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Linkage of sets of cyclic algebras

Adam Chapman

School of Computer Science, Academic College of Tel-Aviv-Yaffo, Rabenu Yeruham St., P.O.B. 8401, Yaffo, 6818211, Israel

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ABSTRACT

Let p be a prime integer and F the function field in two algebraically independent variables over a smaller field F_0. We prove that if $\mathrm{char}(F_0) = p \geqslant 3$ then there exist $p^2 - 1$ cyclic algebras of degree p over F that have no maximal subfield in common, and if $\mathrm{char}(F_0) = 0$ then there exist p^2 cyclic algebras of degree p over F that have no maximal subfield in common.

1. Introduction

A cyclic algebra of prime degree p over a field F takes the form

$$(\alpha, \beta)_{p,F} = F\langle x, y : x^p = \alpha, y^p = \beta, yxy^{-1} = \rho x\rangle,$$

for some $\alpha, \beta \in F^\times$ when $\mathrm{char}(F) \neq p$ and F contains a primitive pth root of unity ρ. This algebra is a division algebra if $\alpha \notin (F^\times)^p$ and β is not a norm in the field extension $F[\sqrt[p]{\alpha}]/F$, and otherwise it is the matrix algebra $M_p(F)$. When $\mathrm{char}(F) = p$, a cyclic algebra of degree p over F takes the form

$$[\alpha, \beta)_{p,F} = F\langle x, y : x^p - x = \alpha, y^p = \beta, yxy^{-1} = x + 1\rangle,$$

for some $\alpha \in F$ and $\beta \in F^\times$. This algebra is a division algebra if $\alpha \notin \wp(F) = \{\lambda^p - \lambda : \lambda \in F\}$ and β is not a norm in the field extension $F[x : x^p - x = \alpha]/F$, and otherwise it is the matrix algebra $M_p(F)$. These

E-mail address: adam1chapman@yahoo.com.

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algebras won their significance for being the generators of ${}_pBr(F)$ (see [12] and [10, Chapter 9]). These algebras are called "quaternion algebras" when $p = 2$.

We say that cyclic algebras $A_1, \dots, A_\ell$ of degree p over F are linked if they share a common maximal subfield. We say that ${}_pBr(F)$ is ℓ-linked if every ℓ cyclic algebras of degree p over F are linked.

The linkage properties of such algebras demonstrate a deeper phenomenon yet to be fully understood: clearly if A and B are linked then $A \otimes B$ is not a division algebra, but for quaternion algebras the converse holds true as well. This means that ${}_2Br(F)$ is 2-linked if and only if its symbol length is $\leqslant 1$ (i.e., every class is represented by a single quaternion algebra). Moreover, if ${}_2Br(F)$ is 2-linked then the u-invariant of F is either 0, 1, 2, 4 or 8 ([8] and [5]), and for nonreal fields F, ${}_2Br(F)$ is 3-linked if and only if $u(F) \leqslant 4$ (see [2] and [6]).

For local fields F, ${}_pBr(F)$ is clearly ℓ-linked for any ℓ. It follows from the local-global principle (e.g., see [7] and [13, Proposition 15]) that for global fields F, ${}_pBr(F)$ is ℓ-linked for any ℓ too. A question was raised ([2]) on whether function fields $F = F_0(\alpha, \beta)$ in two algebraically independent variables over algebraically closed fields F_0 also satisfy this property. It was answered in the negative for quaternion algebras ([7] for $\mathrm{char}(F) = 0$ and [3] for $\mathrm{char}(F) = 2$), showing that for such fields ${}_2Br(F)$ is not 4-linked.

In the current paper, we extend this observation to cyclic algebras of odd prime degree p over $F = F_0(\alpha, \beta)$, showing that when $\mathrm{char}(F_0) = p$, the group ${}_pBr(F)$ is not $(p^2 - 1)$-linked, and when $\mathrm{char}(F_0) = 0$, the group ${}_pBr(F)$ is not p^2-linked.

2. Characteristic p

Lemma 2.1. *Let $A = [\alpha, \beta)_{p,F}$ be a cyclic algebra of degree p generated by x and y over a field F of $\mathrm{char}(F) = p$, and write $\mathrm{Tr} : A \to F$ for its reduced trace map. Then for any $\lambda = \sum_{i=0}^{p-1} \sum_{j=0}^{p-1} c_{i,j} x^i y^j \in A$, $\mathrm{Tr}(\lambda) = -c_{p-1,0}$.*

Proof. For each $j \in \{1, \dots, p-1\}$, every element v in $F(x)y^j$ satisfies $v^p \in F$, and so $\mathrm{Tr}(v) = 0$. The problem therefore reduces to calculating $\mathrm{Tr}_{K/F}(x^i)$ where $K = F(x)$ with x a root of the irreducible polynomial $X^p - X - \alpha$ in $F[X]$. For this, let $L_{x^i} : K \to K$ be the F-linear transformation given by multiplication by x^i. For $i \in \{1, 2, \dots, p-1\}$, the matrix $[L_{x^i}]$ of L_{x^i} relative to the F-basis $\{1, x, \dots, x^{p-1}\}$ of K has the following form: a diagonal of 1-s starting at the $(i+1, 1)$-entry (i.e., row $i+1$ and column 1), a diagonal of α-s starting at the $(1, p-i+1)$-entry, a 1 directly below each α, and all other entries are 0-s. Thus, $\mathrm{Tr}_{K/F}(x^i) = \mathrm{Tr}[L_{x^i}] = 0$ for $i \in \{1, \dots, p-2\}$, while $\mathrm{Tr}_{K/F}(x^{p-1}) = \mathrm{Tr}[L_{x^{p-1}}] = p - 1$. $\square$

Remark 2.2. The last statement appeared in [4, Remark 2.2], but we provided here a simpler proof which was suggested by an anonymous colleague. Note that the trace argument works in a more general setting, in any characteristic and for roots x of any irreducible polynomial $X^n - X - \alpha$ for any natural number n.

Theorem 2.3. *Let p be an odd prime, F_0 a field of $\mathrm{char}(F_0) = p$ and $F = F_0(\alpha, \beta)$ the function field in two algebraically independent variables α and β over F_0. Then there exist $p^2 - 1$ cyclic algebras of degree p over F that share no maximal subfield.*

Proof. Note that F is endowed with the right-to-left $(\alpha^{-1}, \beta^{-1})$-adic valuation, which we denote by $\mathfrak{v}$. This is in fact the restriction to the standard rank 2 valuation on $F_0(\!(\alpha^{-1})\!)(\!(\beta^{-1})\!)$. Write Γ_F for the value group of F with respect to $\mathfrak{v}$. Note $\Gamma_F = \mathbb{Z} \times \mathbb{Z}$. For each $(i, j) \in I = \{0, 1, \dots, p-1\}^{\times 2} \setminus \{(0, 0)\}$, write

$$A_{i,j} = \begin{cases} [\alpha^i \beta^j, \beta)_{p,F} & i \neq 0 \\ [\beta^j, \alpha)_{p,F} & i = 0. \end{cases}$$

Since the values of $\alpha^i\beta^j$ and β when $i \neq 0$ are negative and $\mathbb{F}_p$-independent in $\Gamma_F/p\Gamma_F$, the valuation $\mathfrak{v}$ extends to $A_{i,j}$ and $A_{i,j}$ is totally ramified over F with value group $\Gamma_{A_{i,j}} = \frac{1}{p}\mathbb{Z} \times \frac{1}{p}\mathbb{Z}$ (see [14]). For a similar argument, the valuation extends also to $A_{0,j}$, when $j \neq 0$, and $A_{0,j}$ is totally ramified over F with value group $\Gamma_{A_{0,j}} = \frac{1}{p}\mathbb{Z} \times \frac{1}{p}\mathbb{Z}$ (see also [4, Remark 3.2]). Write $V_{i,j}$ for the subspace of trace zero elements of $A_{i,j}$. It follows from Lemma 2.1 that $\mathfrak{v}(V_{i,j})/\Gamma_F \not\ni (\frac{i}{p}, \frac{j}{p})$, because writing x and y for the standard generators of $A_{i,j}$, $V_{i,j} = \mathrm{Span}_F\{x^k y^\ell : (k,\ell) \in \{0,1,\dots,p-1\}^{\times 2} \setminus \{(p-1,0)\}\}$, the values of the $x^k y^\ell$-s are distinct modulo Γ_F and none is congruent to $(\frac{i}{p}, \frac{j}{p})$. Therefore the intersection of all the $\mathfrak{v}(V_{i,j})$-s modulo Γ_F is trivial, which means that $\bigcap_{(i,j)\in I} \mathfrak{v}(V_{i,j}) = \Gamma_F$.

Now, suppose the contrary, that the algebras above share a maximal subfield K. Since K is a subfield of each $A_{i,j}$, it is totally ramified over F. Write W for its subspace of elements of trace 0. Then $\dim_F W$ is at least $p-1$. Since $W \subseteq \bigcap_{(i,j)\in I} V_{i,j}$, the values of all the nonzero elements in W are in Γ_F. Recall that $p > 2$. We can therefore choose two elements w_1 and w_2 in K whose values are $\mathbb{F}_p$-independent in Γ_K/Γ_F. As a result, they are also linearly independent, and there is a nonzero linear combination of theirs $w_3 \in Fw_1 + Fw_2$ which lives in W. Hence, the value of w_3 is either $\mathfrak{v}(w_1)$ or $\mathfrak{v}(w_2)$. In either case, $\mathfrak{v}(w_3)$ is not in Γ_F, despite the fact that $w_3 \in W$, contradiction. Consequently, the algebras $A_{i,j}$ have no maximal subfield in common. $\square$

Remark 2.4. Theorem 2.3 holds true also if one replaces $F_0(\alpha,\beta)$ with the field of iterated Laurent series $F_0(\!(\alpha^{-1})\!)(\!(\beta^{-1})\!)$ in two variables over F_0 of $\mathrm{char}(F_0) = p$. This demonstrates another difference between the behaviour of Laurent series in the good characteristic and the bad characteristic, because over an algebraically closed field F_0 of $\mathrm{char}(F_0) = 0$, the group ${}_pBr(F)$ for $F = F_0(\!(\alpha^{-1})\!)(\!(\beta^{-1})\!)$ is generated by a single division cyclic algebra of degree p and thus ${}_pBr(F)$ is ℓ-linked for any ℓ.

3. Characteristic 0

For the proof of the main result, we need the following observation about inseparable field extensions:

Lemma 3.1. *Let E be a field of $\mathrm{char}(E) = p > 0$, and $\alpha \in E \setminus E^p$.*

1. *Then the E^p-vector spaces $V_i = \mathrm{Span}_{E^p}\{(\alpha - i)^k : k \in \{1,\dots,p-1\}\}$ for $i \in \{0,\dots,p-1\}$ satisfy $\bigcap_{i=0}^{p-1} V_i = \{0\}$.*
2. *Furthermore, if there is another element $\beta \in E \setminus E^p(\alpha)$, then the vector spaces $W_{i,j}$ given by $W_{i,0} = \mathrm{Span}_{E^p}\{(\alpha-i)^m\beta^n : m,n \in \{0,\dots,p-1\}, (m,n) \neq (0,0)\}$ and $W_{i,j} = \mathrm{Span}_{E^p}\{\alpha^m(\alpha^i\beta - j)^n : m,n \in \{0,\dots,p-1\}, (m,n) \neq (0,0)\}$ for $i \in \{0,\dots,p-1\}$ and $j \in \{1,\dots,p-1\}$, satisfy $\bigcap_{i,j=0}^{p-1} W_{i,j} = \{0\}$.*

Proof. The first statement follows from the fact that an element $v = c_0 + c_1\alpha + \dots + c_{p-1}\alpha^{p-1} \in E^p(\alpha)$ is in V_i if and only if

$$c_0 + c_1 i + c_2 i^2 + \dots + c_{p-1} i^{p-1} = 0.$$

If we assume that $v \in \bigcap_{i=0}^{p-1} V_i$, then the polynomial $c_0 + c_1 X + \dots + c_{p-1} X^{p-1} \in E^p[X]$ has at least p distinct roots in E^p (which are the elements of the subfield $\mathbb{F}_p$), it must be the zero polynomial, i.e., $c_0 = c_1 = \dots = c_{p-1} = 0$.

For the second statement, we first note that $W_{i,0}$ can be written as

$$\begin{aligned} W_{i,0} &= \mathrm{Span}_{E^p(\alpha-i)}\{\beta^k : k \in \{1,\dots,p-1\}\} \oplus \mathrm{Span}_{E^p}\{(\alpha-i)^k : k \in \{1,\dots,p-1\}\} = \\ &= \mathrm{Span}_{E^p(\alpha)}\{\beta^k : k \in \{1,\dots,p-1\}\} \oplus \mathrm{Span}_{E^p}\{(\alpha-i)^k : k \in \{1,\dots,p-1\}\}. \end{aligned}$$

It follows from the first statement that

$$\bigcap_{i=0}^{p-1} W_{i,0} = \mathrm{Span}_{E^p(\alpha)}\{\beta^k : k \in \{1, \dots, p-1\}\}.$$

Now, for any $i \in \{0, \dots, p-1\}$, the space $W_{0,0}$ can also be written as

$$W_{0,0} = \mathrm{Span}_{E^p(\alpha^i\beta)}\{\alpha^k : k \in \{1, \dots, p-1\}\} \oplus \mathrm{Span}_{E^p}\{(\alpha^i\beta - 0)^k : k \in \{1, \dots, p-1\}\},$$

and for any $j \in \{1, \dots, p-1\}$ we can write the space $W_{i,j}$ as

$$\begin{aligned} W_{i,j} &= \mathrm{Span}_{E^p(\alpha^i\beta-j)}\{\alpha^k : k \in \{1, \dots, p-1\}\} \oplus \mathrm{Span}_{E^p}\{(\alpha^i\beta - j)^k : k \in \{1, \dots, p-1\}\} \\ &= \mathrm{Span}_{E^p(\alpha^i\beta)}\{\alpha^k : k \in \{1, \dots, p-1\}\} \oplus \mathrm{Span}_{E^p}\{(\alpha^i\beta - j)^k : k \in \{1, \dots, p-1\}\}. \end{aligned}$$

It follows from the first statement that for any $i \in \{0, \dots, p-1\}$,

$$W_{0,0} \cap \left(\bigcap_{j=1}^{p-1} W_{i,j} \right) = \mathrm{Span}_{E^p(\alpha^i\beta)}\{\alpha^k : k \in \{1, \dots, p-1\}\}.$$

Since the intersection $\bigcap_{(i,j)\in\{0,\dots,p-1\}^{\times 2}} W_{i,j}$ can be written as

$$\bigcap_{(i,j)\in\{0,\dots,p-1\}^{\times 2}} W_{i,j} = \left(\bigcap_{i=0}^{p-1} W_{i,0}\right) \cap \left(\bigcap_{i=1}^{p-1} \left(W_{0,0} \cap \left(\bigcap_{j=1}^{p-1} W_{i,j}\right)\right)\right),$$

we conclude that

$$\bigcap_{(i,j)\in\{0,\dots,p-1\}^{\times 2}} W_{i,j} = \mathrm{Span}_{E^p(\alpha)}\{\beta^k : k \in \{1, \dots, p-1\}\} \cap \left(\bigcap_{i=1}^{p-1} \left(\mathrm{Span}_{E^p(\alpha^i\beta)}\{\alpha^k : k \in \{1, \dots, p-1\}\}\right)\right),$$

and the intersection on the right-hand side is clearly trivial. □

Theorem 3.2. *Let F_0 be a field of $\mathrm{char}(F_0) = 0$ and $F = F_0(\alpha, \beta)$ the function field in two algebraically independent variables over F_0. Then there exist p^2 cyclic algebras of degree p over F that have no maximal subfield in common.*

Proof. Consider the algebras $A_{i,j} = (\gamma_{i,j}, \delta_{i,j})_{p,F}$ for $i, j \in \{0, \dots, p-1\}$ where $(\gamma_{i,j}, \delta_{i,j})$ are given (as elements of $F \times F$) by the formula

$$(\gamma_{i,j}, \delta_{i,j}) = \begin{cases} (\alpha - i, \beta) & (i,j) \in \{0, \dots, p-1\} \times \{0\} \\ (\alpha^i\beta - j, \alpha) & (i,j) \in \{0, \dots, p-1\} \times \{1, \dots, p-1\} \end{cases}$$

In the rest of the proof we can assume that F_0 is algebraically closed. If it is not, we can extend scalars to $F_0^{alg}(\alpha, \beta)$. If the algebras do not have a common maximal subfield under this restriction, they did not have any common maximal subfield from the beginning. Denote by $V_{i,j}$ the subspace of $A_{i,j}$ of elements of trace zero. Every maximal subfield is generated by an element of trace zero, and therefore in order for the algebras to have a common maximal subfield, they must posses nonzero elements of trace zero of the same reduced norm. Write $\varphi_{i,j}$ for the restriction of the reduced norm to $V_{i,j}$, and thus a necessary condition for the algebras to share a maximal subfield is that the forms $\varphi_{i,j}$ for $i,j \in \{0,\dots,p-1\}$ represent a common nonzero value. Now, the p-adic valuation extends from $\mathbb{Q}$ to F_0 with residue field k, and thus to F with residue field $E = k(\alpha,\beta)$. (See [9, Chapter 3] for details.) Since the value group of F_0 is divisible, if the forms represent a common nonzero value, we can suppose the equality $\varphi_{1,1}(v_{1,1}) = \dots = \varphi_{p-1,p-1}(v_{p-1,p-1})$ is obtained for elements $v_{1,1},\dots,v_{p-1,p-1}$ of minimal value 0. If such a solution to the system above exists, then it gives rise to a solution to the system

$$\overline{\varphi_{1,1}(v_{1,1})} = \dots = \overline{\varphi_{p-1,p-1}(v_{p-1,p-1})} \tag{1}$$

and their value in $k(\alpha,\beta)$ is nonzero as the residue of an element of value zero.

The valuation extends from F to the algebras $A_{i,j}$, which are unramified, and their residue algebras are $k(\sqrt[p]{\alpha},\sqrt[p]{\beta})$. (This follows from [15, Proposition 3.38] and the fact that $k(\sqrt[p]{\gamma_{i,j}},\sqrt[p]{\delta_{i,j}}) = k(\sqrt[p]{\alpha},\sqrt[p]{\beta})$ is a degree p^2 field extension of k.) Fixing generators x and y for $A_{i,j}$, the image of the reduced norm of an element $t = \sum_{m=0}^{p}\sum_{n=0}^{p} c_{m,n}x^m y^n$ of value zero in the residue field $k(\alpha,\beta)$ is $\overline{t}^p$ (by [15, Lemma 11.16]), which is $\sum_{m=0}^{p}\sum_{n=0}^{p} c_{m,n}^p\gamma_{i,j}^m\delta_{i,j}^n$. If $\mathrm{Tr}(t)=0$ then $c_{0,0}=0$. Thus, the solution to the system (1) gives rise to a nontrivial intersection of the E^p-vector spaces $W_{i,j} = \mathrm{Span}\{\gamma_{i,j}^m\delta_{i,j}^n : m,n \in \{0,\dots,p-1\},(m,n)\neq(0,0)\}$ for $i,j \in \{0,\dots,p-1\}$. However, they intersect trivially by Lemma 3.1. Hence, the algebras $A_{i,j}$ for $i,j \in \{0,\dots,p-1\}$ share no maximal subfield. □

4. In the opposite direction

It is important to point out what is known about the linkage of ${}_pBr(F)$ for function fields $F = F_0(\alpha,\beta)$ over algebraically closed fields F_0:

- When $p=2$, ${}_2Br(F)$ is 3-linked in any characteristic, so the story is complete in this case, for previous papers have shown that it need not be 4-linked.
- When $p=3$, ${}_3Br(F)$ is 2-linked by an easy argument mentioned in [1] based on [11]. Here we show that it need not be 8-linked in characteristic 3, or 9-linked in characteristic 0. Between 2 and 8 or 9 there is still a significant gap.
- There are no results in this direction for $p>3$ to the author's knowledge. There are results on the related period-index problem but that does not settle the problem yet.

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