Journal of Pure and Applied Algebra 224 (2020) 610–629

Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

www.elsevier.com/locate/jpaa

Cartesian closed exact completions in topology ☆

Maria Manuel Clementino [a,*], Dirk Hofmann [b], Willian Ribeiro [a]

[a] CMUC, Department of Mathematics, University of Coimbra, 3001-501 Coimbra, Portugal
[b] CIDMA, Department of Mathematics, University of Aveiro, 3810-193 Aveiro, Portugal

A R T I C L E I N F O

Article history:
Received 9 November 2018
Received in revised form 28 April 2019
Available online 18 June 2019
Communicated by J. Adámek

MSC:
18B30; 18B35; 18D15; 18D20; 54B30; 54E70

Keywords:
Quantale
Enriched category
(Probabilistic) metric space
Exponentiation
(Weakly) cartesian closed category
Exact completion

A B S T R A C T

Using generalized enriched categories, in this paper we show that Rosický's proof of cartesian closedness of the exact completion of the category of topological spaces can be extended to a wide range of topological categories over **Set**, like metric spaces, approach spaces, ultrametric spaces, probabilistic metric spaces, and bitopological spaces. In order to do so we prove a sufficient criterion for exponentiability of $(\mathbb{T},V)$-categories and show that, under suitable conditions, every injective $(\mathbb{T},V)$-category is exponentiable in $(\mathbb{T},V)$-**Cat**.

1. Introduction

As Lawvere has shown in his celebrated paper [24], when V is a closed category the category V-**Cat** of V-enriched categories and V-functors is also monoidal closed. This result extends neither to the cartesian structure nor to the more general setting of $(\mathbb{T},V)$-categories. Indeed, cartesian closedness of V does not guarantee cartesian closedness of V-**Cat**: take for instance the category of (Lawvere's) metric spaces P_+-**Cat**, where P_+ is the complete real half-line, ordered with the $\geq$ relation, and equipped with the monoidal structure given by addition $+$; P_+ is cartesian closed but P_+-**Cat** is not (see [6] for details); and, even when the monoidal structure of V is the cartesian one, the category $(\mathbb{T},V)$-**Cat** of $(\mathbb{T},V)$-categories and

☆ Research partially supported by Centro de Matemática da Universidade de Coimbra – UID/MAT/00324/2019, by Centro de Investigação e Desenvolvimento em Matemática e Aplicações da Universidade de Aveiro/FCT – UID/MAT/04106/2019, funded by the Portuguese Government through FCT/MCTES and co-funded by the European Regional Development Fund through the Partnership Agreement PT2020. W. Ribeiro also acknowledges the FCT PhD Grant PD/BD/128059/2016.
* Corresponding author.
E-mail addresses: mmc@mat.uc.pt (M.M. Clementino), dirk@ua.pt (D. Hofmann), willian.ribeiro.vs@gmail.com (W. Ribeiro).

https://doi.org/10.1016/j.jpaa.2019.06.003
0022-4049/

$(\mathbb{T},V)$-functors (see [11]) does not need to be cartesian closed, as it is the case of the category **Top** of topological spaces and continuous maps, that is $(\mathbb{U},2)$-**Cat** for $\mathbb{U}$ the ultrafilter monad.

Rosický showed in [30] that **Top** is weakly cartesian closed, and, consequently, that its exact completion is cartesian closed. Weak cartesian closedness of **Top** follows from the existence of enough injectives in its full subcategory $\mathbf{Top}_0$ of $T0$-spaces and the fact that they are exponentiable, and this feature, together with several good properties of **Top**, gives cartesian closedness of its exact completion. More precisely, Rosický has shown in [30] the following theorem.

Theorem 1.1. *Let* $\mathbf{C}$ *be a complete, infinitely extensive and well-powered category with (reg epi, mono)-factorizations such that* $f\times 1$ *is an epimorphism whenever* f *is a regular epimorphism. Then the exact completion of* $\mathbf{C}$ *is cartesian closed provided that* $\mathbf{C}$ *is weakly cartesian closed.*

In this paper we use the setting of $(\mathbb{T},V)$-categories, for a quantale V and a **Set**-monad $\mathbb{T}$ laxly extended to V-**Rel**, to conclude, in a unified way, that several topological categories over **Set** share with **Top** the cartesian closedness of the exact completion. This was recently used by Adámek and Rosický in the study of free completions of categories [2]. In fact, the category $(\mathbb{T},V)$-**Cat** is topological over **Set** [5,11], hence complete and with (reg epi, mono)-factorizations such that $f\times 1$ is an epimorphism whenever f is, and it is infinitely extensive [28]. To assure weak cartesian closedness of $(\mathbb{T},V)$-**Cat** we consider two distinct scenarios, either restricting to the case that V is a frame – so that its monoidal structure is the cartesian one – or considering the case that the lax extension is determined by a $\mathbb{T}$-algebraic structure on V, as introduced in [17] under the name of topological theory. In the latter case the proof generalizes Rosický's proof for $\mathbf{Top}_0$, after observing that, using the Yoneda embedding of [7,18], every separated $(\mathbb{T},V)$-category can be embedded in an injective one, and, moreover, these are exponentiable in $(\mathbb{T},V)$-**Cat**. For general $(\mathbb{T},V)$-categories one proceeds again as in [30], using the fact that the reflection of $(\mathbb{T},V)$-**Cat** into its full subcategory of separated $(\mathbb{T},V)$-categories preserves finite products. As observed by Rosický, the exact completion of **Top** relates to the cartesian closed category of equilogical spaces [3]. Analogously, our approach leads to the study of generalized equilogical spaces, as developed in [29].

The paper is organized as follows. In Section 2 we introduce $(\mathbb{T},V)$-categories and list their properties used throughout the paper. In Section 3 we revisit the exponentiability problem in $(\mathbb{T},V)$-**Cat**, establishing a sufficient criterion for exponentiability which generalizes the results obtained in [17,21]. In Section 4 we study the properties of injective $(\mathbb{T},V)$-categories which will be used in the forthcoming section to conclude that, under suitable assumptions, injective $(\mathbb{T},V)$-categories are exponentiable (Theorem 5.8). This result will allow us to conclude, in Theorem 6.3, that $(\mathbb{T},V)$-**Cat** is weakly cartesian closed, and, finally, thanks to Theorem 1.1, that the exact completion of $(\mathbb{T},V)$-**Cat** is cartesian closed. We conclude our paper with a section on examples, which include, among others, metric spaces, approach spaces, probabilistic metric spaces, and bitopological spaces.

2. The category of $(\mathbb{T},V)$-categories

Throughout V *is a commutative and unital quantale*, i.e. V is a complete lattice with a symmetric and associative tensor product $\otimes$, with unit k and right adjoint hom, so that $u\otimes v\le w$ if, and only if, $v\le \hom(u,w)$, for all $u,v,w\in V$. Further assume that V *is a Heyting algebra*, so that $u\wedge -$ also has a right adjoint, for every $u\in V$. We denote by V-**Rel** the 2-category of V*-relations* (or V-matrices), having as objects sets, as 1-cells V-relations $r: X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$, i.e. maps $r: X\times Y\to V$, and 2-cells $\varphi: r\to r'$ given by componentwise order $r(x,y)\le r'(x,y)$. Composition of 1-cells is given by relational composition. V-**Rel** has an involution, given by transposition: the transpose of $r: X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$ is $r^\circ: Y \relbar\joinrel\mapstochar\joinrel\rightarrow X$ with $r^\circ(y,x)=r(x,y)$.

We fix a *non-trivial monad* $\mathbb{T}=(T,m,e)$ *on* **Set** *satisfying (BC)*, i.e. T preserves weak pullbacks and the naturality squares of the natural transformation m are weak pullbacks (see [9]). In general we do not assume that T preserves products. Later we will make use of the comparison map $\mathrm{can}_{X,Y} : T(X \times Y) \to TX \times TY$ defined by $\mathrm{can}_{X,Y}(\mathfrak{w}) = (T\pi_X(\mathfrak{w}), T\pi_Y(\mathfrak{w}))$ for all $\mathfrak{w} \in T(X \times Y)$, where π_X and π_Y are the product projections. Moreover, we assume that $\mathbb{T}$ *has an extension to* V-**Rel**, which we also denote by $\mathbb{T}$, in the following sense:

– there is a lax functor $T : V\text{-}\mathbf{Rel} \to V\text{-}\mathbf{Rel}$ which extends $T : \mathbf{Set} \to \mathbf{Set}$;
– $T(r^\circ) = (Tr)^\circ$ for all V-relations r;
– the natural transformations $e : 1_{V\text{-}\mathbf{Rel}} \to T$ and $m : T^2 \to T$ become op-lax; that is, for every $r : X \to\!\!\!\!\!\!\shortmid\;\; Y$,

$$e_Y \cdot r \le Tr \cdot e_X, \qquad m_Y \cdot TTr \le Tr \cdot m_X.$$

$$\begin{tikzcd} X \arrow[r, "e_X"] \arrow[d, "r"', "\shortmid" marking] \arrow[dr, phantom, "\le"] & TX \arrow[d, "Tr", "\shortmid" marking] \\ Y \arrow[r, "e_Y"'] & TY \end{tikzcd} \qquad \begin{tikzcd} TTX \arrow[r, "m_X"] \arrow[d, "TTr"', "\shortmid" marking] \arrow[dr, phantom, "\le"] & TX \arrow[d, "Tr", "\shortmid" marking] \\ TTY \arrow[r, "m_Y"'] & TY \end{tikzcd}$$

We note that our conditions are stronger than those used in [22].

A $(\mathbb{T}, V)$-*category* is a pair (X, a), where X is a set and $a : TX \to\!\!\!\!\!\!\shortmid\;\; X$ is a V-relation, such that

$$\begin{tikzcd} X \arrow[r, "e_X"] \arrow[dr, "1_X"'] \arrow[r, phantom, shift right=4, "\le"] & TX \arrow[d, "a", "\shortmid" marking] \\ & X \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} T^2X \arrow[r, "m_X"] \arrow[d, "Ta"', "\shortmid" marking] \arrow[dr, phantom, "\le"] & TX \arrow[d, "a", "\shortmid" marking] \\ TX \arrow[r, "a"', "\shortmid" marking] & X \end{tikzcd}$$

that is, the map $a : TX \times X \to V$ satisfies the conditions:

(R) for each $x \in X$, $k \le a(e_X(x), x)$;
(T) for each $\mathfrak{X} \in T^2X, \mathfrak{x} \in TX, x \in X$, $Ta(\mathfrak{X}, \mathfrak{x}) \otimes a(\mathfrak{x}, x) \le a(m_X(\mathfrak{X}), x)$.

Given $(\mathbb{T}, V)$-categories (X, a), (Y, b), a $(\mathbb{T}, V)$-*functor* $f : (X, a) \to (Y, b)$ is a map $f : X \to Y$ such that

$$\begin{tikzcd} TX \arrow[r, "Tf"] \arrow[d, "a"', "\shortmid" marking] \arrow[dr, phantom, "\le"] & TY \arrow[d, "b", "\shortmid" marking] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

that is, for each $\mathfrak{x} \in TX$ and $x \in X$, $a(\mathfrak{x}, x) \le b(Tf(\mathfrak{x}), f(x))$; f is said to be *fully faithful* when this inequality is an equality.

$(\mathbb{T}, V)$-categories and $(\mathbb{T}, V)$-functors form the category $(\mathbb{T}, V)$-**Cat**. If $(X, a : TX \to\!\!\!\!\!\!\shortmid\;\; X)$ satisfies (R) (and not necessarily (T)), we call it a $(\mathbb{T}, V)$-*graph*. The category $(\mathbb{T}, V)$-**Gph**, of $(\mathbb{T}, V)$-graphs and $(\mathbb{T}, V)$-functors, contains $(\mathbb{T}, V)$-**Cat** as a full reflective subcategory.

We present the examples in detail in the last section. We mention here, however, that the leading examples are obtained when one considers the quantale $2 = (\{0, 1\}, \le, \&, 1)$ and Lawvere's real half-line $P_+ = ([0, \infty], \ge, +, 0)$, the identity monad $\mathbb{I}$ and the ultrafilter monad $\mathbb{U}$ on **Set**. Thus we obtain the following examples:

- $(\mathbb{I}, V)$-**Cat** is the category of V-categories and V-functors; in particular, $(\mathbb{I}, 2)$-**Cat** is the category **Ord** of (pre)ordered sets and monotone maps, while $(\mathbb{I}, P_+)$-**Cat** is the category **Met** of Lawvere's metric spaces and non-expansive maps (see [24]).
- $(\mathbb{U}, 2)$-**Cat** is the category **Top** of topological spaces and continuous maps.
- $(\mathbb{U}, P_+)$-**Cat** is the category **App** of Lowen's approach spaces and non-expansive maps (see [26]).

We recall (see [1, Definition 21.1]) that a functor $G : \mathbf{A} \to \mathbf{B}$ is said to be *topological* if every source $(f_i : B \to GA_i)_{i\in I}$ in **B** has a unique G-initial lift $(\overline{f_i} : A \to A_i)_{i\in I}$. The following was proved in [5] (see also [11]).

Theorem 2.1. *The forgetful functors* $(\mathbb{T}, V)$-**Cat** $\to$ **Set** *and* $(\mathbb{T}, V)$-**Gph** $\to$ **Set** *are topological.*

This shows, in particular, that (see [1, Chapter 21] for details):

- $(\mathbb{T}, V)$-**Cat** is complete and cocomplete.
- Monomorphisms in $(\mathbb{T}, V)$-**Cat** are the morphisms whose underlying map is injective; therefore, since the $(\mathbb{T}, V)$-structures on any set form a set, $(\mathbb{T}, V)$-**Cat** is well-powered.
- Every topological category over **Set** has two factorization systems, (reg epi, mono) and (epi, reg mono); in $(\mathbb{T}, V)$-**Cat** the former one is in general not stable (that is, regular epimorphisms need not be stable under pullback – **Top** is such an example), but the latter one is. Indeed, epimorphisms in $(\mathbb{T}, V)$-**Cat** are the $(\mathbb{T}, V)$-functors which are surjective as maps, the forgetful functor $(\mathbb{T}, V)$-**Cat** $\to$ **Set** preserves pullbacks, and surjective maps are stable under pullback in **Set**. Therefore, as $f \times 1_Z$ is the pullback of $f : X \to Y$ along $\pi_Y : Y \times Z \to Y$, we conclude that $f \times 1_Z$ is an epimorphism provided f is.

$(\mathbb{T}, V)$-**Cat** has a natural structure of 2-category: for $(\mathbb{T}, V)$-functors $f, g : (X, a) \to (Y, b)$, $f \leq g$ if $g \cdot a \leq b \cdot Tf$. This condition can be equivalently written as $k \leq b(e_Y(f(x)), g(x))$ for every $x \in X$ (see [11] for details). We write $f \simeq g$ if $f \leq g$ and $g \leq f$.

Extensivity of $(\mathbb{T}, V)$-**Cat** was studied in [28]:

Theorem 2.2. $(\mathbb{T}, V)$-**Cat** *is infinitely extensive.*

In general $(\mathbb{T}, V)$-**Cat** is not cartesian closed, while $(\mathbb{T}, V)$-**Gph** is. In fact, the following was proved in [10]:

Theorem 2.3. $(\mathbb{T}, V)$-**Gph** *is a quasitopos.*

We also note that the tensor product of V induces a canonical structure c on $X \times Y$ defined by

$$c(\mathfrak{w}, (x, y)) = a(T\pi_X(\mathfrak{w}), x) \otimes b(T\pi_Y(\mathfrak{w}), y),$$

where $\mathfrak{w} \in T(X \times Y)$, $x \in X$, $y \in Y$. We put

$$(X, a) \otimes (Y, b) = (X \times Y, c),$$

and this construction is in an obvious way part of a functor $\otimes : (\mathbb{T}, V)\text{-}\mathbf{Gph} \times (\mathbb{T}, V)\text{-}\mathbf{Gph} \to (\mathbb{T}, V)\text{-}\mathbf{Gph}$. However, the tensor product of two $(\mathbb{T}, V)$-categories is in general not a $(\mathbb{T}, V)$-category (see [17, Lemma 6.1]).

Weak cartesian closedness of $(\mathbb{T}, V)$-**Cat** needs a thorough study of injective $(\mathbb{T}, V)$-categories and some extra conditions. This is the subject of the following sections.

3. Exponentiable $(\mathbb{T}, V)$-categories

Recall that an object C of a category $\mathbf{C}$ with finite products is *exponentiable* whenever the functor $C \times - : \mathbf{C} \to \mathbf{C}$ has a right adjoint. The category $\mathbf{C}$ is *cartesian closed* if every object C of $\mathbf{C}$ is exponentiable. Equivalently, if for each pair of objects A, B of $\mathbf{C}$ there exists an object $\langle A, B\rangle$ and a morphism $\mathrm{ev} : \langle A, B\rangle \times A \to B$ such that, for each morphism $f : C \times A \to B$ there exists a unique morphism $\overline{f} : C \to \langle A, B\rangle$ with $\mathrm{ev} \cdot (\overline{f} \times 1_A) = f$. Dropping uniqueness of $\overline{f}$ gives the notion of *weakly cartesian closed category*.

In this section we present a sufficient condition for a $(\mathbb{T}, V)$-category X to be exponentiable in $(\mathbb{T}, V)$-**Cat**, which generalises [16, Theorem 4.3] and [17, Theorem 6.5]. To start, we recall that $(\mathbb{T}, V)$-**Cat** can be fully embedded into the cartesian closed category $(\mathbb{T}, V)$-**Gph**. Here, for $(\mathbb{T}, V)$-graphs (X, a) and (Y, b), the exponential $\langle (X, a), (Y, b)\rangle$ has as underlying set

$$Z := \{h : (X, a) \times (1, e_1^\circ) \to (Y, b) \mid h \text{ is a } (\mathbb{T}, V)\text{-functor}\},$$

which becomes a $(\mathbb{T}, V)$-graph when equipped with the largest structure b^a making the evaluation map

$$\mathrm{ev} : Z \times X \to Y,\ (h, x) \mapsto h(x)$$

a $(\mathbb{T}, V)$-functor: for $\mathfrak{p} \in TZ$ and $h \in Z$, put

$$b^a(\mathfrak{p}, h) = \bigvee \{v \in V \mid \forall \mathfrak{q} \in (T\pi_Z)^{-1}(\mathfrak{p}), x \in X \;.\; a(T\pi_X(\mathfrak{q}), x) \wedge v \le b(T\,\mathrm{ev}(\mathfrak{q}), h(x))\},$$

where π_X and π_Z are the product projections. Note that the supremum above is even a maximum since $- \wedge -$ distributes over suprema.

Given V-relations $r : X \nrightarrow X'$ and $s : Y \nrightarrow Y'$, we define in V-**Rel** $r \owedge s : X \times Y \nrightarrow X' \times Y'$ by $(r \owedge s)((x, y), (x', y')) = r(x, x') \wedge s(y, y')$. That is, $r \owedge s = (\pi_{X'}^\circ \cdot r \cdot \pi_X) \wedge (\pi_{Y'}^\circ \cdot s \cdot \pi_Y)$ in the ordered set V-**Rel**$(X \times Y, X' \times Y')$.

Theorem 3.1. *Assume that the diagram*

$$\begin{tikzcd} T(X \times Y) \arrow[r, "\mathrm{can}_{X,Y}"] \arrow[d, "T(r \owedge s)"', "\shortmid" marking] & TX \times TY \arrow[d, "(Tr) \owedge (Ts)", "\shortmid" marking] \\ T(X' \times Y') \arrow[r, "\mathrm{can}_{X',Y'}"'] & TX' \times TY' \end{tikzcd} \tag{3.i}$$

commutes, for all V-relations $r : X \nrightarrow X'$ and $s : Y \nrightarrow Y'$. Then a $(\mathbb{T}, V)$-category (X, a) is exponentiable provided that

$$\bigvee_{\mathfrak{x} \in TX} (Ta(\mathfrak{X}, \mathfrak{x}) \wedge u) \otimes (a(\mathfrak{x}, x) \wedge v) \ge a(m_X(\mathfrak{X}), x) \wedge (u \otimes v), \tag{3.ii}$$

for all $\mathfrak{X} \in TTX$, $x \in X$ and $u, v \in V$.

Proof. We show that the $(\mathbb{T}, V)$-graph structure b^a on Z is transitive, for each $(\mathbb{T}, V)$-category (Y, b). To this end, let $\mathfrak{P} \in TTZ$, $\mathfrak{p} \in TZ$, $h \in Z$, $x \in X$ and $\mathfrak{w} \in T(Z \times X)$ with $T\pi_Z(\mathfrak{w}) = m_Z(\mathfrak{P})$. We have to show that

$$(T(b^a)(\mathfrak{P}, \mathfrak{p}) \otimes b^a(\mathfrak{p}, h)) \wedge a(T\pi_X(\mathfrak{w}), x) \le b(T\,\mathrm{ev}(\mathfrak{w}), h(x)).$$

Since m has (BC), there is some $\mathfrak{Q} \in TT(Z \times X)$ with $TT\pi_Z(\mathfrak{Q}) = \mathfrak{P}$ and $m_{Z\times X}(\mathfrak{Q}) = \mathfrak{w}$. Hence, $m_X(TT\pi_X(\mathfrak{Q})) = T\pi_X(\mathfrak{w})$, and we calculate:

$$\begin{aligned}
&(T(b^a)(\mathfrak{P},\mathfrak{p}) \otimes b^a(\mathfrak{p},h)) \wedge a(T\pi_X(\mathfrak{w}),x) \\
&\quad\leq \bigvee_{\mathfrak{x}\in TX} ((T(b^a)(TT\pi_Z(\mathfrak{Q}),\mathfrak{p}) \wedge Ta(TT\pi_X(\mathfrak{Q}),\mathfrak{x})) \otimes (b^a(\mathfrak{p},h) \wedge a(\mathfrak{x},x)) && \text{(by (3.ii))} \\
&\quad\leq \bigvee_{\mathfrak{x}\in TX} \bigvee_{\mathfrak{q}\in \mathrm{can}^{-1}(\mathfrak{p},\mathfrak{x})} T(b^a \owedge a)(T\,\mathrm{can}_{Z,X}(\mathfrak{Q}),\mathfrak{q}) \otimes (b^a \owedge a)(\mathrm{can}_{Z,X}(\mathfrak{q}),(h,x)) && \text{(using (3.i))} \\
&\quad= \bigvee_{\mathfrak{q}\in (T\pi_Z)^{-1}(\mathfrak{p})} T(b^a \owedge a)(T\,\mathrm{can}_{Z,X}(\mathfrak{Q}),\mathfrak{q}) \otimes (b^a \owedge a)(\mathrm{can}_{Z,X}(\mathfrak{q}),(h,x)) \\
&\quad= \bigvee_{\mathfrak{q}\in (T\pi_Z)^{-1}(\mathfrak{p})} T(b^a \times a)(\mathfrak{Q},\mathfrak{q}) \otimes (b^a \times a)(\mathfrak{q},(h,x)) \\
&\quad\leq \bigvee_{\mathfrak{q}\in (T\pi_Z)^{-1}(\mathfrak{p})} Tb(TT\,\mathrm{ev}(\mathfrak{Q}),T\,\mathrm{ev}(\mathfrak{q})) \otimes b(T\,\mathrm{ev}(\mathfrak{q}),h(x)) \\
&\quad\leq b(m_Y \cdot TT\,\mathrm{ev}(\mathfrak{Q}),h(x)) = b(T\,\mathrm{ev}(\mathfrak{w}),h(x)). \quad \square
\end{aligned}$$

Remark 3.2. We note that the inequality $\mathrm{can}_{X',Y'} \cdot T(r \owedge s) \leq ((Tr) \owedge (Ts)) \cdot \mathrm{can}_{X,Y}$ is automatically true. Firstly, this inequality is equivalent to $T(r \owedge s) \leq \mathrm{can}^\circ_{X',Y'} \cdot ((Tr) \owedge (Ts)) \cdot \mathrm{can}_{X,Y}$; secondly,

$$\begin{aligned}
T(r \owedge s) &= T((\pi^\circ_{X'} \cdot r \cdot \pi_X) \wedge (\pi^\circ_{Y'} \cdot s \cdot \pi_Y)) \\
&\leq T(\pi^\circ_{X'} \cdot r \cdot \pi_X) \wedge T(\pi^\circ_{Y'} \cdot s \cdot \pi_Y) \\
&\leq \mathrm{can}^\circ_{X',Y'} \cdot ((Tr) \owedge (Ts)) \cdot \mathrm{can}_{X,Y}\,.
\end{aligned}$$

It is worthwhile to notice that, when V is a frame, that is $\otimes = \wedge$, the condition above is equivalent to

$$\bigvee_{\mathfrak{x}\in TX} Ta(\mathfrak{X},\mathfrak{x}) \wedge a(\mathfrak{x},x) \geq a(m_X(\mathfrak{X}),x),$$

for all $\mathfrak{X} \in TTX$ and $x \in X$. Therefore:

Corollary 3.3. *When V is a frame and (3.i) commutes for all V-relations $r : X \to\!\!\!\!\!\!\mapstochar\;\; X'$ and $s : Y \to\!\!\!\!\!\!\mapstochar\;\; Y'$, a $(\mathbb{T},V)$-category (X,a) is exponentiable provided that*

$$a \cdot m_X = a \cdot Ta.$$

4. Injective and representable $(\mathbb{T},V)$-categories

In this section we recall an important class of $(\mathbb{T},V)$-categories, the so-called *representable* ones. More information on this type of $(\mathbb{T},V)$-categories can be found in [4,22]. We also recall from [7,17,18] that every injective $(\mathbb{T},V)$-category is representable.

Based on the lax extension of the **Set**-monad $\mathbb{T} = (T,m,e)$ to V-**Rel**, $\mathbb{T}$ admits a natural extension to a monad on V-**Cat**, in the sequel also denoted by $\mathbb{T} = (T,m,e)$ (see [32]). Here the functor $T : V\text{-}\mathbf{Cat} \to V\text{-}\mathbf{Cat}$ sends a V-category (X,a_0) to (TX,Ta_0), and $e_X : X \to TX$ and $m_X : TTX \to TX$ become V-functors for each V-category X. The Eilenberg–Moore algebras for this monad can be described as triples (X,a_0,α) where (X,a_0) is a V-category and (X,α) is an algebra for the **Set**-monad $\mathbb{T}$ such that $\alpha : T(X,a_0) \to (X,a_0)$ is a V-functor. For $\mathbb{T}$-algebras (X,a_0,α) and (Y,b_0,β), a map $f : X \to Y$ is a homomorphism $f : (X,a_0,\alpha) \to (Y,b_0,\beta)$ precisely if f preserves both structures, that is, whenever $f : (X,a_0) \to (Y,b_0)$ is a V-functor and $f : (X,\alpha) \to (Y,\beta)$ is a $\mathbb{T}$-homomorphism.

There are canonical adjoint functors

$$\begin{tikzcd} (V\text{-}\mathbf{Cat})^{\mathbb{T}} \arrow[r, bend left, "K"] \arrow[r, phantom, "\top"] & (\mathbb{T},V)\text{-}\mathbf{Cat}. \arrow[l, bend left, "M"] \end{tikzcd}$$

The functor K associates to each $X = (X, a_0, \alpha)$ in $(V\text{-}\mathbf{Cat})^{\mathbb{T}}$ the $(\mathbb{T}, V)$-category $KX = (X, a)$, where $a = a_0 \cdot \alpha$, and keeps morphisms unchanged. Its left adjoint $M : (\mathbb{T}, V)\text{-}\mathbf{Cat} \to (V\text{-}\mathbf{Cat})^{\mathbb{T}}$ sends a $(\mathbb{T}, V)$-category (X, a) to $(TX, Ta \cdot m_X^\circ, m_X)$ and a $(\mathbb{T}, V)$-functor f to Tf. Via the adjunction $M \dashv K$ one obtains a lifting of the **Set**-monad $\mathbb{T} = (T, m, e)$ to a monad on $(\mathbb{T}, V)$-**Cat**, also denoted by $\mathbb{T} = (T, m, e)$.

In this setting we can define 'duals' in $(V\text{-}\mathbf{Cat})^{\mathbb{T}}$ and carry them into $(\mathbb{T}, V)$-**Cat**. Indeed, since $T : V\text{-}\mathbf{Rel} \to V\text{-}\mathbf{Rel}$ commutes with the involution $(-)^\circ$, for every $\mathbb{T}$-algebra $X = (X, a_0, \alpha)$ also (X, a_0°, α) is a $\mathbb{T}$-algebra. Moreover, if (X, a) is a $(\mathbb{T}, V)$-category, we define X^{op} by mapping (X, a) into $(V\text{-}\mathbf{Cat})^{\mathbb{T}}$ via M, dualizing the image in $(V\text{-}\mathbf{Cat})^{\mathbb{T}}$, and then carrying it back to $(\mathbb{T}, V)$-**Cat**; that is,

$$X^{\mathrm{op}} = K((M(X, a))^{\mathrm{op}}) = (TX, m_X \cdot (Ta)^\circ \cdot m_X).$$

Since the monad $\mathbb{T} = (T, m, e)$ on $(\mathbb{T}, V)$-**Cat** is lax idempotent (i.e., of Kock-Zöberlein type), an algebra structure $\alpha : TX \to X$ on a $(\mathbb{T}, V)$-category X is left adjoint to the unit $e_X : X \to TX$. We call a $(\mathbb{T}, V)$-category X *representable* whenever $e_X : X \to TX$ has a left adjoint in $(\mathbb{T}, V)$-**Cat**; equivalently, whenever there is some $(\mathbb{T}, V)$-functor $\alpha : TX \to X$ with $\alpha \cdot e_X \simeq 1_X$, since then

$$e_X \cdot \alpha = T\alpha \cdot e_{TX} \geq T\alpha \cdot Te_X \simeq 1_{TX}.$$

However, a left adjoint $\alpha : TX \to X$ to e_X is in general only a pseudo-algebra structure on X, that is,

$$\alpha \cdot e_X \simeq 1_X \qquad \text{and} \qquad \alpha \cdot T\alpha \simeq \alpha \cdot m_X.$$

For every representable $(\mathbb{T}, V)$-category (X, a), the structure $a : TX \nrightarrow X$ can be decomposed as $a = a_0 \cdot \alpha$, where $a_0 = a \cdot e_X$ denotes the underlying V-category structure.

A $(\mathbb{T}, V)$-category X is *injective* whenever, for each fully faithful $h : A \to B$ in $(\mathbb{T}, V)$-**Cat** and each $(\mathbb{T}, V)$-functor $f : A \to X$, there is a $(\mathbb{T}, V)$-functor $g : B \to X$ with $g \cdot h \simeq f$.

Proposition 4.1. *Every injective $(\mathbb{T}, V)$-category is representable.*

Proof. Let X be an injective $(\mathbb{T}, V)$-category. The $(\mathbb{T}, V)$-functor $e_X : (X, a) \to (TX, Ta \cdot m_X^\circ \cdot m_X)$ is an embedding. Indeed, e_X is injective because the monad T is non-trivial, and it is fully faithful:

$$e_X^\circ \cdot Ta \cdot m_X^\circ \cdot m_X \cdot Te_X \leq a \cdot Ta \cdot m_X^\circ \leq a \cdot m_X \cdot m_X^\circ \leq a.$$

Hence, there is a $(\mathbb{T}, V)$-functor $\alpha : TX \to X$ with $\alpha \cdot e_X \simeq 1_X$, and so X is representable. $\square$

5. Injective $(\mathbb{T}, V)$-categories are exponentiable

In Section 6 we will show that, under some conditions, $(\mathbb{T}, V)$-**Cat** is weakly cartesian closed. Notably, we will use that every $(\mathbb{T}, V)$-category can be embedded into an injective one; which, by the main result of this section, implies that every $(\mathbb{T}, V)$-category can be embedded into an exponentiable one. We hasten to remark that this is easily seen to be fulfilled for $\mathbb{T}$ being the identity monad, witnessed by the *Yoneda embedding* (see [24])

$$y_X : X \to PX := V^{X^{\mathrm{op}}}.$$

Here PX is the free cocompletion of X; being cocomplete, PX is injective.

To treat the general case, we *will consider from now on only* extensions of the monad $\mathbb{T}$ to V-**Rel** given by a $\mathbb{T}$-algebra structure $\xi : TV \to V$ on V, so that we are dealing with a *strict topological theory* in the sense of [17]. In this case, the extension of $T : \mathbf{Set} \to \mathbf{Set}$ to V-**Rel** is defined by

$$\begin{aligned} &Tr : TX \times TY \to V \\ &\qquad\qquad (\mathfrak{x}, \mathfrak{y}) \mapsto \bigvee \left\{ \xi \cdot Tr(\mathfrak{w}) \;\middle|\; \mathfrak{w} \in T(X \times Y), T\pi_X(\mathfrak{w}) = \mathfrak{x}, T\pi_Y(\mathfrak{w}) = \mathfrak{y} \right\} \end{aligned}$$

for each V-relation $r : X \times Y \to V$.

In order to obtain a Yoneda embedding, we consider the $\mathbb{T}$-algebra $(V, \hom, \xi)$ which is mapped by K into the important $(\mathbb{T}, V)$-category $(V, \hom_\xi)$, where $\hom_\xi = \hom \cdot \xi$. The proof of the following result can be found in [7] and [18].

Theorem 5.1. *If the extension of $\mathbb{T}$ to V-**Rel** is induced by a strict topological theory, then, for every $(\mathbb{T}, V)$-category (X, a), the V-relation $a : TX \to\!\!\!\!\!\!+\; X$ defines a $(\mathbb{T}, V)$-functor*

$$a : X^{\mathrm{op}} \otimes X \to (V, \hom_\xi).$$

Moreover, the $\otimes$-exponential mate $y_X = \ulcorner a \urcorner : X \to V^{X^{\mathrm{op}}}$ of a is fully faithful, and the $(\mathbb{T}, V)$-category $PX = V^{X^{\mathrm{op}}}$ is injective.

*In fact, this construction defines a functor $P : (\mathbb{T}, V)$-**Cat** $\to (\mathbb{T}, V)$-**Cat** and $y = (y_X)_X$ is a natural transformation $y : 1_{(\mathbb{T},V)\text{-}\mathbf{Cat}} \to P$.*

Since y_X is fully faithful, when X is injective there exists a $(\mathbb{T}, V)$-functor $\mathrm{Sup}_X : PX \to X$ such that $\mathrm{Sup}_X \cdot y_X \simeq 1_X$. As shown in [18, Theorem 2.7], $\mathrm{Sup}_X \dashv y_X$. Moreover, for each $(\mathbb{T}, V)$-category (X, a), y_X is one-to-one if, and only if, (X, a) is *separated*, i.e. for every $f, g : (Y, b) \to (X, a)$, $f \simeq g$ only if $f = g$ (see [23], for example). It follows immediately that, for an injective $(\mathbb{T}, V)$-functor $f : X \to Y$ where Y is separated, also X is.

Lemma 5.2. *The $\otimes$-exponential Y^X is separated, for every separated $(\mathbb{T}, V)$-category Y and every representable $(\mathbb{T}, V)$-category X; in particular, PX is separated, for every $(\mathbb{T}, V)$-category X.*

Proof. See [23, Corollary 4.12 (2)]. □

Corollary 5.3. *Every separated $(\mathbb{T}, V)$-category embeds into an injective $(\mathbb{T}, V)$-category.*

In Section 2 we introduced the tensor product $X \otimes Y$ of $(\mathbb{T}, V)$-graphs X and Y. We remark that, in the setting of a strict topological theory, $X \otimes Y$ is a $(\mathbb{T}, V)$-category provided that X and Y are so (see [17]).

The result promised in the title of this section was shown in [19, Proposition 2.7] for the special case of $\otimes = \wedge$:

Proposition 5.4. *If the quantale V is a frame and (3.i) commutes for all V-relations $r : X \to\!\!\!\!\!\!+\; X'$ and $s : Y \to\!\!\!\!\!\!+\; Y'$, then every representable $(\mathbb{T}, V)$-category is exponentiable. In particular, in this case every injective $(\mathbb{T}, V)$-category is exponentiable.*

To treat the general case, we will make use of the following conditions:

Assumptions 5.5.

(1) The diagram (3.i) commutes, for all V-relations $r : X \nrightarrow X'$ and $s : Y \nrightarrow Y'$.
(2) For all $u, v, w \in V$,

$$w \wedge (u \otimes v) = \bigvee \{u' \otimes v' \mid u' \leq u, v' \leq v, u' \otimes v' \leq w\};$$

or, equivalently, every injective V-category is exponentiable: see [20, Theorem 5.3].
(3) For every V-relation $a : X \nrightarrow Y$ and $u \in V$,

$$T(a \otimes u) = Ta \otimes u,$$

where $a \otimes u$ is the V-relation defined by $(a \otimes u)(x, y) = a(x, y) \otimes u$.
(4) The maps $V \otimes V \xrightarrow{\otimes} V$ and $X \xrightarrow{(-,u)} X \otimes V$ are $(\mathbb{T}, V)$-functors, for all $u \in V$.

These morphisms induce an interesting action of V on every injective $(\mathbb{T}, V)$-category (X, a) as follows. The $(\mathbb{T}, V)$-functor

$$X^{\mathrm{op}} \otimes X \otimes V \xrightarrow{a \otimes 1} V \otimes V \xrightarrow{\otimes} V$$

induces a $(\mathbb{T}, V)$-functor $\tilde{a} : X \otimes V \to PX$. We denote the composite

$$X \otimes V \xrightarrow{\tilde{a}} PX \xrightarrow{\mathrm{Sup}_X} X$$

by $\oplus$, and

$$X \xrightarrow{(-,u)} X \otimes V \xrightarrow{\tilde{a}} PX \xrightarrow{\mathrm{Sup}_X} X,$$

assigning to each $x \in X$ an element $x \oplus u$ in X, by $- \oplus u$.

Analogously we will write $\mathfrak{x} \oplus u$ for $T(-\oplus u)(\mathfrak{x})$, for every $\mathfrak{x} \in TX$ and $u \in V$. Note that $(\mathbb{T}, V)$-functoriality of $- \oplus u$ can be written as

$$a(\mathfrak{x}, y) \leq a(\mathfrak{x} \oplus u, y \oplus u),$$

for every $\mathfrak{x} \in TX$ and $y \in X$.

Lemma 5.6. *Assuming 5.5 (4), for an injective $(\mathbb{T}, V)$-category (X, a), with $a = a_0 \cdot \alpha$ as usual, the following holds, for every $x, y \in X$, $\mathfrak{x} \in TX$ and $u \in V$:*

(1) $a_0(x \oplus u, y) = \hom(u, a_0(x, y))$;
(2) $a_0(x, y \oplus u) \geq a_0(x, y) \otimes u$;
(3) $a(\mathfrak{x} \oplus u, y) \geq \hom(u, a(\mathfrak{x}, y))$;
(4) $a(\mathfrak{x}, y \oplus u) \geq a(\mathfrak{x}, y) \otimes u$.

Moreover, if, in addition, 5.5 (3) holds, then, for every $\mathfrak{X} \in T^2X$, $\mathfrak{y} \in TX$, $u \in V$,

(5) $Ta(\mathfrak{X}, \mathfrak{y} \oplus u) \geq Ta(\mathfrak{X}, \mathfrak{y}) \otimes u$.

Proof. (1) For every $x, y \in X$ and $u \in V$,

$$\begin{aligned} a_0(x \oplus u, y) &= a_0(\mathrm{Sup}_X(\tilde{a}(x,u)), y) && \text{(by definition of } \oplus) \\ &= [\tilde{a}(x,u), y_X(y)] && \text{(because } \mathrm{Sup}_X \dashv y_X) \\ &= \bigwedge_{\mathfrak{x} \in TX} \hom(\tilde{a}(x,u)(\mathfrak{x}), y_X(y)(\mathfrak{x})) && \text{(by definition of } [\ ,\ ]) \\ &= \bigwedge_{\mathfrak{x} \in TX} \hom(a(\mathfrak{x}, x) \otimes u, a(\mathfrak{x}, y)) && \text{(by definition of } \tilde{a} \text{ and } y_X(y)) \\ &= \hom(u, a_0(x,y)), \end{aligned}$$

because, using the fact that $a = a_0 \cdot \alpha$ and

$$a_0(\alpha(\mathfrak{x}), x) \otimes u \otimes \hom(u, a_0(x,y)) \le a_0(\alpha(\mathfrak{x}), x) \otimes a_0(x,y) \le a_0(\alpha(\mathfrak{x}), y),$$

for $\mathfrak{x} \in TX$, we can conclude that

$$\hom(u, a_0(x,y)) \le \bigwedge_{\mathfrak{x} \in TX} \hom(a_0(\alpha(\mathfrak{x}), x) \otimes u, a_0(\alpha(\mathfrak{x}), y)).$$

Taking $\mathfrak{x} = e_X(x)$, we see that this inequality is in fact an equality as claimed.

(2) Since, by hypothesis, $- \oplus u$ is a $(\mathbb{T}, V)$-functor, and so, in particular, a V-functor $(X, a_0) \to (X, a_0)$,

$$a_0(x,y) \le a_0(x \oplus u, y \oplus u) = \hom(u, a_0(x, y \oplus u)),$$

and then

$$a_0(x,y) \otimes u \le \hom(u, a_0(x, y \oplus u)) \otimes u \le a_0(x, y \oplus u).$$

(3) One has

$$\begin{aligned} k &\le a_0(\alpha(\mathfrak{x}), \alpha(\mathfrak{x})) = a(\mathfrak{x}, \alpha(\mathfrak{x})) \\ &\le a(\mathfrak{x} \oplus u, \alpha(\mathfrak{x}) \oplus u) \\ &= a_0(\alpha(\mathfrak{x} \oplus u), \alpha(\mathfrak{x}) \oplus u). \end{aligned}$$

Using (1) we conclude that

$$\begin{aligned} \hom(u, a(\mathfrak{x}, y)) &= a_0(\alpha(\mathfrak{x}) \oplus u, y) \\ &\le a_0(\alpha(\mathfrak{x} \oplus u), \alpha(\mathfrak{x}) \oplus u) \otimes a_0(\alpha(\mathfrak{x}) \oplus u, y) \\ &\le a_0(\alpha(\mathfrak{x} \oplus u), y) = a(\mathfrak{x} \oplus u, y). \end{aligned}$$

(4) follows directly from (2), while (5) follows from (4). $\square$

Lemma 5.7. *Let $\varphi : V \to W$ be a surjective quantale homomorphism; that is, φ preserves the tensor, the neutral element, and suprema. Then, if V satisfies condition 5.5 (2), so does W.*

Theorem 5.8. *Under Assumptions 5.5, every injective $(\mathbb{T}, V)$-category is exponentiable in $(\mathbb{T}, V)$-***Cat***.*

Proof. Let $\mathfrak{X} \in T^2X$, $x \in X$ and $u, v \in V$. In order to conclude that

$$\bigvee_{\mathfrak{x} \in TX} (Ta(\mathfrak{X}, \mathfrak{x}) \wedge u) \otimes (a(\mathfrak{x}, x) \wedge v) \geq a(m_X(\mathfrak{X}), x) \wedge (u \otimes v),$$

we make use of Hypothesis 5.5 (2). Let $u', v' \in V$ with $u' \leq u$, $v' \leq v$ and $u' \otimes v' \leq a(m_X(\mathfrak{X}), x)$. First we note that

$$\begin{aligned} Ta(\mathfrak{X}, T\alpha(\mathfrak{X}) \oplus u') \wedge u &\geq (Ta(\mathfrak{X}, T\alpha(\mathfrak{X})) \otimes u') \wedge u && \text{(by 5.6 (5))} \\ &= (Ta_0(T\alpha(\mathfrak{X}), T\alpha(\mathfrak{X})) \otimes u') \wedge u \\ &\geq (k \otimes u') \wedge u = u', \end{aligned}$$

and

$$\begin{aligned} a(T\alpha(\mathfrak{X}) \oplus u', x) &\geq \hom(u', a(T\alpha(\mathfrak{X}), x)) && \text{(by 5.6 (3))} \\ &= \hom(u', a_0(\alpha(T\alpha(\mathfrak{X})), x)) \\ &= \hom(u', a_0(\alpha(m_X(\mathfrak{X})), x)) \\ &= \hom(u', a(m_X(\mathfrak{X}), x)). \end{aligned}$$

Now, from $u' \otimes v' \leq a(m_X(\mathfrak{X}), x)$ and $v' \leq v$ we get

$$v' \leq \hom(u', a(m_X(\mathfrak{X}), x)) \wedge v \leq a(T\alpha(\mathfrak{X}) \oplus u', x) \wedge v,$$

hence

$$u' \otimes v' \leq (Ta(\mathfrak{X}, T\alpha(\mathfrak{X}) \oplus u') \wedge u) \otimes (a(T\alpha(\mathfrak{X}) \oplus u', x) \wedge v).$$

Therefore $a(m_X(\mathfrak{X}), x) \wedge (u \otimes v) \leq \bigvee_{\mathfrak{x} \in TX} (Ta(\mathfrak{X}, \mathfrak{x}) \wedge u) \otimes (a(\mathfrak{x}, x) \wedge v)$. $\square$

Remark 5.9. Under Assumptions 5.5, it follows from Lemma 5.2 that the exponential $\langle (X, a), (Y, b) \rangle$ is separated, for all separated injective $(\mathbb{T}, V)$-categories (X, a) and (Y, b). In fact, with $a = a_0 \cdot \alpha$, the epimorphism $(X, \alpha) \to (X, a)$ in $(\mathbb{T}, V)$-**Cat** is mapped to the monomorphism

$$\langle (X, a), (Y, b) \rangle \quad \longrightarrow \quad \langle (X, \alpha), (Y, b) \rangle = (Y, b)^{(X, \alpha)},$$

which proves that $\langle (X, a), (Y, b) \rangle$ is separated.

6. $(\mathbb{T}, V)$-Cat is weakly cartesian closed

Building on the results of the previous section, in this section we show that, under some conditions, $(\mathbb{T}, V)$-**Cat** is weakly cartesian closed. We start by proving this property for the full subcategory $(\mathbb{T}, V)$-$\mathbf{Cat}_{\mathrm{sep}}$ of $(\mathbb{T}, V)$-**Cat** of separated $(\mathbb{T}, V)$-categories.

Theorem 6.1. *Under Assumptions 5.5, $(\mathbb{T}, V)$-$\mathbf{Cat}_{\mathrm{sep}}$ is weakly cartesian closed.*

Proof. For X, Y separated $(\mathbb{T}, V)$-categories, consider the Yoneda embeddings $y_X : X \to PX$ and $y_Y : Y \to PY$, and the exponential $\langle PX, PY \rangle$. The elements of its underlying set can be identified with

$(\mathbb{T},V)$-functors $E\times PX\to PY$ (where $E=(1,e_1^\circ)$ is the generator of $(\mathbb{T},V)$-**Cat**), and the universal morphism $\mathrm{ev}:\langle PX,PY\rangle\times PX\to PY$ with the evaluation map: $\mathrm{ev}(\varphi,\mathfrak{x})=\varphi(\mathfrak{x})$ (where, for simplicity, we identify the set $E\times PX$ with PX). We can therefore consider

$$\ll X,Y\gg=\{\varphi:E\times PX\to PY\mid \varphi(y_X(X))\subseteq y_Y(Y)\},$$

with the initial structure with respect to the inclusion $\iota:\ll X,Y\gg\to\langle PX,PY\rangle$. Moreover, the morphism

$$\ll X,Y\gg\times X\xrightarrow{\ \iota\times y_X\ }\langle PX,PY\rangle\times PX\xrightarrow{\ \mathrm{ev}\ }PY$$

factors through y_Y via a morphism

$$\ll X,Y\gg\times X\xrightarrow{\ \widetilde{\mathrm{ev}}\ }Y.$$

Next we show that this is a weak exponential in $(\mathbb{T},V)$-$\mathbf{Cat}_{\mathrm{sep}}$.

Given any separated $(\mathbb{T},V)$-category Z, and a $(\mathbb{T},V)$-functor $f:Z\times X\to Y$, by injectivity of PY there exists a $(\mathbb{T},V)$-functor $f':Z\times PX\to PY$ making the square below commute. Then, by universality of the evaluation map ev, there exists a unique $(\mathbb{T},V)$-functor $\overline{f}:Z\to\langle PX,PY\rangle$ making the bottom triangle commute.

$$\begin{tikzcd}
Z\times X \arrow[r, "f"] \arrow[d, "1_Z\times y_X"'] & Y \arrow[d, "y_Y"] \\
Z\times PX \arrow[r, "f'"] \arrow[d, "\overline{f}\times 1_{PX}"'] & PY \\
\langle PX,PY\rangle\times PX \arrow[ur, "\mathrm{ev}"'] &
\end{tikzcd}$$

The map $\overline{f}:Z\to\langle PX,PY\rangle$, assigning to each $z\in Z$ a map $\overline{f}(z):PX\to PY$, is such that $\overline{f}(z)(y_X(x))=\mathrm{ev}(\overline{f}(z),y_X(x))=y_Y(f(z,x))$; that is, $\overline{f}(z)(y_X(X))\subseteq y_Y(Y)$, and this means that $\overline{f}(z)\in\ll X,Y\gg$. Hence we can consider the corestriction $\tilde{f}$ of $\overline{f}$ to $\ll X,Y\gg$, which is again a $(\mathbb{T},V)$-functor since $\ll X,Y\gg$ has the initial structure with respect to $\langle PX,PY\rangle$, so that the following diagram commutes.

$$\begin{tikzcd}
\ll X,Y\gg\times X \arrow[r, "\widetilde{\mathrm{ev}}"] & Y \\
Z\times X \arrow[u, "\tilde{f}\times 1_X"] \arrow[ur, "f"'] &
\end{tikzcd}\qquad\square$$

In order to show that $(\mathbb{T},V)$-**Cat** is weakly cartesian closed, we follow the proof of [30]. Hence, first we show that:

Proposition 6.2. *The reflector* $R:(\mathbb{T},V)\text{-}\mathbf{Cat}\to(\mathbb{T},V)\text{-}\mathbf{Cat}_{\mathrm{sep}}$ *preserves finite products.*

Proof. We recall that, for any $(\mathbb{T},V)$-category (X,a), $R(X,a)=(\tilde{X},\tilde{a})$, with $\tilde{X}=X/\sim$, where $x\sim y$ if $k\le a(e_X(x),y)\wedge a(e_X(y),x)$, and $\tilde{a}=\eta_X\cdot a\cdot(T\eta_X)^\circ$, with $\eta_X:X\to\tilde{X}$ the projection. This structure makes η_X both an initial and a final morphism (see [22] for details).

Let $f : R(X \times Y) \to RX \times RY$ be the unique morphism such that $f \cdot \eta_{X\times Y} = \eta_X \times \eta_Y$.

$$\begin{tikzcd}
(X \times Y, c) \arrow[r, "\eta_{X\times Y}"] \arrow[dr, "\eta_X\times\eta_Y"'] & (R(X \times Y), \tilde{c}) \arrow[d, "f"] \\
& (RX \times RY, d)
\end{tikzcd}$$

From $c(e_{X\times Y}(x, y), (x', y')) = a(e_X(x), x') \wedge b(e_Y(y), y')$ it is immediate that $(x, y) \sim (x', y')$ in $X \times Y$ if, and only if, $x \sim x'$ in X and $y \sim y'$ in Y. Therefore, f is a bijection. Assuming the Axiom of Choice, so that T preserves surjections, we have, for every $\mathfrak{z} \in T(R(X \times Y))$, $(x, y) \in X \times Y$,

$$\begin{aligned}
\tilde{c}(\mathfrak{z}, [(x, y)]) &= c(\mathfrak{w}, (x, y)) && (\text{for any } \mathfrak{w} \in (T\eta_{X\times Y})^{-1}(\mathfrak{z}))\\
&= d(T(\eta_X \times \eta_Y)(\mathfrak{w}), ([x], [y])) && (\text{because } \eta_X \times \eta_Y \text{ is initial})\\
&= d(Tf(\mathfrak{z}), ([x], [y]);
\end{aligned}$$

that is, f is initial and therefore an isomorphism. □

Theorem 6.3. *Under Assumptions 5.5, $(\mathbb{T}, V)$-**Cat** is weakly cartesian closed.*

Proof. Given $(\mathbb{T}, V)$-categories (X, a), (Y, b), to build the weak exponential $\ll X, Y \gg$ we will show the *cosolution set condition* for the functor $- \times (X, a)$.

For each $(\mathbb{T}, V)$-functor $f : (Z, c)\times(X, a) \to (Y, b)$ we take its reflection $Rf : RZ\times RX \cong R(Z\times X) \to RY$ and we factorise it through the weak evaluation in $(\mathbb{T}, V)$-$\mathbf{Cat}_{\mathrm{sep}}$, $Rf = \widetilde{\mathrm{ev}} \cdot (\overline{Rf} \times 1_{RX})$, so that in the diagram below the outer rectangle commutes.

Then we define $Z_f = Z/\sim$ by

$$z \sim z' \text{ if both } f(z, x) = f(z', x), \text{ for every } x \in X, \text{ and } \overline{Rf}(\eta_Z(z)) = \overline{Rf}(\eta_Z(z')),$$

and equip it with the final structure for the projection $q_f : Z \to Z_f$. Then $h_f : Z_f \to \ll RX, RY \gg$, with $h_f([z]) = \overline{Rf}(\eta_Z(z))$, is a $(\mathbb{T}, V)$-functor since its composition with q_f is $\overline{Rf} \cdot \eta_Z$ and q_f is final. Then we factorise f via the surjection $q_f \times 1_X : Z \times X \to Z_f \times X$ as in the diagram below. Moreover, the map $\hat{f} : Z_f \times X \to Y$, with $\hat{f}([z], x) = f(z, x)$, is a $(\mathbb{T}, V)$-functor because $\eta_Y \cdot \hat{f} = \widetilde{\mathrm{ev}} \cdot (h_f \times \eta_X)$ is and η_Y is initial.

$$\begin{tikzcd}
Z \times X \arrow[rrr, "f"] \arrow[d, "\eta_Z\times 1_X"'] \arrow[dr, "q_f\times 1_X"] & & & Y \arrow[dd, "\eta_Y"] \\
RZ \times X \arrow[d, "\overline{Rf}\times 1_X"'] & Z_f \times X \arrow[r] \arrow[urr, "\hat{f}"] \arrow[dl, "h_f\times 1_X"] & (\coprod_g Z_g \times X) \cong (\coprod_g Z_g) \times X \arrow[ur, "\mathrm{ev}"] & \\
\ll RX, RY \gg \times X \arrow[r, "1\times\eta_X"'] & \ll RX, RY \gg \times RX \arrow[rr, "\widetilde{\mathrm{ev}}"'] & & RY
\end{tikzcd}$$

Since the cardinality of Z_f is bounded by the cardinality of the set $|\ll RX, RY \gg| \times |Y|^{|X|}$, as witnessed by the injective map

$$\begin{aligned} Z_f &\to |\ll RX, RY \gg| \times |Y|^{|X|}, \\ [z] &\mapsto (\overline{Rf}(\eta_Z(z)), f(z,-)) \end{aligned}$$

there is only a set of possible $(\mathbb{T}, V)$-categories Z_f. Hence we can form its coproduct, as in the diagram above, and consider the induced $(\mathbb{T}, V)$-functor $\mathrm{ev} : (\coprod_g Z_g) \times X \cong \coprod_g (Z_g \times X) \to Y$ (note that the isomorphism follows from extensivity of $(\mathbb{T}, V)$-**Cat**). $\square$

7. Examples

In this section we use Theorem 6.3 to present examples of weakly cartesian closed categories. Hence, in conjunction with the following theorem established in [30], we obtain examples of categories with cartesian closed exact completion since all other conditions of that theorem are trivially satisfied in these examples.

Theorem 7.1. *Let* $\mathbf{C}$ *be a complete, infinitely extensive and well-powered category in which every morphism factorizes as a regular epi followed by a mono, and where* $f \times 1$ *is an epimorphism for every regular epimorphism* $f : A \to B$ *in* $\mathbf{C}$*. Then, if* $\mathbf{C}$ *is weakly cartesian closed, the exact completion* $\mathbf{C}_{\mathrm{ex}}$ *of* $\mathbf{C}$ *is cartesian closed.*

We note that, in order to conclude that $(\mathbb{T}, V)$-**Cat** is weakly cartesian closed, we have to check whether V and $\mathbb{T}$ satisfy Assumptions 5.5.

First we analyse examples where $\mathbb{T}$ is the identity monad. In this particular setting we only have to check that 5.5 (2) holds. The category V-**Cat** is always monoidal closed, as shown in [24]. Therefore, when V is a frame considered as a quantale, then V-**Cat** is cartesian closed. This is the case of 2, and so one concludes that **Ord** *is cartesian closed*. Moreover, for V the lattice $([0,\infty], \geq)$ with $\otimes = \wedge$, V-**Cat** is *the category of ultrametric spaces*, which *is* therefore also *cartesian closed*.

When $V = P_+$, V-**Cat** is the category **Met** of Lawvere's metric spaces [24], which is not cartesian closed (see [6] for details). However, the quantale P_+ satisfies 5.5 (2), and so from Theorem 6.3 it follows that **Met** *is weakly cartesian closed*.

Metric and ultrametric spaces can be also viewed as categories enriched in a quantale based on the complete lattice $[0,1]$ with the usual "less or equal" relation $\leq$, which is isomorphic to $[0,\infty]$ via the map $[0,1] \to [0,\infty]$, $u \mapsto -\ln(u)$ where $-\ln(0) = \infty$. More in detail, we consider the following quantale operations on $[0,1]$ with neutral element 1.

(1) For $\otimes = *$ being the ordinary multiplication, via the isomorphism $[0,1] \simeq [0,\infty]$, this quantale is isomorphic to the quantale P_+, hence $[0,1]$-**Cat** $\simeq$ **Met**.
(2) For the tensor $\otimes = \wedge$ being infimum, the isomorphism $[0,1] \simeq [0,\infty]$ establishes an equivalence between $[0,1]$-**Cat** and the category of ultrametric spaces and non-expansive maps.
(3) Another interesting multiplication on $[0,1]$ is the *Łukasiewicz tensor* $\otimes = \odot$ given by

$$u \odot v = \max(0, u + v - 1).$$

Via the lattice isomorphism $[0,1] \to [0,1]$, $u \mapsto 1 - u$, this quantale is isomorphic to the quantale $[0,1]$ with "greater or equal" relation $\geq$ and tensor $u \otimes v = \min(1, u + v)$ truncated addition. Therefore $[0,1]$-**Cat** is equivalent to the *category of bounded-by-*1 *metric spaces and non-expansive maps*. Moreover, with respect to the "greater or equal" relation and truncated addition on $[0,1]$, the map

$$[0,\infty] \to [0,1],\ u \mapsto \min(1,u)$$

is a surjective quantale morphism; therefore, by Lemma 5.7, also $[0,1]$ with the Łukasiewicz tensor satisfies 5.5 (2).

(4) More generally, every continuous quantale structure $\otimes$ on the lattice $[0,1]$ (with Euclidean topology and the usual "less or equal" relation) with neutral element 1 satisfies 5.5 (2). This can be shown using the fact, proven in [13] and [27], that every such tensor $\otimes : [0,1]\times[0,1] \to [0,1]$ is a combination of the three operations on $[0,1]$ described above. More precise:
   (a) For $u,v \in [0,1]$ and $e \in [0,1]$ idempotent with $u \le e \le v$: $u \otimes v = \min(u,v) = u$.
   (b) For every non-idempotent $u \in [0,1]$, there exist idempotents e and f with $e < u < f$ and such that the interval $[e,f]$ (with the restriction of the tensor on $[0,1]$ and with neutral element f) is isomorphic to $[0,1]$ either with multiplication or Łukasiewicz tensor.

Now let $w,u,v \in [0,1]$. We may assume $u \le v$. If $u \otimes v \le w$, then clearly

$$w \wedge (u \otimes v) = u \otimes v = \bigvee\{u' \otimes v' \mid u' \le u, v' \le v, u' \otimes v' \le w\}.$$

We consider now $w < u \otimes v \le u \le v$. If w is idempotent, then

$$w = w \otimes v, \quad w \le u, \quad v \le v;$$

otherwise there are idempotents e and f with $e < w < f$ and $[e,f]$ is isomorphic to $[0,1]$ either with multiplication or Łukasiewicz tensor.

Case 1: $v \le f$. Then 5.5 (2) holds since $w, u\otimes v, u, v \in [e,f]$.

Case 2: $f < v$. Then $w = w \wedge v = w \otimes v$, $w \le u$ and $v \le v$.

We conclude that $[0,1]$-**Cat** *is weakly cartesian closed*, for every continuous quantale structure $\otimes$ on the lattice $[0,1]$ with neutral element 1.

Now let $V = \Delta$ be the *quantale of distribution functions* (see [20,8] for details). As observed in [20], it verifies 5.5 (2), and so we can conclude from Theorem 6.3 that *the category* Δ-**Cat** *of probabilistic metric spaces and non-expansive maps is weakly cartesian closed.*

When $\mathbb{T}$ is not the identity monad, some further work is need to guarantee Assumptions 5.5.

Theorem 7.2.

(1) The tensor product on the quantale V defines a $(\mathbb{T},V)$-functor $\otimes : V \otimes V \to V$.

(2) Let $u \in V$ satisfying $u\cdot! \ge \xi \cdot Tu$.

$$\begin{array}{ccc} T1 & \xrightarrow{Tu} & TV \\ {\scriptstyle !}\downarrow & \ge & \downarrow{\scriptstyle \xi} \\ 1 & \xrightarrow[u]{} & V \end{array}$$

Then $(-,u) : X \to X \times V$ is a $(\mathbb{T},V)$-functor, for every $(\mathbb{T},V)$-category X.

(3) Let $u \in V$ satisfying $u\cdot! = \xi \cdot Tu$. Then $T(r \otimes u) = (Tr) \otimes u$, for every V-relation $r : X \to\!\!\!\!\!\mapsto Y$.

Proof. The first assertion is [18, Proposition 1.4(1)]. To see (2), assume that $u \in V$ with $u\cdot! \ge \xi \cdot Tu$. Let (X,a) be a $(\mathbb{T},V)$-category, $\mathfrak{x} \in TX$ and $x \in X$. Considering the map $X \xrightarrow{!} 1 \xrightarrow{u} V$, we have to show that

$$a(\mathfrak{x},x) \le a(\mathfrak{x},x) \otimes \hom(T(u\cdot !)(\mathfrak{x}), u),$$

which follows immediately from $u{\cdot}! \geq \xi \cdot Tu$. Finally, to prove (3), let $r : X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$ be a V-relation and $u \in V$ with $u{\cdot}! = \xi \cdot Tu$. Note that the V-relation $r \otimes u : X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$ is given by

$$X \times Y \xrightarrow{\ r\ } V \xrightarrow{\langle 1_V, u\cdot ! \rangle} V \times V \xrightarrow{\ \otimes\ } V.$$

Hence, applying the **Set**-functor T to the functions $r : X \times Y \to V$ and $r \otimes u : X \times Y \to V$, we obtain

$$\begin{aligned} \xi \cdot T(r \otimes u) &= \xi \cdot T(\otimes) \cdot T\langle 1_V, u{\cdot}! \rangle \cdot Tr \\ &= \otimes \cdot (\xi \times \xi) \cdot \mathrm{can}_{X,Y} \cdot T\langle 1_V, u{\cdot}! \rangle \cdot Tr \\ &= \otimes \cdot \langle \xi, u{\cdot}! \cdot \xi \rangle \cdot Tr \\ &= \otimes \cdot \langle 1_V, u{\cdot}! \rangle \cdot \xi \cdot Tr. \end{aligned}$$

Therefore, returning to V-relations, we conclude that $T(r \otimes u) = (Tr) \otimes u$. □

Remark 7.3. If $T1 = 1$, then $u{\cdot}! = \xi \cdot Tu$ for every $u \in V$.

In order to guarantee Assumptions 5.5 (1), we need an extra condition on ξ.

Proposition 7.4. *Assume that*

$$\begin{tikzcd} T(V \times V) \arrow[r, "T(\wedge)"] \arrow[d, "\langle \xi\cdot T\pi_1, \xi\cdot T\pi_2 \rangle"'] \arrow[dr, phantom, "\leq"] & TV \arrow[d, "\xi"] \\ V \times V \arrow[r, "\wedge"'] & V. \end{tikzcd}$$

Then, for all V-relations $r : X \relbar\joinrel\mapstochar\joinrel\rightarrow X'$ and $s : Y \relbar\joinrel\mapstochar\joinrel\rightarrow Y'$,

$$\begin{tikzcd} T(X \times Y) \arrow[r, "\mathrm{can}_{X,Y}"] \arrow[d, "T(r \owedge s)"', "|" marking] \arrow[dr, phantom, "\geq"] & TX \times TY \arrow[d, "Tr \owedge Ts", "|" marking] \\ T(X' \times Y') \arrow[r, "\mathrm{can}_{X',Y'}"'] & TX' \times TY'. \end{tikzcd}$$

Proof. First we note that, from the preservation of weak pullbacks by T, it follows that the commutative diagram

$$\begin{tikzcd} T(A \times B) \arrow[r, "T(f\times g)"] \arrow[d, "\mathrm{can}_{A,B}"'] & T(X \times Y) \arrow[d, "\mathrm{can}_{X,Y}"] \\ TA \times TB \arrow[r, "Tf\times Tg"'] & TX \times TY \end{tikzcd}$$

is also a weak pullback.

Let $\mathfrak{w} \in T(X \times Y)$, $\mathfrak{x}' \in TX'$ and $\mathfrak{y}' \in TY'$. Put $(\mathfrak{x}, \mathfrak{y}) = \mathrm{can}_{X,Y}(\mathfrak{w})$. By the definition of the extension of T and since V is a Heyting algebra,

$$Tr(\mathfrak{x},\mathfrak{x}')\wedge Ts(\mathfrak{y},\mathfrak{y}')=\bigvee\left\{\xi\cdot Tr(\mathfrak{w}_1)\wedge\xi\cdot Ts(\mathfrak{w}_2)\;\middle|\;\begin{matrix}\mathfrak{w}_1\in T(X\times X'):\mathfrak{w}_1\mapsto\mathfrak{x},\mathfrak{w}_1\mapsto\mathfrak{x}'\\ \mathfrak{w}_2\in T(Y\times Y'):\mathfrak{w}_2\mapsto\mathfrak{y},\mathfrak{w}_2\mapsto\mathfrak{y}'\end{matrix}\right\}.$$

Note that in

$$\begin{tikzcd}
 & T(X\times Y\times X'\times Y') \arrow[d, "\cong"'] & & \\
T(X\times Y) \arrow[d, "\mathrm{can}"'] & T(X\times X'\times Y\times Y') \arrow[l, "T(\pi_X\times\pi_Y)"'] \arrow[r, "T(r\times s)"] \arrow[d, "\mathrm{can}"'] & T(V\times V) \arrow[r, "T(\wedge)"] \arrow[d, "\mathrm{can}"'] & TV \arrow[dd, "\xi"] \\
TX\times TY & T(X\times X')\times T(Y\times Y') \arrow[l, "T\pi_X\times T\pi_Y"] \arrow[r, "Tr\times Ts"'] & TV\times TV \arrow[d, "\xi\times\xi"'] \arrow[r, phantom, "\leq"] & \\
 & & V\times V \arrow[r, "\wedge"'] & V
\end{tikzcd}$$

the left hand side is a weak pullback, the middle diagram commutes, and in the right hand side we have "lower path" $\leq$ "upper path" as indicated. Therefore, for such $\mathfrak{w}_1\in T(X\times X')$ and $\mathfrak{w}_2\in T(Y\times Y')$, there exists some $\mathfrak{v}\in T(X\times X'\times Y\times Y')$ which projects to $\mathfrak{w}\in T(X\times Y)$ and to $(\mathfrak{w}_1,\mathfrak{w}_2)\in T(X\times X')\times T(Y\times Y')$. Hence, taking also into account the definition of the V-relation $T(r\circledcirc s)$,

$$\begin{aligned}Tr(\mathfrak{x},\mathfrak{x}')\wedge Ts(\mathfrak{y},\mathfrak{y}')&\leq\bigvee\left\{\xi\cdot T(\wedge)\cdot T(r\times s)(\mathfrak{v})\;\middle|\;\mathfrak{v}\in T(X\times Y\times X'\times Y');\begin{matrix}\mathfrak{v}\mapsto\mathfrak{w}\\ \mathfrak{v}\mapsto\mathfrak{x}',\mathfrak{v}\mapsto\mathfrak{y}'\end{matrix}\right\}\\ &\leq\bigvee\{T(r\circledcirc s)(\mathfrak{w},\mathfrak{w}')\mid\mathfrak{w}'\in T(X'\times Y'),\ \mathrm{can}_{X',Y'}(\mathfrak{w}')=(\mathfrak{x}',\mathfrak{y}')\}.\qquad\Box\end{aligned}$$

Remark 7.5. We note that the inequality

$$\begin{tikzcd}
T(V\times V) \arrow[r, "T(\wedge)"] \arrow[d, "\langle\xi\cdot T\pi_1,\xi\cdot T\pi_2\rangle"'] \arrow[dr, phantom, "\geq"] & TV \arrow[d, "\xi"] \\
V\times V \arrow[r, "\wedge"'] & V
\end{tikzcd}$$

is always true.

Corollary 7.6. *If the quantale V satisfies Assumption 5.5 (2) and the diagrams*

$$\begin{tikzcd}
T(V\times V) \arrow[r, "T(\wedge)"] \arrow[d, "\langle\xi\cdot T\pi_1,\xi\cdot T\pi_2\rangle"'] & TV \arrow[d, "\xi"] \\
V\times V \arrow[r, "\wedge"'] & V
\end{tikzcd}
\quad\text{and}\quad
\begin{tikzcd}
T1 \arrow[r, "Tu"] \arrow[d, "!"'] & TV \arrow[d, "\xi"] \\
1 \arrow[r, "u"'] & V
\end{tikzcd}$$

commute, for all $u\in V$, then all Assumptions 5.5 are satisfied.

Let $\mathbb{T}$ be the ultrafilter monad $\mathbb{U}=(U,m,e)$. Then, when V is any of the quantales listed above but Δ, all the needed conditions are satisfied. Therefore, in particular we can conclude that:

Examples 7.7.

(1) *The category* **Top** $= (\mathbb{U}, 2)$-**Cat** *of topological spaces and continuous maps is weakly cartesian closed* (as shown by Rosický in [30]).
(2) *The category* **App** $= (\mathbb{U}, P_+)$-**Cat** *of approach spaces and non-expansive maps is weakly cartesian closed.*
(3) In fact, for each continuous quantale structure on the lattice $([0, 1], \leq) \simeq ([0, \infty], \geq)$, $(\mathbb{U}, [0, 1])$-**Cat** is weakly cartesian closed. In particular, *the category of non-Archimedean approach spaces and non-expansive maps studied in [12] is weakly cartesian closed.*
(4) If V is a completely distributive complete lattice with $\otimes = \wedge$, then, with

$$\xi : UV \to V,\ \mathfrak{x} \mapsto \bigwedge_{A\in\mathfrak{x}} \bigvee A,$$

all the conditions of Theorem 6.3 are satisfied (see [17, Theorem 3.3]) and therefore $(\mathbb{U}, V)$-**Cat** is weakly cartesian closed. In particular, with $V = P2$ being the powerset of a 2-element set, we obtain that *the category* **BiTop** *of bitopological spaces and bicontinuous maps is weakly cartesian closed* (see [22]).

Remark 7.8. For $V = \Delta$ the quantale of distribution functions, we do not know whether there is an appropriate compact Hausdorff topology $\xi : UV \to V$ satisfying the conditions of this section.

Now let $\mathbb{T}$ be the free monoid monad $\mathbb{W} = (W, m, e)$. For each quantale V, we consider

$$\xi : WV \to V,\ (v_1, \dots, v_n) \mapsto v_1 \otimes \cdots \otimes v_n,\ () \mapsto k$$

which induces the extension $W : V\text{-}\mathbf{Rel} \to V\text{-}\mathbf{Rel}$ sending $r : X \to\!\!\!\!\!\to Y$ to the V-relation $Wr : WX \to\!\!\!\!\!\to WY$ given by

$$Wr((x_1, \dots, x_n), (y_1, \dots, y_m)) = \begin{cases} r(x_1, y_1) \otimes \cdots \otimes r(x_n, y_n) & \text{if } n = m, \\ \perp & \text{if } n \neq m. \end{cases}$$

The category $(\mathbb{W}, 2)$-**Cat** is equivalent to the category **MultiOrd** of *multi-ordered sets* and their morphisms (see [22]), more generally, $(\mathbb{W}, V)$-categories can be interpreted as multi-V-categories and their morphisms. The representable multi-ordered sets are precisely the ordered monoids, which is a special case of [14,15] describing monoidal categories as representable multi-categories (see also [4]). We recall that the separated injective multi-ordered sets are precisely the quantales (see [25] and also [31]), and we conclude:

Proposition 7.9. *Every quantale is exponentiable in* **MultiOrd**.

Theorem 7.10. *If the quantale V is a frame, then* $(\mathbb{W}, V)$-**Cat** *is weakly cartesian closed. In particular,* **MultiOrd** *is weakly cartesian closed.*

Finally, for a monoid $(H, \cdot, h)$, we consider the monad $\mathbb{H} = (- \times H, m, e)$, with $m_X : X \times H \times H \to X \times H$ given by $m_X(x, a, b) = (x, a \cdot b)$ and $e_X : X \to X \times H$ given by $e_X(x) = (x, h)$. Here we consider

$$\xi : V \times H \to V,\ (v, a) \mapsto v,$$

which leads to the extension $- \times H : V\text{-}\mathbf{Rel} \to V\text{-}\mathbf{Rel}$ sending the V-relation $r : X \to\!\!\!\!\!\to Y$ to the V-relation $r \times H : X \times H \to\!\!\!\!\!\to Y \times H$ with

$$r \times H((x,a),(y,b)) = \begin{cases} r(x,y) & \text{if } a = b, \\ \perp & \text{if } a \neq b. \end{cases}$$

In particular, $(\mathbb{H}, 2)$-categories can be interpreted as *H-labelled ordered sets* and *equivariant maps*.

For every quantale V and every $v : 1 \to V$, the diagrams

$$\begin{tikzcd} V \times V \times H \arrow[r, "\wedge \times 1_H"] \arrow[d, "\pi_{1,2}"'] & V \times H \arrow[d, "\xi = \pi_1"] \\ V \times V \arrow[r, "\wedge"'] & V \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} 1 \times H \arrow[r, "v \times 1_H"] \arrow[d, "!"'] & V \times H \arrow[d, "\xi"] \\ 1 \arrow[r, "v"'] & V \end{tikzcd}$$

commute, therefore we obtain:

Theorem 7.11. *For every quantale V satisfying Assumption 5.5 (2), the category $(\mathbb{H}, V)$-**Cat** is weakly cartesian closed.*

References

[1] Jiří Adámek, Horst Herrlich, George E. Strecker, Abstract and Concrete Categories: The Joy of Cats, Pure and Applied Mathematics (New York), John Wiley & Sons Inc., New York, 1990, Republished in: Reprints in Theory Appl. Categ. 17 (2006) 1–507.

[2] Jiří Adámek, Jiří Rosický, How nice are free completions of categories?, Topol. Appl. (2019) (to appear), arXiv:1806.02524 [math.CT].

[3] Andrej Bauer, Lars Birkedal, Dana S. Scott, Equilogical spaces, Theor. Comput. Sci. 315 (1) (2004) 35–59.

[4] Dimitri Chikhladze, Maria Manuel Clementino, Dirk Hofmann, Representable $(\mathbb{T}, \mathcal{V})$-categories, Appl. Categ. Struct. 23 (6) (January 2015) 829–858, http://www.mat.uc.pt/preprints/ps/p1247.pdf.

[5] Maria Manuel Clementino, Dirk Hofmann, Topological features of lax algebras, Appl. Categ. Struct. 11 (3) (June 2003) 267–286, http://www.mat.uc.pt/preprints/ps/p0109.ps.

[6] Maria Manuel Clementino, Dirk Hofmann, Exponentiation in V-categories, Topol. Appl. 153 (16) (October 2006) 3113–3128.

[7] Maria Manuel Clementino, Dirk Hofmann, Lawvere completeness in topology, Appl. Categ. Struct. 17 (2) (August 2009) 175–210, arXiv:0704.3976 [math.CT].

[8] Maria Manuel Clementino, Dirk Hofmann, The rise and fall of V-functors, Fuzzy Sets Syst. 321 (August 2017) 29–49, http://www.mat.uc.pt/preprints/ps/p1606.pdf.

[9] Maria Manuel Clementino, Dirk Hofmann, George Janelidze, The monads of classical algebra are seldom weakly cartesian, J. Homotopy Relat. Struct. 9 (1) (November 2014) 175–197, http://www.mat.uc.pt/preprints/ps/p1246.pdf.

[10] Maria Manuel Clementino, Dirk Hofmann, Walter Tholen, Exponentiability in categories of lax algebras, Theory Appl. Categ. 11 (15) (2003) 337–352, http://www.mat.uc.pt/preprints/ps/p0302.pdf.

[11] Maria Manuel Clementino, Walter Tholen, Metric, topology and multicategory—a common approach, J. Pure Appl. Algebra 179 (1–2) (April 2003) 13–47.

[12] Eva Colebunders, Karen Van Opdenbosch, Topological properties of non-Archimedean approach spaces, Theory Appl. Categ. 32 (41) (2017) 1454–1484.

[13] William M. Faucett, Compact semigroups irreducibly connected between two idempotents, Proc. Am. Math. Soc. 6 (5) (May 1955) 741–747.

[14] Claudio Hermida, Representable multicategories, Adv. Math. 151 (2) (May 2000) 164–225.

[15] Claudio Hermida, From coherent structures to universal properties, J. Pure Appl. Algebra 165 (1) (November 2001) 7–61.

[16] Dirk Hofmann, Exponentiation for unitary structures, Topol. Appl. 153 (16) (October 2006) 3180–3202.

[17] Dirk Hofmann, Topological theories and closed objects, Adv. Math. 215 (2) (November 2007) 789–824.

[18] Dirk Hofmann, Injective spaces via adjunction, J. Pure Appl. Algebra 215 (3) (March 2011) 283–302, arXiv:0804.0326 [math.CT].

[19] Dirk Hofmann, The enriched Vietoris monad on representable spaces, J. Pure Appl. Algebra 218 (12) (December 2014) 2274–2318, arXiv:1212.5539 [math.CT].

[20] Dirk Hofmann, Carla D. Reis, Probabilistic metric spaces as enriched categories, Fuzzy Sets Syst. 210 (January 2013) 1–21, arXiv:1201.1161 [math.GN].

[21] Dirk Hofmann, Gavin J. Seal, Exponentiable approach spaces, Houst. J. Math. 41 (3) (2015) 1051–1062, arXiv:1304.6862 [math.GN].

[22] Maria Manuel Clementino, Eva Colebunders, Dirk Hofmann, Robert Lowen, Rory Lucyshyn-Wright, Gavin J. Seal, Walter Tholen, Monoidal Topology. A Categorical Approach to Order, Metric, and Topology, in: Dirk Hofmann, Gavin J. Seal, Walter Tholen (Eds.), Encyclopedia of Mathematics and its Applications, vol. 153, Cambridge University Press, Cambridge, July 2014.

[23] Dirk Hofmann, Walter Tholen, Lawvere completion and separation via closure, Appl. Categ. Struct. 18 (3) (November 2010) 259–287, arXiv:0801.0199 [math.CT].
[24] F. William Lawvere, Metric spaces, generalized logic, and closed categories, Rend. Semin. Mat. Fis. Milano 43 (1) (2002) 135–166, Republished in: Reprints in Theory Appl. Categ. 1 (2002) 1–37.
[25] Joachim Lambek, Michael Barr, John F. Kennison, Robert Raphael, Injective hulls of partially ordered monoids, Theory Appl. Categ. 26 (13) (2012) 338–348.
[26] Robert Lowen, Approach Spaces: The Missing Link in the Topology-Uniformity-Metric Triad, Oxford Mathematical Monographs, Oxford University Press, Oxford, 1997.
[27] Paul S. Mostert, Allen L. Shields, On the structure of semi-groups on a compact manifold with boundary, Ann. Math. (2) 65 (1) (January 1957) 117–143.
[28] Mojgan Mahmoudi, Christoph Schubert, Walter Tholen, Universality of coproducts in categories of lax algebras, Appl. Categ. Struct. 14 (3) (June 2006) 243–249.
[29] Willian Ribeiro, On generalized equilogical spaces, Topol. Appl. (2019) (to appear), arXiv:1811.08240 [math.CT].
[30] Jiří Rosický, Cartesian closed exact completions, J. Pure Appl. Algebra 142 (3) (October 1999) 261–270.
[31] Gavin J. Seal, Order-adjoint monads and injective objects, J. Pure Appl. Algebra 214 (6) (June 2010) 778–796.
[32] Walter Tholen, Ordered topological structures, Topol. Appl. 156 (12) (July 2009) 2148–2157.