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Tridiagonal pairs of q-Racah type and the q-tetrahedron algebra

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ABSTRACT

Let $\mathbb{F}$ denote a field, and let V denote a vector space over $\mathbb{F}$ with finite positive dimension. We consider an ordered pair of $\mathbb{F}$-linear maps $A : V \to V$ and $A^* : V \to V$ such that (i) each of A, A^* is diagonalizable; (ii) there exists an ordering $\{V_i\}_{i=0}^d$ of the eigenspaces of A such that $A^*V_i \subseteq V_{i-1} + V_i + V_{i+1}$ for $0 \leq i \leq d$, where $V_{-1} = 0$ and $V_{d+1} = 0$; (iii) there exists an ordering $\{V_i^*\}_{i=0}^\delta$ of the eigenspaces of A^* such that $AV_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^*$ for $0 \leq i \leq \delta$, where $V_{-1}^* = 0$ and $V_{\delta+1}^* = 0$; (iv) there does not exist a subspace U of V such that $AU \subseteq U$, $A^*U \subseteq U$, $U \neq 0$, $U \neq V$. We call such a pair a tridiagonal pair on V. We assume that A, A^* belongs to a family of tridiagonal pairs said to have q-Racah type. There is an infinite-dimensional algebra $\boxtimes_q$ called the q-tetrahedron algebra; it is generated by four copies of $U_q(\mathfrak{sl}_2)$ that are related in a certain way. Using A, A^* we construct two $\boxtimes_q$-module structures on V. In this construction the two main ingredients are the double lowering map $\psi : V \to V$ due to Sarah Bockting-Conrad, and a certain invertible map $W : V \to V$ motivated by the spin model concept due to V. F. R. Jones.

1. Introduction

This paper is about a linear-algebraic object called a tridiagonal pair, and its relationship to a certain infinite-dimensional algebra $\boxtimes_q$ called the q-tetrahedron algebra. Before we explain our purpose in detail, we first define a tridiagonal pair. We will use the following terms. Let $\mathbb{F}$ denote a field, and let V denote a vector space over $\mathbb{F}$ with finite positive dimension. Let $\mathrm{End}(V)$ denote the algebra consisting of the $\mathbb{F}$-linear maps from V to V. For $A \in \mathrm{End}(V)$ and a subspace $U \subseteq V$, we call U an *eigenspace* of A whenever $U \neq 0$ and there exists $\theta \in \mathbb{F}$ such that $U = \{v \in V | Av = \theta v\}$; in this case θ is the *eigenvalue* of A associated with U. We say that A is *diagonalizable* whenever V is spanned by the eigenspaces of A.

Definition 1.1. (See [15, Definition 1.1].) Let V denote a vector space over $\mathbb{F}$ with finite positive dimension. By a *tridiagonal pair* (or *TD pair*) on V, we mean an ordered pair A, A^* of elements in $\mathrm{End}(V)$ that satisfy the following four conditions.

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(i) Each of A, A^* is diagonalizable.
(ii) There exists an ordering $\{V_i\}_{i=0}^d$ of the eigenspaces of A such that

$$A^*V_i \subseteq V_{i-1} + V_i + V_{i+1} \qquad (0 \leq i \leq d), \tag{1}$$

where $V_{-1} = 0$ and $V_{d+1} = 0$.
(iii) There exists an ordering $\{V_i^*\}_{i=0}^\delta$ of the eigenspaces of A^* such that

$$AV_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^* \qquad (0 \leq i \leq \delta), \tag{2}$$

where $V_{-1}^* = 0$ and $V_{\delta+1}^* = 0$.
(iv) There does not exist a subspace U of V such that $AU \subseteq U$, $A^*U \subseteq U$, $U \neq 0$, $U \neq V$.

The TD pair A, A^* is said to be *over* $\mathbb{F}$. We call V the *underlying vector space*.

Note 1.2. According to a common notational convention, A^* denotes the conjugate-transpose of A. We are not using this convention. In a TD pair A, A^* the linear maps A and A^* are arbitrary subject to (i)–(iv) above.

We refer the reader to [29] for background information on TD pairs. In that article, the introduction summarizes the origin of the TD pair concept in algebraic graph theory, and Section 19 gives a comprehensive discussion of the current state of the art.

In order to motivate our results, we recall some basic facts about TD pairs. Let A, A^* denote a TD pair on V, as in Definition 1.1. By [15, Lemma 4.5] the integers d and δ from (ii), (iii) are equal; we call this common value the *diameter* of the pair. For $0 \leq i \leq d$ let θ_i (resp. θ_i^*) denote the eigenvalue of A (resp. A^*) for the eigenspace V_i (resp. V_i^*). By [15, Theorem 11.1] the scalars

$$\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i}, \qquad \frac{\theta_{i-2}^* - \theta_{i+1}^*}{\theta_{i-1}^* - \theta_i^*}$$

are equal and independent of i for $2 \leq i \leq d-1$. For this recurrence the solutions can be given in closed form [15, Theorem 11.2]. The "most general" solution is called q-Racah, and will be described shortly.

By construction, the vector space V has a direct sum decomposition into the eigenspaces $\{V_i\}_{i=0}^d$ of A and the eigenspaces $\{V_i^*\}_{i=0}^d$ of A^*. The vector space V has two more direct sum decompositions of interest, called the first split decomposition $\{U_i\}_{i=0}^d$ and second split decomposition $\{U_i^\Downarrow\}_{i=0}^d$. By [15, Theorem 4.6] the first split decomposition satisfies

$$U_0 + U_1 + \cdots + U_i = V_0^* + V_1^* + \cdots + V_i^*,$$
$$U_i + U_{i+1} + \cdots + U_d = V_i + V_{i+1} + \cdots + V_d$$

for $0 \leq i \leq d$. By [15, Theorem 4.6] the second split decomposition satisfies

$$U_0^\Downarrow + U_1^\Downarrow + \cdots + U_i^\Downarrow = V_0^* + V_1^* + \cdots + V_i^*,$$
$$U_i^\Downarrow + U_{i+1}^\Downarrow + \cdots + U_d^\Downarrow = V_0 + V_1 + \cdots + V_{d-i}$$

for $0 \leq i \leq d$. By [15, Theorem 4.6],

$$(A-\theta_i I)U_i \subseteq U_{i+1}, \qquad (A^*-\theta_i^* I)U_i \subseteq U_{i-1},$$
$$(A-\theta_{d-i} I)U_i^{\Downarrow} \subseteq U_{i+1}^{\Downarrow}, \qquad (A^*-\theta_i^* I)U_i^{\Downarrow} \subseteq U_{i-1}^{\Downarrow}$$

for $0 \leq i \leq d$, where $U_{-1}=0$, $U_{d+1}=0$ and $U_{-1}^{\Downarrow}=0$, $U_{d+1}^{\Downarrow}=0$.

We now describe the q-Racah case. The TD pair A, A^* is said to have q-Racah type whenever there exist nonzero $a, b, q \in \mathbb{F}$ such that $q^4 \neq 1$ and

$$\theta_i = aq^{d-2i} + a^{-1}q^{2i-d}, \qquad \theta_i^* = bq^{d-2i} + b^{-1}q^{2i-d}$$

for $0 \leq i \leq d$. The name q-Racah comes from a connection to the q-Racah polynomials; see [1], [31, Section 18], [32, Example 5.3]. For the rest of this section, assume that A, A^* has q-Racah type with $d \geq 1$.

In this paper, our purpose is to turn the vector space V into a module for the q-tetrahedron algebra $\boxtimes_q$. Over the next few paragraphs, we will describe the maps that get used and explain what $\boxtimes_q$ is all about.

We introduce an invertible $W \in \mathrm{End}(V)$ such that for $0 \leq i \leq d$, V_i is an eigenspace of W with eigenvalue $(-1)^i a^i q^{i(d-i)}$. We remark that the idea behind W goes back to the spin model concept introduced by V. F. R. Jones [25]. In the interval since then, the idea was developed in the context of association schemes [3,22,23,28], distance-regular graphs [2,13], the subconstituent algebra [9,10], spin Leonard pairs [11], and Leonard triples [12,29,37]. See [30] for a comprehensive description of W in the context of spin models, distance-regular graphs, and spin Leonard pairs. We also remark that W^2 is closely related to the Lusztig automorphism of the q-Onsager algebra [4,38]; indeed $W^2 = H$ where H is from [39, Section 3]. In the present paper, we will obtain a number of identities involving $W^{\pm 1}$; for example

$$W = \sum_{i=0}^{d} \frac{(-1)^i q^{i^2}(A-\theta_0 I)(A-\theta_1 I)\cdots(A-\theta_{i-1}I)}{(q^2;q^2)_i (a^{-1}q^{1-d};q^2)_i},$$
$$W^{-1} = \sum_{i=0}^{d} \frac{(-1)^i q^{-i^2}(A-\theta_0 I)(A-\theta_1 I)\cdots(A-\theta_{i-1}I)}{(q^{-2};q^{-2})_i (aq^{d-1};q^{-2})_i}.$$

For the above sums, the denominator notation is explained in Section 6.

Next we recall the maps K, B. Following [19, Section 1.1], we define $K, B \in \mathrm{End}(V)$ such that for $0 \leq i \leq d$, U_i (resp. $U_i^{\Downarrow}$) is an eigenspace of K (resp. B) with eigenvalue q^{d-2i}. The maps K, B are invertible. By [19, Section 1.1],

$$\frac{qKA - q^{-1}AK}{q-q^{-1}} = aK^2 + a^{-1}I, \qquad \frac{qBA - q^{-1}AB}{q-q^{-1}} = a^{-1}B^2 + aI.$$

By [6, Theorem 9.9],

$$aK^2 - \frac{a^{-1}q - aq^{-1}}{q-q^{-1}}KB - \frac{aq - a^{-1}q^{-1}}{q-q^{-1}}BK + a^{-1}B^2 = 0.$$

We will show that

$$A = aW^{-1}KW + a^{-1}WK^{-1}W^{-1},$$
$$\frac{qW^{-1}KWK^{-1} - q^{-1}K^{-1}W^{-1}KW}{q-q^{-1}} = I,$$
$$\frac{qW^{-2}KW^2K^{-1} - q^{-1}K^{-1}W^{-2}KW^2}{q-q^{-1}} = I$$

and also

$$A = a^{-1}W^{-1}BW + aWB^{-1}W^{-1},$$
$$\frac{qW^{-1}BWB^{-1} - q^{-1}B^{-1}W^{-1}BW}{q - q^{-1}} = I,$$
$$\frac{qW^{-2}BW^2B^{-1} - q^{-1}B^{-1}W^{-2}BW^2}{q - q^{-1}} = I.$$

Next we discuss the maps M, N, Q. Following [7, Section 6] and [39, Section 7] we define

$$M = \frac{aK - a^{-1}B}{a - a^{-1}}, \qquad N = \frac{a^{-1}K^{-1} - aB^{-1}}{a^{-1} - a}.$$

We will show that $W^{-1}MW = WNW^{-1}$; denote this common value by Q. We will show that Q is diagonalizable with eigenvalues $\{q^{d-2i}\}_{i=0}^d$; in particular Q is invertible.

Next we recall the double lowering map ψ, due to Sarah Bockting-Conrad [5]. By [5, Lemma 11.2, Corollary 15.3],

$$\psi U_i \subseteq U_{i-1}, \qquad \psi U_i^{\Downarrow} \subseteq U_{i-1}^{\Downarrow} \qquad (0 \leq i \leq d).$$

By [6, Theorem 9.8], ψ is equal to each of

$$\frac{I - BK^{-1}}{q(aI - a^{-1}BK^{-1})}, \qquad \frac{I - KB^{-1}}{q(a^{-1}I - aKB^{-1})},$$
$$\frac{q(I - K^{-1}B)}{aI - a^{-1}K^{-1}B}, \qquad \frac{q(I - B^{-1}K)}{a^{-1}I - aB^{-1}K}.$$

Expanding on [7, Lemma 6.8], we will show that

$$\psi + \frac{qAM^{-1} - q^{-1}M^{-1}A}{q^2 - q^{-2}} = \frac{a + a^{-1}}{q + q^{-1}}I,$$
$$\psi + \frac{qN^{-1}A - q^{-1}AN^{-1}}{q^2 - q^{-2}} = \frac{a + a^{-1}}{q + q^{-1}}I.$$

We will also show that A commutes with $\psi - Q^{-1}$, and that

$$W\psi W^{-1} + Q^{-1} = \psi + M^{-1}, \qquad W^{-1}\psi W + Q^{-1} = \psi + N^{-1}.$$

Next we recall the Casimir element Λ. Following [6, Lemma 7.2] we define

$$\Lambda = \psi(A - aK - a^{-1}K^{-1}) + q^{-1}K + qK^{-1}.$$

We call Λ the Casimir element, because A, A^* induce a certain action of the quantum group $U_q(\mathfrak{sl}_2)$ on V [6, Lemma 7.1] for which the action of the Casimir element of $U_q(\mathfrak{sl}_2)$ coincides with Λ [6, Lemma 7.2]. It is shown in [6, Lemmas 7.3, 8.3, 9.1] that Λ commutes with each of A, K, B, ψ. By this and the construction, Λ commutes with W, M, N, Q. We will show that

$$M^{-1} + \frac{q\psi A - q^{-1}A\psi}{q^2 - q^{-2}} = \frac{\Lambda}{q + q^{-1}},$$
$$N^{-1} + \frac{qA\psi - q^{-1}\psi A}{q^2 - q^{-2}} = \frac{\Lambda}{q + q^{-1}}.$$

We will also show that

$$(\psi - Q^{-1})((q+q^{-1})I - A) = (a+a^{-1})I - \Lambda.$$

Next we recall the q-tetrahedron algebra $\boxtimes_q$. This infinite-dimensional algebra was introduced in [16], and used to study the TD pairs of q-geometric type. See [17,18,20,27,36,41] for subsequent work. The algebra $\boxtimes_q$ is defined by generators and relations. To describe the generators, let $\mathbb{Z}_4 = \mathbb{Z}/4\mathbb{Z}$ denote the cyclic group of order 4. The algebra $\boxtimes_q$ has eight generators

$$\{x_{ij} \mid i, j \in \mathbb{Z}_4, \ \ j - i = 1 \text{ or } j - i = 2\}.$$

The defining relations for $\boxtimes_q$ are given in Definition 15.1 below. In [16] we described the finite-dimensional irreducible $\boxtimes_q$-modules, under the assumption that q is not a root of unity. As explained in [20, Section 1] these modules are a tensor product of evaluation $\boxtimes_q$-modules. In [20] we described the evaluation $\boxtimes_q$-modules in detail. In the present paper we introduce a type of $\boxtimes_q$-module, said to be t-segregated; here t is a nonzero scalar parameter. These $\boxtimes_q$-modules are a bit more general than the evaluation $\boxtimes_q$-modules; see Section 16 below. We will show that on a t-segregated $\boxtimes_q$-module, the following four elements coincide and commute with everything in $\boxtimes_q$:

$$t(x_{01}x_{23} - 1) + qx_{30} + q^{-1}x_{12}, \qquad t^{-1}(x_{12}x_{30} - 1) + qx_{01} + q^{-1}x_{23},$$
$$t(x_{23}x_{01} - 1) + qx_{12} + q^{-1}x_{30}, \qquad t^{-1}(x_{30}x_{12} - 1) + qx_{23} + q^{-1}x_{01}.$$

Let Υ denote the common value of the above four elements.

We now describe our two main results. In this description, we refer to the above TD pair A, A^* on V that has q-Racah type. In our first main result, we show that V becomes an a-segregated $\boxtimes_q$-module on which the $\boxtimes_q$-generators act as follows:

generator	x_{01}	x_{12}	x_{23}	x_{30}
action on V	$W^{-1}KW$	$WK^{-1}W^{-1}$	$Q^{-1} + W\psi W^{-1}$	$Q^{-1} + W^{-1}\psi W$

generator	x_{02}	x_{13}	x_{20}	x_{31}
action on V	Q	K^{-1}	Q^{-1}	K

Moreover $\Upsilon = \Lambda$ on V. In our second main result, we show that V becomes an a^{-1}-segregated $\boxtimes_q$-module on which the $\boxtimes_q$-generators act as follows:

generator	x_{01}	x_{12}	x_{23}	x_{30}
action on V	$W^{-1}BW$	$WB^{-1}W^{-1}$	$Q^{-1} + W\psi W^{-1}$	$Q^{-1} + W^{-1}\psi W$

generator	x_{02}	x_{13}	x_{20}	x_{31}
action on V	Q	B^{-1}	Q^{-1}	B

Moreover $\Upsilon = \Lambda$ on V.

This paper is organized as follows. In Section 2 we recall the notion of a tridiagonal system. In Section 3 we recall the q-Dolan/Grady relations and discuss their basic properties. In Section 4 we introduce a certain map that makes it easier to discuss elements in $\mathrm{End}(V)$ that commute with A. In Section 5 we introduce the element W, and in Section 6 we display some identities involving $W^{\pm 1}$. In Section 7 we discuss the elements K, B and in Section 8 we discuss the elements M, N, Q. In Section 9 we describe how W, K, B, Q

are related using the concept of an equitable triple. In Section 10 we discuss the double lowering map ψ, and in Section 11 we discuss the Casimir element Λ. In Section 12 we describe the element $\psi - Q$ in some detail. In Section 13 we discuss how W, K are related and how W, B are related. In Section 14 we recall the algebra $U_q(\mathfrak{sl}_2)$ in its equitable presentation. In Section 15 we recall the q-tetrahedron algebra $\boxtimes_q$, and in Sections 16, 17 we discuss the t-segregated $\boxtimes_q$-modules. In Section 18 we give our main results, which are Theorems 18.1, 18.2. In Section 19 we give some suggestions for future research.

2. Tridiagonal systems

We now begin our formal argument. When working with a TD pair, it is often convenient to consider a closely related object called a TD system. We will review this notion after some notational comments. Recall the natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$ and integers $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$. Recall the field $\mathbb{F}$. Every vector space discussed in this paper is over $\mathbb{F}$. Every algebra discussed in this paper is associative, over $\mathbb{F}$, and has a multiplicative identity. A subalgebra has the same multiplicative identity as the parent algebra. For the rest of this paper, V denotes a vector space with finite positive dimension. Recall the algebra $\mathrm{End}(V)$ from above Definition 1.1. Let A denote a diagonalizable element in $\mathrm{End}(V)$. Let $\{V_i\}_{i=0}^d$ denote an ordering of the eigenspaces of A. For $0 \leq i \leq d$ let θ_i denote the eigenvalue of A for V_i. Define $E_i \in \mathrm{End}(V)$ such that $(E_i - I)V_i = 0$ and $E_i V_j = 0$ for $j \neq i$ $(0 \leq j \leq d)$. Thus E_i is the projection from V onto V_i. We call E_i the *primitive idempotent* of A corresponding to V_i (or θ_i). Observe that (i) $V_i = E_i V$ $(0 \leq i \leq d)$; (ii) $E_i E_j = \delta_{i,j} E_i$ $(0 \leq i, j \leq d)$; (iii) $I = \sum_{i=0}^d E_i$; (iv) $A = \sum_{i=0}^d \theta_i E_i$; (v) $AE_i = \theta_i E_i = E_i A$ $(0 \leq i \leq d)$. Moreover

$$E_i = \prod_{\substack{0 \leq j \leq d \\ j \neq i}} \frac{A - \theta_j I}{\theta_i - \theta_j} \qquad (0 \leq i \leq d). \tag{3}$$

Let $\mathcal{D}$ denote the subalgebra of $\mathrm{End}(V)$ generated by A. Note that $\{A^i\}_{i=0}^d$ is a basis for the vector space $\mathcal{D}$, and $\prod_{i=0}^d (A - \theta_i I) = 0$. Moreover $\{E_i\}_{i=0}^d$ is a basis for the vector space $\mathcal{D}$. Now let A, A^* denote a TD pair on V, as in Definition 1.1. An ordering of the eigenspaces of A (resp. A^*) is said to be *standard* whenever it satisfies (1) (resp. (2)). We comment on the uniqueness of the standard ordering. Let $\{V_i\}_{i=0}^d$ denote a standard ordering of the eigenspaces of A. By [15, Lemma 2.4], the ordering $\{V_{d-i}\}_{i=0}^d$ is also standard and no further ordering is standard. A similar result holds for the eigenspaces of A^*. An ordering of the primitive idempotents of A (resp. A^*) is said to be *standard* whenever the corresponding ordering of the eigenspaces of A (resp. A^*) is standard.

Definition 2.1. (See [15, Definition 2.1].) By a *tridiagonal system* (or *TD system*) on V we mean a sequence

$$\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$$

that satisfies (i)–(iii) below:

(i) A, A^* is a TD pair on V;
(ii) $\{E_i\}_{i=0}^d$ is a standard ordering of the primitive idempotents of A;
(iii) $\{E_i^*\}_{i=0}^d$ is a standard ordering of the primitive idempotents of A^*.

The TD system Φ is said to be *over* $\mathbb{F}$. We call V the *underlying vector space*.

Let $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a TD system on V. Then the following is a TD system on V:

$$\Phi^{\Downarrow} = (A; \{E_{d-i}\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d).$$

For any object f attached to Φ, let $f^{\Downarrow}$ denote the corresponding object attached to $\Phi^{\Downarrow}$.

Definition 2.2. Let $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a TD system on V. For $0 \leq i \leq d$ let θ_i (resp. θ_i^*) denote the eigenvalue of A (resp. A^*) for the eigenspace E_iV (resp. E_i^*V). We call $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta_i^*\}_{i=0}^d$) the *eigenvalue sequence* (resp. *dual eigenvalue sequence*) of Φ.

Referring to Definition 2.2, we emphasize that $\{\theta_i\}_{i=0}^d$ are mutually distinct, and $\{\theta_i^*\}_{i=0}^d$ are mutually distinct. By [15, Theorem 11.1] the expressions

$$\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i}, \qquad \frac{\theta_{i-2}^* - \theta_{i+1}^*}{\theta_{i-1}^* - \theta_i^*}$$

are equal and independent of i for $2 \leq i \leq d-1$. For this recurrence the solutions can be expressed in closed form [15, Theorem 11.2]. The "most general" solution is called q-Racah, and described below.

Definition 2.3. Let Φ denote a TD system on V, with eigenvalue sequence $\{\theta_i\}_{i=0}^d$ and dual eigenvalue sequence $\{\theta_i^*\}_{i=0}^d$. Then Φ is said to have *q-Racah type* whenever there exist nonzero $a, b, q \in \mathbb{F}$ such that $q^4 \neq 1$ and

$$\theta_i = aq^{d-2i} + a^{-1}q^{2i-d}, \qquad \theta_i^* = bq^{d-2i} + b^{-1}q^{2i-d} \tag{4}$$

for $0 \leq i \leq d$.

From now until the end of Section 13, we fix a TD system $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ on V that has q-Racah type, with eigenvalue sequence $\{\theta_i\}_{i=0}^d$ and dual eigenvalue sequence $\{\theta_i^*\}_{i=0}^d$ as in Definition 2.3. To avoid trivialities we assume that $d \geq 1$. Let $\mathcal{D}$ denote the subalgebra of $\mathrm{End}(V)$ generated by A.

We mention some basic results for later use.

Lemma 2.4. *The following hold:*

(i) $q^{2i} \neq 1$ *for* $1 \leq i \leq d$;
(ii) *neither of* a^2, b^2 *is among* $q^{2d-2}, q^{2d-4}, \ldots, q^{2-2d}$.

Proof. Use the sentence below Definition 2.2, along with Definition 2.3. □

Lemma 2.5. *For* $1 \leq i \leq d$,

$$\frac{q\theta_{i-1} - q^{-1}\theta_i}{q^2 - q^{-2}} = aq^{d-2i+1}, \qquad \frac{q\theta_i - q^{-1}\theta_{i-1}}{q^2 - q^{-2}} = a^{-1}q^{2i-d-1}.$$

Proof. By the form of the eigenvalue expressions in (4). □

Lemma 2.6. *For* $0 \leq i, j \leq d$ *such that* $|i - j| = 1$,

$$\frac{q\theta_i - q^{-1}\theta_j}{q^2 - q^{-2}} \, \frac{q\theta_j - q^{-1}\theta_i}{q^2 - q^{-2}} = 1. \tag{5}$$

Proof. By Lemma 2.5. □

Let $X \in \mathrm{End}(V)$. Using $I = E_0 + \cdots + E_d$ we have

$$X = IXI = \sum_{i=0}^{d} \sum_{j=0}^{d} E_i X E_j.$$

With this result we routinely obtain the following three lemmas.

Lemma 2.7. *For $X \in \mathrm{End}(V)$ the following are equivalent:*

(i) $E_i X E_j = 0$ *for* $0 \leq i, j \leq d$;
(ii) $X = 0$.

Lemma 2.8. *For $X \in \mathrm{End}(V)$ the following are equivalent:*

(i) $E_i X E_j = 0$ *if* $i \neq j$ $(0 \leq i, j \leq d)$;
(ii) $X E_i V \subseteq E_i V$ *for* $0 \leq i \leq d$;
(iii) A *commutes with* X.

Lemma 2.9. *For $X \in \mathrm{End}(V)$ the following are equivalent:*

(i) $E_i X E_j = 0$ *if* $|i - j| > 1$ $(0 \leq i, j \leq d)$;
(ii) *for* $0 \leq i \leq d$,

$$X E_i V \subseteq E_{i-1} V + E_i V + E_{i+1} V,$$

where $E_{-1} = 0$ *and* $E_{d+1} = 0$.

Definition 2.10. Referring to Lemma 2.9, we say that X *acts on the eigenspaces of A in a tridiagonal fashion* whenever the equivalent conditions (i), (ii) hold.

Example 2.11. The elements I, A, A^* act on the eigenspaces of A in a tridiagonal fashion.

3. The q-Dolan/Grady relations

We continue to discuss the TD system $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ on V that has q-Racah type. In this section we consider how A and A^* are related. Recall the notation

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} \qquad\qquad n \in \mathbb{Z}.$$

For elements X, Y in any algebra, their commutator and q-commutator are given by

$$[X, Y] = XY - YX, \qquad\qquad [X, Y]_q = qXY - q^{-1}YX.$$

Note that

$$[X, [X, Y]_q]_{q^{-1}} = X^2 Y - (q^2 + q^{-2}) XYX + YX^2,$$
$$[X, [X, [X, Y]_q]_{q^{-1}}] = X^3 Y - [3]_q X^2 YX + [3]_q XYX^2 - YX^3.$$

Lemma 3.1. (See [15, Theorem 10.1].) *Referring to the TD system* Φ*,*

$$[A,[A,[A,A^*]_q]_{q^{-1}}] = (q^2-q^{-2})^2[A^*,A], \tag{6}$$

$$[A^*,[A^*,[A^*,A]_q]_{q^{-1}}] = (q^2-q^{-2})^2[A,A^*]. \tag{7}$$

The relations (6), (7) are called the *q-Dolan/Grady relations*. In the following results we explore their meaning.

Lemma 3.2. *For* $X \in \mathrm{End}(V)$ *the following are equivalent:*

(i) $[A,[A,[A,X]_q]_{q^{-1}}] = (q^2-q^{-2})^2[X,A]$;
(ii) A *commutes with*

$$X + \frac{[A,[A,X]_q]_{q^{-1}}}{(q^2-q^{-2})^2}. \tag{8}$$

Proof. By the definition of the commutator map. □

Proposition 3.3. *Let* X *denote an element of* $\mathrm{End}(V)$ *that acts on the eigenspaces of* A *in a tridiagonal fashion. Then* X *satisfies the equivalent conditions* (i), (ii) *in Lemma 3.2.*

Proof. We show that X satisfies Lemma 3.2(ii). Let Δ denote the expression in (8). To show that A commutes with Δ, by Lemma 2.8 it suffices to show that $E_i\Delta E_j = 0$ if $i \neq j$ $(0 \leq i,j \leq d)$. Let i,j be given with $i \neq j$. We have $E_i\Delta E_j = E_iXE_jc_{ij}$ where

$$c_{ij} = 1 + \frac{q\theta_i - q^{-1}\theta_j}{q^2-q^{-2}}\,\frac{q^{-1}\theta_i - q\theta_j}{q^2-q^{-2}}.$$

If $|i-j| > 1$ then $E_iXE_j = 0$. If $|i-j| = 1$ then $c_{ij} = 0$ by (5). In any case $E_i\Delta E_j = 0$. The result follows. □

Later in the paper, we will encounter pairs of elements in $\mathrm{End}(V)$ that are related to each other in the following way.

Proposition 3.4. *For* $X,Y \in \mathrm{End}(V)$ *the following are equivalent:*

(i) X *acts on the eigenspaces of* A *in a tridiagonal fashion, and* A *commutes with*

$$Y + \frac{qXA - q^{-1}AX}{q^2-q^{-2}}. \tag{9}$$

(ii) Y *acts on the eigenspaces of* A *in a tridiagonal fashion, and* A *commutes with*

$$X + \frac{qAY - q^{-1}YA}{q^2-q^{-2}}. \tag{10}$$

Proof. Let C (resp. D) denote the expression in (9) (resp. (10)). Note that

$$X + \frac{[A,[A,X]_q]_{q^{-1}}}{(q^2-q^{-2})^2} = D - \frac{[A,C]_q}{q^2-q^{-2}}, \tag{11}$$

$$Y + \frac{[A,[A,Y]_q]_{q^{-1}}}{(q^2-q^{-2})^2} = C - \frac{[D,A]_q}{q^2-q^{-2}}. \tag{12}$$

(i) $\Rightarrow$ (ii) From the form of C, we see that Y acts on the eigenspaces of A in a tridiagonal fashion. Next we show that A commutes with D. By assumption, X acts on the eigenspaces of A in a tridiagonal fashion. By this and Proposition 3.3, A commutes with the expression on the left in (11). By assumption A commutes with C, so A commutes with $[A,C]_q$. By these comments and (11), A commutes with D.
(ii) $\Rightarrow$ (i) From the form of D, we see that X acts on the eigenspaces of A in a tridiagonal fashion. Next we show that A commutes with C. By assumption, Y acts on the eigenspaces of A in a tridiagonal fashion. By this and Proposition 3.3, A commutes with the expression on the left in (12). By assumption A commutes with D, so A commutes with $[D,A]_q$. By these comments and (12), A commutes with C. □

4. The map $X \mapsto X^\vee$

We continue to discuss the TD system $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ on V that has q-Racah type. In this section, we introduce a map that will make it easier to discuss the elements in $\mathrm{End}(V)$ that commute with A.

Definition 4.1. For $X \in \mathrm{End}(V)$ define

$$X^\vee = \sum_{i=0}^d E_i X E_i. \tag{13}$$

Note that the map $\mathrm{End}(V) \to \mathrm{End}(V)$, $X \to X^\vee$ is $\mathbb{F}$-linear. Note also that A commutes with $X^\vee$ for all $X \in \mathrm{End}(V)$. Next comes a stronger statement.

Lemma 4.2. *For $X \in \mathrm{End}(V)$ the following are equivalent:*

(i) *A commutes with X;*
(ii) $X = X^\vee$.

Proof. By Lemma 2.8. □

Lemma 4.3. *For $X \in \mathrm{End}(V)$ the following* (i)–(v) *hold:*

(i) $(AX)^\vee = AX^\vee$;
(ii) $(XA)^\vee = AX^\vee$;
(iii) $[A,X]^\vee = 0$;
(iv) $([A,X]_q)^\vee = (q-q^{-1})AX^\vee$;
(v) $([A,X]_{q^{-1}})^\vee = -(q-q^{-1})AX^\vee$.

Proof. (i), (ii) For the given equation each side is equal to $\sum_{i=0}^d \theta_i E_i X E_i$.
(iii)–(v) By (i), (ii) above. □

Lemma 4.4. *Let X denote an element of $\mathrm{End}(V)$ that acts on the eigenspaces of A in a tridiagonal fashion. Then*

$$X + \frac{[A,[A,X]_q]_{q^{-1}}}{(q^2-q^{-2})^2} = \left(I - \frac{A^2}{(q+q^{-1})^2}\right) X^\vee. \tag{14}$$

Proof. Let Δ denote the expression on the left in (14). The element A commutes with Δ by Proposition 3.3, so $\Delta = \Delta^\vee$ by Lemma 4.2. By Lemma 4.3, $\Delta^\vee$ is equal to the expression on the right in (14). The result follows. □

5. The element W

We continue to discuss the TD system $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ on V that has q-Racah type. In this section we introduce a certain invertible $W \in \mathrm{End}(V)$, and discuss how W is related to the map $X \to X^\vee$ from Section 4.

Definition 5.1. Define

$$t_i = (-1)^i a^i q^{i(d-i)} \qquad (0 \leq i \leq d).$$

Note that $t_0 = 1$ and $t_d = (-1)^d a^d$.

Note 5.2. We caution the reader that the scalar t_i in Definition 5.1 is a square root of the scalar t_i in [39, Lemma 3.15].

Lemma 5.3. *We have $t_i \neq 0$ for $0 \leq i \leq d$. Moreover*

$$t_i/t_{i-1} = -aq^{d-2i+1} \qquad (1 \leq i \leq d).$$

Proof. By Definition 5.1. □

Lemma 5.4. *For $0 \leq i, j \leq d$ such that $|i-j| = 1$,*

$$\frac{t_j}{t_i} + \frac{q\theta_i - q^{-1}\theta_j}{q^2 - q^{-2}} = 0. \tag{15}$$

Proof. Use Lemmas 2.5, 5.3. □

Definition 5.5. Define

$$W = \sum_{i=0}^d t_i E_i,$$

where the scalars $\{t_i\}_{i=0}^d$ are from Definition 5.1.

Lemma 5.6. *The element W is invertible. Moreover*

$$W^{-1} = \sum_{i=0}^d t_i^{-1} E_i.$$

Proof. By Lemma 5.3 and Definition 5.5. □

Note 5.7. We acknowledge that the element W appeared earlier in the context of spin models [30, Definition 14.2]; spin Leonard pairs [11, Theorem 1.18], [30, Lemma 6.16]; and Leonard triples [12, Lemma 2.10], [29, Definition 16.3], [37, Definition 8.1]. In the context of TD pairs, we have $W^2 = H$ where H is from [39, Section 3].

Lemma 5.8. *We have* $W^{\pm 1} \in \mathcal{D}$.

Proof. By (3) and Definition 5.5. □

By Lemma 5.8 we see that $W^{\pm 1}$ are polynomials in A. These polynomials will be made explicit in Section 6.

Lemma 5.9. *For* $X \in \mathrm{End}(V)$,

$$(W^{-1}XW)^{\vee} = X^{\vee} = (WXW^{-1})^{\vee}.$$

Proof. By Lemma 4.3(i), (ii) and since $W^{\pm 1} \in \mathcal{D}$. □

Proposition 5.10. *Let* X *denote an element of* $\mathrm{End}(V)$ *that acts on the eigenspaces of* A *in a tridiagonal fashion. Then both*

$$W^{-1}XW + \frac{qAX - q^{-1}XA}{q^2 - q^{-2}} = \left(I + \frac{A}{q + q^{-1}}\right) X^{\vee}, \tag{16}$$

$$WXW^{-1} + \frac{qXA - q^{-1}AX}{q^2 - q^{-2}} = \left(I + \frac{A}{q + q^{-1}}\right) X^{\vee}. \tag{17}$$

Proof. We first verify (16). Let Δ denote the expression on the left in (16). We first show that A commutes with Δ. By Lemma 2.8 it suffices to show that $E_i \Delta E_j = 0$ if $i \neq j$ $(0 \leq i, j \leq d)$. Let i, j be given with $i \neq j$. Using the expression on the left in (16) we have $E_i \Delta E_j = E_i X E_j c_{ij}$, where

$$c_{ij} = \frac{t_j}{t_i} + \frac{q\theta_i - q^{-1}\theta_j}{q^2 - q^{-2}}.$$

If $|i - j| > 1$ then $E_i X E_j = 0$. If $|i - j| = 1$ then $c_{ij} = 0$ by (15). In any case $E_i \Delta E_j = 0$. Therefore A commutes with Δ, so $\Delta = \Delta^{\vee}$ by Lemma 4.2. Using Lemmas 4.3(iv), 5.9 we see that $\Delta^{\vee}$ is equal to the expression on the right in (16). We have verified (16). One similarly verifies (17). □

Corollary 5.11. *Let* X *denote an element of* $\mathrm{End}(V)$ *that acts on the eigenspaces of* A *in a tridiagonal fashion. Then*

$$WXW^{-1} - W^{-1}XW = \frac{[A, X]}{q - q^{-1}}. \tag{18}$$

Proof. Subtract (16) from (17). □

The following is a variation on [38, Corollary 2.3].

Corollary 5.12. *Let* X *denote an element of* $\mathrm{End}(V)$ *that acts on the eigenspaces of* A *in a tridiagonal fashion. Then*

$$W^{-2}XW^2 = X + \frac{[A, [A, X]_q]}{(q - q^{-1})(q^2 - q^{-2})}, \tag{19}$$

$$W^2XW^{-2} = X + \frac{[A, [A, X]_{q^{-1}}]}{(q - q^{-1})(q^2 - q^{-2})}. \tag{20}$$

Proof. We first obtain (19). Recall that A commutes with W. In (18), multiply each side on the left by W^{-1} and the right by W. This yields

$$X - W^{-2}XW^2 = \frac{[A, W^{-1}XW]}{q - q^{-1}}. \tag{21}$$

By Proposition 5.10,

$$[A, W^{-1}XW] + \frac{[A, [A, X]_q]}{q^2 - q^{-2}} = 0. \tag{22}$$

Combining (21), (22) we obtain (19). We similarly obtain (20). □

Proposition 5.13. *Let X, Y denote elements of* End(V) *that satisfy the equivalent conditions* (i), (ii) *in Proposition 3.4. Then*

$$WXW^{-1} - Y = X - W^{-1}YW,$$

and this common value commutes with A.

Proof. By (17), A commutes with

$$WXW^{-1} + \frac{qXA - q^{-1}AX}{q^2 - q^{-2}}. \tag{23}$$

Combining (9), (23) we see that A commutes with $WXW^{-1} - Y$. We mentioned earlier that W is a polynomial in A. So W commutes with $WXW^{-1} - Y$. Consequently

$$WXW^{-1} - Y = W^{-1}(WXW^{-1} - Y)W = X - W^{-1}YW. \quad \square$$

We have a comment.

Lemma 5.14. *We have*

(i) $t_i^{\Downarrow} = t_{d-i}/t_d$ *for* $0 \leq i \leq d$;
(ii) $W^{\Downarrow} = t_d^{-1}W$.

Proof. (i) Each side is equal to $(-1)^i a^{-i} q^{i(d-i)}$.
(ii) By Definition 5.5 and the construction,

$$W^{\Downarrow} = \sum_{i=0}^{d} t_i^{\Downarrow} E_{d-i} = t_d^{-1} \sum_{i=0}^{d} t_{d-i} E_{d-i} = t_d^{-1} W. \quad \square$$

6. Some identities involving $W^{\pm 1}$

We continue to discuss the TD system $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ on V that has q-Racah type. Recall the element W from Definition 5.5. In this section we obtain some identities involving $W^{\pm 1}$. Using these identities we express $W^{\pm 1}$ as a polynomial in A.

We recall some notation. For $c, z \in \mathbb{F}$ define

$$(c; z)_n = (1 - c)(1 - cz) \cdots (1 - cz^{n-1}) \qquad (n \in \mathbb{N}).$$

We will be discussing basic hypergeometric series, using the notation of [14,26].

Lemma 6.1. *For $0 \leq r \leq s \leq d$,*

$$\frac{t_s}{t_r} = \sum_{i=0}^{s-r} \frac{(-1)^i q^{i^2} (\theta_s - \theta_r)(\theta_s - \theta_{r+1}) \cdots (\theta_s - \theta_{r+i-1})}{(q^2;q^2)_i (a^{-1} q^{2r+1-d}; q^2)_i}, \tag{24}$$

$$\frac{t_r}{t_s} = \sum_{i=0}^{s-r} \frac{(-1)^i q^{-i^2} (\theta_s - \theta_r)(\theta_s - \theta_{r+1}) \cdots (\theta_s - \theta_{r+i-1})}{(q^{-2};q^{-2})_i (a q^{d-2r-1}; q^{-2})_i}. \tag{25}$$

Proof. To verify (24), evaluate the left-hand side using Definition 5.1 and the right-hand side using (4). The result becomes a special case of the basic Chu/Vandermonde summation formula [14, p. 354]:

$$(-1)^{s-r} a^{s-r} q^{(s-r)(d-r-s)} = {}_2\phi_1 \left(\begin{matrix} q^{2s-2r}, a^2 q^{2d-2r-2s} \\ a q^{d-2r-1} \end{matrix} \middle| q^{-2}; q^{-2} \right).$$

We have verified (24). To obtain (25) from (24), replace $q \mapsto q^{-1}$ and $a \mapsto a^{-1}$. □

Proposition 6.2. *For $0 \leq r \leq d$ the following holds on $E_r V + E_{r+1} V + \cdots + E_d V$:*

$$W = t_r \sum_{i=0}^{d-r} \frac{(-1)^i q^{i^2} (A - \theta_r I)(A - \theta_{r+1} I) \cdots (A - \theta_{r+i-1} I)}{(q^2;q^2)_i (a^{-1} q^{2r+1-d}; q^2)_i}, \tag{26}$$

$$W^{-1} = t_r^{-1} \sum_{i=0}^{d-r} \frac{(-1)^i q^{-i^2} (A - \theta_r I)(A - \theta_{r+1} I) \cdots (A - \theta_{r+i-1} I)}{(q^{-2};q^{-2})_i (a q^{d-2r-1}; q^{-2})_i}. \tag{27}$$

Proof. To verify (26), use Definition 5.5 and (24) to see that for $r \leq s \leq d$ the $E_s V$-eigenvalue for either side of (26) is equal to t_s. We have verified (26). To verify (27), use Lemma 5.6 and (25) to see that for $r \leq s \leq d$ the $E_s V$-eigenvalue for either side of (27) is equal to t_s^{-1}. We have verified (27). □

Proposition 6.3. *The following holds on V:*

$$W = \sum_{i=0}^{d} \frac{(-1)^i q^{i^2} (A - \theta_0 I)(A - \theta_1 I) \cdots (A - \theta_{i-1} I)}{(q^2;q^2)_i (a^{-1} q^{1-d}; q^2)_i},$$

$$W^{-1} = \sum_{i=0}^{d} \frac{(-1)^i q^{-i^2} (A - \theta_0 I)(A - \theta_1 I) \cdots (A - \theta_{i-1} I)}{(q^{-2};q^{-2})_i (a q^{d-1}; q^{-2})_i}.$$

Proof. Set $r = 0$ in Proposition 6.2. □

We mention a variation on Lemma 6.1 and Propositions 6.2, 6.3.

Lemma 6.4. *For $0 \leq r \leq s \leq d$,*

$$\frac{t_r}{t_s} = \sum_{i=0}^{s-r} \frac{(-1)^i q^{i^2} (\theta_r - \theta_s)(\theta_r - \theta_{s-1}) \cdots (\theta_r - \theta_{s-i+1})}{(q^2;q^2)_i (a q^{d-2s+1}; q^2)_i} \tag{28}$$

$$\frac{t_s}{t_r} = \sum_{i=0}^{s-r} \frac{(-1)^i q^{-i^2} (\theta_r - \theta_s)(\theta_r - \theta_{s-1}) \cdots (\theta_r - \theta_{s-i+1})}{(q^{-2};q^{-2})_i (a^{-1} q^{2s-d-1}; q^{-2})_i} \tag{29}$$

Proof. To verify (28), evaluate the left-hand side using Definition 5.1 and the right-hand side using (4). The result becomes a special case of the basic Chu/Vandermonde summation formula [14, p. 354]:

$$(-1)^{s-r}a^{r-s}q^{(r-s)(d-r-s)} = {}_2\phi_1\left(\begin{matrix} q^{2s-2r}, a^{-2}q^{2r+2s-2d} \\ a^{-1}q^{2s-d-1} \end{matrix} \middle| q^{-2}; q^{-2}\right).$$

We have verified (28). To obtain (29) from (28), replace $q \mapsto q^{-1}$ and $a \mapsto a^{-1}$. □

Proposition 6.5. *For $0 \leq s \leq d$ the following holds on $E_0V + E_1V + \cdots + E_sV$:*

$$W = t_s \sum_{i=0}^{s} \frac{(-1)^i q^{i^2}(A-\theta_s I)(A-\theta_{s-1}I)\cdots(A-\theta_{s-i+1}I)}{(q^2;q^2)_i (aq^{d-2s+1};q^2)_i},$$

$$W^{-1} = t_s^{-1} \sum_{i=0}^{s} \frac{(-1)^i q^{-i^2}(A-\theta_s I)(A-\theta_{s-1}I)\cdots(A-\theta_{s-i+1}I)}{(q^{-2};q^{-2})_i (a^{-1}q^{2s-d-1};q^{-2})_i}.$$

Proof. Similar to the proof of Proposition 6.2. □

Proposition 6.6. *The following holds on V:*

$$W = t_d \sum_{i=0}^{d} \frac{(-1)^i q^{i^2}(A-\theta_d I)(A-\theta_{d-1}I)\cdots(A-\theta_{d-i+1}I)}{(q^2;q^2)_i (aq^{1-d};q^2)_i},$$

$$W^{-1} = t_d^{-1} \sum_{i=0}^{d} \frac{(-1)^i q^{-i^2}(A-\theta_d I)(A-\theta_{d-1}I)\cdots(A-\theta_{d-i+1}I)}{(q^{-2};q^{-2})_i (a^{-1}q^{d-1};q^{-2})_i}.$$

Proof. Set $s = d$ in Proposition 6.5. □

7. The elements K, B

We continue to discuss the TD system $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ on V that has q-Racah type. In this section we recall the elements $K, B \in \mathrm{End}(V)$ and discuss their basic properties.

Definition 7.1. By a *decomposition of V* we mean a sequence $\{V_i\}_{i=0}^d$ of nonzero subspaces whose direct sum is V.

For $0 \leq i \leq d$ define

$$U_i = (E_0^*V + E_1^*V + \cdots + E_i^*V) \cap (E_iV + E_{i+1}V + \cdots + E_dV).$$

By [15, Theorem 4.6] the sequence $\{U_i\}_{i=0}^d$ is a decomposition of V. We call $\{U_i\}_{i=0}^d$ the *first split decomposition* of V. By [15, Theorem 4.6] the following hold for $0 \leq i \leq d$:

$$E_0^*V + E_1^*V + \cdots + E_i^*V = U_0 + U_1 + \cdots + U_i, \tag{30}$$

$$E_iV + E_{i+1}V + \cdots + E_dV = U_i + U_{i+1} + \cdots + U_d. \tag{31}$$

Also by [15, Theorem 4.6],

$$(A-\theta_i I)U_i \subseteq U_{i+1} \qquad (0 \leq i \leq d-1), \quad (A-\theta_d I)U_d = 0, \tag{32}$$

$$(A^*-\theta_i^* I)U_i \subseteq U_{i-1} \qquad (1 \leq i \leq d), \quad (A^*-\theta_0^* I)U_0 = 0. \tag{33}$$

By these remarks and our comments above Definition 2.2, we obtain the following. For $0 \leq i \leq d$,

$$U_i^{\Downarrow} = (E_0^* V + E_1^* V + \cdots + E_i^* V) \cap (E_0 V + E_1 V + \cdots + E_{d-i} V).$$

The sequence $\{U_i^{\Downarrow}\}_{i=0}^d$ is a decomposition of V. We call $\{U_i^{\Downarrow}\}_{i=0}^d$ the *second split decomposition* of V. For $0 \leq i \leq d$,

$$E_0^* V + E_1^* V + \cdots + E_i^* V = U_0^{\Downarrow} + U_1^{\Downarrow} + \cdots + U_i^{\Downarrow}, \tag{34}$$

$$E_0 V + E_1 V + \cdots + E_{d-i} V = U_i^{\Downarrow} + U_{i+1}^{\Downarrow} + \cdots + U_d^{\Downarrow}. \tag{35}$$

We have

$$(A - \theta_{d-i} I) U_i^{\Downarrow} \subseteq U_{i+1}^{\Downarrow} \qquad (0 \leq i \leq d-1), \quad (A - \theta_0 I) U_d^{\Downarrow} = 0,$$
$$(A^* - \theta_i^* I) U_i^{\Downarrow} \subseteq U_{i-1}^{\Downarrow} \qquad (1 \leq i \leq d), \quad (A^* - \theta_0^* I) U_0^{\Downarrow} = 0.$$

Definition 7.2. (See [19, Section 1.1].) Define $K \in \mathrm{End}(V)$ such that for $0 \leq i \leq d$, U_i is an eigenspace for K with eigenvalue q^{d-2i}. Define $B = K^{\Downarrow}$. So for $0 \leq i \leq d$, $U_i^{\Downarrow}$ is an eigenspace for B with eigenvalue q^{d-2i}.

By construction K, B are invertible. The elements A, K, B are related as follows. By [19, Section 1.1],

$$\frac{qKA - q^{-1}AK}{q - q^{-1}} = aK^2 + a^{-1} I, \qquad \frac{qBA - q^{-1}AB}{q - q^{-1}} = a^{-1} B^2 + aI. \tag{36}$$

By [6, Theorem 9.9],

$$aK^2 - \frac{a^{-1}q - aq^{-1}}{q - q^{-1}} KB - \frac{aq - a^{-1}q^{-1}}{q - q^{-1}} BK + a^{-1} B^2 = 0. \tag{37}$$

The equations (36), (37) can be reformulated as follows. By [8, Lemma 12.12],

$$\frac{qAK^{-1} - q^{-1}K^{-1}A}{q - q^{-1}} = a^{-1}K^{-2} + aI, \qquad \frac{qAB^{-1} - q^{-1}B^{-1}A}{q - q^{-1}} = aB^{-2} + a^{-1} I. \tag{38}$$

By [6, Theorem 9.10],

$$a^{-1}K^{-2} - \frac{a^{-1}q - aq^{-1}}{q - q^{-1}} K^{-1}B^{-1} - \frac{aq - a^{-1}q^{-1}}{q - q^{-1}} B^{-1}K^{-1} + aB^{-2} = 0. \tag{39}$$

We now bring in W.

Lemma 7.3. (See [39, Proposition 6.1].) *We have*

$$W^{-2}KW^2 = a^{-1}A - a^{-2}K^{-1}, \qquad W^{-2}BW^2 = aA - a^2 B^{-1}, \tag{40}$$

$$W^2 K^{-1} W^{-2} = aA - a^2 K, \qquad W^2 B^{-1} W^{-2} = a^{-1}A - a^{-2}B. \tag{41}$$

Proposition 7.4. *We have*

$$\frac{qW^{-2}KW^2K^{-1} - q^{-1}K^{-1}W^{-2}KW^2}{q - q^{-1}} = I, \tag{42}$$

$$\frac{qKW^2K^{-1}W^{-2} - q^{-1}W^2K^{-1}W^{-2}K}{q - q^{-1}} = I \tag{43}$$

and

$$\frac{qW^{-2}BW^2B^{-1} - q^{-1}B^{-1}W^{-2}BW^2}{q - q^{-1}} = I, \tag{44}$$

$$\frac{qBW^2B^{-1}W^{-2} - q^{-1}W^2B^{-1}W^{-2}B}{q - q^{-1}} = I. \tag{45}$$

Proof. To verify (42), eliminate $W^{-2}KW^2$ using the equation on the left in (40), and evaluate the result using the equation on the left in (38). To obtain (43), multiply each side of (42) on the left by W^2 and the right by W^{-2}. The equations (44), (45) are similarly verified. □

Proposition 7.5. *We have*

(i) $A = aW^{-1}KW + a^{-1}WK^{-1}W^{-1}$;
(ii) $A = a^{-1}W^{-1}BW + aWB^{-1}W^{-1}$.

Proof. (i) In the equation $W^{-2}KW^2 = a^{-1}A - a^{-2}K^{-1}$, multiply each side on the left by W and the right by W^{-1}. Evaluate the result using the fact that A, W commute.
(ii) In the equation $W^{-2}BW^2 = aA - a^2B^{-1}$, multiply each side on the left by W and the right by W^{-1}. Evaluate the result using the fact that A, W commute. □

Corollary 7.6. *We have*

$$W^{-1}\frac{aK - a^{-1}B}{a - a^{-1}}W = W\frac{a^{-1}K^{-1} - aB^{-1}}{a^{-1} - a}W^{-1}. \tag{46}$$

Proof. Compare the two equations in Proposition 7.5. □

8. The elements M, N, Q

We continue to discuss the TD system $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ on V that has q-Racah type. Recall the maps K, B from Definition 7.2. In this section we use K, B to define some elements $M, N, Q \in \mathrm{End}(V)$ that will play a role in our theory.

Following [7, Section 6] and [39, Section 7] we define

$$M = \frac{aK - a^{-1}B}{a - a^{-1}}, \qquad N = \frac{a^{-1}K^{-1} - aB^{-1}}{a^{-1} - a}. \tag{47}$$

By [7, Lemma 8.1], M is diagonalizable with eigenvalues $\{q^{d-2i}\}_{i=0}^d$. The same holds for N by Corollary 7.6. The elements M, N are invertible. By construction $M^{\Downarrow} = M$ and $N^{\Downarrow} = N$. Also by construction,

$$KNB = M = BNK. \tag{48}$$

Lemma 8.1. *Each of M^{-1}, N^{-1} acts on the eigenspaces of A in a tridiagonal fashion.*

Proof. This holds for M^{-1} by [7, Lemma 10.3]. It holds for N^{-1}, by Corollary 7.6 and since A commutes with W. □

Definition 8.2. By Corollary 7.6 we have

$$W^{-1}MW = WNW^{-1}; \tag{49}$$

this common value will be denoted by Q.

Lemma 8.3. *The element Q is diagonalizable, with eigenvalues $\{q^{d-2i}\}_{i=0}^d$. Moreover Q is invertible.*

Proof. By Definition 8.2 and the comments above Lemma 8.1. □

Lemma 8.4. *We have $Q^{\Downarrow} = Q$.*

Proof. By Lemma 5.14(ii) and Definition 8.2, along with the last sentence before Lemma 8.1. □

Lemma 8.5. *The element Q^{-1} acts on the eigenspaces of A in a tridiagonal fashion.*

Proof. By Lemma 8.1, Definition 8.2, and since A commutes with W. □

9. Equitable triples

We continue to discuss the TD system $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ on V that has q-Racah type. The two equations in Proposition 7.5 each express A as a sum of two terms. In this section we describe how these terms are related to the element Q from Definition 8.2. To facilitate this description, we will use the notion of an equitable triple.

Definition 9.1. (See [39, Definition 7.3].) An *equitable triple on V* is a 3-tuple X, Y, Z of invertible elements in $\mathrm{End}(V)$ such that

$$\frac{qXY - q^{-1}YX}{q - q^{-1}} = I, \qquad \frac{qYZ - q^{-1}ZY}{q - q^{-1}} = I, \qquad \frac{qZX - q^{-1}XZ}{q - q^{-1}} = I.$$

Equitable triples are related to the quantum group $U_q(\mathfrak{sl}_2)$; see Section 14 below and [21], [33], [34].

Lemma 9.2. (See [39, Proposition 7.4].) *Each of the following* (i)–(iv) *is an equitable triple:*

(i) $aA - a^2K, M^{-1}, K$;
(ii) $a^{-1}A - a^{-2}B, M^{-1}, B$;
(iii) $K^{-1}, N^{-1}, a^{-1}A - a^{-2}K^{-1}$;
(iv) $B^{-1}, N^{-1}, aA - a^2B^{-1}$.

Proposition 9.3. *Each of the following* (i), (ii) *is an equitable triple:*

(i) $WK^{-1}W^{-1}, Q^{-1}, W^{-1}KW$;
(ii) $WB^{-1}W^{-1}, Q^{-1}, W^{-1}BW$.

Proof. (i) Define $X = aA - a^2K$, $Y = M^{-1}$, $Z = K$. By Lemma 9.2(i) the three-tuple X, Y, Z is an equitable triple. Therefore the three-tuple $W^{-1}XW$, $W^{-1}YW$, $W^{-1}ZW$ is an equitable triple. Using Proposition 7.5(i) we obtain $W^{-1}XW = WK^{-1}W^{-1}$. By construction $W^{-1}YW = Q^{-1}$ and $W^{-1}ZW = W^{-1}KW$. The result follows.
(ii) Similar to the proof of (i) above, using the equitable triple from Lemma 9.2(ii). □

10. The double lowering map ψ

We continue to discuss the TD system $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ on V that has q-Racah type. In this section we recall the double lowering map ψ and discuss its basic properties.

Recall the maps K, B from Definition 7.2. By [6, Lemma 9.7], each of the following is invertible:

$$aI - a^{-1}BK^{-1}, \qquad a^{-1}I - aKB^{-1},$$
$$aI - a^{-1}K^{-1}B, \qquad a^{-1}I - aB^{-1}K.$$

Lemma 10.1. (See [6, Theorem 9.8].) *The following coincide:*

$$\frac{I - BK^{-1}}{q(aI - a^{-1}BK^{-1})}, \qquad \frac{I - KB^{-1}}{q(a^{-1}I - aKB^{-1})},$$
$$\frac{q(I - K^{-1}B)}{aI - a^{-1}K^{-1}B}, \qquad \frac{q(I - B^{-1}K)}{a^{-1}I - aB^{-1}K}.$$

Definition 10.2. (See [7, Definition 4.3].) Define $\psi \in \mathrm{End}(V)$ to be the common value of the four expressions in Lemma 10.1.

Lemma 10.3. (See [6, Lemma 5.4].) *Both*

$$K\psi = q^2\psi K, \qquad B\psi = q^2\psi B.$$

Lemma 10.4. (See [5, Corollary 15.2].) *We have* $\psi^{\Downarrow} = \psi$.

Lemma 10.5. (See [5, Lemma 11.2, Corollary 15.3].) *We have*

$$\psi U_i \subseteq U_{i-1}, \qquad \psi U_i^{\Downarrow} \subseteq U_{i-1}^{\Downarrow} \qquad (0 \leq i \leq d),$$

where $U_{-1} = 0$ *and* $U_{-1}^{\Downarrow} = 0$.

Motivated by Lemma 10.5, the map ψ is often called the *double lowering map* for Φ.

Lemma 10.6. (See [5, Corollary 15.4].) *The element* ψ *acts on the eigenspaces of* A *in a tridiagonal fashion.*

Lemma 10.7. (See [7, Lemma 6.4].) *The element* M^{-1} *is equal to each of the following:*

$$K^{-1}(I - a^{-1}q\psi), \qquad (I - a^{-1}q^{-1}\psi)K^{-1},$$
$$B^{-1}(I - aq\psi), \qquad (I - aq^{-1}\psi)B^{-1}.$$

Lemma 10.8. *The element* N^{-1} *is equal to each of the following:*

$$K(I - aq^{-1}\psi), \qquad (I - aq\psi)K,$$
$$B(I - a^{-1}q^{-1}\psi), \qquad (I - a^{-1}q\psi)B.$$

Proof. Use (48) and Lemma 10.7. □

The equation (50) below appears in [7, Lemma 6.8]; we will give a short proof for the sake of completeness.

Proposition 10.9. *We have*

$$\psi + \frac{qAM^{-1} - q^{-1}M^{-1}A}{q^2 - q^{-2}} = \frac{a + a^{-1}}{q + q^{-1}} I, \tag{50}$$

$$\psi + \frac{qN^{-1}A - q^{-1}AN^{-1}}{q^2 - q^{-2}} = \frac{a + a^{-1}}{q + q^{-1}} I. \tag{51}$$

Proof. We first obtain (50). Abbreviate $X = aA - a^2K$ and $Y = M^{-1}$. The elements X, Y are the first two terms in the equitable triple from Lemma 9.2(i). So $qXY - q^{-1}YX = (q - q^{-1})I$. In this equation, eliminate the products KM^{-1}, $M^{-1}K$ using the equations $KM^{-1} = 1 - a^{-1}q\psi$ and $M^{-1}K = 1 - a^{-1}q^{-1}\psi$ from Lemma 10.7. This yields (50). The equation (51) is similarly obtained, using the last two terms in the equitable triple from Lemma 9.2(iii). □

11. The Casimir element Λ

We continue to discuss the TD system $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ on V that has q-Racah type. In this section we recall the Casimir element Λ, and discuss its basic properties.

Lemma 11.1. (See [6, Lemmas 7.2, 8.2, 9.1].) *The following coincide:*

$$\psi(A - aK - a^{-1}K^{-1}) + q^{-1}K + qK^{-1},$$
$$(A - aK - a^{-1}K^{-1})\psi + qK + q^{-1}K^{-1},$$
$$\psi(A - a^{-1}B - aB^{-1}) + q^{-1}B + qB^{-1},$$
$$(A - a^{-1}B - aB^{-1})\psi + qB + q^{-1}B^{-1}.$$

Definition 11.2. Let Λ denote the common value of the four expressions in Lemma 11.1.

Lemma 11.3. *The element Λ commutes with each of $A, W, K, B, M, N, Q, \psi$.*

Proof. It was shown in [6, Lemma 7.3, 8.3, 9.1] that Λ commutes with A, K, B, ψ. Now Λ commutes with W, M, N, Q by Lemma 5.8, line (47), and Definition 8.2. □

Motivated by [6, Lemmas 7.1,7.2] and Lemma 11.3, we call Λ the *Casimir element* for Φ.

Lemma 11.4. *We have $\Lambda^{\Downarrow} = \Lambda$.*

Proof. By Lemmas 10.4, 11.1 and $K^{\Downarrow} = B$. □

Lemma 11.5. *We have*

(i) $A\psi = \Lambda - qN^{-1} - q^{-1}M^{-1}$;
(ii) $\psi A = \Lambda - q^{-1}N^{-1} - qM^{-1}$.

Proof. (i) By Definition 11.2 we have $\Lambda = (A - aK - a^{-1}K^{-1})\psi + qK + q^{-1}K^{-1}$; evaluate this equation using $M^{-1} = K^{-1}(1 - a^{-1}q\psi)$ and $N^{-1} = K(1 - aq^{-1}\psi)$.
(ii) Similar to the proof of (i) above. □

Proposition 11.6. *We have*

$$M^{-1} + \frac{q\psi A - q^{-1}A\psi}{q^2 - q^{-2}} = \frac{\Lambda}{q + q^{-1}}, \tag{52}$$

$$N^{-1} + \frac{qA\psi - q^{-1}\psi A}{q^2 - q^{-2}} = \frac{\Lambda}{q + q^{-1}}. \tag{53}$$

Proof. Use Lemma 11.5. □

12. The element $\psi - Q^{-1}$

We continue to discuss the TD system $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ on V that has q-Racah type. In this section we investigate the element $\psi - Q^{-1}$, where Q is from Definition 8.2 and ψ is from Definition 10.2.

Lemma 12.1. *Each of the following pairs satisfy the equivalent conditions* (i), (ii) *in Proposition 3.4:*

(i) ψ, M^{-1};
(ii) N^{-1}, ψ.

Proof. By Lemmas 8.1, 10.6, 11.3 and Propositions 10.9, 11.6. □

Proposition 12.2. *The element A commutes with $\psi - Q^{-1}$. Moreover,*

$$W\psi W^{-1} + Q^{-1} = \psi + M^{-1}, \qquad W^{-1}\psi W + Q^{-1} = \psi + N^{-1}. \tag{54}$$

Proof. By Proposition 5.13, Definition 8.2, and Lemma 12.1. □

Proposition 12.3. *We have*

$$(\psi - Q^{-1})((q + q^{-1})I - A) = (a + a^{-1})I - \Lambda.$$

Proof. The element A commutes with I and Λ. So by Lemma 4.2, $I = I^\vee$ and $\Lambda = \Lambda^\vee$. By Proposition 12.2, A commutes with $\psi - Q^{-1}$. So by Lemma 4.2,

$$\psi - Q^{-1} = \psi^\vee - (Q^{-1})^\vee. \tag{55}$$

By Lemma 5.9 and Definition 8.2,

$$(Q^{-1})^\vee = (M^{-1})^\vee. \tag{56}$$

For the equation (50), apply the map $\vee$ to each side and evaluate the result using Lemma 4.3 along with $I = I^\vee$; this yields

$$\psi^\vee + \frac{A(M^{-1})^\vee}{q + q^{-1}} = \frac{a + a^{-1}}{q + q^{-1}} I. \tag{57}$$

For the equation (52), apply the map $\vee$ to each side and evaluate the result using Lemma 4.3 along with $\Lambda = \Lambda^\vee$; this yields

$$(M^{-1})^\vee + \frac{A\psi^\vee}{q + q^{-1}} = \frac{\Lambda}{q + q^{-1}}. \tag{58}$$

To finish the proof, subtract (58) from (57) and evaluate the result using (55), (56). □

13. How W, K are related and how W, B are related

We continue to discuss the TD system $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ on V that has q-Racah type. In Proposition 7.4 we showed how W^2, K are related and how W^2, B are related. In the present section we show how W, K are related and how W, B are related. We will use an identity from Section 6.

Proposition 13.1. *We have*

$$aW^{-1}KW - qI = K(aI - qWK^{-1}W^{-1}), \tag{59}$$

$$aW^{-1}KW - q^{-1}I = (aI - q^{-1}WK^{-1}W^{-1})K. \tag{60}$$

Proof. We first obtain (59). To this end, it is convenient to make a change of variables. In (59), eliminate $W^{-1}KW$ using Proposition 7.5(i), and in the result eliminate A using

$$R = A - aK - a^{-1}K^{-1}. \tag{61}$$

This yields

$$(a^{-1}I - qK)(WK^{-1} - K^{-1}W) = RW. \tag{62}$$

We will verify (62) after a few comments. Recall the first split decomposition $\{U_i\}_{i=0}^d$ of V. By Definition 7.2, $K = q^{d-2i}I$ on U_i for $0 \leq i \leq d$. So for $0 \leq i \leq d$ the following holds on U_i:

$$aK + a^{-1}K^{-1} = \theta_i I. \tag{63}$$

By (61), (63) we find that for $0 \leq i \leq d$ the following holds on U_i:

$$R = A - \theta_i I. \tag{64}$$

By (32) and (64),

$$RU_i \subseteq U_{i+1} \qquad (0 \leq i \leq d-1), \qquad RU_d = 0. \tag{65}$$

By (65) and the construction,

$$RK = q^2 KR.$$

For $0 \leq r \leq d$ we show that (62) holds on U_r. Using (64), (65) we find that for $0 \leq i \leq d - r$ the following holds on U_r:

$$R^i = (A - \theta_r I)(A - \theta_{r+1} I) \cdots (A - \theta_{r+i-1} I). \tag{66}$$

Also by (65) we have $R^{d-r+1} = 0$ on U_r. By (31),

$$E_r V + E_{r+1} V + \cdots + E_d V = U_r + U_{r+1} + \cdots + U_d.$$

The above subspace contains U_r, so by Proposition 6.2 and (66) the following holds on U_r:

$$W = t_r \sum_{i=0}^{d-r} \frac{(-1)^i q^{i^2} R^i}{(q^2; q^2)_i (a^{-1} q^{2r+1-d}; q^2)_i}.$$

We may now argue that on U_r,

$$\begin{aligned}
(a^{-1}I - qK)(WK^{-1} - K^{-1}W) &= (a^{-1}I - qK)t_r \sum_{i=0}^{d-r} \frac{(-1)^i q^{i^2}(R^iK^{-1} - K^{-1}R^i)}{(q^2;q^2)_i(a^{-1}q^{2r+1-d};q^2)_i} \\
&= (a^{-1}I - qK)t_r \sum_{i=0}^{d-r} \frac{(-1)^i q^{i^2} R^iK^{-1}(1-q^{2i})}{(q^2;q^2)_i(a^{-1}q^{2r+1-d};q^2)_i} \\
&= (a^{-1}I - qK)t_r \sum_{i=1}^{d-r} \frac{(-1)^i q^{i^2} R^iK^{-1}(1-q^{2i})}{(q^2;q^2)_i(a^{-1}q^{2r+1-d};q^2)_i} \\
&= t_r \sum_{i=1}^{d-r} \frac{(-1)^i q^{i^2} R^i(a^{-1}I - q^{1-2i}K)K^{-1}(1-q^{2i})}{(q^2;q^2)_i(a^{-1}q^{2r+1-d};q^2)_i} \\
&= -t_r \sum_{i=1}^{d-r} \frac{(-1)^i q^{i^2} R^i q^{1-2i}(1-a^{-1}q^{2r-d+2i-1})(1-q^{2i})}{(q^2;q^2)_i(a^{-1}q^{2r+1-d};q^2)_i} \\
&= t_r \sum_{i=1}^{d-r} \frac{(-1)^{(i-1)} q^{(i-1)^2} R^i}{(q^2;q^2)_{i-1}(a^{-1}q^{2r+1-d};q^2)_{i-1}} \\
&= t_r \sum_{i=0}^{d-r-1} \frac{(-1)^i q^{i^2} R^{i+1}}{(q^2;q^2)_i(a^{-1}q^{2r+1-d};q^2)_i} \\
&= t_r \sum_{i=0}^{d-r} \frac{(-1)^i q^{i^2} R^{i+1}}{(q^2;q^2)_i(a^{-1}q^{2r+1-d};q^2)_i} \\
&= RW.
\end{aligned}$$

We have obtained (62), and (59) follows. Next we obtain (60). Let E denote the equation obtained by adding (59) to the equation in Proposition 7.5(i). Then $W^{-1}EW$ minus the equation in Proposition 7.5(i) is equal to (60) times $a^{-1}qK^{-1}$. This gives (60). □

Corollary 13.2. *We have*

$$\frac{qW^{-1}KWK^{-1} - q^{-1}K^{-1}W^{-1}KW}{q - q^{-1}} = I, \tag{67}$$

$$\frac{qKWK^{-1}W^{-1} - q^{-1}WK^{-1}W^{-1}K}{q - q^{-1}} = I. \tag{68}$$

Proof. To obtain (68), subtract (59) from (60). To obtain (67), multiply each side of (68) on the left by W^{-1} and the right by W. □

Proposition 13.3. *We have*

$$aI - qW^{-1}BW = (aWB^{-1}W^{-1} - qI)B, \tag{69}$$

$$aI - q^{-1}W^{-1}BW = B(aWB^{-1}W^{-1} - q^{-1}I). \tag{70}$$

Proof. Apply Proposition 13.1 to $\Phi^{\Downarrow}$, and use $K^{\Downarrow} = B$ along with Lemma 5.14(ii). □

Corollary 13.4. *We have*

$$\frac{qW^{-1}BWB^{-1} - q^{-1}B^{-1}W^{-1}BW}{q - q^{-1}} = I, \tag{71}$$

$$\frac{qBWB^{-1}W^{-1} - q^{-1}WB^{-1}W^{-1}B}{q - q^{-1}} = I. \tag{72}$$

Proof. Apply Corollary 13.2 to $\Phi^{\Downarrow}$, and use $K^{\Downarrow} = B$ along with Lemma 5.14(ii). □

14. The algebra $U_q(\mathfrak{sl}_2)$

In the previous sections we discussed a TD system Φ of q-Racah type. For the next four sections, we turn our attention to some algebras and their modules. In Section 18 we will return our attention to Φ. From now until the end of Section 17, fix $0 \neq q \in \mathbb{F}$ such that $q^4 \neq 1$. In this section we recall the algebra $U_q(\mathfrak{sl}_2)$ in its equitable presentation. For more information on this presentation, see [21,33–35].

Definition 14.1. (See [21, Section 2].) The algebra $U_q(\mathfrak{sl}_2)$ is defined by generators $x, y^{\pm 1}, z$ and relations $yy^{-1} = 1 = y^{-1}y$,

$$\frac{qxy - q^{-1}yx}{q - q^{-1}} = 1, \qquad \frac{qyz - q^{-1}zy}{q - q^{-1}} = 1, \qquad \frac{qzx - q^{-1}xz}{q - q^{-1}} = 1. \tag{73}$$

We call $x, y^{\pm 1}, z$ the *equitable generators* of $U_q(\mathfrak{sl}_2)$.

Lemma 14.2. (See [33, Lemma 2.15].) *The following coincide:*

$$qx + q^{-1}y + qz - qxyz, \qquad q^{-1}x + qy + q^{-1}z - q^{-1}zyx,$$
$$qy + q^{-1}z + qx - qyzx, \qquad q^{-1}y + qz + q^{-1}x - q^{-1}xzy,$$
$$qz + q^{-1}x + qy - qzxy, \qquad q^{-1}z + qx + q^{-1}y - q^{-1}yxz.$$

Definition 14.3. Let $\mathbf{\Lambda}$ denote the common value of the six expressions in Lemma 14.2. We call $\mathbf{\Lambda}$ the *Casimir element* of $U_q(\mathfrak{sl}_2)$.

Lemma 14.4. *The element $\mathbf{\Lambda}$ generates the center of $U_q(\mathfrak{sl}_2)$. Moreover $\{\mathbf{\Lambda}^i\}_{i \in \mathbb{N}}$ forms a basis for this center, provided that q is not a root of unity.*

Proof. By [24, Lemma 2.7, Proposition 2.18] and [33, Lemma 2.15]. □

Next we discuss the elements ν_x, ν_y, ν_z of $U_q(\mathfrak{sl}_2)$. Rearranging the relations (73) we obtain

$$q(1 - xy) = q^{-1}(1 - yx), \qquad q(1 - yz) = q^{-1}(1 - zy), \qquad q(1 - zx) = q^{-1}(1 - xz).$$

Definition 14.5. (See [33, Definition 3.1].) Define

$$\nu_x = q(1 - yz) = q^{-1}(1 - zy),$$
$$\nu_y = q(1 - zx) = q^{-1}(1 - xz),$$
$$\nu_z = q(1 - xy) = q^{-1}(1 - yx).$$

By Definition 14.5,

$$xy = 1 - q^{-1}\nu_z, \qquad yx = 1 - q\nu_z,$$
$$yz = 1 - q^{-1}\nu_x, \qquad zy = 1 - q\nu_x,$$
$$zx = 1 - q^{-1}\nu_y, \qquad xz = 1 - q\nu_y.$$

It follows that

$$\frac{[x,y]}{q-q^{-1}} = \nu_z, \qquad \frac{[y,z]}{q-q^{-1}} = \nu_x, \qquad \frac{[z,x]}{q-q^{-1}} = \nu_y.$$

By [33, Lemma 3.5],

$$x\nu_y = q^2\nu_y x, \qquad y\nu_z = q^2\nu_z y, \qquad z\nu_x = q^2\nu_x z,$$
$$\nu_z x = q^2 x\nu_z, \qquad \nu_x y = q^2 y\nu_x, \qquad \nu_y z = q^2 z\nu_y.$$

By [33, Lemma 3.7],

$$\frac{[x,\nu_x]}{q-q^{-1}} = y - z, \qquad \frac{[y,\nu_y]}{q-q^{-1}} = z - x, \qquad \frac{[z,\nu_z]}{q-q^{-1}} = x - y.$$

By [33, Lemma 3.10],

$$\frac{[\nu_x,\nu_y]_q}{q-q^{-1}} = 1 - z^2, \qquad \frac{[\nu_y,\nu_z]_q}{q-q^{-1}} = 1 - x^2, \qquad \frac{[\nu_z,\nu_x]_q}{q-q^{-1}} = 1 - y^2.$$

15. The q-tetrahedron algebra $\boxtimes_q$

In this section we recall the q-tetrahedron algebra $\boxtimes_q$ and review some of its properties. For more information on this algebra, see [16–18,20,27,36,41].

Let $\mathbb{Z}_4 = \mathbb{Z}/4\mathbb{Z}$ denote the cyclic group of order 4.

Definition 15.1. [16, Definition 6.1]. Let $\boxtimes_q$ denote the algebra defined by generators

$$\{x_{ij} \mid i,j \in \mathbb{Z}_4, \ \ j - i = 1 \text{ or } j - i = 2\} \tag{74}$$

and the following relations:

(i) For $i,j \in \mathbb{Z}_4$ such that $j - i = 2$,

$$x_{ij}x_{ji} = 1. \tag{75}$$

(ii) For $i,j,k \in \mathbb{Z}_4$ such that $(j-i, k-j)$ is one of $(1,1)$, $(1,2)$, $(2,1)$,

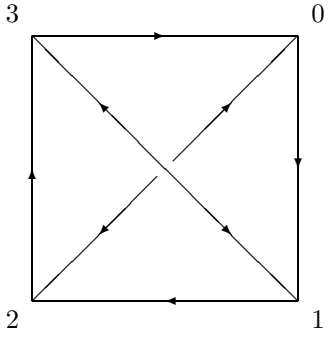
$$\frac{qx_{ij}x_{jk} - q^{-1}x_{jk}x_{ij}}{q-q^{-1}} = 1. \tag{76}$$

(iii) For $i,j,k,\ell \in \mathbb{Z}_4$ such that $j - i = k - j = \ell - k = 1$,

$$x_{ij}^3 x_{k\ell} - [3]_q x_{ij}^2 x_{k\ell} x_{ij} + [3]_q x_{ij} x_{k\ell} x_{ij}^2 - x_{k\ell} x_{ij}^3 = 0. \tag{77}$$

We call $\boxtimes_q$ the *q-tetrahedron algebra*. The elements (74) are called the *standard generators* of $\boxtimes_q$. The relations (77) are called the *q-Serre relations*.

We just gave a presentation of $\boxtimes_q$ by generators and relations. We find it illuminating to describe this presentation with a diagram. This diagram is a directed graph with vertex set $\mathbb{Z}_4$. Each standard generator x_{ij} is represented by a directed arc from vertex i to vertex j. The diagram looks as follows:



The defining relations for $\boxtimes_q$ can be read off the diagram as follows. For any two arcs with the same endpoints and pointing in the opposite direction, the corresponding generators are inverses. For any two arcs that create a directed path of length two, the corresponding generators r, s satisfy

$$\frac{qrs - q^{-1}sr}{q - q^{-1}} = 1.$$

For any two arcs that are distinct and parallel (horizontal or vertical), the corresponding generators satisfy the q-Serre relations.

Lemma 15.2. *There exists an automorphism ρ of $\boxtimes_q$ that sends each standard generator x_{ij} to $x_{i+1,j+1}$. Moreover $\rho^4 = 1$.*

Proof. By Definition 15.1. □

Lemma 15.3. (See [27, Proposition 4.3].) *For $i \in \mathbb{Z}_4$ there exists an algebra homomorphism $\kappa_i : U_q(\mathfrak{sl}_2) \to \boxtimes_q$ that sends*

$$x \mapsto x_{i+2,i+3}, \qquad y \mapsto x_{i+3,i+1}, \qquad y^{-1} \mapsto x_{i+1,i+3}, \qquad z \mapsto x_{i+1,i+2}.$$

This homomorphism is injective.

Recall the Casimir element $\mathbf{\Lambda}$ of $U_q(\mathfrak{sl}_2)$, from Definition 14.3.

Definition 15.4. For $i \in \mathbb{Z}_4$ let Υ_i denote the image of $\mathbf{\Lambda}$ under the injection κ_i from Lemma 15.3.

The elements Υ_i from Definition 15.4 are not central in $\boxtimes_q$. However, we do have the following.

Lemma 15.5. *For $i \in \mathbb{Z}_4$ the element Υ_i commutes with each of*

$$x_{i+2,i+3}, \qquad x_{i+3,i+1}, \qquad x_{i+1,i+3}, \qquad x_{i+1,i+2}.$$

Proof. By Lemma 15.3 and since $\mathbf{\Lambda}$ is central in $U_q(\mathfrak{sl}_2)$. □

16. The t-segregated $\boxtimes_q$-modules

We continue to discuss the algebra $\boxtimes_q$ from Definition 15.1. In [20] we introduced a type of $\boxtimes_q$-module, called an evaluation module. An evaluation module comes with a nonzero scalar parameter t, called the evaluation parameter. In [20, Lemmas 9.4, 9.5] we showed that on a t-evaluation $\boxtimes_q$-module the standard generators satisfy ten attractive equations involving the commutator map and t; these equations are given in Definition 16.1 below. It turns out that there exist nonevaluation $\boxtimes_q$-modules on which the ten equations are satisfied; this fact motivates the following definition.

Definition 16.1. For $0 \neq t \in \mathbb{F}$, a $\boxtimes_q$-module is called *t-segregated* whenever it is nonzero, finite-dimensional, and the following equations hold on the module:

$$t(x_{01} - x_{23}) = \frac{[x_{30}, x_{12}]}{q - q^{-1}}, \qquad t^{-1}(x_{12} - x_{30}) = \frac{[x_{01}, x_{23}]}{q - q^{-1}} \tag{78}$$

and

$$t(x_{01} - x_{02}) = \frac{[x_{30}, x_{02}]}{q - q^{-1}}, \qquad t^{-1}(x_{12} - x_{13}) = \frac{[x_{01}, x_{13}]}{q - q^{-1}}, \tag{79}$$

$$t(x_{23} - x_{20}) = \frac{[x_{12}, x_{20}]}{q - q^{-1}}, \qquad t^{-1}(x_{30} - x_{31}) = \frac{[x_{23}, x_{31}]}{q - q^{-1}} \tag{80}$$

and

$$t^{-1}(x_{30} - x_{20}) = \frac{[x_{20}, x_{01}]}{q - q^{-1}}, \qquad t(x_{01} - x_{31}) = \frac{[x_{31}, x_{12}]}{q - q^{-1}}, \tag{81}$$

$$t^{-1}(x_{12} - x_{02}) = \frac{[x_{02}, x_{23}]}{q - q^{-1}}, \qquad t(x_{23} - x_{13}) = \frac{[x_{13}, x_{30}]}{q - q^{-1}}. \tag{82}$$

Recall from Definition 15.4 the elements $\{\Upsilon_i\}_{i \in \mathbb{Z}_4}$ in $\boxtimes_q$. We next consider how these elements act on a t-segregated $\boxtimes_q$-module. Our results on this topic are given in Lemmas 16.2–16.4 below. These lemmas are proven in [20, Lemmas 9.11, 9.12, 9.14] for a t-evaluation $\boxtimes_q$-module; however the proofs essentially use only the ten equations in Definition 16.1 and consequently apply to every t-segregated $\boxtimes_q$-module.

Lemma 16.2. *Let V denote a t-segregated $\boxtimes_q$-module. Then the action of Υ_i on V is independent of $i \in \mathbb{Z}_4$. Denote this common action by Υ. Then on V,*

$$\Upsilon = t(x_{01}x_{23} - 1) + qx_{30} + q^{-1}x_{12}, \qquad \Upsilon = t^{-1}(x_{12}x_{30} - 1) + qx_{01} + q^{-1}x_{23},$$
$$\Upsilon = t(x_{23}x_{01} - 1) + qx_{12} + q^{-1}x_{30}, \qquad \Upsilon = t^{-1}(x_{30}x_{12} - 1) + qx_{23} + q^{-1}x_{01}.$$

By Lemmas 15.5, 16.2 we find that on a t-segregated $\boxtimes_q$-module, the element Υ commutes with everything in $\boxtimes_q$.

Lemma 16.3. *On a t-segregated $\boxtimes_q$-module,*

$$\Upsilon = (q + q^{-1})x_{30} + t\left(\frac{qx_{01}x_{23} - q^{-1}x_{23}x_{01}}{q - q^{-1}} - 1\right),$$
$$\Upsilon = (q + q^{-1})x_{01} + t^{-1}\left(\frac{qx_{12}x_{30} - q^{-1}x_{30}x_{12}}{q - q^{-1}} - 1\right),$$

$$\Upsilon = (q+q^{-1})x_{12} + t\bigg(\frac{qx_{23}x_{01} - q^{-1}x_{01}x_{23}}{q-q^{-1}} - 1\bigg),$$
$$\Upsilon = (q+q^{-1})x_{23} + t^{-1}\bigg(\frac{qx_{30}x_{12} - q^{-1}x_{12}x_{30}}{q-q^{-1}} - 1\bigg).$$

Lemma 16.4. *Let V denote a t-segregated $\boxtimes_q$-module. Then x_{01}, x_{23} satisfy the following on V:*

$$\begin{aligned} x_{01}^2 x_{23} - (q^2+q^{-2})x_{01}x_{23}x_{01} + x_{23}x_{01}^2 \\ \quad = -(q-q^{-1})^2(1+t^{-1}\Upsilon)x_{01} + (q-q^{-1})(q^2-q^{-2})t^{-1}, \\ x_{23}^2 x_{01} - (q^2+q^{-2})x_{23}x_{01}x_{23} + x_{01}x_{23}^2 \\ \quad = -(q-q^{-1})^2(1+t^{-1}\Upsilon)x_{23} + (q-q^{-1})(q^2-q^{-2})t^{-1}. \end{aligned}$$

Moreover x_{12}, x_{30} satisfy the following on V:

$$\begin{aligned} x_{12}^2 x_{30} - (q^2+q^{-2})x_{12}x_{30}x_{12} + x_{30}x_{12}^2 \\ \quad = -(q-q^{-1})^2(1+t\Upsilon)x_{12} + (q-q^{-1})(q^2-q^{-2})t, \\ x_{30}^2 x_{12} - (q^2+q^{-2})x_{30}x_{12}x_{30} + x_{12}x_{30}^2 \\ \quad = -(q-q^{-1})^2(1+t\Upsilon)x_{30} + (q-q^{-1})(q^2-q^{-2})t. \end{aligned}$$

We remark that the relations in Lemma 16.4 are the Askey-Wilson relations [40,42].

17. How to construct a t-segregated $\boxtimes_q$-module

We continue to discuss the algebra $\boxtimes_q$ from Definition 15.1. In the previous section, we introduced the concept of a t-segregated $\boxtimes_q$-module. In this section, we show how to construct a t-segregated $\boxtimes_q$-module, starting with a nonzero finite-dimensional $U_q(\mathfrak{sl}_2)$-module and a bit more.

Throughout this section, V denotes a nonzero finite-dimensional $U_q(\mathfrak{sl}_2)$-module.

Assumption 17.1. Let $0 \neq t \in \mathbb{F}$. Assume that there exists an invertible $w \in \mathrm{End}(V)$ such that on V,

$$tz - q = w(t - qx), \qquad tz - q^{-1} = (t - q^{-1}x)w. \tag{83}$$

Under Assumption 17.1, we will turn V into a t-segregated $\boxtimes_q$-module.

Lemma 17.2. *Under Assumption 17.1, the following holds on the $U_q(\mathfrak{sl}_2)$-module V:*

$$\begin{aligned} xw &= 1 - qtz + qtw, \\ wx &= 1 - q^{-1}tz + q^{-1}tw, \\ w^{-1}z &= 1 - qt^{-1}x + qt^{-1}w^{-1}, \\ zw^{-1} &= 1 - q^{-1}t^{-1}x + q^{-1}t^{-1}w^{-1}. \end{aligned}$$

Proof. Use (83). □

Recall the elements ν_x, ν_y, ν_z of $U_q(\mathfrak{sl}_2)$ from Definition 14.5.

Lemma 17.3. *Under Assumption 17.1, the following holds on the $U_q(\mathfrak{sl}_2)$-module V:*

$$w\nu_z = qw - qy + tzy - twy,$$
$$\nu_z w = q^{-1}w - q^{-1}y + tyz - tyw,$$
$$\nu_x w^{-1} = qw^{-1} - qy + t^{-1}yx - t^{-1}yw^{-1},$$
$$w^{-1}\nu_x = q^{-1}w^{-1} - q^{-1}y + t^{-1}xy - t^{-1}w^{-1}y.$$

Proof. Use Definition 14.5 and Lemma 17.2. □

Proposition 17.4. *Under Assumption 17.1, V becomes a t-segregated $\boxtimes_q$-module on which the $\boxtimes_q$-generators act as follows:*

generator	x_{01}	x_{12}	x_{23}	x_{30}	x_{02}	x_{13}	x_{20}	x_{31}
action on V	z	x	$y + t^{-1}\nu_z$	$y + t\nu_x$	y^{-1}	w^{-1}	y	w

Moreover $\Upsilon = \mathbf{\Lambda}$ on V.

Proof. It is trivial to check that the relations (75) hold on V. Using Lemmas 17.2, 17.3 and the various relations below Definition 14.5, one routinely checks that the relations (76) and (78)–(82) hold on V. Next we check that the q-Serre relations (77) hold on V. Using Definitions 14.3, 14.5 one finds that on V,

$$\mathbf{\Lambda} = t(x_{01}x_{23} - 1) + qx_{30} + q^{-1}x_{12}, \qquad \mathbf{\Lambda} = t^{-1}(x_{12}x_{30} - 1) + qx_{01} + q^{-1}x_{23},$$
$$\mathbf{\Lambda} = t(x_{23}x_{01} - 1) + qx_{12} + q^{-1}x_{30}, \qquad \mathbf{\Lambda} = t^{-1}(x_{30}x_{12} - 1) + qx_{23} + q^{-1}x_{01}.$$

In other words, the four relations in Lemma 16.2 hold on V with $\Upsilon = \mathbf{\Lambda}$. Using this result, one finds that the relations in Lemmas 16.3, 16.4 hold on V with $\Upsilon = \mathbf{\Lambda}$. By these comments, the relations (77) hold on V. The last assertion of the proposition statement follows from the construction. □

18. The main results

In Sections 2–13 we discussed a TD system $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ on V that has q-Racah type. In this section, we return our attention to Φ. Recall the scalar a from Definition 2.3.

Theorem 18.1. *For the above TD system Φ, the underlying vector space V becomes an a-segregated $\boxtimes_q$-module on which the $\boxtimes_q$-generators act as follows:*

generator	x_{01}	x_{12}	x_{23}	x_{30}
action on V	$W^{-1}KW$	$WK^{-1}W^{-1}$	$Q^{-1} + W\psi W^{-1}$	$Q^{-1} + W^{-1}\psi W$

generator	x_{02}	x_{13}	x_{20}	x_{31}
action on V	Q	K^{-1}	Q^{-1}	K

Moreover $\Upsilon = \Lambda$ on V.

Proof. By Proposition 9.3(i) and Definition 14.1, the vector space V becomes a $U_q(\mathfrak{sl}_2)$-module on which

$$x = WK^{-1}W^{-1}, \qquad y = Q^{-1}, \qquad z = W^{-1}KW. \tag{84}$$

Define $w = K$, and note that w is invertible. Also define $t = a$. The element w satisfies (83) by Proposition 13.1. Assumption 17.1 is now satisfied, so Proposition 17.4 applies. Next we show that on V,

$$y + t^{-1}\nu_z = Q^{-1} + W\psi W^{-1}, \qquad y + t\nu_x = Q^{-1} + W^{-1}\psi W. \tag{85}$$

Since $y = Q^{-1}$ on V and also $t = a$, it suffices to show that on V,

$$a^{-1}\nu_z = W\psi W^{-1}, \qquad a\nu_x = W^{-1}\psi W. \tag{86}$$

The equation on the left in (86) is obtained using $\nu_z = q(1 - xy)$ with x, y from (84), along with $Q^{-1} = WN^{-1}W^{-1}$ and $K^{-1}N^{-1} = I - aq^{-1}\psi$. The equation on the right in (86) is obtained using $\nu_x = q(1 - yz)$ with y, z from (84), along with $Q^{-1} = W^{-1}M^{-1}W$ and $M^{-1}K = I - a^{-1}q^{-1}\psi$. We have shown that (85) holds on V. It remains to show that $\Upsilon = \Lambda$ on V. To do this, by Proposition 17.4 it suffices to show that $\mathbf{\Lambda} = \Lambda$ on V. On V,

$$\begin{aligned}\mathbf{\Lambda} &= qx + q^{-1}y + qz - qxyz \\ &= qx_{12} + q^{-1}x_{20} + qx_{01} - qx_{12}x_{20}x_{01}.\end{aligned}$$

By Proposition 7.5(i) and the construction, $A = ax_{01} + a^{-1}x_{12}$ on V. The generators x_{01}, x_{12} commute with $\mathbf{\Lambda}$ on V, so A commutes with $\mathbf{\Lambda}$ on V. The element W is a polynomial in A, so W commutes with $\mathbf{\Lambda}$ on V. We may now argue that on V,

$$\begin{aligned}\mathbf{\Lambda} &= W(qx_{12} + q^{-1}x_{20} + qx_{01} - qx_{12}x_{20}x_{01})W^{-1} \\ &= W(qx_{12}(1 - x_{20}x_{01}) + q^{-1}x_{20} + qx_{01})W^{-1} \\ &= qW^2K^{-1}W^{-2}(I - M^{-1}K) + q^{-1}M^{-1} + qK \\ &= q(aA - a^2K)(I - M^{-1}K) + q^{-1}M^{-1} + qK \\ &= (A - aK)\psi + q^{-1}K^{-1}(1 - a^{-1}q\psi) + qK \\ &= (A - aK - a^{-1}K^{-1})\psi + qK + q^{-1}K^{-1} \\ &= \Lambda. \quad \square\end{aligned}$$

Theorem 18.2. *For the above TD system Φ, the underlying vector space V becomes an a^{-1}-segregated $\boxtimes_q$-module on which the $\boxtimes_q$-generators act as follows:*

generator	x_{01}	x_{12}	x_{23}	x_{30}
action on V	$W^{-1}BW$	$WB^{-1}W^{-1}$	$Q^{-1} + W\psi W^{-1}$	$Q^{-1} + W^{-1}\psi W$

generator	x_{02}	x_{13}	x_{20}	x_{31}
action on V	Q	B^{-1}	Q^{-1}	B

Moreover $\Upsilon = \Lambda$ on V.

Proof. Apply Theorem 18.1 to $\Phi^{\Downarrow}$, and use $B = K^{\Downarrow}$ along with Lemmas 5.14(ii), 8.4, 10.4, 11.4. $\square$

Note 18.3. Referring to the tables in Theorems 18.1, 18.2, for the action of x_{23} and x_{30} an alternative description is given in Proposition 12.2; see also Proposition 12.3.

19. Suggestions for future research

In Sections 2–13 and 18 we discussed a TD system $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ on V that has q-Racah type. In this section we give some open problems concerning Φ. To motivate the first problem, we have some comments. Recall the map R from (61). Define

$$R^- = WK^{-1}W^{-1} - K^{-1}, \qquad R^+ = W^{-1}KW - K.$$

By Proposition 13.1,

$$R^+ = -a^{-1}qKR^- = -a^{-1}q^{-1}R^-K. \tag{87}$$

Therefore

$$R^- = -aq^{-1}K^{-1}R^+ = -aqR^+K^{-1}. \tag{88}$$

By (87) and (88),

$$R^-K = q^2KR^-, \qquad R^+K = q^2KR^+.$$

Consequently

$$R^{\pm}U_i \subseteq U_{i+1} \qquad (0 \leq i \leq d-1), \qquad R^{\pm}U_d = 0.$$

Using Proposition 7.5(i) and (61),

$$R = aR^+ + a^{-1}R^-. \tag{89}$$

Using (87) or (88),

$$R^-R^+ = q^2R^+R^-.$$

Problem 19.1. Investigate the algebraic and combinatorial significance of $R^{\pm}$.

On the $\boxtimes_q$-module V in Theorem 18.1 we have $A = ax_{01} + a^{-1}x_{12}$, and on the $\boxtimes_q$-module V in Theorem 18.2 we have $A = a^{-1}x_{01} + ax_{12}$. On these modules the value of $b^{-1}x_{23} + bx_{30}$ is the same, and it is natural to guess that this common value is equal to A^*. It turns out that this guess is false, but it does seem likely that $A^* - b^{-1}x_{23} - bx_{30}$ is important in some way. This motivates the next problem.

Problem 19.2. Define

$$\mathcal{L} = A^* - b^{-1}(M^{-1} + \psi) - b(N^{-1} + \psi).$$

Show that $\mathcal{L}K = q^{-2}K\mathcal{L}$ and $\mathcal{L}B = q^{-2}B\mathcal{L}$. Show that

$$\mathcal{L}U_i \subseteq U_{i-1} \qquad (1 \leq i \leq d), \qquad \mathcal{L}U_0 = 0,$$
$$\mathcal{L}U_i^{\Downarrow} \subseteq U_{i-1}^{\Downarrow} \qquad (1 \leq i \leq d), \qquad \mathcal{L}U_0^{\Downarrow} = 0.$$

Show that $\mathcal{L}\psi = \psi\mathcal{L}$. Investigate how $\mathcal{L}$ is related to $R^{\pm}$ above.

Problem 19.3. Find a relation involving only $K^{\pm 1}$ and $Q^{\pm 1}$.

Problem 19.4. How do A^* and WA^*W^{-1} act on each other's eigenspaces? It seems that the pair A^*, WA^*W^{-1} is not a TD pair in general. Find necessary and sufficient conditions on A, A^* for the pair A^*, WA^*W^{-1} to be a TD pair.

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