



A prime-characteristic analogue of a theorem of Hartshorne-Polini [☆]



Nicholas Switala, Wenliang Zhang ^{*}

Department of Mathematics, Statistics, and Computer Science, University of Illinois at Chicago, 322 SEO (M/C 249), 851 S. Morgan Street, Chicago, IL 60607, United States of America

ARTICLE INFO

Article history:

Received 29 October 2019
Received in revised form 23 March 2020

Available online 6 April 2020
Communicated by S. Iyengar

MSC:

Primary: 13A35; secondary: 13A02; 13D45

Keywords:

Matlis duality
Local cohomology
Frobenius modules
 \mathcal{F} -modules

ABSTRACT

Let R be an F -finite Noetherian regular ring containing an algebraically closed field k of positive characteristic, and let M be an \mathcal{F} -finite \mathcal{F} -module over R in the sense of Lyubeznik (for example, any local cohomology module of R). We prove that the \mathbb{F}_p -dimension of the space of \mathcal{F} -module morphisms $M \rightarrow E(R/\mathfrak{m})$ (where \mathfrak{m} is any maximal ideal of R and $E(R/\mathfrak{m})$ is the R -injective hull of R/\mathfrak{m}) is equal to the k -dimension of the Frobenius stable part of $\text{Hom}_R(M, E(R/\mathfrak{m}))$. This is a positive-characteristic analogue of a recent result of Hartshorne and Polini for holonomic \mathcal{D} -modules in characteristic zero.

© 2020 Elsevier B.V. All rights reserved.

1. Introduction

In the study of finiteness properties of local cohomology modules there has been an emerging theme: the parallel between (holonomic) \mathcal{D} -modules in characteristic zero and (\mathcal{F} -finite) \mathcal{F} -modules in characteristic $p > 0$ (cf. [13], [11]). This paper continues the line of research under the same theme: we prove \mathcal{F} -module analogues of results obtained by Hartshorne and Polini in [4] and by the authors in [12] for \mathcal{D} -modules over formal power series or polynomial rings. The theory of (\mathcal{F} -finite) \mathcal{F} -modules will be reviewed in the next section. We begin by recalling the results for holonomic \mathcal{D} -modules in characteristic zero.

[☆] The first author gratefully acknowledges NSF support through grant DMS-1604503. The second author is partially supported by the NSF through grant DMS-1606414 and CAREER grant DMS-1752081.

^{*} Corresponding author.

E-mail addresses: nswitala@uic.edu (N. Switala), wlzhang@uic.edu (W. Zhang).

Theorem 1.1.

(a) [4, Corollary 5.2], [10, Theorem 5.1] Let $R = k[[x_1, \dots, x_n]]$ where k is a field of characteristic zero, let $\mathfrak{m} \subseteq R$ be the maximal ideal, and let $\mathcal{D} = \mathcal{D}(R, k)$ be the ring of k -linear differential operators on R . Denote by E the R -module $H_{\mathfrak{m}}^n(R)$, which is an R -injective hull of k . If M is a holonomic \mathcal{D} -module, then

$$\begin{aligned} \dim_k H_{\mathrm{dR}}^n(M) &= \dim_k H_{\mathrm{dR}}^0(D(M)) = \dim_k \mathrm{Hom}_{\mathcal{D}}(M, E) \\ &= \max\{t \in \mathbb{N} \mid \exists \text{ a } \mathcal{D}\text{-module surjection } M \rightarrow E^t\}, \end{aligned}$$

where $H_{\mathrm{dR}}^i(-)$ denotes the de Rham cohomology of a \mathcal{D} -module, and $D(-) = \mathrm{Hom}_R(-, E)$ is the Matlis dual functor.

(b) [12, Theorem 5.3] Let $R = k[x_1, \dots, x_n]$ where k is a field of characteristic zero, let $\mathfrak{m} \subseteq R$ be the irrelevant maximal ideal, and let $\mathcal{D} = \mathcal{D}(R, k)$ be the ring of k -linear differential operators on R . Denote by E the R -module $H_{\mathfrak{m}}^n(R)$, which (with the correct choice of grading) is a graded R -injective hull of k . If M is a finitely generated graded \mathcal{D} -module with finite-dimensional de Rham cohomology spaces (for example, a graded holonomic \mathcal{D} -module), then

$$\begin{aligned} \dim_k H_{\mathrm{dR}}^n(M) &= \dim_k H_{\mathrm{dR}}^0(*\mathcal{D}(M)) = \dim_k \mathrm{Hom}_{\mathcal{D}}(M, E) \\ &= \max\{t \in \mathbb{N} \mid \exists \text{ a } \mathcal{D}\text{-module surjection } M \rightarrow E^t\}, \end{aligned}$$

where $*\mathcal{D}(-) = *\mathrm{Hom}_R(-, E)$ is the graded Matlis dual functor.

In Theorem 1.1(b), a *graded* \mathcal{D} -module is a graded R -module on which the operators $\partial_j = \frac{\partial}{\partial x_j} \in \mathcal{D}$ act as graded k -linear maps of degree -1 . If $I \subseteq k[[x_1, \dots, x_n]]$ is an ideal (resp. $I \subseteq k[x_1, \dots, x_n]$ is a homogeneous ideal), the local cohomology modules $H_I^i(R)$ are holonomic (resp. graded holonomic) \mathcal{D} -modules, and so Theorem 1.1 can be applied to them.

The main result of this paper, Theorem A, is an \mathcal{F} -module analogue of Theorem 1.1. The finiteness condition analogous to the holonomicity of \mathcal{D} -modules is the F -finiteness of \mathcal{F} -modules, and local cohomology modules satisfy this condition. De Rham cohomology for \mathcal{D} -modules is ill-behaved in positive characteristic, so any analogue of Theorem 1.1 will require a replacement for $H_{\mathrm{dR}}^0(D(M))$. The desired replacement turns out to be the (Frobenius) stable part of $D(M)$. (See section 2 below for the relevant definitions.) The hypotheses of Theorem A are more general than those of Theorem 1.1 and include both the cases of polynomial and formal power series rings; observe that there is no need to state and prove a graded version separately.

Theorem A (Corollary 3.5). Let R be a regular Noetherian ring containing an algebraically closed field k of characteristic $p > 0$, and let $\mathfrak{m} \subseteq R$ be any maximal ideal. Let $E = E_R(R/\mathfrak{m})$ be the R -injective hull of R/\mathfrak{m} , and denote by $D(-)$ the exact functor $\mathrm{Hom}_R(-, E)$ on the category of R -modules. Assume that the Frobenius $F : R \rightarrow R$ is a finite morphism. If M is an \mathcal{F} -finite \mathcal{F} -module over R , then

$$\begin{aligned} \dim_{\mathbb{F}_p} \mathrm{Hom}_{\mathcal{F}}(M, E) &= \dim_k D(M)_s \\ &= \max\{t \in \mathbb{N} \mid \exists \text{ an } \mathcal{F}\text{-module surjection } M \rightarrow E^t\}, \end{aligned}$$

where $(-)_s$ denotes the stable part of a Frobenius module, and $\mathrm{Hom}_{\mathcal{F}}$ denotes the \mathbb{F}_p -space of \mathcal{F} -module morphisms.

In section 2, we collect the necessary preliminary material on Frobenius modules and \mathcal{F} -modules. Much of this section recalls definitions and results due to Hartshorne and Speiser, Lyubeznik, and Bhatt and Lurie, but Proposition 2.12 appears to us to be new. Theorem A is proved in section 3.

Acknowledgments

The first author thanks Rankeya Datta, Linquan Ma, and Mircea Mustață for helpful discussions.

2. Frobenius modules and \mathcal{F} -modules

We begin with some notation and conventions. All rings considered in this paper are commutative with identity 1. Except in section 1, all rings are of characteristic $p > 0$ unless otherwise noted. Throughout this section, A denotes such a ring, and we reserve the letter R for *regular* Noetherian rings; we will repeat these assumptions in the hypotheses of definitions and theorems. All local rings are assumed to be Noetherian.

We denote by F (or F_A if the context demands) the *Frobenius* endomorphism $F : A \rightarrow A$ defined by $F(a) = a^p$ for all $a \in A$. If M is an A -module, we can consider the A -modules F^*M and F_*M . The A -module F_*M has the same underlying Abelian group as M , with A -action defined by $a * m = a^p m$. On the other hand, as an Abelian group, F^*M can be expressed as $F_*A \otimes_A M$, where the notation means that we form the tensor product by regarding A as a right A -module via the Frobenius. Explicitly, for $a, b \in A$ and $m \in M$, we have $a(b \otimes m) = ab \otimes m$ and $a \otimes bm = ab^p \otimes m$.

The following is just the well-known adjunction between restriction and extension of scalars; we record it here separately so as to have a specific reference for the formulas in the sequel.

Proposition 2.1. *Let A be a ring of characteristic $p > 0$, and let M be an A -module. There is a bijective correspondence between A -linear maps $M \rightarrow F_*M$ and A -linear maps $F^*M \rightarrow M$.*

Proof. If $\varphi : M \rightarrow F_*M$ is an A -linear map, the corresponding A -linear map $\psi : F^*M = F_*A \otimes_A M \rightarrow M$ is defined by $\psi(a \otimes m) = a\varphi(m)$. Conversely, if $\psi : F^*M = F_*A \otimes_A M \rightarrow M$ is an A -linear map, the corresponding A -linear map $\varphi : M \rightarrow F_*M$ is defined by $\varphi(m) = \psi(1 \otimes m)$. \square

The main objects of this paper are A -modules equipped with A -linear maps to or from their pushforwards and pullbacks along the Frobenius F_A . *Frobenius modules* over A (that is, A -modules M equipped with a choice of A -linear map $M \rightarrow F_*M$) were studied by Hartshorne and Speiser in [5] and, more recently, by Bhatt and Lurie in [1]. On the other hand, \mathcal{F} -modules (that is, A -modules M equipped with a choice of A -linear *isomorphism* $M \rightarrow F^*M$), also known as *unit Frobenius modules*, were introduced by Lyubeznik in [9] and studied further by Emerton and Kisin in [3] and Bhatt and Lurie in [1], in the case where A is regular and Noetherian. (By a celebrated theorem of Kunz [8, Theorem 2.1], the functor F^* is *exact* under these hypotheses, and this exactness is crucial to the theory of \mathcal{F} -modules.)

We now proceed to give the basic definitions and relationships between these objects.

Definition 2.2. Let A be a ring of characteristic $p > 0$. A *Frobenius module* over A is a pair (M, φ_M) where M is an A -module and $\varphi_M : M \rightarrow F_*M$ is an A -linear map. (When there is no danger of confusion, we sometimes write φ for φ_M ; we also sometimes refer simply to M as a “Frobenius module”.)

An A -linear map $M \rightarrow F_*M$ is the same thing as an additive map $\varphi_M : M \rightarrow M$ such that $\varphi_M(am) = a^p\varphi_M(m)$ for all $a \in A$ and $m \in M$. In particular, the iterates φ_M^i for $i \geq 0$ make sense. Frobenius modules over A form a category Mod_A^{Fr} , where a morphism $(M, \varphi_M) \rightarrow (N, \varphi_N)$ is an A -linear map $f : M \rightarrow N$ such that $\varphi_N \circ f = F_*f \circ \varphi_M$. We denote by $\text{Hom}_{A[F]}(M, N)$ the \mathbb{F}_p -space of Frobenius module morphisms

$(M, \varphi_M) \rightarrow (N, \varphi_N)$. The reason for the notation is that a Frobenius module over A is the same thing as a left module over the non-commutative ring $A[F]$ generated over A by the symbol F , subject to the relations $Fa = a^p F$ for all $a \in A$.

Definition 2.3. Let A be a ring of characteristic $p > 0$, and let (M, φ_M) be a Frobenius module over A .

- (a) The (Frobenius) *fixed part* $M^{\varphi=1}$ of M is the \mathbb{F}_p -subspace $\{m \in M \mid \varphi_M(m) = m\} \subseteq M$.
- (b) Suppose that A contains a perfect field k of characteristic $p > 0$. The (Frobenius) *stable part* M_s of M is the k -subspace $\cap_{i \geq 0} \varphi_M^i(M) \subseteq M$.

If k is any field of characteristic $p > 0$, the only solutions $\lambda \in k$ to the equation $\lambda^p = \lambda$ are the elements of the prime subfield $\mathbb{F}_p \subseteq k$. Therefore the fixed part $M^{\varphi=1}$ can only be an \mathbb{F}_p -subspace. If k is perfect, the iterated images $\varphi_M^i(M)$ are k -subspaces of M , and so the same is true for M_s .

It is clear that $M^{\varphi=1} \subseteq M_s$. Under stronger hypotheses on k , we can say something more about the relationship between the fixed and stable parts:

Proposition 2.4. [2, Exp. XXII, Corollaire 1.1.10] Let A be a ring containing an algebraically closed field k of characteristic $p > 0$. Let (M, φ_M) be a Frobenius module over A . If M_s is a finite-dimensional k -space, then there is an isomorphism

$$k \otimes_{\mathbb{F}_p} M^{\varphi=1} \cong M_s$$

of k -spaces. In particular, there exists a k -basis $\{m_1, \dots, m_l\}$ of M_s such that $\varphi_M(m_i) = m_i$ for $i = 1, \dots, l$.

Proposition 2.4 implies that if M_s is a finite-dimensional k -space, then $M^{\varphi=1}$ is a finite-dimensional \mathbb{F}_p -space (of the same dimension). The converse, however, is not true, as the following example shows.

Example 2.5. Let $R = k[x]$ where k is a perfect field of characteristic $p > 0$, let F_R be the Frobenius endomorphism on R , and consider the *perfection* $R^{1/p^\infty} = k[x^{1/p^\infty}]$, that is, the colimit

$$\varinjlim (R \xrightarrow{F_R} R \xrightarrow{F_R} R \xrightarrow{F_R} \dots).$$

The Frobenius endomorphism $F_{R^{1/p^\infty}}$ is bijective. Therefore, if we regard $(R^{1/p^\infty}, F_{R^{1/p^\infty}})$ as a Frobenius module over R , we have $(R^{1/p^\infty})_s = R^{1/p^\infty}$, which is not a finite-dimensional k -space. However, $(R^{1/p^\infty})^{\varphi=1}$ is simply \mathbb{F}_p , a one-dimensional \mathbb{F}_p -space. This is true whether or not k is algebraically closed.

The following result of Hartshorne and Speiser provides one useful case in which the finiteness of the stable part is known (and so Proposition 2.4 applies).

Theorem 2.6. [5, Theorem 1.12] Let A be a local ring containing a perfect field k of characteristic $p > 0$, and let (M, φ_M) be a Frobenius module over A . If M is an Artinian A -module, then M_s is a finite-dimensional k -space, and the induced map $\varphi : M_s \rightarrow M_s$ is bijective.

If (M, φ_M) is a Frobenius module over A , we have by Proposition 2.1 an A -linear map $\psi_M : F^*M \rightarrow M$. When ψ_M is an isomorphism, M is called a *unit* Frobenius module by some authors [3, 1]. In Lyubeznik's [9], which deals only with the case of a regular Noetherian ring R , unit Frobenius modules over R are called \mathcal{F} -modules (or \mathcal{F}_R -modules). We will follow Lyubeznik's notation and terminology.

Definition 2.7. Let R be a regular Noetherian ring of characteristic $p > 0$. An \mathcal{F} -module over R (or \mathcal{F}_R -module) is a pair (M, θ_M) where M is an R -module and $\theta_M : M \xrightarrow{\sim} F^*M$ is an R -module isomorphism,

called the *structure morphism*. (When there is no danger of confusion, we sometimes refer simply to M as an “ \mathcal{F} -module”.)

Of course, if (M, θ_M) is an \mathcal{F}_R -module, then (M, φ_M) is a Frobenius module over R , where $\varphi_M : M \rightarrow F_*M$ is the R -linear map that corresponds via adjunction to $\psi_M = \theta_M^{-1}$.

There is a category $\mathcal{F}_R\text{-Mod}$ of \mathcal{F} -modules over R , where a morphism $(M, \theta_M) \rightarrow (N, \theta_N)$ is an R -linear map $f : M \rightarrow N$ such that $\theta_N \circ f = F^*f \circ \theta_M$. We denote by $\text{Hom}_{\mathcal{F}}(M, N)$ the \mathbb{F}_p -space of \mathcal{F} -module morphisms $(M, \theta_M) \rightarrow (N, \theta_N)$. In particular, we can speak of \mathcal{F} -submodules: an \mathcal{F} -submodule N of an \mathcal{F} -module (M, θ_M) is an R -submodule $N \subseteq M$ such that the restriction $\theta_M|_N$ is an isomorphism $N \xrightarrow{\sim} F^*N$ (since R is regular, F^* is exact and so F^*N can always be identified with an R -submodule of F^*M).

In the fruitful analogy between \mathcal{D} -modules in characteristic zero and \mathcal{F} -modules in positive characteristic, the finiteness condition of holonomicity for \mathcal{D} -modules corresponds to the condition of “ \mathcal{F} -finiteness” defined below (in particular, local cohomology modules provide examples of each). Loosely speaking, an \mathcal{F} -finite \mathcal{F} -module is one built from a finitely generated R -module by repeatedly applying the functor F^* and passing to a colimit.

Definition 2.8. Let R be a regular Noetherian ring of characteristic $p > 0$, and let (M, θ_M) be an \mathcal{F} -module. We say that M is \mathcal{F} -finite if there exists a finitely generated R -module M' and an R -linear map $\beta : M' \rightarrow F^*M'$ such that

$$\varinjlim (M' \xrightarrow{\beta} F^*M' \xrightarrow{F^*\beta} (F^*)^2M' \rightarrow \dots) \cong M,$$

and the structure morphism θ_M is induced by taking the colimit over l of $(F^*)^l\beta : (F^*)^lM' \rightarrow (F^*)^{l+1}M'$. In this case we call M' a *generator* of M and β a *generating morphism*.

Example 2.9. Let R be a regular Noetherian ring of characteristic $p > 0$. The following are the most relevant examples of \mathcal{F} -finite \mathcal{F} -modules for the purposes of this paper.

- (a) R itself is an \mathcal{F} -finite \mathcal{F} -module. The corresponding Frobenius module structure is given by $\varphi_R = F_R$, and id_R is an \mathcal{F} -module generating morphism for R . Moreover, R is a simple \mathcal{F} -module, since \mathcal{F} -submodules of R are ideals $I \subseteq R$ such that the natural surjection $R/I^{[p]} \rightarrow R/I$ is an isomorphism (here $I^{[p]}$ is the ideal generated by all p th powers of elements of I), and as R is Noetherian, this can only happen if $I = (0)$ or $I = R$.
- (b) If $I \subseteq R$ is an ideal and $i \geq 0$, the local cohomology module $H_I^i(R)$ is an \mathcal{F} -finite \mathcal{F} -module [9, Example 2.2(b)].
- (c) If $\mathfrak{m} \subseteq R$ is a maximal ideal, then the R -injective hull $E = E_R(R/\mathfrak{m})$ of the R -module R/\mathfrak{m} is an \mathcal{F} -finite \mathcal{F} -module [11, Proposition 5.4(d)]. Moreover, E is a simple \mathcal{F} -module, since R/\mathfrak{m} is a simple R -module as well as an \mathcal{F} -module generator of E .

Proposition 2.10. Let R be a regular Noetherian ring of characteristic $p > 0$, and let J be an injective R -module. Denote by $D_J(-)$ the exact functor $\text{Hom}_R(-, J)$ on the category of R -modules. There are R -module homomorphisms

$$\delta_M : F^*D_J(M) \rightarrow D_J(F^*M)$$

for all R -modules M , functorial in M . Furthermore, if M is a finitely generated R -module, then the δ_M is an isomorphism.

Proof. There is a functorial R -module homomorphism

$$F^*D_J(M) = F_*R \otimes_R \operatorname{Hom}_R(M, J) \xrightarrow{\delta_1} \operatorname{Hom}_R(F_*R \otimes_R M, F_*R \otimes_R J)$$

defined via

$$(\delta_1(r \otimes \varphi))(s \otimes m) = rs \otimes \varphi(m)$$

Since R is regular, F_*R is a flat R -module. Hence, δ_1 is an isomorphism when M is finitely generated.

By [6, Proposition 1.5], since R is Gorenstein, $J \cong F^*J$ as R -modules. Fix a choice of R -module isomorphism $\theta_J^{-1} : F_*R \otimes_R J \rightarrow J$ which induces an isomorphism

$$\widetilde{\theta}^{-1}_J : \operatorname{Hom}_R(F_*R \otimes_R M, F_*R \otimes_R J) \xrightarrow{\sim} \operatorname{Hom}_R(F_*R \otimes_R M, J) = D_J(F^*M).$$

For M an R -module, we define

$$\delta_M = \widetilde{\theta}^{-1}_J \circ \delta_1.$$

Then $\delta_M : F^*D_J(M) \rightarrow D_J(F^*M)$ is an isomorphism when M is finitely generated. \square

Given two \mathcal{F} -modules over R (say M and N) we can regard them as Frobenius modules, and consider the \mathbb{F}_p -space of \mathcal{F} -module (resp. Frobenius module) morphisms between them. We show in Proposition 2.12 below that not only are these two sets of morphisms the same, but that this set arises as the fixed part of a certain Frobenius module structure on $\operatorname{Hom}_R(M, N)$ itself, explained next.

Remark 2.11. Let R be a regular Noetherian ring of characteristic $p > 0$. Let (M, θ_M) and (N, θ_N) be \mathcal{F} -modules over R . Then $\operatorname{Hom}_R(M, N)$ admits a natural Frobenius module structure as follows. Define $\varphi : \operatorname{Hom}_R(M, N) \rightarrow F_*\operatorname{Hom}_R(M, N)$ by

$$\varphi(f) = \theta_N^{-1} \circ (\operatorname{id}_R \otimes f) \circ \theta_M$$

for each $f \in \operatorname{Hom}_R(M, N)$. It is clear that φ is additive; it remains to show it is R -linear. Given any $r \in R$, we have

$$\varphi(rf) = \theta_N^{-1} \circ (\operatorname{id}_R \otimes rf) \circ \theta_M = \theta_N^{-1} \circ (\mu_{r^p} \otimes f) \circ \theta_M = \mu_{r^p} \circ \theta_N^{-1} \circ (\operatorname{id}_R \otimes f) \circ \theta_M = r^p \varphi(f) = r * \varphi(f)$$

for all $f \in \operatorname{Hom}_R(M, N)$, where μ_s (for any $s \in R$) denotes multiplication by s . It follows that φ is R -linear, and hence it provides a Frobenius module structure on $\operatorname{Hom}_R(M, N)$.

Proposition 2.12. Let R be a regular Noetherian ring of characteristic $p > 0$. Let (M, θ_M) and (N, θ_N) be \mathcal{F} -modules over R . Regard M (resp. N) as a Frobenius module via the R -linear map $\varphi_M : M \rightarrow F_*M$ (resp. $\varphi_N : N \rightarrow F_*N$) corresponding via adjunction to θ_M^{-1} (resp. θ_N^{-1}). Then

$$\operatorname{Hom}_{R[F]}(M, N) = \operatorname{Hom}_{\mathcal{F}}(M, N) = \operatorname{Hom}_R(M, N)^{\varphi=1}$$

as \mathbb{F}_p -subspaces of $\operatorname{Hom}_R(M, N)$, where the Frobenius module structure φ on $\operatorname{Hom}_R(M, N)$ is defined as in Remark 2.11.

Proof. The first equality has nothing to do with the choice of φ . Let $f \in \text{Hom}_R(M, N)$ be given. On the one hand, the map f belongs to $\text{Hom}_{R[F]}(M, N)$ if and only if $\varphi_N \circ f = F_* f \circ \varphi_M$. On the other hand, f belongs to $\text{Hom}_{\mathcal{F}}(M, N)$ if and only if $\theta_N \circ f = F^* f \circ \theta_M$; equivalently, $f \circ \theta_M^{-1} = \theta_N^{-1} \circ F^* f$. We have

$$f(\theta_M^{-1}(r \otimes m)) = f(r\varphi_M(m)) = rf(\varphi_M(m))$$

and

$$\theta_N^{-1}(F^* f(r \otimes m)) = \theta_N^{-1}(r \otimes f(m)) = r\varphi_N(f(m))$$

for all $r \in R$ and $m \in M$, and so the equality $f \circ \theta_M^{-1} = \theta_N^{-1} \circ F^* f$ is equivalent to $\varphi_N \circ f = F_* f \circ \varphi_M$. Thus $\text{Hom}_{R[F]}(M, N) = \text{Hom}_{\mathcal{F}}(M, N)$.

For the second equality, observe that a map $f \in \text{Hom}_R(M, N)$ belongs to the fixed part $\text{Hom}_R(M, N)^{\varphi=1}$ if and only if $f = \theta_N^{-1} \circ (\text{id}_R \otimes f) \circ \theta_M$, or equivalently, $\theta_N \circ f = F^* f \circ \theta_M$. This is exactly the criterion for f to be an \mathcal{F} -module morphism, completing the proof. \square

Remark 2.13. Let R be a regular Noetherian ring of characteristic $p > 0$, and let (M, θ_M) be an \mathcal{F} -module over R . By Example 2.9(a), the R -module R itself is an \mathcal{F} -module with corresponding Frobenius module structure given by $\varphi_R = F_R$. Under the canonical identification of M with $\text{Hom}_R(R, M)$, the Frobenius module structure on $\text{Hom}_R(R, M)$ provided by Remark 2.11 coincides with the Frobenius module structure φ_M corresponding by adjunction to the given \mathcal{F} -module structure on M itself. Indeed, if $f : R \rightarrow M$ is defined by $f(1) = m$, then the composite $R \xrightarrow{\theta_R} F^* R \xrightarrow{\text{id}_R \otimes f} F^* M \xrightarrow{\theta_M^{-1}} M$ maps $1 \mapsto 1 \otimes 1 \mapsto 1 \otimes m \mapsto \varphi_M(m)$.

3. Proof of the main theorem

Lemma 3.1. Let R be a regular Noetherian ring of characteristic $p > 0$, and let J be an injective R -module. Denote by $D_J(-)$ the exact functor $\text{Hom}_R(-, J)$ on the category of R -modules. If M is an R -module, then any R -module homomorphism $M \rightarrow F^* M$ induces a Frobenius module structure on $D_J(M)$.

Proof. Apply the functor D_J to the given map, obtaining an R -linear map $D_J(F^* M) \rightarrow D_J(M)$. Precomposition with the map δ_M defined in Proposition 2.10 gives an R -linear map $F^* D_J(M) \rightarrow D_J(M)$, which corresponds by adjunction (Proposition 2.1) to an R -linear map $D_J(M) \rightarrow F_* D_J(M)$, the desired Frobenius module structure. \square

In particular, if M is an \mathcal{F} -module (resp. a generator of an \mathcal{F} -finite \mathcal{F} -module), then for any injective R -module J , $D_J(M)$ has a Frobenius module structure obtained by applying Lemma 3.1 to the structure morphism θ_M (resp. the generating morphism β).

Definition 3.2. Let A be a ring of characteristic $p > 0$. We say that A is F -finite¹ if $F_* A$ is a finitely generated A -module.

For example, if k is a perfect field of characteristic $p > 0$, the rings $k[x_1, \dots, x_n]$ and $k[[x_1, \dots, x_n]]$ are F -finite. We recall the well-known facts that if R is an F -finite regular Noetherian ring of characteristic $p > 0$, then R is F -split (meaning that the R -module homomorphism $R \rightarrow F_* R$ defined by the Frobenius admits a section) and $F_* R$ is locally free as an R -module (because it is finitely generated as well as flat).

¹ Note the distinction between F -finiteness, a property of a ring, and \mathcal{F} -finiteness, a property of an \mathcal{F} -module.

Theorem 3.3. *Let R be an \mathcal{F} -finite regular Noetherian ring containing an algebraically closed field k of characteristic $p > 0$, and let J be an injective R -module. Denote by $D_J(-)$ the exact functor $\text{Hom}_R(-, J)$ on the category of R -modules. Assume that the following conditions are satisfied:*

- (i) J , which is an \mathcal{F} -module by [6, Proposition 1.5], is simple as an \mathcal{F} -module;
- (ii) for every finitely generated R -module M' equipped with a choice of R -module homomorphism $M' \rightarrow F^*M'$, the stable part $D_J(M')_s$ (which is defined by Lemma 3.1) is a finite-dimensional k -space, and the Frobenius structure on $D_J(M')$ restricts to a bijection on $D_J(M')_s$.

Then, for each \mathcal{F} -finite \mathcal{F} -module M , the following numbers are all equal (and, in particular, are all finite):

- (1) the \mathbb{F}_p -dimension of $\text{Hom}_{R[F]}(M, J) = \text{Hom}_{\mathcal{F}}(M, J)$,
- (2) the \mathbb{F}_p -dimension of $D_J(M)^{\varphi=1}$,
- (3) the k -dimension of $D_J(M)_s$,
- (4) the k -dimension of $D_J(M')_s$, where M' is any \mathcal{F} -module generator of M ,
- (5) the maximal integer t such that there exists a surjective \mathcal{F} -module morphism (equivalently, surjective Frobenius module morphism) $M \rightarrow J^t$.

Proof. Let M be an \mathcal{F} -finite \mathcal{F} -module over R , and let $\beta : M' \rightarrow F^*M'$ be an \mathcal{F} -module generating morphism for M . We have already proved the equality of (1) and (2) above, in Proposition 2.12. Since M' is a finitely generated R -module, our condition (ii) implies that $D_J(M')_s$ is a finite-dimensional k -space, that is, that (4) is finite. If we can prove the equality of (3) and (4) and hence the finiteness of (3), then the equality of (2) and (3) will follow from Proposition 2.4. Therefore we need only prove the equality of (1) and (5) as well as the equality of (3) and (4).

We begin with the equality of (1) and (5). Suppose first that there exists an \mathcal{F} -module surjection $M \rightarrow J^t$. Post-composing it with each of the t coordinate projections $J^t \rightarrow J$ produces t \mathbb{F}_p -linearly independent \mathcal{F} -module morphisms $M \rightarrow J$.

Conversely, assume there are t such \mathbb{F}_p -linearly independent \mathcal{F} -module morphisms $\varphi_1, \dots, \varphi_t : M \rightarrow J$; we wish to construct an \mathcal{F} -module surjection $M \rightarrow J^t$, or equivalently, an \mathcal{F} -submodule $N \subseteq M$ such that M/N is isomorphic to J^t . Since J is a simple \mathcal{F} -module by our condition (i), each φ_i must itself be surjective, since its image is a non-zero \mathcal{F} -submodule of J . Set $M_i = \ker(\varphi_i)$, an \mathcal{F} -submodule of M , for all i . Since $M/M_i \cong J$, we must have $M_i + M_j = M$ whenever $i \neq j$. We claim that $M/(\cap_{i=1}^t M_i) \cong J^t$, which will complete the proof (take $N = \cap_{i=1}^t M_i$); we do this by showing, by induction on j , that $M/(\cap_{i=1}^j M_i) \cong J^j$ for $1 \leq j \leq t$. This assertion is obvious for $j = 1$. Now suppose that for some $1 \leq j < t$ we know that $M/(\cap_{i=1}^j M_i) \cong J^j$ as \mathcal{F} -modules. We cannot have $\cap_{i=1}^j M_i \subseteq M_{j+1}$, since otherwise φ_{j+1} would factor through $M \rightarrow M/(\cap_{i=1}^j M_i)$ and hence would lie in the \mathbb{F}_p -span of $\varphi_1, \dots, \varphi_j$, a contradiction. Therefore $\cap_{i=1}^j M_i \not\subseteq M_{j+1}$, and so $\cap_{i=1}^j M_i + M_{j+1} = M$ by the simplicity. But then the short exact sequence

$$0 \rightarrow M/(\cap_{i=1}^{j+1} M_i) \rightarrow M/(\cap_{i=1}^j M_i) \oplus M/M_{j+1} \rightarrow M/(M_{j+1} + \cap_{i=1}^j M_i) \rightarrow 0$$

of \mathcal{F} -modules implies that $M/(\cap_{i=1}^{j+1} M_i) \cong M/(\cap_{i=1}^j M_i) \oplus M/M_{j+1} \cong J^j \oplus J \cong J^{j+1}$ by the induction hypothesis, as desired.

Finally, we prove the equality of (3) and (4). By definition,

$$M = \varinjlim (M' \xrightarrow{\beta} F^*M' \xrightarrow{F^*\beta} (F^*)^2M' \rightarrow \dots)$$

and the \mathcal{F} -module structure on M is induced by β and its F^* -iterates. Applying $D_J(-)$, we find

$$D_J(M) \cong \varprojlim (\dots \rightarrow D_J((F^*)^2M') \xrightarrow{D_J(F^*\beta)} D_J(F^*M') \xrightarrow{D(\beta)} D_J(M')).$$

Since R is F -finite, not only M' but also $(F^*)^l M'$ for all $l \geq 0$ are finitely generated R -modules. Therefore, using Proposition 2.10 to identify $D((F^*)^l M')$ with $(F^*)^l D_J(M')$ for all $l \geq 0$, we can rewrite the limit as

$$D_J(M) \cong \varprojlim (\cdots \rightarrow (F^*)^2 D_J(M') \rightarrow F^* D_J(M') \rightarrow D_J(M')).$$

Since $(F^*)^l$ and $(F^l)^*$ are isomorphic functors, this is exactly the *leveling functor* of [5, p. 47]. (In the notation of [5], we have $D_J(M) = G(D_J(M'))$.) It follows from the proof of [5, Proposition 1.2(b)] (see Remark 3.4 below) that

$$D_J(M)_s \cong \varprojlim (\cdots \rightarrow k \otimes_{F_k^2} D_J(M')_s \rightarrow k \otimes_{F_k} D_J(M')_s \rightarrow D_J(M')_s),$$

where $F_k : k \rightarrow k$ is the Frobenius endomorphism of k , and the maps

$$k \otimes_{F_k^{l+1}} D_J(M')_s \rightarrow k \otimes_{F_k^l} D_J(M')_s$$

are given by the identity on the first tensor factor and the restriction of the map $D_J(M') \rightarrow D_J(M')$ defining the Frobenius module structure on $D_J(M')$ in the second. But by our condition (ii), this last map restricts to a bijection from $D_J(M')_s$ to itself. That is, the displayed limit can be identified with $D_J(M')_s$, so that $D_J(M)_s \cong D_J(M')_s$ as k -spaces, completing the proof of the equality of (3) and (4) and therefore the proof of the theorem. \square

Remark 3.4. In the proof of Theorem 3.3 above, we appealed to [5, Proposition 1.2(b)]. This proposition is stated in [5] only for a ring R of characteristic $p > 0$ such that $F_* R$ is a *free* R -module, a hypothesis that is stronger than ours. However, examining the proof of [5, Proposition 1.2(b)], it is clear that this hypothesis is only used in the form of the following consequence: if M is an R -module and $m, m' \in M$ are such that $1 \otimes m = 1 \otimes m'$ in $F^* M = F_* R \otimes_R M$, then $m = m'$. But since we assumed in Theorem 3.3 that R is regular and F -finite, it is also F -split, from which the previous statement is immediate.

The following corollary of Theorem 3.3, which identifies a class of injective modules for which the hypotheses of the theorem are satisfied, is the main result of this paper.

Corollary 3.5. *Let R be a regular, F -finite Noetherian ring containing an algebraically closed field k of characteristic $p > 0$, and let $\mathfrak{m} \subseteq R$ be any maximal ideal. Let $E = E_R(R/\mathfrak{m})$ be the R -injective hull of R/\mathfrak{m} , and denote by $D(-)$ the exact functor $\mathrm{Hom}_R(-, E)$ on the category of R -modules. Then, for each \mathcal{F} -finite \mathcal{F} -module M , the following numbers are all equal (and, in particular, are all finite):*

- (1) the \mathbb{F}_p -dimension of $\mathrm{Hom}_{R[F]}(M, E) = \mathrm{Hom}_{\mathcal{F}}(M, E)$,
- (2) the \mathbb{F}_p -dimension of $D(M)^{\varphi=1}$,
- (3) the k -dimension of $D(M)_s$,
- (4) the k -dimension of $D(M')_s$, where M' is any \mathcal{F} -module generator of M ,
- (5) the maximal integer t such that there exists a surjective \mathcal{F} -module morphism (equivalently, surjective Frobenius module morphism) $M \rightarrow E^t$.

Proof. By Example 2.9(c), E is a simple \mathcal{F} -module, so condition (i) of Theorem 3.3 is satisfied. Now let M' be a finitely generated R -module equipped with a choice of R -module homomorphism $M' \rightarrow F^* M'$. There exists an R -linear surjection $R^l \rightarrow M'$ for some $l \geq 0$; applying the exact functor D , we obtain an R -linear injection $D(M') \rightarrow D(R^l) = E^l$. Since E (and hence E^l) is an Artinian R -module supported only at \mathfrak{m} , the same is true of $D(M')$, so $D(M')$ has a natural structure as a module over the \mathfrak{m} -adic completion

$\widehat{R}^{\mathfrak{m}}$ of R . In fact, $D(M')$ is a Frobenius module over $\widehat{R}^{\mathfrak{m}}$, with the Frobenius structure given by the same underlying additive map $D(M') \rightarrow D(M')$ defined by Lemma 3.1. The ring $\widehat{R}^{\mathfrak{m}}$ is local (since \mathfrak{m} is maximal) and contains an algebraically closed (hence perfect) field k ; moreover, $D(M')$ is Artinian as an $\widehat{R}^{\mathfrak{m}}$ -module. Therefore, Theorem 2.6 applies. We conclude that $D(M')_s$ is a finite-dimensional k -space and the Frobenius structure on $D(M')$ restricts to a bijection on $D(M')_s$, so condition (ii) of Theorem 3.3 is satisfied. The corollary now follows from Theorem 3.3 applied to $J = E$. \square

The proof of the equality of (1) and (5) in Corollary 3.5 works in characteristic zero as well, replacing “ \mathcal{F} -finite \mathcal{F} -module” with “holonomic \mathcal{D} -module”. Therefore we obtain an alternate proof of the fact that if $R = k[x_1, \dots, x_n]$ or $k[[x_1, \dots, x_n]]$ where k is a field of characteristic zero, and M is a holonomic $\mathcal{D}(R, k)$ -module, then $\dim_k \operatorname{Hom}_{\mathcal{D}}(M, E)$ is equal to the maximal integer t for which there exists a \mathcal{D} -linear surjection $M \rightarrow E^t$. This statement is part of [4, Corollary 5.2]. An easier “dual” statement is the following [12, Lemma 2.3]: $\dim_k \operatorname{Hom}_{\mathcal{D}}(R, M)$ is equal to the maximal integer t for which there exists a \mathcal{D} -linear injection $R^t \rightarrow M$. We can prove a version of this in the Frobenius module setting, as part of a “dual” version of Corollary 3.5. Note, however, that Theorem 3.6 has a finite-dimensionality hypothesis whose analogue is not needed (because it is automatically satisfied) in Corollary 3.5.

Theorem 3.6. *Let R be a regular Noetherian ring containing an algebraically closed field k of characteristic $p > 0$, and let M be an \mathcal{F} -finite \mathcal{F} -module over R . Assume that M_s is a finite-dimensional k -space. Then the following numbers are all equal (and, in particular, are all finite):*

- (1) the \mathbb{F}_p -dimension of $\operatorname{Hom}_{R[F]}(R, M) = \operatorname{Hom}_{\mathcal{F}}(R, M)$,
- (2) the \mathbb{F}_p -dimension of $M^{\varphi=1}$,
- (3) the k -dimension of M_s ,
- (4) the maximal integer t such that there exists an injective \mathcal{F} -module morphism (equivalently, injective Frobenius module morphism) $R^t \rightarrow M$.

Proof. By Remark 2.13, we can identify M with $\operatorname{Hom}_R(R, M)$ as Frobenius modules over R , and therefore we can identify $M^{\varphi=1}$ with $\operatorname{Hom}_R(R, M)^{\varphi=1}$ as \mathbb{F}_p -spaces. Therefore the equality of (1) and (2) follows from Proposition 2.12. We have assumed that M_s is a finite-dimensional k -space, so the equality of (2) and (3) follows from Proposition 2.4. Finally, the proof of the equality of (1) and (4) is essentially dual to the proof of the equality of (1) and (5) in Theorem 3.3, using the fact that R is a simple \mathcal{F} -module (Example 2.9(a)). The arguments are similar enough that we omit the details, providing a sketch. An \mathcal{F} -module injection $R^t \rightarrow M$ gives rise to t \mathbb{F}_p -linearly independent \mathcal{F} -module morphisms $R \rightarrow M$ by pre-composition with the coordinate inclusions; conversely, given t distinct isomorphic copies of R (say M_1, \dots, M_t) as \mathcal{F} -submodules of M , it can be shown (since all M_i are simple \mathcal{F} -submodules) that the sum $\sum_{i=1}^j M_i \subseteq M$ is a *direct* sum for $j = 1, \dots, t$ by induction on j , and the case $j = t$ is the desired assertion. \square

Question 3.7. If R is a regular Noetherian ring containing an algebraically closed field k of characteristic $p > 0$, and M is an \mathcal{F} -finite \mathcal{F} -module over R , is M_s a finite-dimensional k -space?

A positive answer to Question 3.7 would, of course, permit us to remove the finite-dimensionality hypothesis in Theorem 3.6, since Proposition 2.4 would apply.

We end with the following remarks.

Remark 3.8. Let $X \subset \mathbb{P}_k^n$ be a projective scheme over an algebraically closed field k of characteristic p and let $I \subset R = k[x_0, \dots, x_n]$ be its defining ideal. Applying Corollary 3.5 to the \mathcal{F} -finite \mathcal{F} -module $H_I^j(R)$, one has

$$\begin{aligned}
\dim_{\mathbb{F}_p} \operatorname{Hom}_{\mathcal{F}}(H_I^{n-j}(R), E) &= \dim_k \operatorname{Hom}_R(H_I^{n-j}(R), E)_s \\
&= \dim_k \operatorname{Hom}_R(\operatorname{Ext}_R^{n-j}(R/I, R), E)_s \\
&= \dim_k H_{\mathfrak{m}}^{j+1}(R/I)_s \\
&= \dim_k H^j(X, \mathcal{O}_X)_s
\end{aligned}$$

According to [5, proposition 5.1], $\dim_k H^j(X, \mathcal{O}_X)_s = \dim_{\mathbb{F}_p} H_{\text{ét}}^j(X, \mathbb{F}_p)$. Hence

$$\dim_{\mathbb{F}_p} \operatorname{Hom}_{\mathcal{F}}(H_I^{n-j}(R), E) = \dim_{\mathbb{F}_p} H_{\text{ét}}^j(X, \mathbb{F}_p).$$

Assume that the local cohomology module $H_I^{n-j}(R)$ is supported only at \mathfrak{m} (e.g. when X is Cohen-Macaulay). Then using the previous paragraph one may recover [5, Corollary 2.4] as follows:

$$H_I^{n-j}(R) \cong E^{\lambda_j}, \text{ where } \lambda_j = \dim_{\mathbb{F}_p}(H_{\text{ét}}^j(X, \mathbb{F}_p)).$$

Remark 3.9. Let X, R, I be the same as in Remark 3.8 and, additionally, assume that X is irreducible and each local ring $\mathcal{O}_{X,x}$ is F -rational for each point $x \in X$. Our Remark 3.8 implies that $H_I^{n-\dim(X)}(R)$ admits a \mathcal{F} -module quotient that is isomorphic to E^{λ_d} where $\lambda_d = \dim_{\mathbb{F}_p}(H_{\text{ét}}^{\dim(X)}(X, \mathbb{F}_p))$. This does not fully recover the prime-characteristic analogue of [4, Theorem 6.4] in the case when $i = n - \dim(X)$; the length differs by one from the desired analogous result. The reason is that the simple \mathcal{F} -submodule of $H_I^{n-\dim(X)}(R)$ does not admit any non-zero \mathcal{F} -module morphism to E . However, it follows directly from [7, Theorem 4.3] that $H_I^{n-\dim(X)}(R)$ admits a simple \mathcal{F} -submodule H_0 such that $H_I^{n-\dim(X)}(R)/H_0 \cong E^{\lambda_d}$.

Acknowledgement

The authors thank the referee for helpful comments and suggestions.

References

- [1] B. Bhatt, J. Lurie, A Riemann-Hilbert correspondence in positive characteristic, arXiv:1711.04148, 2017.
- [2] P. Deligne, N. Katz (Eds.), Groupes de Monodromie en Géométrie Algébrique (SGA 7), vol. 2, Lecture Notes in Mathematics, vol. 340, Springer-Verlag, Berlin, 1973.
- [3] M. Emerton, M. Kisin, The Riemann-Hilbert Correspondence for Unit F -Crystals, Astérisque, vol. 293, Société Mathématique de France, Paris, 2004.
- [4] R. Hartshorne, C. Polini, Simple D -module components of local cohomology modules, J. Algebra (2016), in press, arXiv:1606.01278.
- [5] R. Hartshorne, R. Speiser, Local cohomological dimension in characteristic p , Ann. Math. 105 (1) (1977) 45–79.
- [6] C. Huneke, R. Sharp, Bass numbers of local cohomology modules, Trans. Am. Math. Soc. 339 (2) (1993) 765–779.
- [7] Mordechai Katzman, Linqun Ma, Ilya Smirnov, Wenliang Zhang, D -module and F -module length of local cohomology modules, Trans. Am. Math. Soc. 370 (12) (2018) 8551–8580.
- [8] E. Kunz, Characterizations of regular local rings of characteristic p , Am. J. Math. 91 (1969) 772–784.
- [9] G. Lyubeznik, F -modules: applications to local cohomology and D -modules in characteristic $p > 0$, J. Reine Angew. Math. 491 (1997) 65–130.
- [10] N. Switala, On the de Rham homology and cohomology of a complete local ring in equicharacteristic zero, Compos. Math. 153 (10) (2017) 2075–2146.
- [11] N. Switala, W. Zhang, A dichotomy for the injective dimension of F -finite F -modules and holonomic D -modules, Commun. Algebra (2017), in press, arXiv:1708.06481. A special issue in honor of Gennady Lyubeznik on occasion of his 60th birthday.
- [12] N. Switala, W. Zhang, Duality and de Rham cohomology for graded D -modules, Adv. Math. 340 (2018) 1141–1165.
- [13] Wenliang Zhang, On injective dimension of F -finite F -modules and holonomic D -modules, Bull. Lond. Math. Soc. 49 (4) (2017) 593–603.