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On quantum toroidal algebra of type A_1

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ABSTRACT

In this paper we introduce a new quantum algebra which specializes to the 2-toroidal Lie algebra of type A_1. We prove that this quantum toroidal algebra has a natural triangular decomposition, a (topological) Hopf algebra structure and a vertex operator realization.

1. Introduction

Let $\dot{\mathfrak{g}}$ be a finite dimensional simple Lie algebra over $\mathbb{C}$. The universal central extension $\mathfrak{t}(\dot{\mathfrak{g}})$ of the 2-loop algebra $\dot{\mathfrak{g}} \otimes \mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}]$, called the toroidal Lie algebra, has a celebrated presentation given by Moody-Rao-Yokonuma [9] for constructing the vertex representation for $\mathfrak{t}(\dot{\mathfrak{g}})$. In understanding the Langlands reciprocity for algebraic surfaces, Ginzburg-Kapranov-Vasserot [3] introduced a notion of quantum toroidal algebra $\mathcal{U}_\hbar(\dot{\mathfrak{g}}_{tor})$ associated to $\dot{\mathfrak{g}}$. The algebra $\mathcal{U}_\hbar(\dot{\mathfrak{g}}_{tor})$ specializes to the Moody-Rao-Yokonuma presentation of $\mathfrak{t}(\dot{\mathfrak{g}})$ in general, except for $\dot{\mathfrak{g}}$ in type A_1 when $\mathcal{U}_\hbar(\dot{\mathfrak{g}}_{tor})$ specializes to a *proper* quotient of the latter [2].

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The theory of quantum toroidal algebras has been extensively studied, especially with a rich representation theory developed by Hernandez [5,6] and others, see [7] for a survey. One notices that two major structural properties of $\mathcal{U}_\hbar(\dot{\mathfrak{g}}_{tor})$ have played a fundamental role in Hernandez's work: the triangular decomposition and the (deformed) Drinfeld coproduct.

Let A be the generalized Cartan matrix associated to the affine Lie algebra $\mathfrak{g}$ of $\dot{\mathfrak{g}}$. When A is symmetric, by using the vertex operators calculus, Jing introduced in [8] a quantum affinization algebra $\mathcal{U}_\hbar(\hat{\mathfrak{g}})$ associated to $\mathfrak{g}$. Meanwhile, it is remarkable that finite dimensional representations of $\mathcal{U}_\hbar(\hat{\mathfrak{g}})$ were studied by Nakajima in [10] using powerful geometric approach of quiver varieties. If A is of simply-laced type, then $\mathcal{U}_\hbar(\hat{\mathfrak{g}})$ is nothing but the quantum toroidal algebra $\mathcal{U}_\hbar(\dot{\mathfrak{g}}_{tor})$. However, for the case that A is not of simply-laced type, the definition of $\mathcal{U}_\hbar(\hat{\mathfrak{g}})$ is slightly different from that of $\mathcal{U}_\hbar(\dot{\mathfrak{g}}_{tor})$. Explicitly, one notices that $A_1^{(1)}$ is the unique symmetric but non-simply-laced affine generalized Cartan matrix. In this case, the defining currents $x_0^\pm(z), x_1^\pm(z)$ in $\mathcal{U}_\hbar(\hat{\mathfrak{g}})$ satisfy the relation

$$(z-q^{\mp 2}w)(z-w)x_0^\pm(z)x_1^\pm(w)=(q^{\mp 2}z-w)(z-w)x_1^\pm(w)x_0^\pm(z), \tag{1.1}$$

which appeared naturally in calculation of quantum vertex operators [8] and equivariant K-homology of quiver varieties [10]. In particular, $\mathcal{U}_\hbar(\hat{\mathfrak{g}})$ specializes to the toroidal Lie algebra $\mathfrak{t}(\dot{\mathfrak{g}})$ of type A_1 as the classical limit of (1.1) holds in $\mathfrak{t}(\dot{\mathfrak{g}})$. On the other hand, in $\mathcal{U}_\hbar(\dot{\mathfrak{g}}_{tor})$ these two currents satisfy the relation

$$(z-q^{\mp 2}w)x_0^\pm(z)x_1^\pm(w)=(q^{\mp 2}z-w)x_1^\pm(w)x_0^\pm(z). \tag{1.2}$$

This stronger relation was needed in verifying the compatibility with affine quantum Serre relations in $\mathcal{U}_\hbar(\dot{\mathfrak{g}}_{tor})$ so that it processes a canonical triangular decomposition [5]. For the algebra $\mathcal{U}_\hbar(\hat{\mathfrak{g}})$, we only know that it has a weak form of triangular decomposition [10].

From now on, we assume that $\dot{\mathfrak{g}}$ is of type A_1. The main goal of this paper is to define a "middle" quantum algebra

$$\mathcal{U}_\hbar(\hat{\mathfrak{g}}) \twoheadrightarrow \mathcal{U} \twoheadrightarrow \mathcal{U}_\hbar(\dot{\mathfrak{g}}_{tor})$$

of $\mathcal{U}_\hbar(\hat{\mathfrak{g}})$ and $\mathcal{U}_\hbar(\dot{\mathfrak{g}}_{tor})$, and prove that this new quantum toroidal algebra $\mathcal{U}$ processes the "good" properties enjoyed by both $\mathcal{U}_\hbar(\hat{\mathfrak{g}})$ and $\mathcal{U}_\hbar(\dot{\mathfrak{g}}_{tor})$. Explicitly, we first introduce in Section 2 a quantum algebra $\mathcal{U}$ which specializes to the toroidal Lie algebra $\mathfrak{t}(\dot{\mathfrak{g}})$. By definition, $\mathcal{U}$ is the quotient algebra of $\mathcal{U}_\hbar(\hat{\mathfrak{g}})$ obtained by modulo the relation

$$[x_0^\pm(z_1),(z_2-q^{\mp 2}w)x_0^\pm(z_2)x_1^\pm(w)-(q^{\mp 2}z_2-w)x_1^\pm(w)x_0^\pm(z_2)]=0. \tag{1.3}$$

One notices that $\mathcal{U}_\hbar(\dot{\mathfrak{g}}_{tor})$ is a quotient algebra of $\mathcal{U}$ as the relation (1.2) implies the relations (1.1) and (1.3). In Section 3, we prove that $\mathcal{U}$ admits a triangular decomposition (see Theorem 3.1). In Section 4, we prove that $\mathcal{U}$ has a deformed Drinfeld coproduct (see Theorem 4.1). As in [4], this allows us to define a (topological) Hopf algebra structure on $\mathcal{U}$ (see Theorem 4.2). As usual, the crucial step in establishing Theorems 3.1 and 4.1 is to check the compatibility with affine quantum Serre relations, in which the new relation (1.3) appeared naturally (see (3.10) and (4.6)). Finally, in Section 5 we point out that the quantum vertex operators constructed in [8] satisfy the relation (1.3), so we obtain a vertex representation for $\mathcal{U}$.

Throughout this paper, we denote by $\mathbb{C}[[\hbar]]$ the ring of complex formal series in one variable $\hbar$. By a $\mathbb{C}[[\hbar]]$-algebra, we mean a topological algebra over $\mathbb{C}[[\hbar]]$ with respect to the $\hbar$-adic topology. For $n,k,s\in\mathbb{Z}$ with $0\le k\le s$, we denote the usual quantum numbers as follows

$$[n]_q=\frac{q^n-q^{-n}}{q-q^{-1}},\quad [s]_q!=[s]_q[s-1]_q\cdots[1]_q,\quad \binom{s}{k}_q=\frac{[s]_q!}{[k]_q![s-k]_q!},$$

where

$$q = \exp(\hbar) \in \mathbb{C}[[\hbar]].$$

2. Quantum toroidal algebra of type A_1

In this section we introduce a new quantum algebra which specializes to toroidal Lie algebra of type A_1. Let

$$A = (a_{ij})_{i,j=0,1} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} \tag{2.1}$$

be the generalized Cartan matrix of type $A_1^{(1)}$. For $i, j = 0, 1$, let

$$g_{ij}(z) = \frac{q^{a_{ij}} - z}{1 - q^{a_{ij}} z} \tag{2.2}$$

be the formal Taylor series at $z = 0$. The following is the main object of this paper:

Definition 2.1. The quantum toroidal algebra $\mathcal{U}$ is the $\mathbb{C}[[\hbar]]$-algebra topologically generated by the elements

$$h_{i,n},\ x_{i,n}^{\pm},\ c \quad i = 0, 1,\ n \in \mathbb{Z}, \tag{2.3}$$

and subject to the relations in terms of generating functions in z:

$$\phi_i^{\pm}(z) = q^{\pm h_{i,0}} \exp\left(\pm(q - q^{-1}) \sum_{\pm n > 0} h_{i,n} z^{-n}\right), \quad x_i^{\pm}(z) = \sum_{n \in \mathbb{Z}} x_{i,n}^{\pm} z^{-n}.$$

The relations are:

(Q1) c is central, $[\phi_i^{\pm}(z), \phi_j^{\pm}(w)] = 0$,

(Q2) $\phi_i^{+}(z)\phi_j^{-}(w) = \phi_j^{-}(w)\phi_i^{+}(z) g_{ij}(q^{c}w/z)^{-1} g_{ij}(q^{-c}w/z)$,

(Q3) $\phi_i^{+}(z)x_j^{\pm}(w) = x_j^{\pm}(w)\phi_i^{+}(z) g_{ij}(q^{\mp\frac{1}{2}c}w/z)^{\pm 1}$,

(Q4) $\phi_i^{-}(z)x_j^{\pm}(w) = x_j^{\pm}(w)\phi_i^{-}(z) g_{ji}(q^{\mp\frac{1}{2}c}z/w)^{\mp 1}$,

(Q5) $[x_i^{+}(z), x_j^{-}(w)] = \dfrac{\delta_{ij}}{q - q^{-1}}\left(\phi_i^{+}(zq^{-\frac{1}{2}c})\delta\left(\dfrac{q^{c}w}{z}\right) - \phi_i^{-}(zq^{\frac{1}{2}c})\delta\left(\dfrac{q^{-c}w}{z}\right)\right)$,

(Q6) $(z - q^{\pm 2}w)x_i^{\pm}(z)x_i^{\pm}(w) = (q^{\pm 2}z - w)x_i^{\pm}(w)x_i^{\pm}(z)$,

(Q7) $(z - q^{\mp 2}w)(z - w)x_i^{\pm}(z)x_j^{\pm}(w) = (q^{\mp 2}z - w)(z - w)x_j^{\pm}(w)x_i^{\pm}(z)$,

(Q8) $[x_i^{\pm}(z_1), ((z_2 - q^{\mp 2}w)x_i^{\pm}(z_2)x_j^{\pm}(w) - (q^{\mp 2}z - w)x_j^{\pm}(w)x_i^{\pm}(z_2))] = 0$,

(Q9) $\displaystyle\sum_{\sigma \in S_3} \sum_{r=0}^{3} (-1)^r \binom{3}{r}_q x_i^{\pm}(z_{\sigma(1)}) \cdots x_i^{\pm}(z_{\sigma(r)}) x_j^{\pm}(w) x_i^{\pm}(z_{\sigma(r+1)}) \cdots x_i^{\pm}(z_{\sigma(3)}) = 0$,

where $i, j = 0, 1$ with $i \neq j$ in (Q7), (Q8), (Q9) and $\delta(z) = \sum_{n \in \mathbb{Z}} z^n$ is the usual δ-function.

Remark 2.2. As indicated in Introduction, in literature there have been two other definitions of quantum toroidal algebra of type A_1: the algebra $\mathcal{U}_\hbar(\hat{\mathfrak{g}})$ introduced in [8,10] and the algebra $\mathcal{U}_\hbar(\dot{\mathfrak{g}}_{tor})$ introduced in [3,5]. By definition, the algebra $\mathcal{U}_\hbar(\hat{\mathfrak{g}})$ is the $\mathbb{C}[[\hbar]]$-algebra topologically generated by the elements as in (2.3) with relations (Q1)-(Q7) and (Q9), while the algebra $\mathcal{U}_\hbar(\dot{\mathfrak{g}}_{tor})$ is the $\mathbb{C}[[\hbar]]$-algebra topologically generated by the elements as in (2.3) with relations (Q1)-(Q6), (Q9) and the following relation

$$(z-q^{\mp 2}w)x_i^{\pm}(z)x_j^{\pm}(w)=(q^{\mp 2}z-w)x_j^{\pm}(w)x_i^{\pm}(z),\quad i\neq j\in\{0,1\}. \tag{2.4}$$

By definition our new algebra $\mathcal{U}$ is a quotient algebra of $\mathcal{U}_\hbar(\hat{\mathfrak{g}})$, while $\mathcal{U}_\hbar(\dot{\mathfrak{g}}_{tor})$ is a quotient of $\mathcal{U}$.

Now we recall the definition of the toroidal Lie algebra of type A_1. Let $\mathcal{K}$ be the $\mathbb{C}$-vector space spanned by the symbols

$$t_1^{m_1}t_2^{m_2}\mathrm{k}_i,\quad i=1,2,\ m_1,m_2\in\mathbb{Z}$$

subject to the relations

$$m_1t_1^{m_1}t_2^{m_2}\mathrm{k}_1+m_2t_1^{m_1}t_2^{m_2}\mathrm{k}_2=0.$$

Let $\dot{\mathfrak{g}}=\mathfrak{sl}_2(\mathbb{C})$ be the simple Lie algebra of type A_1 and $\langle\cdot,\cdot\rangle$ the Killing form on $\dot{\mathfrak{g}}$. The toroidal Lie algebra (see [9])

$$\mathfrak{t}=\mathfrak{t}(\dot{\mathfrak{g}})=\left(\dot{\mathfrak{g}}\otimes\mathbb{C}[t_1^{\pm1},t_2^{\pm1}]\right)\oplus\mathcal{K}$$

is the universal central extension of the double loop algebra $\dot{\mathfrak{g}}\otimes\mathbb{C}[t_1^{\pm1},t_2^{\pm1}]$, where $\mathcal{K}$ is the center space and

$$[x\otimes t_1^{m_1}t_2^{m_2},y\otimes t_1^{n_1}t_2^{n_2}]=[x,y]\otimes t_1^{m_1+n_1}t_2^{m_2+n_2}+\langle x,y\rangle(\sum_{i=1}^{2}m_it_1^{m_1+n_1}t_2^{m_2+n_2}\mathrm{k}_i),$$

for $x,y\in\dot{\mathfrak{g}}$ and $m_1,m_2,n_1,n_2\in\mathbb{Z}$.

Let $\{e^+,\alpha,e^-\}$ be a standard $\mathfrak{sl}_2$-triple in $\dot{\mathfrak{g}}$, that is,

$$[e^+,e^-]=\alpha,\quad [\alpha,e^{\pm}]=\pm2e^{\pm}.$$

For $i=0,1$ and $m\in\mathbb{Z}$, set

$$\alpha_{1,m}=\alpha\otimes t_2^m,\ \alpha_{0,m}=t_2^mk_1-\alpha\otimes t_2^m,\ e_{1,m}^{\pm}=e^{\pm}\otimes t_2^m,\ e_{0,m}^{\pm}=e^{\mp}\otimes t_1^{\pm1}t_2^m.$$

Note that these elements generate the algebra $\mathfrak{t}$.

Following [9], we have:

Proposition 2.3. *The toroidal Lie algebra $\mathfrak{t}$ is abstractly generated by the elements $\alpha_{i,m},e_{i,m}^{\pm},\mathrm{k}_2$ for $i=0,1,m\in\mathbb{Z}$ with relations*

(L1) $[\mathrm{k}_2,\mathfrak{t}]=0,\ [\alpha_{i,m},\alpha_{j,n}]=a_{ij}\delta_{m+n,0}m\mathrm{k}_2,$

(L2) $[\alpha_{i,m},e_{j,n}^{\pm}]=\pm a_{ij}e_{j,m+n},$

(L3) $[e_{i,m}^{+},e_{j,n}^{-}]=\delta_{ij}\left(\alpha_{j,m+n}+m\delta_{m+n,0}\mathrm{k}_2\right),$

(L4) $(z-w)[e_i^{\pm}(z),e_i^{\pm}(w)]=0,$

$$\begin{aligned}
&\text{(L5)} \quad (z-w)^2[e_i^{\pm}(z), e_j^{\pm}(w)] = 0, \quad i \neq j,\\
&\text{(L6)} \quad (z_2-w)[e_i^{\pm}(z_1), [e_i^{\pm}(z_2), e_j^{\pm}(w)]] = 0, \quad i \neq j,\\
&\text{(L7)} \quad [e_i^{\pm}(z_1), [e_i^{\pm}(z_2), [e_i^{\pm}(z_3), e_j^{\pm}(w)]]] = 0, \quad i \neq j,
\end{aligned}$$

where $i, j = 0, 1,\ m, n \in \mathbb{Z}$ *and* $e_i^{\pm}(z) = \sum_{n\in\mathbb{Z}} e_{i,n}^{\pm} z^{-n}$.

Proof. Denote by $\mathcal{L}$ the Lie algebra abstractly generated by the elements $\alpha_{i,m}, e_{i,m}^{\pm}, \mathrm{k}_2$ for $i = 0, 1, m \in \mathbb{Z}$ with relations (L1)-(L7). One easily checks that the relations (L1)-(L7) hold in $\mathfrak{t}$ and so we have a surjective Lie homomorphism ψ from $\mathcal{L}$ to $\mathfrak{t}$. On the other hand, denote by $\mathcal{L}'$ the Lie algebra abstractly generated by the elements $\alpha_{i,m}, e_{i,m}^{\pm}, \mathrm{k}_2$ for $i = 0, 1, m \in \mathbb{Z}$ with relations (L1)-(L4) and (L7). Then there is a quotient map, say φ, from $\mathcal{L}'$ to $\mathcal{L}$. It was proved in [9] that the surjective homomorphism $\psi \circ \varphi : \mathcal{L}' \to \mathcal{L} \to \mathfrak{t}$ is an isomorphism, noting that the relation (L4) is equivalent to the relation $[e_i^{\pm}(z), e_i^{\pm}(w)] = 0$. This in turn implies that the map ψ is an isomorphism, as required. □

By combining Definition 2.1 with Proposition 2.3, we have the following result.

Theorem 2.4. *The classical limit* $\mathcal{U}/\hbar\mathcal{U}$ *of* $\mathcal{U}$ *is isomorphic to the universal enveloping algebra* $\mathcal{U}(\mathfrak{t})$ *of the toroidal Lie algebra* $\mathfrak{t}$.

Proof. Let $\mathbb{C}\langle S\rangle$ be the free $\mathbb{C}$-algebra generated by the set

$$S = \{h_{i,n},\, x_{i,n}^{\pm},\, c \,|\, i = 0, 1,\, n \in \mathbb{Z}\},$$

and let ψ be the surjective $\mathbb{C}$-algebra homomorphism from $\mathbb{C}\langle S\rangle$ to $\mathcal{U}(\mathfrak{t})$ determined by

$$h_{i,n} \mapsto \alpha_{i,n}, \quad x_{i,n}^{\pm} \mapsto e_{i,n}^{\pm}, \quad c \mapsto \mathrm{k}_2,$$

for $i = 0, 1,\ n \in \mathbb{Z}$. Set $\mathcal{U}^f = (\mathbb{C}\langle S\rangle)[[\hbar]]$, and let $\bar{I}$ be the closure (for the $\hbar$-adic topology) of the two-sided ideal I of $\mathcal{U}^f$ generated by the coefficients in (Q1)-(Q9). Then by definition we have $\mathcal{U} = \mathcal{U}^f/\bar{I}$. We extend ψ to a $\mathbb{C}$-algebra homomorphism from $\mathcal{U}^f$ to $\mathcal{U}(\mathfrak{t})$ such that $\psi(\hbar) = 0$. Note that, via the homomorphism ψ, the relations (Q1)-(Q9) in $\mathcal{U}^f$ are exactly the relations (L1)-(L7) of $\mathfrak{t}$. This implies that $\psi(I) = 0$ and hence $\psi(\bar{I}) = 0$ (as $\psi(\hbar) = 0$ and $I/\hbar I = \bar{I}/\hbar\bar{I}$). Therefore, we obtain a surjective $\mathbb{C}$-algebra homomorphism from $\mathcal{U}$ to $\mathcal{U}(\mathfrak{t})$. Furthermore, this homomorphism induces a surjective $\mathbb{C}$-algebra homomorphism from $\mathcal{U}/\hbar\mathcal{U}$ to $\mathcal{U}(\mathfrak{t})$, which we also denote as ψ.

On the other hand, we have a surjective $\mathbb{C}$-algebra homomorphism φ from $\mathcal{U}(\mathfrak{t})$ to $\mathcal{U}/\hbar\mathcal{U}$ determined by

$$\alpha_{i,n} \mapsto h_{i,n} + \hbar\mathcal{U}, \quad e_{i,n}^{\pm} \mapsto x_{i,n}^{\pm} + \hbar\mathcal{U}, \quad \mathrm{k}_2 \mapsto c + \hbar\mathcal{U},$$

for $i = 0, 1,\ n \in \mathbb{Z}$. It is obvious that φ is the inverse of ψ, so the theorem is proved. □

Remark 2.5. From the proof of Proposition 2.3, one knows that the algebra $\mathcal{U}_\hbar(\hat{\mathfrak{g}})$ also specializes to $\mathfrak{t}$. On the other hand, it is straightforward to see that the current

$$(z-w)[e_0^{\pm}(z), e_1^{\pm}(w)]$$

is nonzero in $\mathfrak{t}$ and its components lie in the space $\bar{\mathcal{K}} = \sum_{m_1\in\mathbb{Z}}(\mathbb{C}t_1^{m_1}t_2\mathrm{k}_1 + \mathbb{C}t_1^{m_1}t_2^{-1}\mathrm{k}_1)$. Thus, the algebra $\mathcal{U}_\hbar(\mathfrak{g}_{tor})$ specializes to the quotient algebra $\mathfrak{t}/\bar{\mathcal{K}}$ of $\mathfrak{t}$ (cf. [2]).

3. Triangular decomposition of $\mathcal{U}$

In this section, we prove that $\mathcal{U}$ has a triangular decomposition. By a triangular decomposition of a $\mathbb{C}[[\hbar]]$-algebra A, we mean a datum of three closed $\mathbb{C}[[\hbar]]$-subalgebras (A^-, H, A^+) of A such that the multiplication $x^- \otimes h \otimes x^+ \mapsto x^- h x^+$ induces an $\mathbb{C}[[\hbar]]$-module isomorphism from $A^- \widehat{\otimes} H \widehat{\otimes} A^+$ to A. Here and henceforth, for two $\mathbb{C}[[\hbar]]$-modules U, V, the notation $U \widehat{\otimes} V$ stands for the $\hbar$-adically completed tensor product of U and V.

Let $\mathcal{U}^+$ (resp. $\mathcal{U}^-$; resp. $\mathcal{H}$) be the closed subalgebra of $\mathcal{U}$ generated by $x^+_{i,m}$ (resp. $x^-_{i,m}$; resp. $h_{i,m}$, c). The following is the main result of this section:

Theorem 3.1. *$(\mathcal{U}^-, \mathcal{H}, \mathcal{U}^+)$ is a triangular decomposition of $\mathcal{U}$. Moreover, $\mathcal{U}^+$ (resp. $\mathcal{U}^-$; resp $\mathcal{H}$) is isomorphic to the $\mathbb{C}[[\hbar]]$-algebra topologically generated by $x^+_{i,m}$ (resp. $x^-_{i,m}$; resp. $h_{i,m}$, c) subject to the relations (Q6)-(Q9) with "+" (resp. (Q6)-(Q9) with "−"; resp. (Q1), (Q2)).*

The rest of this section is devoted to proving Theorem 3.1. We first introduce some algebras related to $\mathcal{U}$ that will be used later on.

Definition 3.2. Let $\widetilde{\mathcal{U}}$ be the $\mathbb{C}[[\hbar]]$-algebra topologically generated by the elements in (2.3) with defining relations (Q1)-(Q5), $\widehat{\mathcal{U}}$ the quotient algebra of $\widetilde{\mathcal{U}}$ modulo the relations (Q6), (Q7), and $\bar{\mathcal{U}}$ the quotient algebra of $\widehat{\mathcal{U}}$ modulo the relation (Q8).

Denote by $\widetilde{\mathcal{U}}^+$ (resp. $\widetilde{\mathcal{U}}^-$; resp. $\widetilde{\mathcal{H}}$) the closed subalgebra of $\widetilde{\mathcal{U}}$ generated by $x^+_{i,m}$ (resp. $x^-_{i,m}$; resp. $h_{i,m}$, c). The following result is standard.

Lemma 3.3. *$(\widetilde{\mathcal{U}}^-, \widetilde{\mathcal{H}}, \widetilde{\mathcal{U}}^+)$ is a triangular decomposition of $\widetilde{\mathcal{U}}$. Moreover, $\widetilde{\mathcal{U}}^+$ (resp. $\widetilde{\mathcal{U}}^-$) is isomorphic to the $\mathbb{C}[[\hbar]]$-algebra topologically free generated by $x^+_{i,m}$ (resp. $x^-_{i,m}$) and $\widetilde{\mathcal{H}}$ is isomorphic to the $\mathbb{C}[[\hbar]]$-algebra topologically generated by $h_{i,m}$, c with relations (Q1), (Q2).*

The following result was proved in (the proof of) [5, Lemma 8].

Lemma 3.4. *For $i, j, k = 0, 1$, the following hold in $\widetilde{\mathcal{U}}$:*

$$[(z - q^{\pm a_{ij}} w) x_i^\pm(z) x_j^\pm(w) - (q^{\pm a_{ij}} z - w) x_j^\pm(w) x_i^\pm(z), x_k^\mp(w_0)] = 0. \tag{3.1}$$

Similarly, we have:

Lemma 3.5. *For $i, j, k = 0, 1$ with $i \neq j$, the following hold in $\widehat{\mathcal{U}}$:*

$$[[x_i^\pm(z_1), (z_2 - q^{\mp 2} w) x_i^\pm(z_2) x_j^\pm(w) - (q^{\mp 2} z_2 - w) x_j^\pm(w) x_i^\pm(z_2)], x_k^\mp(w_0)] = 0. \tag{3.2}$$

Proof. Let i, j be as in the hypothesis. We first prove that for $\eta = \pm$,

$$[\phi_i^\eta(q^{\mp \frac{\eta}{2} c} z_1), \big((z_2 - q^{\mp 2} w) x_i^\pm(z_2) x_j^\pm(w) - (q^{\mp 2} z_2 - w) x_j^\pm(w) x_i^\pm(z_2)\big)] = 0. \tag{3.3}$$

Indeed, it follows from (Q3) and (Q7) that

$$\begin{aligned} &[\phi_i^+(q^{\mp \frac{1}{2} c} z_1), \big((z_2 - q^{\mp 2} w) x_i^\pm(z_2) x_j^\pm(w) - (q^{\mp 2} z_2 - w) x_j^\pm(w) x_i^\pm(z_2)\big)] \\ = \quad & \left(\frac{q^{\pm 2} z_1 - z_2}{z_1 - q^{\pm 2} z_2} \frac{q^{\mp 2} z_1 - w}{z_2 - q^{\mp 2} w} - 1 \right) \end{aligned}$$

$$
\begin{aligned}
&\cdot \left((z_2-q^{\mp 2}w)x_i^{\pm}(z_2)x_j^{\pm}(w)-(q^{\mp 2}z_2-w)x_j^{\pm}(w)x_i^{\pm}(z_2)\right)\phi_i^{+}(q^{\mp\frac{1}{2}c}z_1)\\
=&\frac{(q^{\pm 2}-q^{\mp 2})z_1(z_2-w)}{(z_1-q^{\pm 2}z_2)(z_1-q^{\mp 2}w)}\\
&\cdot \left((z_2-q^{\mp 2}w)x_i^{\pm}(z_2)x_j^{\pm}(w)-(q^{\mp 2}z_2-w)x_j^{\pm}(w)x_i^{\pm}(z_2)\right)\phi_i^{+}(q^{\mp\frac{1}{2}c}z_1)\\
=&0.
\end{aligned}
$$

Similarly, for the case $\eta=-$, (3.3) follows from (Q4) and (Q7). Now, in view of (3.3) and (Q5), we have

$$
[[x_i^{\pm}(z_1),x_k^{\mp}(w_0)],(z_2-q^{\mp 2}w)x_i^{\pm}(z_2)x_j^{\pm}(w)-(q^{\mp 2}z_2-w)x_j^{\pm}(w)x_i^{\pm}(z_2)]=0. \tag{3.4}
$$

This together with (3.1) (with $i\neq j$) gives (3.2), which proves the lemma. □

Lemma 3.6. *For $i,j,k=0,1$ with $i\neq j$, the following equations hold in $\bar{\mathcal{U}}$:*

$$
\begin{aligned}
&\sum_{\sigma\in S_3}\sum_{r=0}^{3}(-1)^r\binom{3}{r}_q x_i^{\pm}(z_{\sigma(1)})\cdots x_i^{\pm}(z_{\sigma(r)})\phi_j^{\eta}(q^{\mp\eta\frac{1}{2}c}w)\\
&\quad\times x_i^{\pm}(z_{\sigma(r+1)})\cdots x_i^{\pm}(z_{\sigma(3)})=0,
\end{aligned} \tag{3.5}
$$

$$
\begin{aligned}
&\sum_{\sigma\in S_3}\sum_{r=0}^{3}(-1)^r\binom{3}{r}_q \xi_i(z_{\sigma(1)})\cdots \xi_i(z_{\sigma(r)})x_j^{\pm}(w)\\
&\quad\times \xi_i(z_{\sigma(r+1)})\cdots \xi_i(z_{\sigma(3)})=0,
\end{aligned} \tag{3.6}
$$

where $\eta=\pm$, $\xi_i(z_p)=x_i^{\pm}(z_p)$ if $p\neq 1$ and $\xi_i(z_1)=\phi_i^{\eta}(q^{\mp\eta\frac{1}{2}c}z_1)$.

Proof. Equation (3.5) can be proved as that of [5, Eq. (20)] and we omit the details. For (3.6), we will prove the case of $\eta=+$, the other case $\eta=-$ is similar. Denote by $R^{\pm}$ the LHS of (3.6). Then it follows from the relations (Q3), (Q4) that

$$
\begin{aligned}
R^{\pm}=D^{\pm}\sum_{\pi\in S_2}\sum_{r=1}^{3}&P_r(z_1,z_{\pi(2)},z_{\pi(3)},w,q^{\pm 1})x_i^{\pm}(z_{\pi(2)})\cdots x_i^{\pm}(z_{\pi(r)})x_j^{\pm}(w)\\
&\times x_i^{\pm}(z_{\pi(r+1)})\cdots x_i^{\pm}(z_{\pi(3)})\phi_i^{+}(q^{\mp\frac{1}{2}c}z_1),
\end{aligned}
$$

where S_2 acts on the set $\{2,3\}$ and for $1\le r\le 3$,

$$
D^{\pm}=\frac{1}{z_1-q^{\mp 2}w}\prod_{2\le a\le 3}\frac{1}{z_1-q^{\pm 2}z_a},
$$

and

$$
\begin{aligned}
&P_r(z_1,z_2,z_3,w,q)\\
=&\binom{3}{r}_q(-1)^r\sum_{p=1}^{r}\prod_{2\le a\le p}\left(z_1-q^2z_a\right)\prod_{p<a\le 3}\left(q^2z_1-z_a\right)\left(q^{a_{ij}}z_1-w\right)\\
&+\binom{3}{r-1}_q(-1)^{r-1}\sum_{p=r}^{3}\prod_{2\le a\le p}\left(z_1-q^2z_a\right)\prod_{p<a\le 3}\left(q^2z_1-z_a\right)\left(z_1-q^{a_{ij}}w\right).
\end{aligned}
$$

It was proved in [5, Lemma 6] that

$$P_1(z_1,z_2,z_3,w,q)=(z_2-q^2w)f_3^{(1)}(z_1,z_3,w,q)+(z_3-q^{-2}z_2)f_2^{(1)}(z_1,w,q), \tag{3.7}$$

$$P_2(z_1,z_2,z_3,w,q)=(w-q^2z_2)f_2^{(2)}(z_1,z_3,w,q)+(z_3-q^2w)f_3^{(2)}(z_1,z_2,w,q), \tag{3.8}$$

$$P_3(z_1,z_2,z_3,w,q)=(w-q^2z_3)f_3^{(3)}(z_1,z_2,w,q)+(z_3-q^{-2}z_2)f_2^{(3)}(z_1,w,q), \tag{3.9}$$

where $f_2^{(r)}$ and $f_3^{(r)}$ are some polynomials of degree at most 1 in the variables z_1, z_2, w.

In view of (3.7), (3.8), (3.9) and (Q6), we have

$$\sum_{\pi\in S_2} P_r(z_1,z_{\pi(2)},z_{\pi(3)},w,q^{\pm1})x_i^{\pm}(z_{\pi(2)})x_i^{\pm}(z_{\pi(3)})x_j^{\pm}(w)\phi_i^{+}(q^{\mp\frac{1}{2}c}z_1)=0,$$

$$\sum_{\pi\in S_2} P_r(z_1,z_{\pi(2)},z_{\pi(3)},w,q^{\pm1})x_j^{\pm}(w)x_i^{\pm}(z_{\pi(2)})x_i^{\pm}(z_{\pi(3)})\phi_i^{+}(q^{\mp\frac{1}{2}c}z_1)=0.$$

This implies that all the terms in $R^{\pm}$ which contain the polynomials $f_a^{(r)}$ with $a\neq r,3$ can be erased. Thus, we obtain

$$\begin{aligned} R^{\pm}=&D^{\pm}\sum_{\pi\in S_2}(z_{\pi(2)}-q^{\pm2}w)f_3^{(1)}(z_1,z_{\pi(3)},w,q^{\pm1})x_j^{\pm}(w)x_i^{\pm}(z_{\pi(2)})x_i^{\pm}(z_{\pi(3)})\phi_i^{+}(q^{\mp\frac{1}{2}c}z_1)\\ &-D^{\pm}\sum_{\pi\in S_2}(q^{\pm2}z_{\pi(2)}-w)f_2^{(2)}(z_1,z_{\pi(3)},w,q^{\pm1})x_i^{\pm}(z_{\pi(2)})x_j^{\pm}(w)x_i^{\pm}(z_{\pi(3)})\phi_i^{+}(q^{\mp\frac{1}{2}c}z_1)\\ &+D^{\pm}\sum_{\pi\in S_2}(z_{\pi(3)}-q^{\pm2}w)f_3^{(2)}(z_1,z_{\pi(2)},w,q^{\pm1})x_i^{\pm}(z_{\pi(2)})x_j^{\pm}(w)x_i^{\pm}(z_{\pi(3)})\phi_i^{+}(q^{\mp\frac{1}{2}c}z_1)\\ &-D^{\pm}\sum_{\pi\in S_2}(q^{\pm2}z_{\pi(3)}-w)f_3^{(3)}(z_1,z_{\pi(2)},w,q^{\pm1})x_i^{\pm}(z_{\pi(2)})x_i^{\pm}(z_{\pi(3)})x_j^{\pm}(w)\phi_i^{+}(q^{\mp\frac{1}{2}c}z_1). \end{aligned}$$

A straightforward calculation shows that

$$\begin{aligned} f_3^{(1)}(z_1,z_3,w,q)=f_2^{(2)}(z_1,z_3,w,q)=Q(z_1-z_3),\\ f_3^{(2)}(z_1,z_2,w,q)=f_3^{(3)}(z_1,z_2,w,q)=-Q(z_1-z_2), \end{aligned}$$

where

$$Q=(q^{-4}-q^4+q^{-2}-q^2)z_1.$$

Then we have

$$\begin{aligned} R^{\pm}=&\pm QD^{\pm}\sum_{\pi\in S_2}(z_{\pi(2)}-q^2w)(z_1-z_{\pi(3)})x_j^{\pm}(w)x_i^{\pm}(z_{\pi(2)})x_i^{\pm}(z_{\pi(3)})\phi_i^{+}(q^{\mp\frac{1}{2}c}z_1)\\ &\mp QD^{\pm}\sum_{\pi\in S_2}(q^2z_{\pi(2)}-w)(z_1-z_{\pi(3)})x_i^{\pm}(z_{\pi(2)})x_j^{\pm}(w)x_i^{\pm}(z_{\pi(3)})\phi_i^{+}(q^{\mp\frac{1}{2}c}z_1)\\ &\pm QD^{\pm}\sum_{\pi\in S_2}(z_{\pi(3)}-q^2w)(z_1-z_{\pi(2)})x_i^{\pm}(z_{\pi(2)})x_j^{\pm}(w)x_i^{\pm}(z_{\pi(3)})\phi_i^{+}(q^{\mp\frac{1}{2}c}z_1)\\ &\mp QD^{\pm}\sum_{\pi\in S_2}(q^2z_{\pi(3)}-w)(z_1-z_{\pi(2)})x_i^{\pm}(z_{\pi(2)})x_i^{\pm}(z_{\pi(3)})x_j^{\pm}(w)\phi_i^{+}(q^{\mp\frac{1}{2}c}z_1) \end{aligned}$$

$$\begin{aligned} &= \mp QD^{\pm}q^{\pm 2}\sum_{\pi\in S_2}(z_1 - z_{\pi(2)})\big[x_i^{\pm}(z_{\pi(2)}), (z_{\pi(3)} - q^{\mp 2}w)x_i^{\pm}(z_{\pi(3)})x_j^{\pm}(w) \\ &\qquad - (q^{\mp 2}z_{\pi(3)} - w)x_j^{\pm}(w)x_i^{\pm}(z_{\pi(3)})\big]\phi_i^{+}(q^{\mp\frac{1}{2}c}z_1) \\ &= 0, \end{aligned} \tag{3.10}$$

where the last equation follows from (Q8). □

As in the proof of [5, Lemma 10], it is obvious that Lemma 3.6 implies the following result.

Lemma 3.7. *For $i, j, k = 0, 1$ with $i \neq j$, the following equations hold in $\bar{\mathcal{U}}$:*

$$[\sum_{\sigma\in S_3}\sum_{r=0}^{3}(-1)^r\binom{3}{r}_q x_i^{\pm}(z_{\sigma(1)})\cdots x_i^{\pm}(z_{\sigma(r)})x_j^{\pm}(w)x_i^{\pm}(z_{\sigma(r+1)})\cdots x_i^{\pm}(z_{\sigma(3)}), x_k^{\mp}(w_0)] = 0.$$

Proof of Theorem 3.1. We first recall a general result of triangular decompositions (cf. [5, Lemma 4]). Let A be a completed and separated $\mathbb{C}[[\hbar]]$-algebra and (A^-, H, A^+) a triangular decomposition of A. Let B^+ and B^- be respectively a closed two-sided ideal of A^+ and A^-, and let B be the closed ideal of A generated by $B^+ + B^-$. Set $C = A/B$ and denote by $C^{\pm}$ the image of $B^{\pm}$ in C. Assume that $AB^+ \subset B^+A$ and $B^-A \subset AB^-$. Then (C^+, H, C^-) is a triangular decomposition of C and $C^{\pm}$ are isomorphic to $A^{\pm}/B^{\pm}$. In view of this criterion, Theorem 3.1 follows from Lemmas 3.3, 3.4, 3.5 and 3.7. □

Remark 3.8. It was proved in [5] that the algebra $\mathcal{U}_\hbar(\dot{\mathfrak{g}}_{tor})$ has a triangular decomposition as in Theorem 3.1. That is, $\mathcal{U}$ and $\mathcal{U}_\hbar(\dot{\mathfrak{g}}_{tor})$ are two different choices of the quotient algebras of $\mathcal{U}_\hbar(\hat{\mathfrak{g}})$ with a triangular decomposition.

4. Hopf algebra structure

In this section, we give a Hopf algebra structure on $\mathcal{U}$. For a $\mathbb{C}[[\hbar]]$-module M and $n \in \mathbb{N}$, we denote that

$$M^{\widehat{\otimes}^n} = \underbrace{M\widehat{\otimes}M\widehat{\otimes}\cdots\widehat{\otimes}M}_{n\text{-copies}}. \tag{4.1}$$

Let u, v be formal variables. Motivated by the deformed Drinfeld coproduct given in [6], we have:

Theorem 4.1. *There exists a unique $\mathbb{C}[[\hbar]]$-algebra homomorphism $\Delta_u : \mathcal{U} \to \big(\mathcal{U}^{\widehat{\otimes}^2}\big)((u))$ defined as follows $(i = 0, 1)$*

$$\begin{aligned} &\text{(Co1)} \qquad \Delta_u(c) = c\otimes 1 + 1\otimes c, \\ &\text{(Co2)} \qquad \Delta_u\left(\phi_i^{\pm}(z)\right) = \phi_i^{\pm}(zq^{\pm\frac{c_2}{2}})\otimes\phi_i^{\pm}(zu^{-1}q^{\mp\frac{c_1}{2}}), \\ &\text{(Co3)} \qquad \Delta_u\left(x_i^{+}(z)\right) = x_i^{+}(z)\otimes 1 + \phi_i^{-}(zq^{\frac{c_1}{2}})\otimes x_i^{+}(zu^{-1}q^{c_1}), \\ &\text{(Co4)} \qquad \Delta_u\left(x_i^{-}(z)\right) = 1\otimes x_i^{-}(zu^{-1}) + x_i^{-}(zq^{c_2})\otimes\phi_i^{+}(zu^{-1}q^{\frac{c_2}{2}}), \end{aligned}$$

where $c_1 = c\otimes 1$ and $c_2 = 1\otimes c$. Moreover, as $\mathbb{C}[[\hbar]]$-algebra homomorphisms $\mathcal{U} \to \big(\mathcal{U}^{\widehat{\otimes}^3}\big)((u, v))$, we have

$$(\mathrm{Id}\otimes\Delta_v)\circ\Delta_u = (\Delta_u\otimes\mathrm{Id})\circ\Delta_{uv},$$

and, as $\mathbb{C}[[\hbar]]$-algebra homomorphisms $\mathcal{U} \to \big(\mathcal{U}^{\widehat{\otimes}^2}\big)((u))$, we have

$$(\mathrm{Id}\otimes\epsilon)\circ\Delta_u=\mathrm{Id},\quad(\epsilon\otimes\mathrm{Id})\circ\Delta_u=\mathrm{Id}_u,$$

where $\mathrm{Id}_u:\mathcal{U}\to\mathcal{U}\otimes\mathbb{C}((u))$ *is the* $\mathbb{C}[[\hbar]]$*-algebra homomorphism determined by*

$$\text{(Id1)}\qquad \mathrm{Id}_u(c)=c,\quad \mathrm{Id}_u(\phi_i^{\pm}(z))=\phi_i^{\pm}(zu^{-1}),$$
$$\text{(Id2)}\qquad \mathrm{Id}_u(x_i^{\pm}(z))=x_i^{\pm}(zu^{-1}),$$

and $\epsilon:\mathcal{U}\to\mathbb{C}[[\hbar]]$ *is the* $\mathbb{C}[[\hbar]]$*-algebra homomorphism determined by*

$$\text{(CoU)}\qquad \epsilon\left(\phi_i^{\pm}(z)\right)=1,\quad \epsilon\left(x_i^{\pm}(z)\right)=0=\epsilon(c).$$

Before proving Theorem 4.1, we remark that the above gives a (topological) Hopf algebra structure on $\mathcal{U}$ (by Δ_1, the "limit" of Δ_u at $u=1$). However, one notices that Δ_1 is not a well-defined $\mathbb{C}[[\hbar]]$-algebra homomorphism from $\mathcal{U}$ to $\mathcal{U}^{\widehat{\otimes}^2}$, so we need to introduce certain topological completions of $\mathcal{U}$ and $\mathcal{U}^{\widehat{\otimes}^2}$ as in [4]. Explicitly, let $\mathcal{F}$ be the free $\mathbb{C}[[\hbar]]$-algebra topologically generated by the set (2.3). Now give $h_{i,\pm n}$ degree n for $n>0$, and all other elements degree 0. We extend the degree to all the elements of the algebra by summation on the monomials. For $k\geq 0$, let $\mathcal{F}_k$ be the $\hbar$-adically closed ideal of $\mathcal{F}$ generated by elements of degree greater than k. Then we obtain an inverse system of $\mathbb{C}[[\hbar]]$-algebras $(\mathcal{F}/\mathcal{F}_k,p_k)$, where p_k is the natural projection $\mathcal{F}/\mathcal{F}_k\to\mathcal{F}/\mathcal{F}_{k-1}$. Denote by $\mathcal{F}_c$ the $\hbar$-adic completion of the inverse limit $\varprojlim\mathcal{F}/\mathcal{F}_k$. Note that $\mathcal{F}_c$ is a complete and separated algebra over $\mathbb{C}[[\hbar]]$ with inverse limit topology. Let K be the closed ideal of $\mathcal{F}_c$ generated by the relations (Q1)-(Q10). Set

$$\mathcal{U}_c=\mathcal{F}_c/K,$$

a completion of $\mathcal{U}$. Note that there is a canonical injection from $\mathcal{U}$ to $\mathcal{U}_c$.

Now, we consider the space $\mathcal{U}^{\widehat{\otimes}^2}$. In this case, we view $\mathcal{F}$ as a $\mathbb{Z}$-graded algebra by giving $x_{i,\pm n}^+$, $x_{i,\pm n}^-$, $h_{i,\pm n}$ degree n for $n\geq 0$, and give other generators degree 0. Denote by $\mathcal{F}_k'$ the closed two sided ideal of $\mathcal{F}$ of elements of degree at least k. One notices that $\mathcal{F}_k'$ is a strict subset of $\mathcal{F}_k$. Let $\mathcal{F}\bar{\otimes}\mathcal{F}$ be the topological completion of the inverse limit

$$\mathcal{F}\widehat{\otimes}\mathcal{F}/\mathcal{F}_k'\widehat{\otimes}\mathcal{F}_k'.$$

Then $\mathcal{F}\bar{\otimes}\mathcal{F}$ is also a complete and separated algebra over $\mathbb{C}[[\hbar]]$. Define $\mathcal{U}_c\widetilde{\otimes}\mathcal{U}_c$ to be the quotient algebra of $\mathcal{F}\bar{\otimes}\mathcal{F}$ modulo the closure of $K\widehat{\otimes}\mathcal{F}+\mathcal{F}\widehat{\otimes}K$. It is easy to see that there is a canonical injection from $\mathcal{U}\widehat{\otimes}\mathcal{U}$ to $\mathcal{U}_c\widetilde{\otimes}\mathcal{U}_c$. Using these completions, we deduce from Theorem 4.1 that $(\mathcal{U}_c,\Delta_1,\epsilon)$ carries a $\mathbb{C}[[\hbar]]$-bialgebra structure. Furthermore, by the same argument as in the proof of [1, Theorem 2.1], we have the following result.

Theorem 4.2. $\mathcal{U}_c$ *is a Hopf algebra with coproduct* Δ_1*, counit* ϵ *and the antipode* S *defined by* $(i=0,1)$

$$S(c)=-c,\qquad S\left(x_i^+(z)\right)=-\phi_i^-(zq^{-\frac{c}{2}})^{-1}x_i^+(zq^{-c}),$$
$$S\left(x_i^-(z)\right)=-x_i^-(zq^{-c})\phi_i^+(zq^{-\frac{c}{2}}),\quad S\left(\phi_i^{\pm}(z)\right)=\phi_i^{\pm}(z)^{-1}.$$

The rest of this section is devoted to proving Theorem 4.1. Recall the algebras $\widehat{\mathcal{U}}$ and $\bar{\mathcal{U}}$ introduced in Definition 3.2. Firstly, we have the following straightforward result.

Lemma 4.3. *(Co1)-(Co4) defines a unique* $\mathbb{C}[[\hbar]]$*-algebra homomorphism* $\widehat{\Delta}_u:\widehat{\mathcal{U}}\to\big(\widehat{\mathcal{U}}^{\widehat{\otimes}^2}\big)((u))$.

Furthermore, we have

Lemma 4.4. *$\widehat{\Delta}_u$ induces a $\mathbb{C}[[\hbar]]$-algebra homomorphism from $\bar{\Delta}_u : \bar{\mathcal{U}} \to \left(\bar{\mathcal{U}}^{\widehat{\otimes}^2}\right)((u))$.*

Proof. Fix any $i \neq j \in \{0,1\}$ and denote by $I_{ij}^{\pm}$ the LHS of the relation (Q8). We need to show that $\widehat{\Delta}_u(I_{ij}^{\eta}) = 0$ with $\eta = \pm$. We will show the case $\eta = +$, as the case $\eta = -$ is similar and thus omitted. Set

$$x_{ij}^{+}(z,w) = (z - q^{-2}w)x_i^{+}(z)x_j^{+}(w) - (q^{-2}z - w)x_j^{+}(w)x_i^{+}(z),$$
$$A_{ij}^{+}(z,w) = (z - q^{-2}w)\widehat{\Delta}_u(x_i^{+}(z))\widehat{\Delta}_u(x_j^{+}(w)) - (q^{-2}z - w)\widehat{\Delta}_u(x_j^{+}(w))\widehat{\Delta}_u(x_i^{+}(z)).$$

A straightforward calculation shows that

$$A_{ij}^{+}(z,w) = x_{ij}^{+}(z,w) \otimes 1 + q^{-c_1}u\phi_i^{-}(zq^{\frac{c_1}{2}})\phi_j^{-}(wq^{\frac{c_1}{2}}) \otimes x_{ij}^{+}(zu^{-1}q^{c_1}, wu^{-1}q^{c_1}).$$

Using this, we obtain

$$\begin{aligned}
&[\widehat{\Delta}_u(x_i^{+}(z_1)), A_{ij}^{+}(z_2,w)] = [x_i^{+}(z_1), x_{ij}^{+}(z_2,w)] \otimes 1 \\
&+[\phi_i^{-}(z_1q^{\frac{c_1}{2}}), x_{ij}^{+}(z_2,w)] \otimes x_i^{+}(z_1u^{-1}q^{c_1}) + q^{-c_1}u\phi_i^{-}(z_1q^{\frac{c_1}{2}})\phi_i^{-}(z_2q^{\frac{c_1}{2}})\phi_j^{-}(wq^{\frac{c_1}{2}}) \\
&\otimes [x_i^{+}(z_1u^{-1}q^{c_1}), x_{ij}^{+}(z_2u^{-1}q^{c_1}, wu^{-1}q^{c_1})] \\
&+ q^{-c_1}u[x_i^{+}(z_1), \phi_i^{-}(z_2q^{\frac{c_1}{2}})\phi_j^{-}(wq^{\frac{c_1}{2}})] \otimes x_{ij}^{+}(z_2u^{-1}q^{c_1}, wu^{-1}q^{c_1})
\end{aligned} \tag{4.2}$$

By applying (Q4) one gets that

$$\begin{aligned}
&[\phi_i^{-}(z_1q^{\frac{c}{2}}), x_{ij}^{+}(z_2,w)] \\
=&\phi_i^{-}(z_1q^{\frac{c}{2}})x_{ij}^{+}(z_2,w)\left(1 - g_{ji}(z_1/w)g_{ii}(z_1/z_2)\right) \\
=&\frac{(q^2 - q^{-2})z_1(z_2 - w)}{(w - q^{-2}z_1)(z_2 - q^2z_1)}\phi_i^{-}(z_1q^{\frac{c}{2}})x_{ij}^{+}(z_2,w),
\end{aligned} \tag{4.3}$$

$$\begin{aligned}
&[x_i^{+}(z_1), \phi_i^{-}(z_2q^{\frac{c}{2}})\phi_j^{-}(wq^{\frac{c}{2}})] \\
=&\phi_i^{-}(z_2q^{\frac{c}{2}})\phi_j^{-}(wq^{\frac{c}{2}})x_i^{+}(z_1)\left(g_{ji}(z_1/w)g_{ii}(z_1/z_2) - 1\right) \\
=&-\frac{(q^2 - q^{-2})z_1(z_2 - w)}{(w - q^{-2}z_1)(z_2 - q^2z_1)}\phi_i^{-}(z_2q^{\frac{c}{2}})\phi_j^{-}(wq^{\frac{c}{2}})x_i^{+}(z_1).
\end{aligned} \tag{4.4}$$

Recall from (Q7) that

$$(z - w)x_{ij}^{+}(z,w) = 0 = (z - w)x_{ij}^{+}(zu^{-1}q^{c_1}, wu^{-1}q^{c_1}). \tag{4.5}$$

Combining (4.2), (4.3), (4.4) and (4.5), we deduce from (Q8) that

$$[\widehat{\Delta}_u(x_i^{+}(z_1)), A_{ij}^{+}(z_2,w)] = 0.$$

This implies that $\widehat{\Delta}_u(I_{ij}^{+}) = 0$, as required. □

To continue the discussion, we need to introduce some notations. For $0 \leq s \leq 3$, we denote by $S_{3,s}$ the set of $(s, 3-s)$-shuffles in S_3, that is

$$S_{3,s} = \{\sigma \in S_3 \mid \sigma(a) < \sigma(b), \text{ for } a < b \leq s \text{ or } s < a < b\}.$$

As a convention, we let $\sigma(a) = a$ for any $a < 1$ or $a > 3$. For $0 \leq s \leq 3$ and $\sigma \in S_{3,s}$, we define two partitions

$$P^1_{0,\sigma} \cup \cdots \cup P^1_{3-s,\sigma} \quad \text{and} \quad P^1_{0,\sigma} \cup \cdots \cup P^1_{s+1,\sigma}$$

of the set $\{0, 1, 2, 3\}$, where

$$\begin{aligned}
P^1_{0,\sigma} &= \{p \in \mathbb{Z} \mid 0 \leq p < \sigma(s+1)\},\\
P^1_{k,\sigma} &= \{p \in \mathbb{Z} \mid \sigma(s+k) \leq p < \sigma(s+k+1)\} \quad \text{for } 0 < k < 3-s,\\
P^1_{3-s,\sigma} &= \{p \in \mathbb{Z} \mid \sigma(3) \leq p \leq 3\} \quad \text{if } s < 3,\\
P^2_{0,\sigma} &= \{p \in \mathbb{Z} \mid 0 \leq p < \sigma(1)\} \quad \text{if } s > 0,\\
P^2_{k,\sigma} &= \{p \in \mathbb{Z} \mid \sigma(k) \leq p < \sigma(k+1)\} \quad \text{for } 0 < k < s,\\
P^2_{s,\sigma} &= \{p \in \mathbb{Z} \mid \sigma(s) \leq p \leq 3\}.
\end{aligned}$$

Furthermore, for $0 \leq s \leq 3$ and $0 \leq k \leq 3-s$, set

$$\begin{aligned}
T^1_{s,k}(z_1, z_2, z_3, w) = &\sum_{\sigma \in S_{3,s}} \sum_{r \in P^1_{k,\sigma}} \binom{3}{r}_q (-1)^r \prod_{a \leq s < b, \sigma(a) < \sigma(b)} (q^2 z_a - z_b)\\
&\cdot \prod_{a \leq s < b, \sigma(a) > \sigma(b)} (z_a - q^2 z_b) \prod_{a \leq s, \sigma(a) \leq r} (q^{-2} z_a - w) \prod_{a \leq s, \sigma(a) > r} (z_a - q^{-2} w),
\end{aligned}$$

and for $0 \leq s \leq 3$ and $0 \leq k \leq s$, set

$$\begin{aligned}
T^2_{s,k}(z_1, z_2, z_3, w) = &\sum_{\sigma \in S_{3,s}} \sum_{r \in P^2_{k,\sigma}} \binom{3}{r}_q (-1)^r \prod_{a \leq s < b, \sigma(a) < \sigma(b)} (q^2 z_a - z_b)\\
&\cdot \prod_{a \leq s < b, \sigma(a) > \sigma(b)} (z_a - q^2 z_b) \prod_{a > s, \sigma(a) > r} (q^{-2} w - z_a) \prod_{a > s, \sigma(a) \leq r} (w - q^{-2} z_a).
\end{aligned}$$

We have:

Lemma 4.5. *(1) For $0 \leq k \leq 3$, one has that*

$$T^1_{0,k}(z_1, z_2, z_3, w) = \binom{3}{k}_q (-1)^k.$$

(2) There exist polynomials $f_{1,0}(z_1, z_2), f_{1,1}(z_1, z_2), f_{1,2}(z_1, z_2) \in \mathbb{C}[[\hbar]][z_1, z_2]$ such that

$$\begin{aligned}
T^1_{1,0}(z_1, z_2, z_3, w) &= f_{1,0}(z_1, w)(z_2 - q^2 z_3) - f_{1,1}(z_1, z_3)(q^{-2} z_2 - w),\\
T^1_{1,1}(z_1, z_2, z_3, w) &= f_{1,1}(z_1, z_3)(z_2 - q^{-2} w) + f_{1,1}(z_1, z_2)(q^{-2} z_3 - w),\\
T^1_{1,2}(z_1, z_2, z_3, w) &= -f_{1,1}(z_1, z_2)(z_3 - q^{-2} w) + f_{1,2}(z_1, w)(z_2 - q^2 z_3).
\end{aligned}$$

.

(3) There exist polynomials

$$f_{2,0}(z_1, z_2, z_3, w), f_{2,1}(z_1, z_2), f_{2,2}(z_1, z_2, z_3, w) \in \mathbb{C}[[\hbar]][z_1, z_2, z_3, w]$$

such that

$$f_{2,k}(z_1, z_2, z_3, w) = f_{2,k}(z_2, z_1, z_3, w), \quad k = 0, 2,$$

and

$$T^1_{2,0}(z_1,z_2,z_3,w) = f_{2,0}(z_1,z_2,z_3,w)(z_1-q^2z_2) + f_{2,1}(z_1,z_2)(z_3-z_2)(q^{-2}z_3-w),$$
$$T^1_{2,1}(z_1,z_2,z_3,w) = f_{2,2}(z_1,z_2,z_3,w)(z_1-q^2z_2) - f_{2,1}(z_1,z_2)(z_3-z_2)(z_3-q^{-2}w).$$

(4) There exist polynomials $f_{3,0}(z_1,z_2), f_{3,1}(z_1,z_2) \in \mathbb{C}[[\hbar]][z_1,z_2]$ such that

$$T^1_{3,0}(z_1,z_2,z_3,w) = f_{3,0}(z_3,w)(z_1-q^2z_2) + f_{3,1}(z_1,w)(z_2-q^2w).$$

(5) $T^2_{s,k}(z_1,z_2,z_3,w) = (-1)^s T^1_{3-s,k}(z_3,z_1,z_2,w)$ if $s<3$.
(6) $T^1_{3,0}(z_1,z_2,z_3,w) = -T^2_{0,0}(z_1,z_2,z_3,w)$.

Proof. The lemma follows from the following straightforward facts:

$$\begin{aligned}
&T^1_{0,k}(z_1,z_2,z_3,w) = T^2_{3,k}(z_1,z_2,z_3,w) = \binom{3}{k}_q(-1)^k,\\
&T^1_{1,0}(z_1,z_2,z_3,w) = -T^2_{2,0}(z_2,z_3,z_1,w)\\
&\quad = (q^4-1)z_1\big((q^{-4}z_1-w)(z_2-q^2z_3) - [3]_{q^2}(z_1-z_3)(q^{-2}z_2-w)\big),\\
&T^1_{1,1}(z_1,z_2,z_3,w) = -T^2_{2,1}(z_2,z_3,z_1,w)\\
&\quad = (q^4-1)z_1[3]_{q^2}\big((z_1-z_3)(z_2-q^{-2}w) + (z_1-z_2)(q^{-2}z_3-w)\big),\\
&T^1_{1,2}(z_1,z_2,z_3,w) = -T^2_{2,2}(z_2,z_3,z_1,w)\\
&\quad = (q^4-1)z_1\big(-[3]_{q^2}(z_1-z_2)(z_3-q^{-2}w) + (z_1-q^{-4}w)(z_2-q^2z_3)\big),\\
&T^1_{2,0}(z_1,z_2,z_3,w) = T^2_{1,0}(z_2,z_3,z_1,w)\\
&\quad = z_1z_2(1-q^{-2})(q^4-1)[3]_{q^2}(z_3-w)(q^{-2}z_3-w) + (1-q^{-4})(z_1-q^2z_2)\\
&\quad \times\big(q^2[3]_{q^2}(z_3-w)(q^4z_1z_2-z_3w) - z_3(1+q^{-2})(q^2z_1-z_3)(q^2z_2-z_3)\big),\\
&T^1_{2,1}(z_1,z_2,z_3,w) = T^2_{1,1}(z_2,z_3,z_1,w)\\
&\quad = -z_1z_2(1-q^{-2})(q^4-1)[3]_{q^2}(z_3-w)(z_3-q^{-2}w) - (1-q^{-4})(z_1-q^2z_2)\\
&\quad \times\big(q^2[3]_{q^2}(z_3-w)(z_1z_2-q^4z_3w) - z_3(1+q^{-2})(z_1-q^2z_3)(z_2-q^2z_3)\big),\\
&T^1_{3,0}(z_1,z_2,z_3,w) = -T^2_{0,0}(z_1,z_2,z_3,w)\\
&\quad = w(1-q^{-4})\big(-(z_3-q^{-4}w)(z_1-q^2z_2) + (q^{-4}z_1-w)(z_2-q^2z_3)\big). \quad \square
\end{aligned}$$

For $i \neq j \in \{0,1\}$, denote by $J^{\pm}_{ij}$ the LHS of (Q9).

Lemma 4.6. *For $i \neq j \in \{0,1\}$, we have*

$$\begin{aligned}
\bar{\Delta}_u(J^+_{ij}) = &\sum_{\tau\in S_3}\sum_{s=0}^{3}\sum_{k=0}^{3-s} \frac{T^1_{s,k}(z_{\tau(1)},z_{\tau(2)},z_{\tau(3)},w)}{\prod_{a\le s<b}(z_{\tau(a)}-q^2z_{\tau(b)})\prod_{a\le s}(z_{\tau(a)}-q^{-2}w)}\\
&\cdot\widetilde{\phi}^-_i(z_{\tau(s+1)})\cdots\widetilde{\phi}^-_i(z_{\tau(3)})\widetilde{\phi}^-_j(w)x^+_i(z_{\tau(1)})\cdots x^+_i(z_{\tau(s)})\\
&\otimes \widetilde{x}^+_i(z_{\tau(s+1)})\cdots\widetilde{x}^+_i(z_{\tau(k)})\widetilde{x}^+_j(w)\widetilde{x}^+_i(z_{\tau(k+1)})\cdots\widetilde{x}^+_i(z_{\tau(3)})\\
&+\sum_{\tau\in S_3}\sum_{s=0}^{3}\sum_{k=0}^{s} \frac{T^2_{s,k}(z_{\tau(1)},z_{\tau(2)},z_{\tau(3)},w)}{\prod_{a\le s<b}(z_{\tau(a)}-q^2z_{\tau(b)})\prod_{a>s}(w-q^{-2}z_{\tau(a)})}
\end{aligned}$$

$$\cdot\widetilde{\phi}_i^-(z_{\tau(s+1)})\cdots\widetilde{\phi}_i^-(z_{\tau(3)})x_i^+(z_{\tau(1)})\cdots x_i^+(z_{\tau(k)})x_j^+(w)$$
$$\cdot x_i^+(z_{\tau(k+1)})\cdots x_i^+(z_{\tau(s)})\otimes\widetilde{x}_i^+(z_{\tau(s+1)})\cdots\widetilde{x}_i^+(z_{\tau(3)}),$$

where $\widetilde{\phi}_k^-(z)=\phi_k^-(zq^{\frac{c_1}{2}})$ *and* $\widetilde{x}_k^+(z)=x_k^+(zu^{-1}q^{c_1})$ *for* $k=i,j$.

Proof. For $\sigma\in S_{3,s}$, we define

$$\xi_\sigma(z_a)=\begin{cases}x_i^+(z_a), & \text{if } \sigma^{-1}(a)\le s,\\ \widetilde{\phi}_i^-(z_a), & \text{if } \sigma^{-1}(a)>s,\end{cases}\qquad \widetilde{\xi}_\sigma(z_a)=\begin{cases}\widetilde{x}_i^+(z_a), & \text{if } \sigma^{-1}(a)>s,\\ 1, & \text{if } \sigma^{-1}(a)\le s.\end{cases}$$

For $0\le r,s\le 3$ and $\sigma\in S_{3,s}$, set

$$\begin{aligned}\xi_{r,\sigma}(z_1,z_2,z_3,w)=&\xi_\sigma(z_1)\cdots\xi_\sigma(z_r)\phi_j^-(wq^{\frac{c}{2}})\xi_\sigma(z_{r+1})\cdots\xi(z_3),\\ \eta_{r,\sigma}(z_1,z_2,z_3,w)=&\xi_\sigma(z_1)\cdots\xi_\sigma(z_r)x_j^+(w)\xi_\sigma(z_{r+1})\cdots\xi_\sigma(z_3),\\ \widetilde{\xi}_\sigma(z_1,z_2,z_3)=&\widetilde{\xi}_\sigma(z_1)\widetilde{\xi}_\sigma(z_2)\widetilde{\xi}_\sigma(z_3),\\ \widetilde{\eta}_{r,\sigma}(z_1,z_2,z_3,w)=&\widetilde{\xi}_\sigma(z_1)\cdots\widetilde{\xi}_\sigma(z_r)x_j^+(wu^{-1}q^{c_1})\widetilde{\xi}_\sigma(z_{r+1})\cdots\widetilde{\xi}_\sigma(z_3).\end{aligned}$$

And for $i_1,i_2,\dots,i_n\in\{0,1\}$, $\zeta_{i_r}(z_r)=x_{i_r}^+(z_r)$ or $\phi_{i_r}^-(z_rq^{\frac{c}{2}})$, we define an ordered product

$$\circ\!\!\!\circ\,\zeta_{i_1}(z_1)\cdots\zeta_{i_n}(z_n)\,\circ\!\!\!\circ$$

by moving $\phi_{i_r}^-(z_rq^{\frac{c}{2}})$ to the left. Then it follows from (Q4) that

$$\xi_{r,\sigma}(z_1,z_2,z_3,w)=\prod_{a\le s<b}g_{ii}(z_{\sigma(b)}/z_{\sigma(a)})\prod_{\substack{a\le s\\ \sigma(a)\le r}}g_{ij}(w/z_{\sigma(a)})\,\circ\!\!\!\circ\,\xi_{r,\sigma}(z_1,z_2,z_3,w)\,\circ\!\!\!\circ\,,$$

$$\eta_{r,\sigma}(z_1,z_2,z_3,w)=\prod_{a\le s<b}g_{ii}(z_{\sigma(b)}/z_{\sigma(a)})\prod_{\substack{a\le s\\ \sigma(a)>r}}g_{ji}(z_{\sigma(a)}/w)\,\circ\!\!\!\circ\,\eta_{r,\sigma}(z_1,z_2,z_3,w)\,\circ\!\!\!\circ\,.$$

Moreover, it is straightforward to see that

$$\begin{aligned}&\Delta_u(x_i^+(z_1))\cdots\Delta_u(x_i^+(z_r))\Delta_u(x_j^+(w))\Delta_u(x_i^+(z_{r+1}))\cdots\Delta_u(x_i^+(z_3))\\ =&\sum_{s=0}^{3}\sum_{\sigma\in S_{3,s}}\prod_{a\le s<b}g_{ii}(z_{\sigma(b)}/z_{\sigma(a)})\prod_{\substack{a\le s\\ \sigma(a)\le r}}g_{ij}(w/z_{\sigma(a)})\,\circ\!\!\!\circ\,\xi_{r,\sigma}(z_1,z_2,z_3,w)\,\circ\!\!\!\circ\otimes\widetilde{\eta}_{r,\sigma}(z_1,z_2,z_3,w)\\ &+\sum_{s=0}^{3}\sum_{\sigma\in S_{3,s}}\prod_{a\le s<b}g_{ii}(z_{\sigma(b)}/z_{\sigma(a)})\prod_{\substack{a\le s\\ \sigma(a)>r}}g_{ji}(z_{\sigma(a)}/w)\,\circ\!\!\!\circ\,\eta_{r,\sigma}(z_1,z_2,z_3,w)\,\circ\!\!\!\circ\otimes\widetilde{\xi}_\sigma(z_1,z_2,z_3).\end{aligned}$$

Now the lemma follows from a direct calculation and the following facts:

$$\begin{aligned}&\circ\!\!\!\circ\,\xi_{r_1,\sigma}(z_{\tau(1)},z_{\tau(2)},z_{\tau(3)},w)\,\circ\!\!\!\circ\otimes\widetilde{\eta}_{r_1,\sigma}(z_{\tau(1)},z_{\tau(2)},z_{\tau(3)},w)\\ =&\circ\!\!\!\circ\,\xi_{0,1}(z_{\tau\sigma(1)},z_{\tau\sigma(2)},z_{\tau\sigma(3)},w)\,\circ\!\!\!\circ\otimes\widetilde{\eta}_{p^1_{k,\sigma},1}(z_{\tau\sigma(1)},z_{\tau\sigma(2)},z_{\tau\sigma(3)},w),\\ &\circ\!\!\!\circ\,\eta_{r_2,\sigma}(z_{\tau(1)},z_{\tau(2)},z_{\tau(3)},w)\,\circ\!\!\!\circ\otimes\widetilde{\xi}_\sigma(z_{\tau(1)},z_{\tau(2)},z_{\tau(3)},w)\end{aligned}$$

$$= {}^{\circ}_{\circ}\, \eta_{p^2_{k,\sigma},1}(z_{\tau\sigma(1)}, z_{\tau\sigma(2)}, z_{\tau\sigma(3)}, w)\, {}^{\circ}_{\circ} \otimes \widetilde{\xi}_1(z_{\tau\sigma(1)}, z_{\tau\sigma(2)}, z_{\tau\sigma(3)}, w),$$

where $\tau \in S_3$, $\sigma \in S_{3,s}$ $(0 \le s \le 3)$, $r_a \in P^a_{k,\sigma}$ $(a = 1, 2)$ and $p^a_{k,\sigma}$ is the minimal element in $P^a_{k,\sigma}$ $(a = 1, 2)$. □

Proof of Theorem 4.1. In view of Lemmas 4.3 and 4.4, it suffices to show that

$$\bar{\Delta}_u(J^{\pm}_{ij}) = 0, \quad \text{for } i \ne j \in \{0, 1\}.$$

Let $i \ne j \in \{0, 1\}$ be fixed. From Lemma 4.6 and (1), (5) of Lemma 4.5, it follows that

$$\begin{aligned}
\bar{\Delta}_u(J^+_{ij}) = &\sum_{\tau\in S_3}\sum_{s=1}^{3}\sum_{k=0}^{3-s} \frac{T^1_{s,k}(z_{\tau(1)}, z_{\tau(2)}, z_{\tau(3)}, w)}{\prod_{a\le s<b}(z_{\tau(a)} - q^2 z_{\tau(b)})\prod_{a\le s}(z_{\tau(a)} - q^{-2}w)}\\
&\cdot\widetilde{\phi}^-_i(z_{\tau(s+1)})\cdots\widetilde{\phi}^-_i(z_{\tau(3)})\widetilde{\phi}^-_j(w)x^+_i(z_{\tau(1)})\cdots x^+_i(z_{\tau(s)})\\
&\otimes \widetilde{x}^+_i(z_{\tau(s+1)})\cdots\widetilde{x}^+_i(z_{\tau(k)})\widetilde{x}^+_j(w)\widetilde{x}^+_i(z_{\tau(k+1)})\cdots\widetilde{x}^+_i(z_{\tau(3)})\\
&+\sum_{\tau\in S_3}\sum_{s=0}^{2}\sum_{k=0}^{s} \frac{T^2_{s,k}(z_{\tau(1)}, z_{\tau(2)}, z_{\tau(3)}, w)}{\prod_{a\le s<b}(z_{\tau(a)} - q^2 z_{\tau(b)})\prod_{a>s}(w - q^{-2}z_{\tau(a)})}\\
&\cdot\widetilde{\phi}^-_i(z_{\tau(s+1)})\cdots\widetilde{\phi}^-_i(z_{\tau(3)})x^+_i(z_{\tau(1)})\cdots x^+_i(z_{\tau(k)})x^+_j(w)\\
&\cdot x^+_i(z_{\tau(k+1)})\cdots x^+_i(z_{\tau(s)})\otimes\widetilde{x}^+_i(z_{\tau(s+1)})\cdots\widetilde{x}^+_i(z_{\tau(3)}).
\end{aligned}$$

Combining this with Lemma 4.5 and (Q6)-(Q7), a similar argument of Lemma 3.6 shows that

$$\begin{aligned}
\bar{\Delta}_u(J^+_{ij}) = &\sum_{\tau\in S_3} \frac{f_{1,1}(z_{\tau(1)}, z_{\tau(3)})}{\prod_{a\le s<b}(z_{\tau(a)} - q^2 z_{\tau(b)})\prod_{a\le s}(z_{\tau(a)} - q^{-2}w)}\widetilde{\phi}^-_i(z_{\tau(2)})\widetilde{\phi}^-_i(z_{\tau(3)})\widetilde{\phi}^-_j(w)\\
&\cdot x^+_i(z_{\tau(1)})\otimes\big[(z_{\tau(2)} - q^{-2}w)\widetilde{x}^+_i(z_{\tau(2)})\widetilde{x}^+_j(w) - (q^{-2}z_{\tau(2)} - w)\widetilde{x}^+_j(w)\widetilde{x}^+_i(z_{\tau(2)}), \widetilde{x}^+_i(z_{\tau(3)})\big]\\
&+\sum_{\tau\in S_3} \frac{f_{1,1}(z_{\tau(3)}, z_{\tau(2)})}{\prod_{a\le s<b}(z_{\tau(a)} - q^2 z_{\tau(b)})\prod_{a>s}(w - q^{-2}z_{\tau(a)})}\widetilde{\phi}^-_i(z_{\tau(3)})\\
&\cdot\big[(z_{\tau(1)} - q^{-2}w)x^+_i(z_{\tau(1)})x^+_j(w) - (q^{-2}z_{\tau(1)} - w)x^+_j(w)x^+_i(z_{\tau(1)}), x^+_i(z_{\tau(2)})\big]\otimes\widetilde{x}^+_i(z_{\tau(3)}).
\end{aligned} \tag{4.6}$$

Then it follows from (Q8) that $\bar{\Delta}_u(J^+_{ij}) = 0$. The proof of $\bar{\Delta}_u(J^-_{ij}) = 0$ is similar. Therefore, we complete the proof of Theorem 4.1. □

Remark 4.7. Let $\widetilde{\mathcal{U}}_\hbar(\dot{\mathfrak{g}}_{tor})$ be the quotient algebra of $\widetilde{\mathcal{U}}$ modulo the relations (2.4) and (Q6). It was shown in [5, Proposition 29] that the action (Co1)-(Co4) defines a unique algebra homomorphism $\widetilde{\Delta}_u : \widetilde{\mathcal{U}}_\hbar(\dot{\mathfrak{g}}_{tor}) \to \big(\widetilde{\mathcal{U}}_\hbar(\dot{\mathfrak{g}}_{tor})^{\widehat{\otimes}^2}\big)((u))$. Recall that $\mathcal{U}_\hbar(\dot{\mathfrak{g}}_{tor})$ is the quotient algebra of $\widetilde{\mathcal{U}}_\hbar(\dot{\mathfrak{g}}_{tor})$ modulo the relation (Q9). As the relations (Q6)-(Q8) also hold in $\mathcal{U}_\hbar(\dot{\mathfrak{g}}_{tor})$, it follows from (the proof of) Theorem 4.1 that $\widetilde{\Delta}_u$ is compatible with the affine quantum Serre relation (Q9). This implies that $\widetilde{\Delta}_u$ induces an algebra homomorphism from $\mathcal{U}_\hbar(\dot{\mathfrak{g}}_{tor})$ to $\big(\mathcal{U}_\hbar(\dot{\mathfrak{g}}_{tor})^{\widehat{\otimes}^2}\big)((u))$, as expected in [5, Remark 6].

5. Vertex representation of $\mathcal{U}$

In this section we show the vertex representation for $\mathcal{U}_\hbar(\hat{\mathfrak{g}})$ given in [8] induces a representation for $\mathcal{U}$. We first recall the vertex representation given in [8]. Let

$$\mathcal{S} = \mathbb{C}[h_{i,n} \mid i = 0, 1, n < 0][[\hbar]] \tag{5.1}$$

be the symmetric $\mathbb{C}[[\hbar]]$-algebra in the variable $h_{i,n}$, which is topologically free as a $\mathbb{C}[[\hbar]]$-module. It is known that there is an $\mathcal{H}$-action on $\mathcal{S}$ such that $c = 1$, $h_{i,0} = 0$, $h_{i,-n}$ = multiplication operator and $h_{i,n}$ = annihilation operator subject to the relation

$$[h_{i,m}, h_{j,-n}] = \frac{\delta_{m+n,0}}{n}[na_{ij}]_q \frac{q^{nc} - q^{-nc}}{q - q^{-1}},$$

where $i = 0, 1$ and $n > 0$. From now on, we take c and $h_{i,n}$ $(i = 0, 1,\ n \in \mathbb{Z})$ as operators on $\mathcal{S}$. For $i = 0, 1$, we introduce the following fields on $\mathcal{S}$

$$h_i^{\pm}(z) = \sum_{\pm m>0} h_{i,m} z^{-m}, \quad E_i^{\pm}(z) = \exp\left(\mp \sum_{m>0} \frac{h_{i,\pm m}}{[m]_q} z^{\mp m}\right).$$

Let $Q = \mathbb{Z}\alpha_0 \oplus \mathbb{Z}\alpha_1$ be a rank 2 lattice equipped with a semi-positive form $\langle \cdot, \cdot \rangle$ determined by $\langle \alpha_i, \alpha_j \rangle = a_{ij}$ for $i, j = 0, 1$. We fix a 2-cocycle $\varepsilon : Q \times Q \to \{\pm 1\}$ such that

$$\varepsilon(\alpha, \beta)\varepsilon(\beta, \alpha)^{-1} = C(\alpha, \beta), \quad \varepsilon(\alpha, 0) = 1 = \varepsilon(0, \alpha),$$

where

$$C : Q \times Q \to \{\pm 1\}, \quad (\alpha, \beta) \mapsto (-1)^{\langle \alpha, \beta \rangle}.$$

Denote by $\mathbb{C}_\varepsilon[Q]$ the ε-twisted group $\mathbb{C}$-algebra of Q, which by definition has a designated basis $\{e_\alpha \mid \alpha \in Q\}$ such that $e_\alpha \cdot e_\beta = \varepsilon(\alpha, \beta)e_{\alpha+\beta}$ for $\alpha, \beta \in Q$. Then $\mathbb{C}_\hbar\{Q\} = \mathbb{C}_\varepsilon[Q][[\hbar]]$ becomes an $\mathcal{H}$-module under the action that c and $h_{i,n}$ act trivially for $i = 0, 1$, $n \neq 0$, and $h_{i,0}$ $(i = 0, 1)$ act by

$$h_{i,0} \cdot e_\beta = \langle \beta, \alpha_i \rangle e_\beta \quad \text{for } \beta \in Q.$$

For $i = 0, 1$, define a linear operator $z^{h_{i,0}} : \mathbb{C}_\hbar\{Q\} \to \mathbb{C}_\hbar\{Q\}[[z, z^{-1}]]$ by

$$z^{h_{i,0}} \cdot e_\beta = z^{\langle \beta, \alpha_i \rangle} e_\beta \quad \text{for } \beta \in Q.$$

We define the Fock space

$$V = \mathcal{S} \widehat{\otimes} \mathbb{C}_\hbar\{Q\} \tag{5.2}$$

to be the tensor product $\mathcal{H}$-module. Recall the vertex operators define in [8]:

$$X_i^{\pm}(z) = \sum_{n\in\mathbb{Z}} X_{i,n} z^{-n-1} = E_i^-(q^{\mp\frac{1}{2}} z)^{\pm 1} E_i^+(q^{\pm\frac{1}{2}} z)^{\pm 1} e_{\pm\alpha_i} z^{\pm h_{i,0}}. \tag{5.3}$$

As usual, the normal ordered product $\circ\!\!\circ\, X_i^{\pm}(z) X_j^{\pm}(w) \,\circ\!\!\circ$ is defined by moving annihilation operators $h_{i,n}$ $(i = 0, 1,\ n \geq 0)$ to the right. For $i, j \in I$, it was proved in [8, (3.6)] that

$$X_i^{\pm}(z) X_j^{\pm}(w) = \,\circ\!\!\circ\, X_i^{\pm}(z) X_j^{\pm}(w) \,\circ\!\!\circ\, (z - q^{\mp 1} w)_{q^2}^{a_{ij}}, \tag{5.4}$$

where for $n \in \mathbb{N}$,

$$(1 - x)_{q^2}^{n} = (1 - q^{1-n}x)(1 - q^{3-n}x) \cdots (1 - q^{n-1}x),$$
$$(1 - x)_{q^2}^{-n} = 1/(1 - x)_{q^2}^{n}, \quad (z - w)_{q^2}^{\pm n} = (1 - w/z)_{q^2}^{\pm n} z^{\pm n},$$

and $(1-x)_{q^2}^{-n}$ is understood as power series in x. Moreover, it was proved in [8, (3.8)] that (the bilinear form $\langle\cdot,\cdot\rangle$ is denoted as $(\cdot|\cdot)$ therein)

$$
{}^{\circ}_{\circ} X_i^{\pm}(z) X_j^{\pm}(w) {}^{\circ}_{\circ} = (-1)^{\langle\alpha_i,\alpha_j\rangle} {}^{\circ}_{\circ} X_j^{\pm}(w) X_i^{\pm}(z) {}^{\circ}_{\circ}. \tag{5.5}
$$

This together with the fact that $\langle\alpha_i,\alpha_j\rangle = a_{ij} \in 2\mathbb{Z}$ (see (2.1)) gives that

$$
{}^{\circ}_{\circ} X_i^{\pm}(z) X_j^{\pm}(w) {}^{\circ}_{\circ} = {}^{\circ}_{\circ} X_j^{\pm}(w) X_i^{\pm}(z) {}^{\circ}_{\circ}. \tag{5.6}
$$

We have:

Theorem 5.1. *There is a $\mathcal{U}$-module structure on the Fock space V with the action given by*

$$
c \mapsto 1, \quad h_{i,n} \mapsto h_{i,n} \quad \text{and} \quad x_{i,n}^{\pm} \mapsto X_{i,n}^{\pm},
$$

where $i = 0, 1$ and $n \in \mathbb{Z}$.

Proof. The relations (Q1-Q7) and (Q9) have been checked in [8, Theorem 3.1] and it remains to prove the relation (Q8). For $i \neq j \in \{0,1\}$, it follows from (5.4) that

$$
\begin{aligned}
X_{ij}^{\pm}(z,w) =& (z - q^{\mp 2}w) X_i^{\pm}(z) X_j^{\pm}(w) - (q^{\mp 2}z - w) X_j^{\pm}(w) X_i^{\pm}(z) \\
=& {}^{\circ}_{\circ} X_i^{\pm}(z) X_j^{\pm}(w) {}^{\circ}_{\circ} z^{-1}\delta\left(\frac{w}{z}\right) = {}^{\circ}_{\circ} X_i^{\pm}(w) X_j^{\pm}(w) {}^{\circ}_{\circ} z^{-1}\delta\left(\frac{w}{z}\right).
\end{aligned} \tag{5.7}
$$

It is straightforward to check that

$$
\begin{aligned}
X_i^{\pm}(z)\, {}^{\circ}_{\circ} X_i^{\pm}(w) X_j^{\pm}(w) {}^{\circ}_{\circ} &= {}^{\circ}_{\circ} X_i^{\pm}(z)\, {}^{\circ}_{\circ} X_i^{\pm}(w) X_j^{\pm}(w) {}^{\circ}_{\circ}\, {}^{\circ}_{\circ} \\
= {}^{\circ}_{\circ}\, {}^{\circ}_{\circ} X_i^{\pm}(w) X_j^{\pm}(w) {}^{\circ}_{\circ} X_i^{\pm}(z) {}^{\circ}_{\circ} &= {}^{\circ}_{\circ} X_i^{\pm}(w) X_j^{\pm}(w) {}^{\circ}_{\circ} X_i^{\pm}(z).
\end{aligned} \tag{5.8}
$$

Then it follows from (5.7) and (5.8) that (Q8) holds on V, as required. $\square$

Remark 5.2. From (5.4), it follows that for $i \neq j \in \{0,1\}$,

$$
(z - q^{\mp 2}w) X_i^{\pm}(z) X_j^{\pm}(w) \neq (q^{\mp 2}z - w) X_j^{\pm}(w) X_i^{\pm}(z).
$$

In particular, the action given in Theorem 5.1 cannot induce a $\mathcal{U}_\hbar(\dot{\mathfrak{g}}_{tor})$-module structure on V.

References

[1] J. Ding, K. Iohara, Generalization of Drinfeld quantum affine algebras, Lett. Math. Phys. 41 (1997) 181–193.
[2] B. Enriquez, PBW and duality theorems for quantum groups and quantum current algebras, J. Lie Theory 13 (2003) 21–64.
[3] V. Ginzburg, M. Kapranov, E. Vasserot, Langlands reciprocity for algebraic surfaces, Math. Res. Lett. 2 (1995) 147–160.
[4] P. Grossé, On quantum shuffle and quantum affine algebras, J. Algebra 318 (2007) 495–519.
[5] D. Hernandez, Representations of quantum affinizations and fusion product, Transform. Groups 10 (2005) 163–200.
[6] D. Hernandez, Drinfeld coproduct, quantum fusion tensor category and applications, Proc. Lond. Math. Soc. 95 (2007) 567–608.
[7] D. Hernandez, Quantum toroidal algebras and their representations, Sel. Math. 14 (2009) 701–725.
[8] N. Jing, Quantum Kac-Moody algebras and vertex representations, Lett. Math. Phys. 44 (1998) 261–271.
[9] R. Moody, S.E. Rao, T. Yokonuma, Toroidal Lie algebras and vertex representations, Geom. Dedic. 35 (1990) 283–307.
[10] H. Nakajima, Quiver varieties and finite dimensional representations of quantum affine algebras, J. Am. Math. Soc. 14 (2001) 145–238.