } G, H \in \mathcal{S} & \mathrm{Hom}(G, \Sigma^n H) & \text{is a f.g. } \mathbb{Z}\text{-module} \\
\text{For any pair } G, H \in \widehat{\mathfrak{S}}(\mathcal{S}) & \mathrm{Hom}\left(G, \Sigma^n H\right) & \text{is a f.g. } \mathbb{Z}\text{-module} \\
\text{For any pair } G \in \widehat{\mathfrak{S}}(\mathcal{S}), H \in \mathcal{S} & \mathrm{Hom}\left(G, \Sigma^n H\right) & \text{is a } \mathbb{Q}\text{-vector space}
\end{array}
$$

What's more it's known that $\mathrm{Hom}(G, \Sigma^n G) \neq 0$ for infinitely many n, when G is either $\mathbb{S}$ or $K(\mathbb{Z}, 0)$; these estimates on the non-vanishing of Hom-sets can also be found in Serre [18]. It immediately follows that the categories $\mathcal{S}$ and $\widehat{\mathfrak{S}}(\mathcal{S})$ cannot be triangle equivalent, and that $\mathcal{S} \cap \widehat{\mathfrak{S}}(\mathcal{S}) \cong \mathcal{S} \cap \mathfrak{S}(\mathcal{S}) = \{0\}$.

Example 23. A different example comes about as follows. Let R be a noetherian ring, let $\mathbf{D}^b(R\text{–mod})$ be the derived category whose objects are the bounded complexes of finitely generated R–modules, and let $H : \mathbf{D}^b(R\text{–mod})) \longrightarrow (R\text{–mod})$ be the homological functor taking an object of $\mathbf{D}^b(R\text{–mod})$ to its zeroth cohomology module. We take on $\left[\mathbf{D}^b(R\text{–mod})\right]^{\mathrm{op}}$ the good metric given by applying Example 12(ii) to H^{op}.

Let $\mathbf{D}(R\text{–Mod})$ be the unbounded derived category of all complexes of R–modules. Then the natural inclusion $F : \left[\mathbf{D}^b(R\text{–mod})\right]^{\mathrm{op}} \longrightarrow \left[\mathbf{D}(R\text{–Mod})\right]^{\mathrm{op}}$ is a good extension with respect to the metric. And this can be used to compute $\widehat{\mathfrak{S}}\left(\left[\mathbf{D}^b(R\text{–mod})\right]^{\mathrm{op}}\right)$ as a subcategory of $\left[\mathbf{D}(R\text{–Mod})\right]^{\mathrm{op}}$. It turns out to be $\left[\mathcal{H}^0\left(\mathbf{Perf}(R)\right)\right]^{\mathrm{op}}$, where $\mathcal{H}^0\left(\mathbf{Perf}(R)\right)$ is the derived category whose objects are bounded complexes of finitely generated, projective R–modules. In this particular case there is an inclusion—and Remark 21 tells us that, with $\mathcal{S} = \mathbf{D}^b(R\text{–mod})$, the subcategories $\mathcal{S}^{\mathrm{op}}$ and $\mathfrak{S}(\mathcal{S}^{\mathrm{op}})$ of $\mathfrak{L}(\mathcal{S}^{\mathrm{op}})$ satisfy $\mathfrak{S}(\mathcal{S}^{\mathrm{op}}) \subset \mathcal{S}^{\mathrm{op}}$. Furthermore this inclusion respects the triangulated structure.

The assertions in the paragraph above follow from the far more general [16, Proposition 5.6].

Thus out of the category $\mathbf{D}^b(R\text{–mod})$ we have cooked up its triangulated subcategory $\mathcal{H}^0\left(\mathbf{Perf}(R)\right)$, and hence we also know the quotient

$$
\mathbf{D}_{\mathrm{sing}}(R) \quad = \quad \frac{\mathbf{D}^b(R\text{–mod})}{\mathcal{H}^0\left(\mathbf{Perf}(R)\right)} \ ,
$$

where $\mathbf{D}_{\mathrm{sing}}(R)$ is what's known in the literature as the singularity category of R.

OK: the paragraphs above showed that, given the category $\mathbf{D}^b(R\text{–mod})$ and its metric, then out of the data we can construct the triangulated subcategory $\mathcal{H}^0\left(\mathbf{Perf}(R)\right)$ and the quotient $\mathbf{D}_{\mathrm{sing}}(R)$. The reader might naturally ask if there is a way to construct the metric without appealing to the homological functor $H : \mathbf{D}^b(R\text{–mod}) \longrightarrow (R\text{–mod})$. The answer turns out to be Yes up to equivalence. The equivalence class of the metric can be obtained without using anything other than the triangulated structure on $\mathbf{D}^b(R\text{–mod})$—we will say a little more about this in Remark 24.

It is also possible to construct examples where the inclusion goes the other way, that is $\mathcal{S} \subset \mathfrak{S}(\mathcal{S})$. In fact: starting with the triangulated category $\mathcal{S} = \mathcal{H}^0\left(\mathbf{Perf}(R)\right)$ and the homological functor $H : \mathcal{H}^0\left(\mathbf{Perf}(R)\right) \longrightarrow (R\text{–mod})$, the functor taking a cochain complex to its zeroth cohomology module, we can endow $\mathcal{S}$ with the good metric of Example 12(i). And then it may be computed that $\mathfrak{S}(\mathcal{S}) = \mathbf{D}^b(R\text{–mod})$, and as subcategories of $\mathfrak{L}(\mathcal{S})$ we have an inclusion $\mathcal{S} \subset \mathfrak{S}(\mathcal{S})$ which agrees with the standard triangulated inclusion $\mathcal{H}^0\left(\mathbf{Perf}(R)\right) \subset \mathbf{D}^b(R\text{–mod})$. Once again: the equivalence class of the metric has an intrinsic description, it depends only on the triangulated structure of $\mathcal{S} = \mathcal{H}^0\left(\mathbf{Perf}(R)\right)$. Thus $\mathcal{H}^0\left(\mathbf{Perf}(R)\right)$ also

contains enough data to determine $\mathbf{D}_{\mathrm{sing}}(R)$. See Remark 24 for some elaboration, and [16] for infinitely more detail and much greater generality.

Remark 24. In Example 23 we asserted that, up to equivalence, certain metrics have intrinsic descriptions. It's time to explain this.

Let $\mathcal{S}$ be a triangulated category and let $G \in \mathcal{S}$ be a classical generator—recall: classical generators were defined in Bondal and Van den Bergh [3, 2.1], and we have already met them in Example 22. The functor $\mathrm{Hom}(G,-)$ is a homological functor, and we may apply the formulas of Example 12 to obtain metrics. It is an easy exercise to show that, if $G, G' \in \mathcal{S}$ are two classical generators, then the metrics obtained by applying Example 12 (i), (ii) or (iii) to the functor $\mathrm{Hom}(G,-)$ are (respectively) equivalent to the metrics obtained by applying Example 12 (i), (ii) or (iii) to the functor $\mathrm{Hom}(G',-)$. Thus on any triangulated category with a classical generator there are three intrinsic equivalence classes of metrics. For the sake of definiteness let us call them $\mathcal{L}ength_{(i)}^{\mathrm{Int}}$ $\mathcal{L}ength_{(ii)}^{\mathrm{Int}}$ and $\mathcal{L}ength_{(iii)}^{\mathrm{Int}}$. To spell it out explicitly: to obtain a metric in the equivalence class $\mathcal{L}ength_{(i)}^{\mathrm{Int}}$ you choose a classical generator G for the category $\mathcal{S}$, and then the formula of Example 12(i), applied to the homological functor $\mathrm{Hom}(G,-)$, delivers your metric.

The category $\mathcal{H}^0\big(\mathbf{Perf}(R)\big)$ is known to have a classical generator, more precisely the object $R \in \mathcal{H}^0\big(\mathbf{Perf}(R)\big)$ is a classical generator. To obtain a metric in the equivalence class $\mathcal{L}ength_{(i)}^{\mathrm{Int}}$ we begin by choosing R for our classical generator, and then we apply the formula of Example 12(i) to the homological functor $\mathrm{Hom}(R,-)$. But there is a canonical isomorphism $H(-) = \mathrm{Hom}(R,-)$, between of the functor H of final paragraph of Example 23 and the functor $\mathrm{Hom}(R,-)$. Hence the metric of the final paragraph of Example 23 is in the equivalence class $\mathcal{L}ength_{(i)}^{\mathrm{Int}}$, it is intrinsic up to equivalence.

We also asserted that the metric on $\big[\mathbf{D}^b(R\text{–mod})\big]^{\mathrm{op}}$, which was studied in the opening paragraphs of Example 23, is intrinsic up to equivalence. Here the argument is subtler, partly because (as far as I know) the category $\mathbf{D}^b(R\text{–mod})$ does not have to have a classical generator. The main idea on how to define the metric intrinsically is presented in the opening paragraphs of the introduction to [16].

There are cases in which the category $\mathbf{D}^b(R\text{–mod})$ is known to have a classical generator, and moreover this classical generator is known to satisfy some additional nice properties: it is "strong"—whatever that means—and one has further technical knowledge about it. When this happens to be the case the intrinsic description of the metric simplifies. Concretely: if the ring R is commutative, noetherian, has finite Krull-dimension and every closed subset of $\mathrm{Spec}(R)$ admits a regular alteration, then $\mathbf{D}^b(R\text{–mod})$ has a classical generator, and the metric on $\big[\mathbf{D}^b(R\text{–mod})\big]^{\mathrm{op}}$ given in the first paragraph of Example 23 is in the equivalence class $\mathcal{L}ength_{(ii)}^{\mathrm{Int}}$.

Example 25. In Remark 6 we briefly mentioned the article [8] by Krause. Let us discuss his work a little more fully.

As in the article [16], Krause [8] comes up with a procedure that produces out of $\mathcal{S}$ the category we call $\mathfrak{L}(\mathcal{S})$—Krause's recipe is different from Definition 5(i), but out of a different oven comes exactly the same dish. For the reader interested in looking up Krause [8] for more detail: the category we call $\mathfrak{L}(\mathcal{S})$ is canonically equivalent to what goes by the name $\widehat{\mathcal{S}}$ in [8]. And just as in [16] Krause looks at the special case where $\mathcal{S} = \mathcal{H}^0\big(\mathbf{Perf}(R)\big)$ as in Example 23. With $H : \mathcal{H}^0\big(\mathbf{Perf}(R)\big) \longrightarrow (R\text{–mod})$ the homological functor of the last paragraphs of Example 23, Krause studies the metric of Example 12(iii)—this is where his treatment radically differs from Example 23, where our metric was the one of Example 12(i). Because we now have two metrics let us denote the completion of Example 23 by $\mathfrak{L}_1(\mathcal{S})$ and Krause's completion by $\mathfrak{L}_2(\mathcal{S})$. It isn't difficult to show that, inside the category Mod–$\mathcal{S}$, there is an inclusion $\mathfrak{L}_1(\mathcal{S}) \subset \mathfrak{L}_2(\mathcal{S})$.

Of course the category $\mathfrak{S}_1(\mathcal{S}) \cong \mathbf{D}^b(R\text{–mod})$, being a full subcategory of $\mathfrak{L}_1(\mathcal{S})$, is also a full subcategory of $\mathfrak{L}_2(\mathcal{S})$. And there is an intrinsic description of the subcategory $\mathfrak{S}_1(\mathcal{S}) \subset \mathfrak{L}_2(\mathcal{S})$, the reader can find it in [8]. Krause denotes it $\widehat{\mathcal{S}}^b$, I suppose in our notation it should be $\mathfrak{L}_2(\mathcal{S})^b$.

Next we come to the triangles. With respect to the metric of Example 12(i) the triangles in $\mathfrak{S}_1(\mathcal{S})$ are simply the colimits of Cauchy sequences of triangles in $\mathcal{S}$—subject of course to the restriction that the colimit lies in $\mathfrak{S}_1(\mathcal{S}) \subset \mathfrak{L}_1(\mathcal{S})$. Theorem 15 tells us that, with the triangles defined as above, the category $\mathfrak{S}_1(\mathcal{S})$ is triangulated.

Now let's work out some consequences. Let $F' \longrightarrow F$ be any morphism in $\mathfrak{S}_1(\mathcal{S})$; it can be expressed as the colimit of a Cauchy sequence of morphisms $s'_* \longrightarrow s_*$ in $\mathcal{S}$, where the metric on $\mathcal{S}$ is as in Example 12(i). We may (non-canonically) complete this to a sequence of triangles $s'_* \longrightarrow s_* \longrightarrow s''_* \longrightarrow \Sigma s'_*$ in $\mathcal{S}$, and the reader can easily check that this sequence is Cauchy in the metric of Example 12(i) and the colimit lies in $\mathfrak{S}_1(\mathcal{S})$. By Definition 13 the colimit is a distinguished triangle in $\mathfrak{S}_1(\mathcal{S})$ extending the morphism $F' \longrightarrow F$. But from the axioms of triangulated categories the extension of the morphism $F' \longrightarrow F$ to a distinguished triangle in $\mathfrak{S}_1(\mathcal{S})$ is unique up to non-canonical isomorphism. It follows that the sequences of triangles in $\mathcal{S}$ of the form $s'_* \longrightarrow s_* \longrightarrow s''_* \longrightarrow \Sigma s'_*$, extending the given Cauchy sequence of morphisms $s'_* \longrightarrow s_*$, must all be non-canonically Ind-isomorphic.

This can be proved, but the proof I know relies heavily on the fact that the colimit lies in the subcategory $\mathfrak{S}_1(\mathcal{S}) \subset \mathfrak{L}_1(\mathcal{S})$, which was chosen carefully in terms of the metric. Whereas Krause's definition of $\mathfrak{L}_2(\mathcal{S})^b \subset \mathfrak{L}_2(\mathcal{S})$ makes no mention of the metric.

Now let's compare the Cauchy sequences with respect to the two metrics under consideration. Since the metric of Example 12(i) is finer than the metric of Example 12(iii) there are more Cauchy sequences with respect to Krause's metric—this is what leads to the (proper) inclusion $\mathfrak{L}_1(\mathcal{S}) \subset \mathfrak{L}_2(\mathcal{S})$. As it turns out any Cauchy sequence of morphisms $s'_* \longrightarrow s_*$, with respect to the metric of Example 12(iii) and whose colimit happens to lie in $\mathfrak{L}_1(\mathcal{S}) \subset \mathfrak{L}_2(\mathcal{S})$, is Ind-isomorphic to a Cauchy sequence $t'_* \longrightarrow t_*$ with respect to the metric of Example 12(i). So we might be tempted to guess that the Cauchy sequences of triangles with respect to the metric of Example 12(iii), with colimits in $\mathfrak{S}_1(\mathcal{S}) = \mathfrak{L}_2(\mathcal{S})^b$, will also be Ind-isomorphic to Cauchy sequences of triangles with respect to the metric of Example 12(i). But if we try to produce such an Ind-isomorphism we run into the problem that the mapping cone isn't functorial—the simple-minded approach breaks down. As the definition of $\mathfrak{L}_2(\mathcal{S})^b \subset \mathfrak{L}_2(\mathcal{S})$ doesn't involve the metric I see no sophisticated alternative to the simple-minded method—for all I know there might be Cauchy sequences of triangles, with respect to the metric of Example 12(iii) and with colimit in $\mathfrak{S}_1(\mathcal{S})$, which aren't Ind-isomorphic to Cauchy sequences of triangles with respect to the metric of Example 12(i).

Krause's solution to the problem is to fix an enhancement, and only admit those Cauchy sequences that lift to the chosen enhancement. For the situation at hand a minimal enhancement suffices—it's enough to assume we are working with Keller's towers, see Keller [6] for the original exposition, or his appendix to Krause [8] for a condensed version.

Of course it is possible to apply the machinery surveyed here to Krause's metric—we obtain a triangulated category $\mathfrak{S}_2(\mathcal{S})$ whose triangulated structure is enhancement-free. Using a good extension with respect to the metric and Theorem 20 it can be computed that $\mathfrak{S}_2(\mathcal{S})$ is a proper subcategory of $\mathfrak{S}_1(\mathcal{S}) \cong \mathbf{D}^b(R\text{–mod})$—the objects are those complexes in $\mathbf{D}^b(R\text{–mod})$ which have bounded injective resolutions. For more detail the reader is referred to [16, Example 4.9].

We have said it before but repeat for emphasis: Theorem 20 is at present the only computational tool we have. It would be great to have some more ways to compute $\mathfrak{S}(\mathcal{S})$.

Remark 26. Krause's metric can be defined on other triangulated categories, for example on the homotopy category of finite spectra of Example 22. Thus the category of finite spectra also has two completions $\mathfrak{L}_1(\mathcal{S}) \subset \mathfrak{L}_2(\mathcal{S})$, the one corresponding to the metric of Example 22 being contained in the one with respect to Krause's metric. In Barthel's appendix to Krause's paper [8, Appendix A] there is a proof that, just as in the case of the $\mathcal{S}$ studied in Example 25, for this $\mathcal{S}$ too we have the equality $\mathfrak{S}_1(\mathcal{S}) = \mathfrak{L}_2(\mathcal{S})^b$.

Remark 27. In this survey we've tried to convince the reader that good metrics on triangulated categories can be useful. We've only touched on what's possible—the reader interested in more theorems in this vein is referred to the longer and more extensive survey [15]. To give one instance of a result in [15] which is immediately relevant to our discussion above: Examples 22 and 23 may look quite different, but both can be obtained as special cases of a single, much more general example.

Acknowledgements

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