Journal Pre-proof

Clean group rings over localizations of rings of integers

Yuanlin Li, Qinghai Zhong

Please cite this article as: Y. Li, Q. Zhong, Clean group rings over localizations of rings of integers, *J. Pure Appl. Algebra* (2019), 106284, doi: https://doi.org/10.1016/j.jpaa.2019.106284.

This is a PDF file of an article that has undergone enhancements after acceptance, such as the addition of a cover page and metadata, and formatting for readability, but it is not yet the definitive version of record. This version will undergo additional copyediting, typesetting and review before it is published in its final form, but we are providing this version to give early visibility of the article. Please note that, during the production process, errors may be discovered which could affect the content, and all legal disclaimers that apply to the journal pertain.
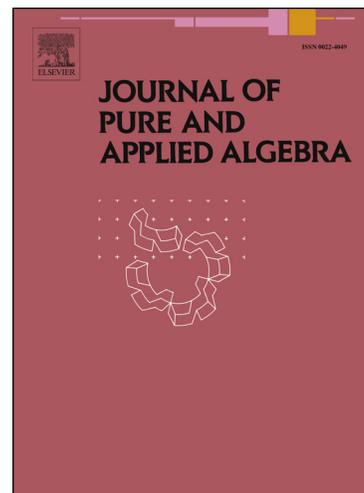
CLEAN GROUP RINGS OVER LOCALIZATIONS OF RINGS OF INTEGERS

YUANLIN LI AND QINGHAI ZHONG

ABSTRACT. A ring R is said to be clean if each element of R can be written as the sum of a unit and an idempotent. In a recent article (J. Algebra, 405 (2014), 168-178), Immormino and McGoven characterized when the group ring $\mathbb{Z}_{(p)}[C_n]$ is clean, where $\mathbb{Z}_{(p)}$ is the localization of the integers at the prime p. In this paper, we consider a more general setting. Let K be an algebraic number field, $\mathcal{O}_K$ be its ring of integers, and R be a localization of $\mathcal{O}_K$ at some prime ideal. We investigate when $R[G]$ is clean, where G is a finite abelian group, and obtain a complete characterization for such a group ring to be clean for the case when $K = \mathbb{Q}(\zeta_n)$ is a cyclotomic field or $K = \mathbb{Q}(\sqrt{d})$ is a quadratic field.

1. INTRODUCTION

All rings considered here are associative with identity $1 \neq 0$. An element of a ring R is called clean if it is the sum of a unit and an idempotent, and a ring R is called clean if each element of R is clean. Clean rings were introduced and related to exchange rings by Nicholson in 1977 [12] and the study of clean rings has attracted a great deal of attention in recent 2 decades. For some fundamental properties about clean rings as well as a nice history of clean rings we suggest the interested reader to check the article [10].

Let G be a multiplicative group. We denote by $R[G]$ the group ring of G over R which is the set of all formal sums

$$\alpha = \sum_{g \in G} \alpha_g g \,,$$

where $\alpha_g \in R$ and the support of α, $\operatorname{supp}(\alpha) = \{g \in G \mid \alpha_g \neq 0\}$, is finite. We let C_n denote the cyclic group of order n. Since a homomorphic image of a clean ring is a clean ring, it follows that it is necessary that R is clean whenever $R[G]$ is.

In this paper, we investigate the question of when a commutative group ring $R[G]$ over a local ring R is clean. We also study when such a group ring is $*$-clean (see next section for the definition of $*$-clean rings). Let $\mathbb{Z}_{(p)}$ denote the localization of the ring $\mathbb{Z}$ of integers at the prime p. In [3], the authors proved that $\mathbb{Z}_{(7)}[C_3]$ is not clean. It then follows that since $\mathbb{Z}_{(p)}$ is a clean ring (as it is local) that R being a commutative clean ring is not sufficient for $R[G]$ to be a clean ring. In a recent paper [7], it was shown that $\mathbb{Z}_{(p)}[C_3]$ is clean if and only if $p \not\equiv 1 \pmod 3$. More generally, the authors gave a complete characterization of when $\mathbb{Z}_{(p)}[C_n]$ is clean. Note that $\mathbb{Z}_{(p)}$ is a local ring between $\mathbb{Z}$ and $\mathbb{Q}$. In this paper, we consider a more general setting. Let $(R, \mathfrak{m})$ be a commutative local ring and we denote $\overline{R} = R/\mathfrak{m}$. Let K be an algebraic number field, $\mathcal{O}_K$ be its ring of integers, and R be a localization of $\mathcal{O}_K$ at some prime ideal $\mathfrak{p}$. We investigate when $R[G]$ is clean, where G is a finite abelian group, and provide a complete characterization for such a group ring to be clean for the case when $K = \mathbb{Q}(\zeta_n)$ is a cyclotomic field or $K = \mathbb{Q}(\sqrt{d})$ is a quadratic field. Our main results are as follows.

Theorem 1.1. *Let $K = \mathbb{Q}(\zeta_n)$ be a cyclotomic field for some $n \in \mathbb{N}$, $\mathcal{O} = \mathbb{Z}[\zeta_n]$ its rings of integers, $\mathfrak{p} \subset \mathcal{O}$ a nonzero prime ideal, and G a finite abelian group. Let p be the prime with $p\mathbb{Z} = \mathfrak{p} \cap \mathbb{Z}$, let n_0 be the maximal positive divisor of n with $p \nmid n_0$, let n_1 be the maximal divisor of $\exp(G)$ with $p \nmid n_1$ and $\gcd(n_1, n_0) = 1$, and let m' be the maximal divisor of $\frac{\mathrm{lcm}(\exp(G), n_0)}{n_0 n_1}$ with $p \nmid m'$.*

Then the group ring $\mathcal{O}_{\mathfrak{p}}[G]$ is clean if and only if $\mathrm{ord}_{n_1} p = \varphi(n_1)$, $\mathrm{ord}_{n_0 m'} p = m' \mathrm{ord}_{n_0} p$, and $\gcd(\mathrm{ord}_{n_1} p, \mathrm{ord}_{n_0 m'} p) = 1$. In particular, if $\exp(G)$ is a divisor of n, then $\mathcal{O}_{\mathfrak{p}}[G]$ is clean.

Note that if $n = 1$ (i.e. $K = \mathbb{Q}$), then $n_0 = 1$ and $m' = 1$. Therefore Theorem 1.1 implies the following corollary which is exactly the main result of [7, Theorem 3.3].

Corollary 1.2. *Let G be a finite abelian group, let p be a prime number, and let n_1 be the maximal divisor of $\exp(G)$ with $p \nmid n_1$. Then $\mathbb{Z}_{(p)}[G]$ is clean if and only if $\mathrm{ord}_{n_1} p = \varphi(n_1)$ (i.e. p is a primitive root of n_1).*

Theorem 1.3. *Let $K = \mathbb{Q}(\sqrt{d})$ be a quadratic field for some non-zero square-free integer $d \neq 1$, $\mathcal{O}$ its rings of integers, $\mathfrak{p} \subset \mathcal{O}$ a nonzero prime ideal, and G a finite abelian group. Let p be the prime with $p\mathbb{Z} = \mathfrak{p} \cap \mathbb{Z}$, let Δ be the discriminant of K, and let n be the maximal positive divisor of $\exp(G)$ with $p \nmid n$.*

1. *If $\Delta \nmid n$, then $\mathcal{O}_{\mathfrak{p}}[G]$ is clean if and only if one of the following holds*
   (a) *$p = 2$ is a primitive root of unity of n and $\Delta \not\equiv 5 \pmod 8$;*
   (b) *$p \neq 2$ is a primitive root of unity of n and $\left(\frac{\Delta}{p}\right) = 1$ or 0;*
   (c) *$n = 2$, $p \neq 2$, and $\left(\frac{\Delta}{p}\right) = -1$, where $\left(\frac{\Delta}{p}\right)$ is the Legendre symbol.*
2. *If $\Delta \mid n$ and $d \equiv 2, 3 \pmod 4$, then $\mathcal{O}_{\mathfrak{p}}[G]$ is clean if and only if $|d|$ is a prime, $n = 4|d|^{\ell}$ with $\ell \in \mathbb{N}$, $p \equiv 3 \pmod 4$, $\left(\frac{\Delta}{p}\right) = 1$, and $\mathrm{ord}_n p = \varphi(n)/2$.*
3. *If $\Delta \mid n$ and $d \equiv 1 \pmod 4$, then $\mathcal{O}_{\mathfrak{p}}[G]$ is clean if and only if one of the following holds*
   (a) *$|d|$ is a prime, $n = |d|^{\ell}$ or $2|d|^{\ell}$ for some $\ell \in \mathbb{N}$, and either $\mathrm{ord}_n p = \frac{2\varphi(n)}{3 + \left(\frac{d}{p}\right)}$, or $\mathrm{ord}_n p = \varphi(n)/2$ with $\left(\frac{d}{p}\right) = -1$ and $d < 0$, where $\left(\frac{d}{p}\right)$ is the Legendre symbol.*
   (b) *$|d| = q$ is a prime, $n = 4q^{\ell}$ with $q \equiv 3 \pmod 4$ and $\ell \in \mathbb{N}$, $\left(\frac{d}{p}\right) = 1$, $p \equiv 3 \pmod 4$, and $\mathrm{ord}_{q^{\ell}} p = q^{\ell-1}(q-1)/2$.*
   (c) *$|d| = q_1$ is a prime, $n = q_1^{\ell_1} q_2^{\ell_2}$ or $2q_1^{\ell_1} q_2^{\ell_2}$ with $q_1 \equiv 3 \pmod 4$, $\ell_1, \ell_2 \in \mathbb{N}$, and q_2 is another odd prime, $\left(\frac{d}{p}\right) = 1$, p is a primitive root of unity of $q_2^{\ell_2}$, $\mathrm{ord}_{q_1^{\ell_1}} p = q_1^{\ell_1-1}(q_1 - 1)/2$, and $\gcd(q_1^{\ell_1-1}(q_1-1)/2, q_2^{\ell_2-1}(q_2-1)) = 1$.*
   (d) *$|d| = q_1 q_2$ is a product of two distinct primes, $n = q_1^{\ell_1} q_2^{\ell_2}$ or $2q_1^{\ell_1} q_2^{\ell_2}$ with $\ell_1, \ell_2 \in \mathbb{N}$, $\left(\frac{d}{p}\right) = 1$, p is a primitive root of unity of $q_1^{\ell_1}$ and $q_2^{\ell_2}$, and $\gcd(q_1^{\ell_1-1}(q_1-1)/2, q_2^{\ell_2-1}(q_2-1)/2) = 1$.*

In Section 2, we collect some necessary knowledge of the structure of the group of units $(\mathbb{Z}/m\mathbb{Z})^{\times}$ and field extension. Furthermore, we give some general characterization theorems for clean and $*$-clean group rings. In Section 3, we deal with group rings over local subrings of cyclotomic fields and provide a proof of Theorem 1.1. In Section 4, we consider group rings over local subrings of quadratic fields and give a proof of Theorem 1.3.

2. Preliminaries

For a finite abelian group G, we denote by $\exp(G)$ the exponent of G. We denote by $\mathbb{N}$ the set of all positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For $n \in \mathbb{N}$, we denote by $\varphi(n)$ the Euler function. Let $n \in \mathbb{N}$ and let $n = p_1^{k_1} \ldots p_s^{k_s}$ be its prime factorization, where $s, k_1, \ldots, k_s \in \mathbb{N}$ and $p_1, \ldots, p_s$ are pair-wise distinct

primes. It is well-known that

$$\varphi(n) = \prod_{i=1}^{s} \varphi(p_i^{k_i}) = \prod_{i=1}^{s} p_i^{k_i-1}(p_i - 1)$$

$$\text{and} \quad (\mathbb{Z}/n\mathbb{Z})^{\times} \cong (\mathbb{Z}/p_1^{k_1}\mathbb{Z})^{\times} \times \ldots \times (\mathbb{Z}/p_s^{k_s}\mathbb{Z})^{\times}.$$

Furthermore,

$$(\mathbb{Z}/p_i^k\mathbb{Z})^{\times} \cong C_{p_i^{k-1}(p_i-1)} \qquad\qquad \text{where } p_i \geq 3,$$

$$(\mathbb{Z}/2^{\ell}\mathbb{Z})^{\times} = \langle -1 \rangle \times \langle 5 \rangle \cong C_2 \oplus \mathbb{C}_{2^{\ell-2}} \qquad\qquad \text{where } \ell \geq 3,$$

$$\text{and} \quad (\mathbb{Z}/4\mathbb{Z})^{\times} \cong C_2.$$

For every $m \in \mathbb{N}$ with $\gcd(m,n) = 1$, we denote by $\operatorname{ord}_n m = \operatorname{ord}_{(\mathbb{Z}/n\mathbb{Z})^{\times}} m$ the multiplicative order of m modulo n. If $\operatorname{ord}_n m = \varphi(n)$, we say m is a primitive root of n and n has a primitive root if and only if $n = 2$, 4, q^{ℓ}, or $2q^{\ell}$, where q is an odd prime and $\ell \in \mathbb{N}$. Let $n_1 \in \mathbb{N}$ be another integer with $\gcd(n_1, m) = 1$. Then

$$\operatorname{ord}_n m \leq \operatorname{ord}_{nn_1} m \leq n_1 \operatorname{ord}_n m,$$

$$\text{and} \quad \operatorname{lcm}(\operatorname{ord}_n m, \operatorname{ord}_{n_1} m) = \operatorname{ord}_{\operatorname{lcm}(n,n_1)} m.$$

Let ζ_n be an nth primitive root of unity over $\mathbb{Q}$. Then $[\mathbb{Q}(\zeta_n) : \mathbb{Q}] = \varphi(n)$. Let m be another positive integer. Then

$$\mathbb{Q}(\zeta_n) \cap \mathbb{Q}(\zeta_m) = \mathbb{Q}(\zeta_{\gcd(n,m)})$$

$$\text{and} \quad \mathbb{Q}(\zeta_n)(\zeta_m) = \mathbb{Q}(\zeta_{\operatorname{lcm}(n,m)}).$$

Let $(R, \mathfrak{m})$ be a commutative local ring and we denote $\overline{R} = R/\mathfrak{m}$. Then $\overline{R}$ is a field and we denote by $\operatorname{char} \overline{R}$ the characteristic of $\overline{R}$. For any polynomial $f(x) = a_n x^n + \ldots + a_0 \in R[x]$, we denote $\overline{f(x)} = \overline{a_n} x^n + \ldots + \overline{a_0} \in \overline{R}[x]$, where $\overline{a_i} = a_i + \mathfrak{m}$ for all $i \in \{0, \ldots, n\}$.

Let R be a ring R and let G be a multiplicative group. Then the group ring of G over R is the ring $R[G]$ of all formal sums

$$\alpha = \sum_{g \in G} \alpha_g g,$$

where $\alpha_g \in R$ and the support of α, $\operatorname{supp}(\alpha) = \{g \in G \mid \alpha_g \neq 0\}$, is finite. Addition is defined componentwise and multiplication is defined by the following way: for $\alpha, \beta \in R[G]$,

$$\alpha\beta = \left(\sum_{g \in G} \alpha_g g\right)\left(\sum_{h \in G} \beta_h h\right) = \sum_{g,h \in G} \alpha_g \beta_h (gh).$$

For more information on the group ring, we refer [11] as a reference.

Theorem 2.1. *Let $(R, \mathfrak{m})$ be a commutative noetherian local ring with $\operatorname{char} \overline{R} = p \geq 0$, let G be a finite abelian group, and let n be the maximal divisor of $\exp(G)$ with $p \nmid n$. Then $R[G]$ is clean if and only if each monic factor of $x^n - 1$ in $\overline{R}[x]$ can be lifted to a monic factor of $x^n - 1$ in $R[x]$.*

Proof. This follows from [7, Proposition 2.1] and [14, Theorem 5.8]. □

Let K be an algebraic number field, $\mathcal{O}$ its rings of integers, and $\mathfrak{p} \subset \mathcal{O}$ a nonzero prime ideal. Then there exists a prime p such that $\mathfrak{p} \cap \mathbb{Z} = p\mathbb{Z}$ and the localization $\mathcal{O}_{\mathfrak{p}}$ is a discrete valuation ring, which implies that $\mathcal{O}_{\mathfrak{p}}[x]$ is a UFD (Unique factorization domain). Furthermore, the norm $N(\mathfrak{p}) = |\mathcal{O}/\mathfrak{p}| = |\overline{\mathcal{O}_{\mathfrak{p}}}|$ is a prime power of p.

Let $\mathbb{F}_q$ be a finite field, where q is a power of some prime p and let $\overline{\zeta_n}$ be the nth primitive root of unity over $\mathbb{F}_p$ with $\gcd(n, q) = 1$. Then $[\mathbb{F}_q(\overline{\zeta_n}) : \mathbb{F}_q] = \operatorname{ord}_n q$. Let F be an arbitrary field and let $f(x)$ be

a polynomial of $F[x]$. If α is a root of $f(x)$, then $[F(\alpha) : F] = \deg(f(x))$ if and only if $f(x)$ is irreducible in $F[x]$.

Theorem 2.2. *Let K be an algebraic number field, $\mathcal{O}$ its rings of integers, $\mathfrak{p} \subset \mathcal{O}$ a nonzero prime ideal, and G a finite abelian group. Then the group ring $\mathcal{O}_{\mathfrak{p}}[G]$ is clean if and only if $[K[\zeta_m] : K] = \mathrm{ord}_m(N(\mathfrak{p}))$ for every positive divisor m of $\exp(G)$ with $p \nmid m$, where ζ_m is an mth primitive root of unity over $\mathbb{Q}$.*

Proof. Let n be the maximal divisor of $\exp(G)$ with $p \nmid \exp(G)$. Since $\mathcal{O}_{\mathfrak{p}}[x]$ is a UFD, we suppose that $x^n - 1 = f_1(x) \cdot \ldots \cdot f_s(x)$, where $s \in \mathbb{N}$ and $f_1(x), \ldots, f_s(x)$ are monic irreducible polynomials in $\mathcal{O}_{\mathfrak{p}}[x]$. Then the Generalized Gauss' Primitive Polynomial Lemma implies that $f_1(x), \ldots, f_s(x)$ are also monic irreducible polynomials in $K[x]$. For every positive divisor m of n, let $\Phi_m(x)$ be the mth cyclotomic polynomial. Then $\Phi_m(x) \in \mathbb{Z}[x] \subset \mathcal{O}_{\mathfrak{p}}[x]$ and

$$x^n - 1 = \prod_{1 < m \,|\, n} \Phi_m(x) = f_1(x) \cdot \ldots \cdot f_s(x).$$

1. Suppose $\mathcal{O}_{\mathfrak{p}}[G]$ is clean. Let $f(x)$ be a monic irreducible factor of $x^n - 1$ in $\mathcal{O}_{\mathfrak{p}}[x]$ and let $\mathfrak{h}(x)$ be a monic irreducible factor of $\overline{f(x)}$ in $\overline{\mathcal{O}_{\mathfrak{p}}}[x]$. By Theorem 2.1, there exists a monic irreducible factor $h(x)$ of $x^n - 1$ in $\mathcal{O}_{\mathfrak{p}}[x]$ such that $\overline{h(x)} = \mathfrak{h}(x)$. If $h(x) \neq f(x)$, it follows by $\mathcal{O}_{\mathfrak{p}}[x]$ is a UFD that $f(x)h(x)$ is a monic factor of $x^n - 1$ in $\mathcal{O}_{\mathfrak{p}}[x]$ and hence $\overline{h(x)}^2$ is a monic factor of $x^n - 1$ in $\overline{\mathcal{O}_{\mathfrak{p}}}[x]$. Since $\gcd(n, p) = 1$, we obtain $x^n - 1 \in \overline{\mathcal{O}_{\mathfrak{p}}}[x]$ has no multiple root in any extension of $\overline{\mathcal{O}_{\mathfrak{p}}}$, a contradiction. Therefore $h(x) = f(x)$ and hence $\overline{f(x)} = \overline{h(x)} = \mathfrak{h}(x)$ is irreducible in $\overline{\mathcal{O}_{\mathfrak{p}}}[x]$.

Let m be a positive divisor of n. It follows from the fact that $\mathcal{O}_{\mathfrak{p}}[x]$ is a UFD that there exists $i \in [1, s]$ such that $f_i(x)$ divides $\Phi_m(x)$ in $\mathcal{O}_{\mathfrak{p}}[x]$. Thus every root of $f_i(x)$ is a mth primitive root of unity in K and hence $[K(\zeta_m) : K] = \deg(f_i(x)) = \deg(\overline{f_i(x)})$. Since $\overline{f_i(x)}$ is irreducible in $\overline{\mathcal{O}_{\mathfrak{p}}}[x]$ and every root of $\overline{f_i(x)}$ is a mth primitive root of unity in $\overline{\mathcal{O}_{\mathfrak{p}}}$, we have $\deg(\overline{f_i(x)}) = [\overline{\mathcal{O}_{\mathfrak{p}}}(\overline{\zeta}_m) : \overline{\mathcal{O}_{\mathfrak{p}}}] = \mathrm{ord}_m N(\mathfrak{p})$, where $\overline{\zeta}_m$ is a mth primitive root of unity over $\mathbb{F}_p$. Therefore $[K(\zeta_m) : K] = \mathrm{ord}_m N(\mathfrak{p})$.

2. Conversely, suppose $[K(\zeta_m) : K] = \mathrm{ord}_m N(\mathfrak{p})$ for every divisor m of n. Let $i \in [1, s]$. Then $f_i(x)$ is a factor of some mth cyclotomic polynomial $\Phi_m(x)$ with $m \,|\, n$. Since $f_i(x)$ is irreducible in $K[x]$, we have $\deg(f_i(x)) = [K(\zeta_m) : K]$ and hence

$$\deg(\overline{f_i(x)}) = \deg(f_i(x)) = [K(\zeta_m) : K] = \mathrm{ord}_m N(\mathfrak{p}) = |\overline{\mathcal{O}_{\mathfrak{p}}}(\overline{\zeta}_m) : \overline{\mathcal{O}_{\mathfrak{p}}}|.$$

Therefore $\overline{f_i(x)}$ is irreducible in $\overline{\mathcal{O}_{\mathfrak{p}}}[x]$ and

$$x^n - 1 = \overline{f_1(x)} \cdot \ldots \cdot \overline{f_s(x)} \in \overline{\mathcal{O}_{\mathfrak{p}}}[x].$$

Let $\mathfrak{h}(x)$ be a monic factor of $x^n - 1 \in \overline{\mathcal{O}_{\mathfrak{p}}}[x]$. Since $\overline{\mathcal{O}_{\mathfrak{p}}}[x]$ is a UFD, there exists a subset $I \subset [1, s]$ such that $\mathfrak{h}(x) = \prod_{i \in I} \overline{f_i(x)}$ and hence $\overline{\prod_{i \in I} f_i(x)} = \mathfrak{h}(x)$. Therefore every monoic factor of $x^n - 1 \in \overline{\mathcal{O}_{\mathfrak{p}}}[x]$ can be lifted to a monioc factor of $x^n - 1 \in \mathcal{O}_{\mathfrak{p}}[x]$. It follows from Theorem 2.1 that $\mathcal{O}_{\mathfrak{p}}[G]$ is clean. $\square$

A ring R is called a $*$-ring if there exists an operation $* : R \to R$ such that $(x + y)^* = x^* + y^*$, $(xy)^* = x^* y^*$, and $(x^*)^* = x$ for all $x, y \in R$. An element $p \in R$ is said to be a projection if $p^* = p = p^2$ and a $*$-ring R is said to be a $*$-clean ring if every element of R is the sum of a unit and a projection. A commutative $*$-ring is $*$-clean if and only if it is clean and every idempotent is a projection ([8, Theorem 2.2]). Let G be an abelian group. With the classical involution

$$* : R[G] \to R[G], \text{given by}$$

$$\left(\sum a_g g\right)^* = \sum a_g g^{-1},$$

the group ring $R[G]$ is a $*$-ring. The question of when a group ring $R[G]$ is $*$-clean has been recently studied by several authors and many interesting results were obtained (see, for examples, [2, 4, 5, 6, 8, 9], for some recent developments). Next we provide a characterization for $\mathcal{O}_{\mathfrak{p}}[G]$ to be $*$-clean.

Theorem 2.3. *Let K be an algebraic number field, $\mathcal{O}$ its rings of integers, $\mathfrak{p} \subset \mathcal{O}$ a nonzero prime ideal with $p\mathbb{Z} = \mathfrak{p} \cap \mathbb{Z}$, and G a finite abelian group with $p \nmid \exp(G)$. If the group ring $\mathcal{O}_{\mathfrak{p}}[G]$ is clean, then $\mathcal{O}_{\mathfrak{p}}[G]$ is $*$-clean if and only if $K[G]$ is $*$-clean.*

Proof. Let $\mathcal{O}_{\mathfrak{p}}[G]$ be clean. Suppose $K[G]$ is $*$-clean. Since every idempotent of $\mathcal{O}_{\mathfrak{p}}[G]$ is an idempotent of $K[G]$, thus every idempotent of $\mathcal{O}_{\mathfrak{p}}[G]$ is a projective. It follows that $\mathcal{O}_{\mathfrak{p}}[G]$ is $*$-clean.

Suppose $\mathcal{O}_{\mathfrak{p}}[G]$ is $*$-clean. Let $\mathcal{O}_{K(\zeta_{\exp(G)})/\mathbb{Q}}$ be the ring of integers of $K(\zeta_{\exp(G)})$ and let I be a prime ideal of $\mathcal{O}_{K(\zeta_{\exp(G)})/\mathbb{Q}}$ with $I \cap \mathcal{O} = \mathfrak{p}$. By [4, The beginning of Section 5] and $p \nmid \exp(G)$, there is a complete family of orthogonal idempotents of $K(\zeta_{\exp(G)})[G]$ which lies in $(\mathcal{O}_{K(\zeta_{\exp(G)})/\mathbb{Q}})_I[G]$. It follows from [4, Lemma 4.3] that every idempotent of $K[G]$ lies in $(\mathcal{O}_{K(\zeta_{\exp(G)})/\mathbb{Q}})_I[G] \cap K[G] = \mathcal{O}_{\mathfrak{p}}[G]$. Since every idempotent of $\mathcal{O}_{\mathfrak{p}}[G]$ is a projection, we obtain every idempotent of $K[G]$ is a projection. Note that $K[G]$ is clean. Thus $K[G]$ is $*$-clean. $\square$

3. Group rings over local subrings of cyclotomic fields

In this section, we investigate when a group ring over a local subring of a cyclotomic field is clean and provide a proof for our main theorem 1.1. We also characterize when such a group ring is $*$-clean. We start with the following lemma which we will use without further mention.

Lemma 3.1. *Let $K = \mathbb{Q}(\zeta_n)$ be a cyclotomic field for some $n \in \mathbb{N}$, $\mathcal{O} = \mathbb{Z}[\zeta_n]$ its rings of integers, and $\mathfrak{p} \subset \mathcal{O}$ a nonzero prime ideal with $\mathfrak{p} \cap \mathbb{Z} = p\mathbb{Z}$ for some prime p. Suppose $n = p^u n_0$ with $p \nmid n_0$. Then $N(\mathfrak{p}) = p^{\operatorname{ord}_{n_0} p}$.*

Proof. This follows by [1, VI.1.12 and VI.1.15]. $\square$

Proof of Theorem 1.1. Let n_2 be the maximal divisor of $\exp(G)$ such that $p \nmid n_2$ and let $n_3 = \frac{n_2}{n_1}$. Since

$$m' = \frac{\operatorname{lcm}(n_2, n_0)}{n_0 n_1} = \frac{\operatorname{lcm}(n_3, n_0)}{n_0},$$

we have $\operatorname{lcm}(n_3, n_0) = n_0 m'$ and $\operatorname{lcm}(n_2, n_0) = n_0 m' n_1$.

Let m be a divisor of n_2. Then

$$[\mathbb{Q}(\zeta_n)(\zeta_m) : \mathbb{Q}(\zeta_n)] = [\mathbb{Q}(\zeta_{\operatorname{lcm}(n,m)} : \mathbb{Q}(\zeta_n)] = \frac{\varphi(\operatorname{lcm}(n,m))}{\varphi(n)} = \frac{\varphi(\operatorname{lcm}(n_0,m))}{\varphi(n_0)}$$

and

$$\operatorname{ord}_m N(\mathfrak{p}) = \operatorname{ord}_m p^{\operatorname{ord}_{n_0} p} = \frac{\operatorname{ord}_m p}{\gcd(\operatorname{ord}_m p, \operatorname{ord}_{n_0} p)} = \frac{\operatorname{lcm}(\operatorname{ord}_{n_0} p, \operatorname{ord}_m p)}{\operatorname{ord}_{n_0} p} = \frac{\operatorname{ord}_{\operatorname{lcm}(n_0,m)} p}{\operatorname{ord}_{n_0} p}.$$

Therefore by Theorem 2.2 that $R[G]$ is clean if and only if

$$\text{for every divisor } m \text{ of } n_2, \quad \text{we have} \quad \frac{\varphi(\operatorname{lcm}(n_0,m))}{\varphi(n_0)} = \frac{\operatorname{ord}_{\operatorname{lcm}(n_0,m)} p}{\operatorname{ord}_{n_0} p}.$$

1. We first suppose that $R[G]$ is clean. Since n_1, n_3 and n_2 are divisors of n_2, we obtain

$$\frac{\varphi(\operatorname{lcm}(n_0, n_1))}{\varphi(n_0)} = \frac{\operatorname{ord}_{\operatorname{lcm}(n_0,n_1)} p}{\operatorname{ord}_{n_0} p},$$

$$\frac{\varphi(\operatorname{lcm}(n_0, n_3))}{\varphi(n_0)} = \frac{\operatorname{ord}_{\operatorname{lcm}(n_0,n_3)} p}{\operatorname{ord}_{n_0} p},$$

$$\text{and} \quad \frac{\varphi(\operatorname{lcm}(n_0, n_2))}{\varphi(n_0)} = \frac{\operatorname{ord}_{\operatorname{lcm}(n_0,n_2)} p}{\operatorname{ord}_{n_0} p}.$$

Since $\gcd(n_1, n_0) = 1$, we obtain

$$\varphi(n_1) = \frac{\varphi(\operatorname{lcm}(n_0, n_1))}{\varphi(n_0)} = \frac{\operatorname{ord}_{\operatorname{lcm}(n_0, n_1)} p}{\operatorname{ord}_{n_0} p} = \frac{\operatorname{lcm}(\operatorname{ord}_{n_0} p, \operatorname{ord}_{n_1} p)}{\operatorname{ord}_{n_0} p} \le \frac{\operatorname{ord}_{n_1} p \operatorname{ord}_{n_0} p}{\operatorname{ord}_{n_0} p} = \operatorname{ord}_{n_1} p \le \varphi(n_1) \, .$$

Then $\operatorname{ord}_{n_1} p = \varphi(n_1)$.

Since

$$m' = \frac{\varphi(n_0 m')}{\varphi(n_0)} = \frac{\varphi(\operatorname{lcm}(n_0, n_3))}{\varphi(n_0)} = \frac{\operatorname{ord}_{\operatorname{lcm}(n_0, n_3)} p}{\operatorname{ord}_{n_0} p} = \frac{\operatorname{ord}_{n_0 m'} p}{\operatorname{ord}_{n_0} p} \le \frac{m' \operatorname{ord}_{n_0} p}{\operatorname{ord}_{n_0} p} = m' \, ,$$

we obtain $\operatorname{ord}_{n_0 m'} p = m' \operatorname{ord}_{n_0} p$.

Since

$$m' \varphi(n_1) = \frac{\varphi(n_0 m' n_1)}{\varphi(n_0)} = \frac{\varphi(\operatorname{lcm}(n_0, n_2))}{\varphi(n_0)} = \frac{\operatorname{ord}_{\operatorname{lcm}(n_0, n_2)} p}{\operatorname{ord}_{n_0} p} = \frac{\operatorname{ord}_{n_0 m' n_1} p}{\operatorname{ord}_{n_0} p}$$

$$= \frac{\operatorname{lcm}(\operatorname{ord}_{n_0 m'} p, \operatorname{ord}_{n_1} p)}{\operatorname{ord}_{n_0} p} \le \frac{\operatorname{ord}_{n_0 m'} p \operatorname{ord}_{n_1} p}{\operatorname{ord}_{n_0} p} = \frac{m' \operatorname{ord}_{n_0} p \varphi(n_1)}{\operatorname{ord}_{n_0} p} = m' \varphi(n_1) \, ,$$

we obtain $\operatorname{lcm}(\operatorname{ord}_{n_0 m'} p, \operatorname{ord}_{n_1} p) = \operatorname{ord}_{n_0 m'} p \operatorname{ord}_{n_1} p$. Thus $\gcd(\operatorname{ord}_{n_0 m'} p, \operatorname{ord}_{n_1} p) = 1$.

2. Conversely, suppose that $\operatorname{ord}_{n_1} p = \varphi(n_1)$, $\operatorname{ord}_{n_0 m'} p = m' \operatorname{ord}_{n_0} p$, and $\gcd(\operatorname{ord}_{n_1} p, \operatorname{ord}_{n_0 m'} p) = 1$. Then for every $m \mid n_2$, we let $m_1 = \gcd(m, n_1)$ and $m_2 = \frac{\operatorname{lcm}(m/m_1, n_0)}{n_0}$. Then $\operatorname{ord}_{m_1} p = \varphi(m_1)$. It follows by $n_0 m_2 \mid n_0 m'$ that $\gcd(\operatorname{ord}_{m_1} p, \operatorname{ord}_{n_0 m_2} p) = 1$ and

$$\operatorname{ord}_{n_0 m'} p = \operatorname{ord}_{n_0 m_2 \frac{m'}{m_2}} p \le \frac{m'}{m_2} \operatorname{ord}_{n_0 m_2} p \le \frac{m'}{m_2} m_2 \operatorname{ord}_{n_0} p = m' \operatorname{ord}_{n_0} p = \operatorname{ord}_{n_0 m'} p \, .$$

Thus $\operatorname{ord}_{n_0 m_2} p = m_2 \operatorname{ord}_{n_0} p$. It follows that

$$\frac{\operatorname{ord}_{\operatorname{lcm}(n_0, m)} p}{\operatorname{ord}_{n_0} p} = \frac{\operatorname{ord}_{m_1 n_0 m_2} p}{\operatorname{ord}_{n_0} p} = \frac{\operatorname{lcm}(\operatorname{ord}_{m_1} p, \operatorname{ord}_{n_0 m_2} p)}{\operatorname{ord}_{n_0} p}$$

$$= \frac{\operatorname{ord}_{m_1} p \operatorname{ord}_{n_0 m_2} p}{\operatorname{ord}_{n_0} p} = \frac{m_2 \operatorname{ord}_{m_1} p \operatorname{ord}_{n_0} p}{\operatorname{ord}_{n_0} p}$$

$$= m_2 \operatorname{ord}_{m_1} p = m_2 \varphi(m_1)$$

$$= \frac{\varphi(m_1 n_0 m_2)}{\varphi(n_0)} = \frac{\varphi(\operatorname{lcm}(n_0, m))}{\varphi(n_0)} \, .$$

Therefore $R[G]$ is clean.

3. In particular, if $\exp(G)$ is a divisor of n, then $n_1 = m' = 1$ and hence $\mathcal{O}_{\mathfrak{p}}[G]$ is clean. $\qquad\square$

Next we characterize when a group ring of a finite abelian group over a local ring $\mathcal{O}_{\mathfrak{p}}$ is $*$-clean. We need the following two propositions.

Proposition 3.2. *Let* $m, n \in \mathbb{N}$. *Then* $\mathbb{Q}(\zeta_m + \zeta_m^{-1})(\zeta_n) = \mathbb{Q}(\zeta_m)(\zeta_n)$ *if and only if* $\gcd(m, n) \ge 3$ *or* $m \le 2$.

Proof. If $m \le 2$, then it is obvious that $\mathbb{Q}(\zeta_m + \zeta_m^{-1})(\zeta_n) = \mathbb{Q}(\zeta_m)(\zeta_n)$. We suppose $m \ge 3$.

Let $K = \mathbb{Q}(\zeta_m + \zeta_m^{-1})$. Then $K \subset K(\zeta_n) \subset \mathbb{Q}(\zeta_m)(\zeta_n) = \mathbb{Q}(\zeta_{\operatorname{lcm}(n,m)})$. Thus $K(\zeta_n) = \mathbb{Q}(\zeta_m)(\zeta_n)$ if and only if $[K(\zeta_n) : K] = [\mathbb{Q}(\zeta_{\operatorname{lcm}(n,m)}) : K]$.

Since $[\mathbb{Q}(\zeta_{\operatorname{lcm}(n,m)}) : K] = \frac{[\mathbb{Q}(\zeta_{\operatorname{lcm}(n,m)}) : \mathbb{Q}]}{[K : \mathbb{Q}]} = \frac{2\varphi(\operatorname{lcm}(m,n))}{\varphi(m)} = \frac{2\varphi(n)}{\varphi(\gcd(m,n))}$ and

$$[K(\zeta_n) : K] = [\mathbb{Q}(\zeta_n) : K \cap \mathbb{Q}(\zeta_n)] = [\mathbb{Q}(\zeta_n) : \mathbb{Q}(\zeta_{\gcd(m,n)})][\mathbb{Q}(\zeta_{\gcd(m,n)}) : K \cap \mathbb{Q}(\zeta_n)]$$

$$= \frac{\varphi(n)}{\varphi(\gcd(m,n))}[\mathbb{Q}(\zeta_{\gcd(m,n)}) : K \cap \mathbb{Q}(\zeta_{\gcd(m,n)})]$$

$$= \frac{\varphi(n)}{\varphi(\gcd(m,n))}[K(\zeta_{\gcd(m,n)}) : K] \, ,$$

we obtain $|K(\zeta_{\gcd(m,n)}) : K| \leq 2$. Moreover, $K(\zeta_n) = \mathbb{Q}(\zeta_m)(\zeta_n)$ if and only if $|K(\zeta_{\gcd(m,n)}) : K| = 2$ if and only if $\zeta_{\gcd(m,n)} \notin K$.

If $\gcd(m,n) \geq 3$, then $\zeta_{\gcd(m,n)} \notin \mathbb{R}$ and hence $\zeta_{\gcd(m,n)} \notin K \subset \mathbb{R}$. It follows that $K(\zeta_n) = \mathbb{Q}(\zeta_m)(\zeta_n)$.

If $\gcd(m,n) \leq 2$, then $\zeta_{\gcd(m,n)} \in \mathbb{Q} \subset K$ and hence $K(\zeta_n) \neq \mathbb{Q}(\zeta_m)(\zeta_n)$. $\square$

Proposition 3.3. *Let G be a finite abelian group and let $n \in \mathbb{N}$. Then $\mathbb{Q}(\zeta_n)[G]$ is $*$-clean if and only if $\exp(G) \geq 3$ and $\gcd(\exp(G), n) \leq 2$.*

Proof. This follows from [4, Theorem 1.2] and Proposition 3.2. $\square$

Theorem 3.4. *Let $K = \mathbb{Q}(\zeta_n)$ be a cyclotomic field for some $n \in \mathbb{N}$, $\mathcal{O} = \mathbb{Z}[\zeta_n]$ its rings of integers, $\mathfrak{p} \subset \mathcal{O}$ a nonzero prime ideal with $p\mathbb{Z} = \mathfrak{p} \cap \mathbb{Z}$, where p is a prime, and G a finite abelian group with $p \nmid \exp(G)$. let n_0 be the maximal divisor of n with $p \nmid n_0$ and let n_1 be the maximal divisor of $\exp(G)$ with $\gcd(n_1, n_0) = 1$. Then the group ring $\mathcal{O}_{\mathfrak{p}}[G]$ is $*$-clean if and only if $\mathrm{ord}_{n_1} p = \varphi(n_1)$, $3 \leq \exp(G) \leq 2n_1$, and $\gcd(\mathrm{ord}_{n_1} p, \mathrm{ord}_{n_0} p) = 1$.*

Proof. 1. Suppose $\mathrm{ord}_{n_1} p = \varphi(n_1)$, $3 \leq \exp(G) \leq 2n_1$, and $\gcd(\varphi(n_1), \mathrm{ord}_{n_0} p) = 1$. Since every prime divisor of $\exp(G)/n_1$ is a divisor of n_0, it follows from $\exp(G)/n_1 \leq 2$ that $(\exp(G)/n_1)$ divides n_0. Hence

$$\frac{\mathrm{lcm}(\exp(G), n_0)}{n_0 n_1} = \frac{\mathrm{lcm}(\exp(G)/n_1, n_0)}{n_0} = 1.$$

Thus by Theorem 1.1 $\mathcal{O}_{\mathfrak{p}}[G]$ is clean. Since $p \nmid \exp(G)$, we have $\gcd(n, \exp(G)) = \gcd(n_0, \exp(G)/n_1) \leq 2$. Thus it follows from Proposition 3.3 that $\mathbb{Q}(\zeta_n)[G]$ is $*$-clean and hence by Theorem 2.3 $\mathcal{O}_{\mathfrak{p}}[G]$ is $*$-clean.

2. Suppose $\mathcal{O}_{\mathfrak{p}}[G]$ is $*$-clean. Let $m' = \frac{\mathrm{lcm}(\exp(G), n_0)}{n_0 n_1}$. Since $\mathcal{O}_{\mathfrak{p}}[G]$ is clean, it follows from Theorem 1.1 that

$$\mathrm{ord}_{n_1} p = \varphi(n_1), \qquad \mathrm{ord}_{n_0 m'} p = m' \mathrm{ord}_{n_0} p, \quad \text{and} \quad \gcd(\varphi(n_1), m' \mathrm{ord}_{n_0} p) = 1.$$

By Theorem 2.3 and Proposition 3.3, we have $\exp(G) \geq 3$ and $\gcd(n, \exp(G)) \leq 2$. Thus $\gcd(n_0, \exp(G)/n_1) \leq 2$. Since every prime divisor of $\exp(G)/n_1$ is a divisor of n_0, we obtain

$$\exp(G) = 2^{\ell} n_1 \text{ for some } \ell \in \mathbb{N}_0.$$

If $\ell \geq 2$, then $n_0 = 2n_0'$ with n_0' is odd which implies that $m' = 2^{\ell-1}$. Thus

$$2^{\ell-1} \mathrm{ord}_{n_0} p = m' \mathrm{ord}_{n_0} p = \mathrm{ord}_{m' n_0} p = \mathrm{lcm}(\mathrm{ord}_{2^{\ell}} p, \mathrm{ord}_{n_0'} p) \leq 2^{\ell-2} \mathrm{ord}_{n_0'} p = 2^{\ell-2} \mathrm{ord}_{n_0} p,$$

a contradiction. Thus $\exp(G) \leq 2n_1$ and $m' = 1$. The assertion follows. $\square$

Next, we provide some ($*$-clean or non $*$-clean) clean group rings in each case of the characterizations of Theorems 1.1 and 3.4.

Example 3.5. Let $K = \mathbb{Q}(\zeta_n)$ be a cyclotomic field for some $n \in \mathbb{N}$, $\mathcal{O} = \mathbb{Z}[\zeta_n]$ its rings of integers, and G a finite abelian group with $\exp(G) \geq 3$.

1. If p is a primitive root of unity of $\exp(G)$, then $\mathbb{Z}_{(p)}[G]$ is $*$-clean.
2. Suppose $\gcd(\exp(G), n) = 1$ and $\exp(G)$ has a primitive root. If there is a prime divisor q of $\varphi(n)$ such that $q \nmid \varphi(\exp(G))$, then there exists $x, y \in \mathbb{N}$ with $\gcd(x, n) = 1$ and $\gcd(y, \exp(G)) = 1$ such that $\mathrm{ord}_n x = q$ and $\mathrm{ord}_{\exp(G)} y = \varphi(\exp(G))$. By Chinese Remainder Theory, there exists $z \in \mathbb{N}$ with $\gcd(z, n \exp(G)) = 1$ such that $\mathrm{ord}_n z = q$ and $\mathrm{ord}_{\exp(G)} z = \varphi(\exp(G))$. By Dirichlet's prime number theorem, there is a prime p such that $p \equiv z \pmod{n \exp(G)}$. Let $\mathfrak{p} \subset \mathcal{O}$ be a prime ideal such that $\mathfrak{p} \cap \mathbb{Z} = p\mathbb{Z}$. Then by Theorem 3.4 $\mathcal{O}_{\mathfrak{p}}[G]$ is $*$-clean.
3. Suppose $\gcd(\exp(G), n) \geq 3$, $\gcd\left(\frac{\exp(G)}{\gcd(\exp(G), n)}, n\right) = 1$, and $\frac{\exp(G)}{\gcd(\exp(G), n)}$ has a primitive root. If there is a prime divisor q of $\varphi(n)$ such that $q \nmid \varphi\left(\frac{\exp(G)}{\gcd(\exp(G), n)}\right)$, then there exists a prime p such that $\gcd(\mathrm{ord}_n p, \mathrm{ord}_{\frac{\exp(G)}{\gcd(\exp(G), n)}} p) = 1$. Let $\mathfrak{p} \subset \mathcal{O}$ be a prime ideal such that $\mathfrak{p} \cap \mathbb{Z} = p\mathbb{Z}$. Then by Theorems 1.1 and 3.4, $\mathcal{O}_{\mathfrak{p}}[G]$ is clean but not $*$-clean.

4. Let $n = 7$, $\exp(G) = 49 \times 3$, and let $\mathfrak{p} \subset \mathcal{O}$ be a prime ideal such that $\mathfrak{p} \cap \mathbb{Z} = 23\mathbb{Z}$. Since $\mathrm{ord}_7 23 = 3$, $\mathrm{ord}_3 23 = 2 = \varphi(3)$, and $\mathrm{ord}_{49} 23 = 21 = 7 \, \mathrm{ord}_7 23$, it follows from Theorems 1.1 and 3.4 $\mathcal{O}_{\mathfrak{p}}[G]$ is clean but not $*$-clean.

4. Group rings over local subrings of quadratic fields

In this section, we investigate when a group ring over a local subring of a quadratic field is clean. Let d be a non-zero square-free integer with $d \neq 1$, $K = \mathbb{Q}(\sqrt{d})$ a quadratic number field,

$$\omega = \begin{cases} \sqrt{d} & \text{if } d \equiv 2,3 \pmod 4, \\ \dfrac{1+\sqrt{d}}{2} & \text{if } d \equiv 1 \pmod 4, \end{cases} \quad \text{and} \quad \Delta = \begin{cases} 4d & \text{if } d \equiv 2,3 \pmod 4, \\ d & \text{if } d \equiv 1 \pmod 4. \end{cases}$$

Then $\mathcal{O}_K = \mathbb{Z}[\omega]$ is the ring of integers and Δ is the discriminant of K.

For an odd prime p and an integer a, we denote by $\left(\frac{a}{p}\right) \in \{-1, 0, 1\}$ the Legendre symbol of a modulo p.

We first provide two useful lemmas.

Lemma 4.1. *Let $d \neq 1$ be a non-zero square-free integer and let Δ be the discriminant of $\mathbb{Q}(\sqrt{d})$. Then $\mathbb{Q}(\sqrt{d}) \subset \mathbb{Q}(\zeta_n)$ if and only if n is a multiple of Δ.*

Proof. This follows from [13, Corollary 4.5.5] $\qquad \square$

Lemma 4.2. *Let $d \neq 1$ be a non-zero square-free integer and let I be a prime ideal of $\mathcal{O}_K$, where $K = \mathbb{Q}(\sqrt{d})$. Suppose Δ is the discriminant of K and $\mathrm{char}\,\mathcal{O}_K/I = p$, where p is a prime.*
  1. *If $p = 2$, then $N(I) = p$ if and only if $\Delta \not\equiv 5 \pmod 8$.*
  2. *If p is odd, then $N(I) = p$ if and only if $\left(\frac{\Delta}{p}\right) = 1$ or 0.*

Proof. This follows by [1, Theorem 22, III.2.1, and V.1.1]. $\qquad \square$

Proof of Theorem 1.3. Let $R = \mathcal{O}_{\mathfrak{p}}$. By Theorem 2.2, we have $R[G] = \mathcal{O}_{\mathfrak{p}}[G]$ is clean if and only if $[\mathbb{Q}(\zeta_m) : \mathbb{Q}(\zeta_m) \cap \mathbb{Q}(\sqrt{d})] = \mathrm{ord}_m(N(\mathfrak{p}))$ for every divisor m of n.

1. Since $\Delta \nmid n$, it follows by Lemma 4.1 that $\mathbb{Q}(\sqrt{d}) \cap \mathbb{Q}(\zeta_m) = \mathbb{Q}$ for every positive divisor m of n.

1.1. Suppose the item 1.(a) or 1.(b) holds. By Lemma 4.2 we have $N(\mathfrak{p}) = p$. Therefore for every divisor m of n, we obtain that p is a primitive root of unity of m and hence

$$[\mathbb{Q}(\zeta_m) : \mathbb{Q}(\zeta_m) \cap \mathbb{Q}(\sqrt{d})] = [\mathbb{Q}(\zeta_m) : \mathbb{Q}] = \varphi(m) = \mathrm{ord}_m p = \mathrm{ord}_m(N(\mathfrak{p})).$$

Thus $R[G]$ is clean.

Suppose the item 1.(c) holds. By Lemma 4.2 we have $N(\mathfrak{p}) = p^2$. Thus

$$[\mathbb{Q}(\zeta_2) : \mathbb{Q}(\zeta_2) \cap \mathbb{Q}(\sqrt{d})] = [\mathbb{Q}(\zeta_2) : \mathbb{Q}] = \varphi(2) = \mathrm{ord}_2 p^2 = \mathrm{ord}_2(N(\mathfrak{p})),$$

whence $R[G]$ is clean.

1.2. Conversely, suppose $R[G]$ is clean. Then $[\mathbb{Q}(\zeta_n) : \mathbb{Q}(\zeta_n) \cap \mathbb{Q}(\sqrt{d})] = [\mathbb{Q}(\zeta_n) : \mathbb{Q}] = \mathrm{ord}_n(N(\mathfrak{p}))$ implies that $\varphi(n) = \mathrm{ord}_n(N(\mathfrak{p}))$. Thus either $N(\mathfrak{p}) = p$ is a primitive root of unity of n, or $N(\mathfrak{p}) = p^2$ and $\mathrm{ord}_n p = \varphi(n)$ is odd, i.e., $n \leq 2$. The assertions follow by Lemma 4.2.

2.1. Suppose that $R[G]$ is clean. Since $\Delta = 4d \nmid 4$, we have $[\mathbb{Q}(\zeta_4) : \mathbb{Q}(\zeta_4) \cap \mathbb{Q}(\sqrt{d})] = [\mathbb{Q}(\zeta_4) : \mathbb{Q}] = \mathrm{ord}_4(N(\mathfrak{p}))$. Thus $\varphi(4) = \mathrm{ord}_4(N(\mathfrak{p}))$ and hence $N(\mathfrak{p}) = p \equiv 3 \pmod 4$. If $p \mid d$, then $p \mid n$, a contradiction. Thus $p \nmid d$ and hence by Lemma 4.2.2 $\left(\frac{\Delta}{p}\right) = 1$.

Since $[\mathbb{Q}(\zeta_n) : \mathbb{Q}(\zeta_n) \cap \mathbb{Q}(\sqrt{d})] = [\mathbb{Q}(\zeta_n) : \mathbb{Q}(\sqrt{d})] = \mathrm{ord}_n(N(\mathfrak{p}))$, we obtain that $\varphi(n)/2 = \mathrm{ord}_n(N(\mathfrak{p})) = \mathrm{ord}_n p$. Since $4 \mid n$, we may assume that $n = 2^\ell n'$ with $\ell \geq 2$ and n' is odd. Thus $(\mathbb{Z}/n\mathbb{Z})^\times \cong (\mathbb{Z}/2^\ell\mathbb{Z})^\times \times (\mathbb{Z}/n'\mathbb{Z})^\times$. Since $(\mathbb{Z}/n\mathbb{Z})^\times$ has an element of order $\varphi(n)/2$, we obtain that $n' = 1$ if $\ell \geq 3$

and n' is a prime power if $\ell = 2$. Thus $n = 4q^\ell$ and $d \mid q^\ell$, where q is a prime. Note that d is square-free. Therefore $|d| = q$ is a prime.

2.2. Conversely, $\left(\frac{\Delta}{p}\right) = 1$ implies that $N(\mathfrak{p}) = p$. Suppose $|d| = 2$ and $n = 2^\ell$ with $\ell \geq 3$. Let m be a positive divisor of n. If $m = 4$, then $|\mathbb{Q}(\zeta_4) : \mathbb{Q}(\zeta_4) \cap \mathbb{Q}(\sqrt{d})| = 2$ and $\mathrm{ord}_4 \, p = 2$ by $p \equiv 3 \pmod 4$. Thus $[\mathbb{Q}(\zeta_4) : \mathbb{Q}(\zeta_4) \cap \mathbb{Q}(\sqrt{d})] = \mathrm{ord}_4 \, p$. If $m = 2^t$ with $t \geq 3$, then $[\mathbb{Q}(\zeta_m) : \mathbb{Q}(\zeta_m) \cap \mathbb{Q}(\sqrt{d})] = 2^{t-2}$ and $\mathrm{ord}_m \, p = m/4 = 2^{t-2}$ by $2^{\ell-2} = \varphi(n)/2 = \mathrm{ord}_n \, p$. Thus $[\mathbb{Q}(\zeta_m) : \mathbb{Q}(\zeta_m) \cap \mathbb{Q}(\sqrt{d})] = \mathrm{ord}_m(N(\mathfrak{p}))$. Putting all these together, we obtain that $R[G]$ is clean.

Suppose $|d| \geq 3$ is a prime and $n = 4|d|^\ell$. Let m be a positive divisor of n. If $m = |d|^t$ for some $1 \leq t \leq \ell$, then $4d \nmid m$ and hence $[\mathbb{Q}(\zeta_m) : \mathbb{Q}(\zeta_m) \cap \mathbb{Q}(\sqrt{d})] = \varphi(m) = |d|^{t-1}(|d| - 1)$. Since p is a primitive root of unity of $|d|^\ell$, we obtain $\varphi(m) = \mathrm{ord}_m \, p$. Therefore $[\mathbb{Q}(\zeta_m) : \mathbb{Q}(\zeta_m) \cap \mathbb{Q}(\sqrt{d})] = \mathrm{ord}_m(N(\mathfrak{p}))$. If $m = 2|d|^t$ for some $1 \leq t \leq \ell$, then $4d \nmid m$ and hence $[\mathbb{Q}(\zeta_m) : \mathbb{Q}(\zeta_m) \cap \mathbb{Q}(\sqrt{d})] = \varphi(m) = |d|^{t-1}(|d| - 1)$. Since p is a primitive root of unity of $|d|^\ell$, we obtain $\varphi(m) = \mathrm{ord}_m \, p$. Therefore $[\mathbb{Q}(\zeta_m) : \mathbb{Q}(\zeta_m) \cap \mathbb{Q}(\sqrt{d})] = \mathrm{ord}_m(N(\mathfrak{p}))$. If $m = 4|d|^t$ for some $1 \leq t \leq \ell$, then $4d \mid m$ and hence $[\mathbb{Q}(\zeta_m) : \mathbb{Q}(\zeta_m) \cap \mathbb{Q}(\sqrt{d})] = \varphi(m)/2 = |d|^{t-1}(|d| - 1)$. Since p is a primitive root of unity of $|d|^\ell$, we obtain $\varphi(m) = \mathrm{ord}_m \, p$. Therefore $[\mathbb{Q}(\zeta_m) : \mathbb{Q}(\zeta_m) \cap \mathbb{Q}(\sqrt{d})] = \mathrm{ord}_m(N(\mathfrak{p}))$. If $m = 4$, then $[\mathbb{Q}(\zeta_4) : \mathbb{Q}(\zeta_4) \cap \mathbb{Q}(\sqrt{2})] = 2$ and $\mathrm{ord}_4 \, p = 2$ as $p \equiv 3 \pmod 4$. Thus $[\mathbb{Q}(\zeta_4) : \mathbb{Q}(\zeta_4) \cap \mathbb{Q}(\sqrt{2})] = \mathrm{ord}_4 \, p$. Putting all these together, we obtain that $R[G]$ is clean.

3. If $p \mid d$, then $p \mid n$, a contradiction. Therefore $\left(\frac{d}{p}\right) = 1$ or -1.

3.1. Let $R[G]$ be clean. Suppose $|d|$ is a prime and $n = |d|^\ell$ or $n = 2|d|^\ell$ for some $\ell \in \mathbb{N}$. If $\left(\frac{d}{p}\right) = -1$, then $N(\mathfrak{p}) = p^2$ and hence $[\mathbb{Q}(\zeta(n)) : \mathbb{Q}(\zeta(n)) \cap \mathbb{Q}(\sqrt{d})] = \mathrm{ord}_n \, p^2$ implies either $\varphi(n) = \mathrm{ord}_n \, p$ or $\mathrm{ord}_n \, p = \varphi(n)/2$ is odd. If $\left(\frac{d}{p}\right) = 1$, then $N(\mathfrak{p}) = p$ and hence $[\mathbb{Q}(\zeta(n)) : \mathbb{Q}(\zeta(n)) \cap \mathbb{Q}(\sqrt{d})] = \mathrm{ord}_n \, p$ implies that $\varphi(n)/2 = \mathrm{ord}_n \, p$. Putting all these together, we have either $\mathrm{ord}_n \, p = \frac{2\varphi(n)}{3 + \left(\frac{d}{p}\right)}$, or $\mathrm{ord}_n \, p = \varphi(n)/2$ is odd with $\left(\frac{d}{p}\right) = -1$. Note that $\varphi(n)/2$ is odd if and only if $|d| \equiv 3 \pmod 4$, i.e., $d < 0$. Therefore (a) holds.

Otherwise, there exists an $m \mid n$ with $m \geq 3$ such that $d \nmid m$. Therefore $[\mathbb{Q}(\zeta_m) : \mathbb{Q}(\zeta_m) \cap Q(\sqrt{d})] = \varphi(m) = \mathrm{ord}_m(N(\mathfrak{p}))$. Since $\varphi(m)$ must be even, we obtain that $N(\mathfrak{p}) = p$ and hence $\left(\frac{d}{p}\right) = 1$. Since $d \mid n$, we have $\varphi(n)/2 = \mathrm{ord}_n \, p$. Therefore $(\mathbb{Z}/n\mathbb{Z})^\times \cong C_{\varphi(n)}$ or $C_2 \oplus C_{\varphi(n)/2}$, which implies that $n = q_1^{\ell_1} q_2^{\ell_2}$ or $2q_1^{\ell_1} q_2^{\ell_2}$ or $4q_1^{\ell_1}$, where q_1, q_2 are distinct odd primes with $q_1 \mid d$, $\ell_1 \in \mathbb{N}$ and $\ell_2 \in \mathbb{N}_0$. Note that d is square-free. Then either $|d| = q_1$ and $n \in \{q_1^{\ell_1} q_2^{\ell_2}, 2q_1^{\ell_1} q_2^{\ell_2}, 4q_1^{\ell_1}\}$ with $\ell_1, \ell_2 \in \mathbb{N}$, or $|d| = q_1 q_2$ and $n \in \{q_1^{\ell_1} q_2^{\ell_2}, 2q_1^{\ell_1} q_2^{\ell_2}\}$ with $\ell_1, \ell_2 \in \mathbb{N}$.

Suppose $|d| = q_1$. If $n = q_1^{\ell_1} q_2^{\ell_2}$ or $2q_1^{\ell_1} q_2^{\ell_2}$, then $[\mathbb{Q}(\zeta_{q_1^{\ell_1}}) : \mathbb{Q}(\zeta_{q_1^{\ell_1}}) \cap Q(\sqrt{d})] = \varphi(q_1^{\ell_1})/2 = \mathrm{ord}_{q_1^{\ell_1}} \, p$ and $[\mathbb{Q}(\zeta_{q_2^{\ell_2}}) : \mathbb{Q}(\zeta_{q_2^{\ell_2}}) \cap Q(\sqrt{d})] = \varphi(q_2^{\ell_2}) = \mathrm{ord}_{q_2^{\ell_2}} \, p$. Since

$$[\mathbb{Q}(\zeta_{q_1^{\ell_1} q_2^{\ell_2}}) : \mathbb{Q}(\zeta_{q_1^{\ell_1} q_2^{\ell_2}}) \cap Q(\sqrt{d})] = \varphi(q_1^{\ell_1} q_2^{\ell_2})/2 = \mathrm{ord}_{q_1^{\ell_1} q_2^{\ell_2}}(p)$$

$$= \mathrm{lcm}(\mathrm{ord}_{q_1^{\ell_1}} \, p, \mathrm{ord}_{q_2^{\ell_2}} \, p) = \frac{\mathrm{ord}_{q_1^{\ell_1}} \, p \, \mathrm{ord}_{q_2^{\ell_2}} \, p}{\gcd(\mathrm{ord}_{q_1^{\ell_1}} \, p, \mathrm{ord}_{q_2^{\ell_2}} \, p)}$$

$$= \frac{\varphi(q_1^{\ell_1})/2 \cdot \varphi(q_2^{\ell_2})}{\gcd(\varphi(q_1^{\ell_1})/2, \varphi(q_2^{\ell_2}))}$$

$$= \frac{\varphi(q_1^{\ell_1} q_2^{\ell_2})/2}{\gcd(q_1^{\ell_1 - 1}(q_1 - 1)/2, q_2^{\ell_2 - 1}(q_2 - 1))},$$

we have $\gcd(q_1^{\ell_1 - 1}(q_1 - 1)/2, q_2^{\ell_2 - 1}(q_2 - 1)) = 1$ which implies that $d = -q_1 \equiv 1 \pmod 4$. Therefore (c) holds. If $n = 4q_1^{\ell_1}$, then $p \neq 2$ and $[\mathbb{Q}(\zeta_{q_1^{\ell_1}}) : \mathbb{Q}(\zeta_{q_1^{\ell_1}}) \cap Q(\sqrt{d})] = \varphi(q_1^{\ell_1})/2 = \mathrm{ord}_{q_1^{\ell_1}} \, p$, $[\mathbb{Q}(\zeta_4) :$

$\mathbb{Q}(\zeta_4) \cap Q(\sqrt{d})] = \varphi(4) = \mathrm{ord}_4\, p$ which implies that $p \equiv 3 \pmod 4$. Since

$$[\mathbb{Q}(\zeta_{4q_1^{\ell_1}}) : \mathbb{Q}(\zeta_{4q_1^{\ell_1}}) \cap Q(\sqrt{d})] = \varphi(4q_1^{\ell_1})/2 = \mathrm{ord}_{4q_1^{\ell_1}}(p)$$

$$= \mathrm{lcm}(\mathrm{ord}_4\, p, \mathrm{ord}_{q_1^{\ell_1}}\, p) = \frac{2\,\mathrm{ord}_{q_1^{\ell_1}}\, p}{\gcd(2, \mathrm{ord}_{q_1^{\ell_1}}\, p)}$$

$$= \frac{\varphi(4q_1^{\ell_1})/2}{\gcd(2, q_1^{\ell_1-1}(q_1 - 1)/2)},$$

we have $\gcd(2, q_1^{\ell_1-1}(q_1 - 1)/2) = 1$, whence $q_1 \equiv 3 \pmod 4$. Therefore (b) holds.

Suppose $|d| = q_1 q_2$. Then $[\mathbb{Q}(\zeta_{q_1^{\ell_1}}) : \mathbb{Q}(\zeta_{q_1^{\ell_1}}) \cap Q(\sqrt{d})] = \varphi(q_1^{\ell_1}) = \mathrm{ord}_{q_1^{\ell_1}}(p)$ and $[\mathbb{Q}(\zeta_{q_2^{\ell_2}}) : \mathbb{Q}(\zeta_{q_2^{\ell_2}}) \cap Q(\sqrt{d})] = \varphi(q_2^{\ell_2}) = \mathrm{ord}_{q_2^{\ell_2}}(p)$, whence p is a primitive root of unity of both $q_1^{\ell_1}$ and $q_2^{\ell_2}$. Since

$$[\mathbb{Q}(\zeta_{q_1^{\ell_1} q_2^{\ell_2}}) : \mathbb{Q}(\zeta_{q_1^{\ell_1} q_2^{\ell_2}}) \cap Q(\sqrt{d})] = \varphi(q_1^{\ell_1} q_2^{\ell_2})/2 = \mathrm{ord}_{q_1^{\ell_1} q_2^{\ell_2}}(p)$$

$$= \mathrm{lcm}(\mathrm{ord}_{q_1^{\ell_1}}\, p, \mathrm{ord}_{q_2^{\ell_2}}\, p) = \frac{\mathrm{ord}_{q_1^{\ell_1}}\, p\, \mathrm{ord}_{q_2^{\ell_2}}\, p}{\gcd(\mathrm{ord}_{q_1^{\ell_1}}\, p, \mathrm{ord}_{q_2^{\ell_2}}\, p)}$$

$$= \frac{\varphi(q_1^{\ell_1})\varphi(q_2^{\ell_2})}{\gcd(\varphi(q_1^{\ell_1}), \varphi(q_2^{\ell_2}))}$$

$$= \frac{\varphi(q_1^{\ell_1} q_2^{\ell_2})}{\gcd(q_1^{\ell_1-1}(q_1 - 1), q_2^{\ell_2-1}(q_2 - 1))},$$

we have $\gcd(q_1^{\ell_1-1}(q_1 - 1)/2, q_2^{\ell_2-1}(q_2 - 1)/2) = 1$. Therefore (d) holds.

3.2. Conversely, suppose that (a) holds. Let m be a positive divisor of n with $m \geq 3$. Then $d \mid m$. If $\left(\frac{d}{p}\right) = 1$, then $N(\mathfrak{p}) = p$ and hence $[\mathbb{Q}(\zeta_m) : \mathbb{Q}(\zeta_m) \cap \mathbb{Q}(\sqrt{d})] = \varphi(m)/2 = \mathrm{ord}_m\, p = \mathrm{ord}_m(N(\mathfrak{p}))$, implying that $R[G]$ is clean. If $\left(\frac{d}{p}\right) = -1$ and either $\mathrm{ord}_n\, p = \varphi(n)$ or $\mathrm{ord}_n\, p = \varphi(n)/2$ with $|d| = -d \equiv 3 \pmod 4$, then $N(\mathfrak{p}) = p^2$, $\mathrm{ord}_n\, p^2 = \varphi(n)/2$, and hence $[\mathbb{Q}(\zeta_m) : \mathbb{Q}(\zeta_m) \cap \mathbb{Q}(\sqrt{d})] = \varphi(m)/2 = \mathrm{ord}_m\, p^2 = \mathrm{ord}_m(N(\mathfrak{p}))$, implying that $R[G]$ is clean.

Suppose that (b) holds. Then $N(\mathfrak{p}) = p$ and $\mathrm{ord}_{q^i}\, p = q^{i-1}(q-1)/2$ is odd for all $i \in [1, \ell]$. Let m be a positive divisor of n with $m \geq 3$. Then $d \mid m$ and hence $[\mathbb{Q}(\zeta_m) : \mathbb{Q}(\zeta_m) \cap \mathbb{Q}(\sqrt{d})] = \varphi(m)/2 = \mathrm{ord}_m\, p = \mathrm{ord}_m(N(\mathfrak{p}))$. Therefore $R[G]$ is clean.

Suppose that (c) holds. Then $N(\mathfrak{p}) = p$, $\mathrm{ord}_{q_1^i}\, p = q_1^{i-1}(q_1 - 1)/2$ is odd, p is a primitive root of q_2^j, and $\gcd(\mathrm{ord}_{q_1^i}\, p, \mathrm{ord}_{q_2^i}\, p) = 1$ for all $i \in [1, \ell_1]$ and $j \in [1, \ell_2]$. Let m be a positive divisor of n with $m \geq 3$. If $m = q_2^t$ or $2q_2^t$ for some $1 \leq t \leq \ell_2$, then $d \nmid m$ and hence $[\mathbb{Q}(\zeta_m) : \mathbb{Q}(\zeta_m) \cap \mathbb{Q}(\sqrt{d})] = \varphi(m) = \mathrm{ord}_m\, p = \mathrm{ord}_m(N(\mathfrak{p}))$. Otherwise $d \mid m$ and hence $[\mathbb{Q}(\zeta_m) : \mathbb{Q}(\zeta_m) \cap \mathbb{Q}(\sqrt{d})] = \varphi(m)/2 = \mathrm{ord}_m\, p = \mathrm{ord}_m(N(\mathfrak{p}))$. Therefore $R[G]$ is clean.

Suppose that (d) holds. Then $N(\mathfrak{p}) = p$. Let m be a positive divisor of n with $m \geq 3$. If $m = q_1^{t_1}$, or $2q_1^{t_1}$, or $q_2^{t_2}$, or $2q_2^{t_2}$ for some $1 \leq t_1 \leq \ell_1$ or some $1 \leq t_2 \leq \ell_2$, then $d \nmid m$ and hence $[\mathbb{Q}(\zeta_m) : \mathbb{Q}(\zeta_m) \cap \mathbb{Q}(\sqrt{d})] = \varphi(m) = \mathrm{ord}_m\, p = \mathrm{ord}_m(N(\mathfrak{p}))$. If $m = q_1^{t_1} q_2^{t_2}$ or $2q_1^{t_1} q_2^{t_2}$, then $d \mid m$ and hence $[\mathbb{Q}(\zeta_m) : \mathbb{Q}(\zeta_m) \cap \mathbb{Q}(\sqrt{d})] = \varphi(m)/2 = \mathrm{ord}_m\, p = \mathrm{ord}_m(N(\mathfrak{p}))$. Therefore $R[G]$ is clean. $\qquad\square$

Next we characterize when such a group ring is $*$-clean. We first prove the following lemma.

Lemma 4.3. *Let $d \neq 1$ be a non-zero square free integer. Then $\mathbb{Q}(\sqrt{d})(\zeta_n + \zeta_n^{-1}) = \mathbb{Q}(\sqrt{d})(\zeta_n)$ if and only if either $\left(d < 0 \text{ and } \Delta \mid n \right)$ or $n \leq 2$, where $n \in \mathbb{N}$ and Δ is the discriminant of $\mathbb{Q}(\sqrt{d})$.*

Proof. If $n \leq 2$, it is obvious that $\mathbb{Q}(\sqrt{d})(\zeta_n + \zeta_n^{-1}) = \mathbb{Q}(\sqrt{d})(\zeta_n)$. Now we let $n \geq 3$.

Suppose that $d < 0$ and $\Delta \mid n$. Then by Lemma 4.1 $\mathbb{Q}(\sqrt{d}) \subset \mathbb{Q}(\zeta_n)$ and hence $\mathbb{Q}(\sqrt{d})(\zeta_n + \zeta_n^{-1}) \subset \mathbb{Q}(\zeta_n)$. Since $n \geq 3$ and $d < 0$, we have

$$[\mathbb{Q}(\zeta_n) : \mathbb{Q}(\zeta_n + \zeta_n^{-1})] = [\mathbb{Q}(\sqrt{d})(\zeta_n + \zeta_n^{-1}) : \mathbb{Q}(\zeta_n + \zeta_n^{-1})] = 2\,.$$

Therefore $\mathbb{Q}(\sqrt{d})(\zeta_n + \zeta_n^{-1}) = \mathbb{Q}(\zeta_n) = \mathbb{Q}(\sqrt{d})(\zeta_n)$.

Suppose that $d > 0$. Then $\mathbb{Q}(\sqrt{d})(\zeta_n + \zeta_n^{-1}) \subset \mathbb{R}$ and by $n \geq 3$, we have $\mathbb{Q}(\sqrt{d})(\zeta_n) \not\subset \mathbb{R}$. Hence $\mathbb{Q}(\sqrt{d})(\zeta_n + \zeta_n^{-1}) \neq \mathbb{Q}(\sqrt{d})(\zeta_n)$.

Suppose that $d < 0$ and $\Delta \nmid n$. Thus by Lemma 4.1 $\sqrt{d} \notin \mathbb{Q}(\zeta_n)$. Therefore $[\mathbb{Q}(\sqrt{d})(\zeta_n) : \mathbb{Q}] = 2\varphi(n)$ and $[\mathbb{Q}(\sqrt{d})(\zeta_n + \zeta_n^{-1}) : \mathbb{Q}] = \varphi(n)$. It follows that $\mathbb{Q}(\sqrt{d})(\zeta_n + \zeta_n^{-1}) \neq \mathbb{Q}(\sqrt{d})(\zeta_n)$. $\qquad\square$

Proposition 4.4. *Let G be a finite abelian group with exponent n. Then $\mathbb{Q}(\sqrt{d})[G]$ is $*$-clean if and only if $n \geq 3$ and either $d > 0$ or $\Delta \nmid n$, where Δ is the discriminant of $\mathbb{Q}(\sqrt{d})$.*

Proof. This result follows from [4, Theorem 1.2] and Lemma 4.3. $\qquad\square$

Theorem 4.5. *Let $K = \mathbb{Q}(\sqrt{d})$ be a quadratic field for some non-zero square-free integer $d \neq 1$, $\mathcal{O}$ its ring of integers, $\mathfrak{p} \subset \mathcal{O}$ a nonzero prime ideal with $p\mathbb{Z} = \mathfrak{p} \cap \mathcal{O}$, and G a finite abelian group with $p \nmid \exp(G)$. Let Δ be the discriminant of the field extension $K/\mathbb{Q}$. Then*

1. *if $d > 0$, then $\mathcal{O}_{\mathfrak{p}}[G]$ is $*$-clean if and only $\mathcal{O}_{\mathfrak{p}}[G]$ is clean and $\exp(G) \geq 3$.*
2. *if $d < 0$, then $\mathcal{O}_{\mathfrak{p}}[G]$ is $*$-clean if and only if $\Delta \nmid \exp(G)$, p is a primitive root of unity of $\exp(G)$, $\exp(G) \geq 3$, and $\left(\frac{\Delta}{p}\right) = 1$ or 0.*

Proof. 1. Let $d > 0$. Suppose $\mathcal{O}_{\mathfrak{p}}[G]$ is clean and $\exp(G) \geq 3$. Then by Proposition 4.4 $\mathbb{Q}(\sqrt{d})[G]$ is $*$-clean. It follows from Theorem 2.3 that $\mathcal{O}_{\mathfrak{p}}[G]$ is $*$-clean.

Conversely, suppose $\mathcal{O}_{\mathfrak{p}}[G]$ is $*$-clean. Then by Theorem 2.3 $\mathbb{Q}(\sqrt{d})[G]$ is $*$-clean. It follows from Proposition 4.4 that $\exp(G) \geq 3$.

2. Let $d < 0$. Suppose that $\Delta \nmid \exp(G)$, p is a primitive root of unity of $\exp(G)$, $\exp(G) \geq 3$, and $\left(\frac{\Delta}{p}\right) = 1$ or 0. Then by Theorem 1.3 $\mathcal{O}_{\mathfrak{p}}$ is clean and by Proposition 4.4 $\mathbb{Q}(\sqrt{d})[G]$ is $*$-clean. It follows from Theorem 2.3 that $\mathcal{O}_{\mathfrak{p}}[G]$ is $*$-clean.

Conversely, suppose $\mathcal{O}_{\mathfrak{p}}[G]$ is $*$-clean. Then by Theorem 2.3 $\mathbb{Q}(\sqrt{d})[G]$ is $*$-clean and hence by Proposition 4.4 $\exp(G) \geq 3$ and $\Delta \nmid \exp(G)$. It follows from Theorem 1.3.1 that p is a primitive root of unity of $\exp(G)$ and $\left(\frac{\Delta}{p}\right) = 1$ or 0. $\qquad\square$

We close the paper with the following examples which provide some ($*$-clean or non $*$-clean) clean group rings for each case of the characterizations of Theorems 1.3 and 4.5.

Example 4.6. 1. Let $\mathcal{O}$ be the ring of integers of $\mathbb{Q}(\sqrt{d})$ and let G be a finite abelian group with $\gcd(\exp(G), d) = 1$, where $d \neq 1$ is a square free integer and $\exp(G) \neq 4$ has a primitive root. Suppose that $d = \delta d_0$ such that d_0 is the maximal odd positive divisor of d. Thus $\delta \in \{-1, 2, -2\}$. For every prime p with $p \equiv 1 \pmod{8d_0}$, we have $\left(\frac{d}{p}\right) = \left(\frac{d_0}{p}\right) = 1$. Since there exists $x \in \mathbb{N}$ with $\gcd(x, \exp(G)) = 1$ such that $\mathrm{ord}_{\exp(G)} x = \varphi(\exp(G))$, for every prime p with $p \equiv x \pmod{\exp(G)}$, we have $\mathrm{ord}_{\exp(G)} p = \varphi(\exp(G))$. Note that $\mathsf{v}_2(\exp(G)) \leq 1$. By Dirichlet's prime number theorem, there is a prime p such that $p \equiv 1 \pmod{8d_0}$ and $p \equiv x \pmod{\exp(G)}$. Let $\mathfrak{p} \subset \mathcal{O}$ be a prime ideal such that $\mathfrak{p} \cap \mathbb{Z} = p\mathbb{Z}$. Then by Theorem 1.3.1 $\mathcal{O}_{\mathfrak{p}}[G]$ is clean. If $\exp(G) \geq 3$, then by Theorem 4.5 $\mathcal{O}_{\mathfrak{p}}[G]$ is $*$-clean.
2. Let $\mathcal{O}$ be the ring of integers of $\mathbb{Q}(\sqrt{-2})$, let $\mathfrak{p} \subset \mathcal{O}$ be a prime ideal with $\mathfrak{p} \cap \mathbb{Z} = 3\mathbb{Z}$, and let G be a finite abelian group with $\exp(G) = 8$. Then Theorem 1.3.2 and Theorem 4.5.2 imply that $\mathcal{O}_{\mathfrak{p}}[G]$ is clean but not $*$-clean.
3. Let $\mathcal{O}$ be the ring of integers of $\mathbb{Q}(\sqrt{3})$, let $\mathfrak{p} \subset \mathcal{O}$ be a prime ideal with $\mathfrak{p} \cap \mathbb{Z} = 11\mathbb{Z}$, and let G be a finite abelian group with $\exp(G) = 12$. Then Theorem 1.3.2 and Theorem 4.5.1 imply that $\mathcal{O}_{\mathfrak{p}}[G]$ is clean as well as $*$-clean.

4. Let $\mathcal{O}$ be the ring of integers of $\mathbb{Q}(\sqrt{5})$, let $\mathfrak{p} \subset \mathcal{O}$ be a prime ideal with $\mathfrak{p} \cap \mathbb{Z} = 19\mathbb{Z}$, and let G be a finite abelian group with $\exp(G) = 5$. Then Theorem 1.3.3.a and Theorem 4.5.1 imply that $\mathcal{O}_{\mathfrak{p}}[G]$ is clean as well as $*$-clean.

5. Let $\mathcal{O}$ be the ring of integers of $\mathbb{Q}(\sqrt{-3})$, let $\mathfrak{p} \subset \mathcal{O}$ be a prime ideal with $\mathfrak{p} \cap \mathbb{Z} = 5\mathbb{Z}$, and let G be a finite abelian group with $\exp(G) = 6$. Then Theorem 1.3.3.a and Theorem 4.5.2 imply that $\mathcal{O}_{\mathfrak{p}}[G]$ is clean but not $*$-clean.

6. Let $\mathcal{O}$ be the ring of integers of $\mathbb{Q}(\sqrt{33})$, let $\mathfrak{p} \subset \mathcal{O}$ be a prime ideal with $\mathfrak{p} \cap \mathbb{Z} = 2\mathbb{Z}$, and let G be a finite abelian group with $\exp(G) = 33$. Then Theorem 1.3.3.d and Theorem 4.5.1 imply that $\mathcal{O}_{\mathfrak{p}}[G]$ is clean as well as $*$-clean.

References

[1] A. Fröhlich and M. J. Taylor , *Algebraic number theory*, cup ed., Cambridge Studies in Advanced Mathematics, Cambridge University Press, 1992.
[2] Y. Gao, J. Chen and Y. Li, *Some *-clean group rings*, Algebra Colloq. **22** (2015) 169-180.
[3] J. Han and W.K. Nicholson, *Extensions of clean rings*, Comm. Algebra **29** (2001) 2589-255.
[4] D. Han, Y.uan Ren, and H. Zhang, *On *-clean rings over abelian groups*, J. Algebra and its Applications **16** (2017) 1750152, 11p.
[5] H. Huang, Y. Li and G. Tang, *On *-clean non-commutative group rings*, J. Algebra Appl. **15** (2016) 1650150, 17p.
[6] H. Huang, Y. Li and P. Yuan, *On *-clean group rings II*, Comm. Algebra **44** (2016) 3171-3181
[7] Nicholas A. Immormino and Warren Wm. McGovern, *Examples of clean commutative group rings*, J. Algebra **405** (2014) 168 – 178.
[8] C.Li and Y. Zhou, *On strongly *-clean rings*, J. Algebra Appl. **10** (2011) 1363-1370.
[9] Y. Li, M.M. Parmenter and P. Yuan, *On *-clean group rings*, J. Algebra Appl. **14** (2015) 1550004, 11p.
[10] W. Wm. McGovern, *Neat rings*, J. Pure Appl. Algebra **205** (2006) 243-265.
[11] C. P. Milies and S. K. Sehgal, *An Introduction to Group Rings*, Netherlands, Kluqver Academic Publishers, 2002.
[12] W.K. Nicholson, *Lifting idempotents and exchange rings*, Trans. Amer. Math. Soc. **220** (1977) 269-278.
[13] Steven H. Weintraub, *Galois theory*, 1 ed., Universitext, Springer, 2006.
[14] S. M. Woods, *Some results on semi-perfect group rings*, Can. J. Math. **26** (1974) 121 – 129.

Department of Mathematics and Statistics, Brock University, 1812 Sir Isaac Brock Way, St. Catharines, Ontario, Canada L2S 3A1 and Faculty of Science, Jiangsu University, Zhenjiang, Jiangsu China
Email address: yli@brocku.ca

Institute for Mathematics and Scientific Computing, University of Graz, NAWI Graz, Heinrichstrasse 36, 8010 Graz, Austria
Email address: qinghai.zhong@uni-graz.at