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Numerical root finding via Cox rings

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A R T I C L E I N F O

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We present a new eigenvalue method for solving a system of Laurent polynomial equations defining a zero-dimensional reduced subscheme of a toric compactification X of $(\mathbb{C} \setminus \{0\})^n$. We homogenize the input equations to obtain a homogeneous ideal I in the Cox ring of X and generalize the eigenvalue, eigenvector theorem for root finding in affine space to compute homogeneous coordinates of the solutions. Several numerical experiments show the effectiveness of the resulting method. In particular, the method outperforms existing solvers in the case of (nearly) degenerate systems with solutions on or near the torus invariant prime divisors.

1. Introduction

Many problems in science and engineering can be solved by finding the solutions of a system of (Laurent) polynomial equations. Here, we consider the important case where the number of solutions to the system is finite. There exist many different approaches to tackle this problem [1–3]. Symbolic tools such as Groebner bases focus on systems with coefficients in $\mathbb{Q}$ or in finite fields [4,5]. For many applications, it is natural to work in finite precision, floating point arithmetic. This is the case, for instance, when the coefficients are known approximately (e.g. from measurements) or when it is sufficient to compute solutions accurately up to a certain number of significant decimal digits. The most important classes of numerical solvers are homotopy algorithms [6–8] and algebraic methods such as resultant based algorithms [9–13] and normal form algorithms [14–18] which rewrite the problem as an eigenvalue problem. Homotopy solvers are very successful for systems with many variables of low degree, whereas algebraic solvers can handle high degree systems in few variables. The algorithm presented in this paper is a new, numerical normal form algorithm for solving square systems of Laurent polynomial equations. The approach distinguishes itself from existing methods by the interpretation of 'solving' the system: we compute the points defined by the input equations

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on a toric compactification X of $(\mathbb{C} \setminus \{0\})^n \simeq T_X \subset X$ via an eigenvalue computation. More specifically, we work in the Cox ring of X to find 'homogeneous' coordinates of the solutions. The motivation is that, even though generically all solutions lie in $(\mathbb{C} \setminus \{0\})^n$, many problems encountered in applications are non-generic with respect to the Newton polytopes of the input equations. Solutions on or near $X \setminus T_X$ cause trouble for the stability of existing numerical algorithms, as we will show in our experiments, and the proposed algorithm is designed to handle such situations. The correctness of the algorithm depends on a conjecture regarding the regularity of a homogeneous ideal in the Cox ring of X. In the remainder of this section, we discuss some applications and give an overview of related work and of our main contributions. We conclude the section with an outline of this paper.

1.1. Applications

The applications we have in mind are problems that can be formulated as polynomial systems in only a few variables.

Many problems in computer vision, such as relative pose problems, require the solution of a system of polynomial equations [19,20]. In this context, there are often several different polynomial formulations for the same problem, with a different number of variables and a different degree of the equations. See [19, Sec. 7.1.3] for a description of a relative pose problem by a square 7-dimensional system (6 quadratics and a cubic in 7 unknowns) and by a square 3-dimensional system (two cubics and a quintic in 3 unknowns).

Another application comes from molecular biology. In [21] the problem of computing all possible conformations of several molecules is written in the form of a polynomial system in only two or three variables.

A problem encountered in many fields of engineering is that of finding the critical points of a function f, not necessarily polynomial, in a bounded domain $\Omega \subset \mathbb{R}^n$. A possible approach is to replace f by a polynomial $\tilde{f}$, computed from samples, which approximates f on Ω and compute the critical points of $\tilde{f}$ instead. The problem is now reduced to a system of polynomial equations, and if $\tilde{f}$ is a good approximation of f in Ω, the solutions in Ω will be good approximations of the critical points of f. It is clear that high degrees lead to better approximations, but also to higher degree polynomial systems. See [12] for an application of this technique to solve one of the SIAM 100-Digit Challenge problems [22].

1.2. Related work

As stated above, solutions on or near the torus invariant prime divisors (i.e. the irreducible components of $X \setminus T_X$) cause trouble for numerical root finding in non-compact solution spaces such as $\mathbb{C}^n$ or $(\mathbb{C} \setminus \{0\})^n$. In practice, for homotopy methods, such solutions are the reason for diverging paths, which often require a lot of unnecessary computational effort. Algebraic solvers such as the algorithms proposed in [17] and [18, §3, §4], as well as the classical resultant algorithms [9, Chapters 3 and 7] for computing multiplication matrices, require invertibility of a certain matrix: see for instance the matrix M_{11} in [9, Chapter 3, §6] or the matrix $N_{|B}$ in [18, Section 2]. In the presence of solutions on special divisors 'at infinity', these matrices are singular. In a numerical context, if these solutions are not exactly *on*, but *near* $X \setminus T_X$, homotopy paths 'diverge' to large solutions, causing scaling and condition problems, and the algebraic algorithms require the inversion of an ill-conditioned matrix, causing large rounding errors. A partial solution is to homogenize the equations and solve the problem in $X = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}, k \geq 1, n_1 + \ldots + n_k = n$, which should be thought of as a compactification of $\mathbb{C}^n$, such that a 'solution' is defined by $n + k$ (multi-)homogeneous coordinates. This technique is used in total degree homotopies [6,23], multihomogeneous homotopies [24, Chapter 8] and in normal form methods such as [18, §5, §6] or [25]. However, depending on the support of the input equations, this standard way of homogenizing may introduce highly singular solutions on the torus invariant divisors, or even destroy 0-dimensionality. More general sparsity structures are taken into account by polyhedral homotopies [26–28], toric or sparse resultants [10,9,29–32] and truncated normal forms [18, §4]. In [33] a method for dealing with diverging paths in a polyhedral homotopy is proposed.

In symbolic computing, modified sparse resultant methods have been introduced for solving degenerate systems symbolically [34,35]. Recently, specialized Groebner basis methods over semigroup algebras have been developed for exploiting sparsity structure [36].

1.3. Contributions

To the best of the author's knowledge, Cox rings (other than the familiar ones corresponding to products of projective spaces) have not been applied for numerical root finding before. To do so may seem like a bad idea, because the dimension of the Cox ring is (possibly much) greater than that of X. However, because of its fine grading by the class group $\mathrm{Cl}(X) = \mathrm{Div}(X)/\sim$ of Weil divisors modulo linear equivalence, this does not affect the computational complexity that much (see Remark 6.1). The input Laurent polynomial equations define a homogeneous ideal I of the Cox ring $S = \bigoplus_{\alpha \in \mathrm{Cl}(X)} S_\alpha$ with respect to this grading (this is detailed in Section 3). We will assume that I defines a zero-dimensional reduced subscheme $V_X(I)$ of X which is contained in its largest simplicial open subset U (see Section 2). The *regularity* $\mathrm{Reg}(I) \subset \mathrm{Cl}(X)$ of this ideal is defined in Section 4. In the same section, we conjecture a degree $\alpha \in \mathrm{Cl}(X)$ that is in $\mathrm{Reg}(I)$ (Conjecture 1). The correctness of the algorithm depends upon this conjecture, which is supported by some weaker results in Section 4 and by experimental evidence in Section 7. For this degree $\alpha \in \mathrm{Reg}(I)$, let $(S/I)_\alpha$ be the degree α part of the graded S-module S/I. We will construct a linear *multiplication map* $M_f : (S/I)_\alpha \to (S/I)_\alpha$ with respect to a rational function f on X which is regular at the roots of I. Here is a simplified version of Theorem 5.1.

Theorem 1.1. *Let $V_X(I) = \{\zeta_1, \ldots, \zeta_\delta\} \subset U$ be reduced and let $\alpha, \alpha_0 \in \mathrm{Cl}(X)$ be such that $\alpha, \alpha + \alpha_0 \in \mathrm{Reg}(I)$ and there exists $h_0 \in S_{\alpha_0}$ such that $\zeta_j \notin V_X(h_0), j = 1, \ldots, \delta$. Then for any $g \in S_{\alpha_0}$, the multiplication map $M_f : (S/I)_\alpha \to (S/I)_\alpha$ with $f = g/h_0$ has eigenvalues $f(\zeta_j)$.*

For every monomial $x^{b_i} \in S_{\alpha_0}$, we compute a multiplication matrix and denote its eigenvalues by $\lambda_{ij}, j = 1, \ldots, \delta$. This way, we reduce the problem of finding Cox coordinates of ζ_j to finding one point on the affine variety defined by the simple binomial system $\{x^{b_i} = \lambda_{ij} \mid x^{b_i} \in S_{\alpha_0}\}$ (Corollary 5.1). This leads to a numerical linear algebra based algorithm for finding Cox coordinates (Algorithm 1). Unlike other numerical methods, the algorithm is robust in the situation where some of the ζ_j are on or near torus invariant prime divisors. We illustrate this in Section 7 with some examples.

1.4. Outline of the paper

The paper is organized as follows. In the next section we discuss some preliminaries on Cox rings and the classical eigenvalue, eigenvector theorem for polynomial root finding. Our problem setup is discussed in detail in Section 3. In Section 4 we introduce homogeneous Lagrange polynomials and their relation to multigraded regularity. Our main result is discussed in detail in Section 5. The resulting algorithm is presented in Section 6. Finally, in Section 7 we work out several numerical examples. Throughout the paper, we work with polynomials, varieties and vector spaces over $\mathbb{C}$.

2. Preliminaries

In this section we give a brief introduction to the classical eigenvalue, eigenvector theorem and to complete toric varieties and their Cox rings. We denote by $V(I) \subset \mathbb{C}^n$ the affine variety of an ideal $I \subset \mathbb{C}[x_1, \ldots, x_n]$ and by $I(Y) \subset \mathbb{C}[x_1, \ldots, x_n]$ the vanishing ideal of a set $Y \subset \mathbb{C}^n$. If I is generated by $f_1, \dots, f_s \in \mathbb{C}[x_1, \dots, x_n]$, we denote $I = \langle f_1, \dots, f_s \rangle$ and $V(I) = V(\langle f_1, \dots, f_s \rangle) = V(f_1, \dots, f_s)$. For a finite dimensional vector space W, $W^\vee = \mathrm{Hom}_{\mathbb{C}}(W, \mathbb{C})$ denotes its dual. For a linear endomorphism $M : W \to W$ of a finite dimensional vector space W, a right eigenpair is $(\lambda, w) \in \mathbb{C} \times (W \setminus \{0\})$ satisfying $M(w) = \lambda w$. Analogously, a left eigenpair is given by $(v, \lambda) \in (W^\vee \setminus \{0\}) \times \mathbb{C}$ satisfying $v \circ M = \lambda v$.

2.1. The classical eigenvalue, eigenvector theorem for polynomial root finding

Let $R = \mathbb{C}[x_1, \dots, x_n]$ be the ring of n-variate polynomials with coefficients in $\mathbb{C}$. Take $f_i \in R, i = 1, \dots, s$ and let $I = \langle f_1, \dots, f_s \rangle$ be a zero-dimensional ideal in R. That is, $V(I) = \{z_1, \dots, z_\delta\}$ consists of $\delta < \infty$ points in $\mathbb{C}^n$. We assume for simplicity that all of the z_i have multiplicity one or, equivalently, that I is radical. By [9, Chapter 2, Lemma 2.9] there exist polynomials $\ell_i \in R, i = 1, \dots, \delta$ such that

$$\ell_i(z_j) = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}.$$

The ℓ_i are called *Lagrange polynomials* with respect to the set $V(I)$. We define $v_j \in (R/I)^\vee$ by $v_j(f + I) = f(z_j)$.

Lemma 2.1. *The map $\psi : R/I \to \mathbb{C}^\delta : f + I \mapsto (v_1(f + I), \dots, v_\delta(f + I))$ is an isomorphism of vector spaces.*

Proof. The map ψ is clearly linear and injective. Surjectivity follows from $\psi(\ell_j + I) = e_j$ with e_j the j-th standard basis vector of $\mathbb{C}^\delta$. $\square$

It follows from Lemma 2.1 that, under our assumptions, $\dim_{\mathbb{C}}(R/I) = \delta$. This is well known, see for instance [4, Chapter 5, §3, Proposition 7]. In particular, the map ψ defines coordinates on R/I and the residue classes of the Lagrange polynomials form a basis of R/I with dual basis $v_j, j = 1, \dots, \delta$. For $g \in R$, define the linear map $M_g : R/I \to R/I : f + I \mapsto fg + I$.

Theorem 2.1 *(Eigenvalue, eigenvector theorem). The left and right eigenpairs of M_g are*

$$(v_j, g(z_j)), \qquad (g(z_j), \ell_j + I), \qquad j = 1, \dots, \delta.$$

Proof. See for instance [9, Chapter 2, Proposition 4.7]. $\square$

Note that by definition, $M_{g_1} \circ M_{g_2} = M_{g_2} \circ M_{g_1}$ for any $g_1, g_2 \in R$. Therefore, after fixing a basis for R/I, the matrices corresponding to any two multiplication maps commute and have common eigenspaces. Theorem 2.1 provides the following algorithm for finding the points in $V(I)$:

1. compute the matrices $M_{x_1}, \dots, M_{x_n}$,
2. find the coordinates of the z_i from their simultaneous eigenvalue decomposition.

For a more detailed exposition on multiplication matrices, we refer the reader to [9, Chapter 2], [2, Chapter 4] and [1, Chapter 2].

2.2. Complete toric varieties and Cox rings

We will restrict ourselves to the discussion of only those aspects of toric varieties that are directly related to this paper. The reader who is unfamiliar with unexplained basic concepts can find an excellent introduction in [37] or [38]. For more information on Cox rings we refer to [37, Chapter 5] and the original paper by Cox [39]. The n-dimensional algebraic torus $(\mathbb{C}^*)^n = (\mathbb{C} \setminus \{0\})^n$ has character lattice $M = \mathrm{Hom}_{\mathbb{Z}}((\mathbb{C}^*)^n, \mathbb{C}^*) \simeq \mathbb{Z}^n$ and cocharacter lattice $N = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z}) \simeq \mathbb{Z}^n$. An element $m \in M$ gives $\chi^m : (\mathbb{C}^*)^n \to \mathbb{C}^*$ such that if m corresponds to $(m_1, \dots, m_n) \in \mathbb{Z}^n$, $\chi^m(t) = t^m = t_1^{m_1} \cdots t_n^{m_n}$. Hence characters can be thought of as Laurent monomials and

$$\mathbb{C}[M] = \bigoplus_{m \in M} \mathbb{C} \cdot \chi^m \simeq \mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}].$$

Following [37], we denote $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R} \simeq \mathbb{R}^n$ and $T_N = N \otimes_{\mathbb{Z}} \mathbb{C}^* = (\mathbb{C}^*)^n$. A complete, normal toric variety X with torus T_N is given by a complete fan Σ in $N_{\mathbb{R}}$ and we will sometimes emphasize this correspondence by writing $X = X_\Sigma$. The set of d-dimensional cones of Σ is denoted $\Sigma(d)$. In particular, we write $\Sigma(1) = \{\rho_1, \dots, \rho_k\}$ for the rays of Σ and $u_i \in N$ for the primitive generator of ρ_i. It is convenient to think of the u_i as column vectors and to define the matrix $F = [u_1 \; u_2 \; \cdots \; u_k] \in \mathbb{Z}^{n \times k}$. We will use F_{ij} for the entry in row i, column j of F, $F_{i,:}$ for the i-th row of F, $F_{:,j} = u_j$ for the j-th column of F and $F^\top$ for the transpose. Every ray ρ_i corresponds to a torus invariant prime divisor D_i on X_Σ and we have $X_\Sigma \setminus (\bigcup_{i=1}^k D_i) = T_{X_\Sigma} \simeq T_N$. The class group $\mathrm{Cl}(X_\Sigma)$ of X_Σ, which is the group of Weil divisors modulo linear equivalence, is generated by the classes $[D_i]$ of the torus invariant prime divisors. The Picard group $\mathrm{Pic}(X_\Sigma) \subset \mathrm{Cl}(X_\Sigma)$ consists of the classes of Weil divisors that are locally principal. Identifying $\bigoplus_{i=1}^k \mathbb{Z} \cdot D_i \simeq \mathbb{Z}^k$ we have a short exact sequence

$$0 \longrightarrow M \xrightarrow{F^\top} \mathbb{Z}^k \longrightarrow \mathrm{Cl}(X_\Sigma) \longrightarrow 0$$

where $\mathbb{Z}^k \longrightarrow \mathrm{Cl}(X_\Sigma)$ sends a torus invariant Weil divisor $\sum_{i=1}^k a_i D_i$ to its class $[\sum_{i=1}^k a_i D_i] \in \mathrm{Cl}(X_\Sigma)$. Taking $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{C}^*)$ and defining the *reductive group* $G = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Cl}(X_\Sigma), \mathbb{C}^*)$ we find that G is the kernel of the map

$$\pi : (\mathbb{C}^*)^k \to T_N : t \mapsto (t^{F_{1,:}}, \dots, t^{F_{n,:}}). \tag{2.1}$$

That is, G is the subgroup of $(\mathbb{C}^*)^k$ given by

$$G = \{g \in (\mathbb{C}^*)^k : g^{F_{i,:}} = 1, i = 1, \dots, n\}$$

and π is constant on G-orbits. Let $S = \mathbb{C}[x_1, \dots, x_k]$ be the polynomial ring in k variables where each of the x_i corresponds to a ray $\rho_i \in \Sigma(1)$. For every cone $\sigma \in \Sigma$, denote by $\sigma(1)$ the rays contained in σ. We are going to associate a monomial in S to each cone in Σ: for $\sigma \in \Sigma$, define $x^{\hat{\sigma}} = \prod_{\rho_i \notin \sigma(1)} x_i$. The *irrelevant ideal* K of Σ (or of X_Σ) is the monomial ideal defined as

$$K = \langle x^{\hat{\sigma}} : \sigma \in \Sigma(n) \rangle \subset S. \tag{2.2}$$

The *exceptional set* of X_Σ is $Z = V(K) \subset \mathbb{C}^k$. The action of G on $(\mathbb{C}^*)^k$ extends to an action on $\mathbb{C}^k \setminus Z$. In [39], Cox proves that there is a good categorical quotient $\pi : \mathbb{C}^k \setminus Z \to X_\Sigma$, constant on G-orbits, such that (2.1) is its restriction to $(\mathbb{C}^*)^k$. By the properties of good categorical quotients we have a bijection

$$\{ \text{ closed } G\text{-orbits in } \mathbb{C}^k \setminus Z \} \leftrightarrow \{ \text{ points in } X_\Sigma \}.$$

Moreover, π is an almost geometric quotient, meaning that there is a Zariski open subset $U \subset X_\Sigma$ such that $\pi_{|\pi^{-1}(U)} : \pi^{-1}(U) \to U$ is a geometric quotient:

$$\{ \; G\text{-orbits in } \pi^{-1}(U) \; \} \leftrightarrow \{ \text{ points in } U \}.$$

The open set U is the toric variety $X_{\Sigma'} \subset X_\Sigma$ corresponding to the subfan $\Sigma' \subset \Sigma$ of simplicial cones of Σ (see [37, proof of Theorem 5.1.11]). Therefore, by the orbit-cone correspondence, $X \setminus U$ is a union of T_N-orbits of codimension at least 3 (cones of dimension 0, 1 or 2 are simplicial). If Σ is simplicial, the nicest possible bijection holds:

$$\{ \ G\text{-orbits in } \mathbb{C}^k \backslash Z \ \} \leftrightarrow \{ \text{ points in } X_\Sigma \ \}.$$

In this case we write $X_\Sigma = (\mathbb{C}^k \setminus Z)/G$.

Example 2.1. The quotient construction of X_Σ is a generalization of the familiar construction of $\mathbb{P}^n$ as the quotient $\mathbb{P}^n = (\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^*$. In this case $S = \mathbb{C}[x_0, \ldots, x_n]$, $K = \langle x_0, \ldots, x_n \rangle$, $Z = \{0\}$ and $G = \mathrm{Hom}_\mathbb{Z}(\mathrm{Cl}(\mathbb{P}^n), \mathbb{C}^*) = \mathrm{Hom}_\mathbb{Z}(\mathbb{Z}, \mathbb{C}^*) = \mathbb{C}^*$ acts by $g \cdot (x_0, \ldots, x_n) = (gx_0, \ldots, gx_n), g \in G$.

The ring S has a natural grading by $\mathrm{Cl}(X_\Sigma)$:

$$\deg(x^a) = \deg(x_1^{a_1} \cdots x_k^{a_k}) = [\sum_{i=1}^k a_i D_i] \in \mathrm{Cl}(X_\Sigma), \qquad S = \bigoplus_{\alpha \in \mathrm{Cl}(X_\Sigma)} S_\alpha, \tag{2.3}$$

where $S_\alpha = \bigoplus_{\deg(x^a)=\alpha} \mathbb{C} \cdot x^a$. In fact, the only nonzero graded pieces correspond to 'positive' degrees, and one can write

$$\mathrm{Cl}(X_\Sigma)_+ = \{\alpha \in \mathrm{Cl}(X_\Sigma) \mid \alpha = n_1 \deg(x_1) + \cdots + n_k \deg(x_k), n_i \in \mathbb{N}\}, \quad S = \bigoplus_{\alpha \in \mathrm{Cl}(X_\Sigma)_+} S_\alpha.$$

Similarly, we denote $\mathrm{Pic}(X_\Sigma)_+ = \mathrm{Cl}(X_\Sigma)_+ \cap \mathrm{Pic}(X_\Sigma)$. The graded pieces correspond to vector spaces of global sections of divisorial sheaves, that is, for $\alpha \in \mathrm{Cl}(X_\Sigma)$ with $\alpha = [D], D = \sum_{i=1}^k a_i D_i$,

$$S_\alpha \simeq \Gamma(X_\Sigma, \mathscr{O}_{X_\Sigma}(D)) \simeq \bigoplus_{F^\top m + a \geq 0} \mathbb{C} \cdot \chi^m. \tag{2.4}$$

Here the direct sum ranges over all m such that elementwise, $F^\top m + a \geq 0$, that is, $\langle u_i, m \rangle + a_i \geq 0, i = 1, \ldots, k$ where $\langle \cdot, \cdot \rangle$ is the natural pairing between N and M. Denoting $x^{F^\top m + a} = x_1^{\langle u_1, m \rangle + a_1} \cdots x_k^{\langle u_k, m \rangle + a_k}$, the isomorphism (2.4) is given by

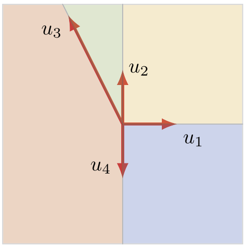
$$\sum_{F^\top m + a \geq 0} c_m \chi^m \mapsto \sum_{F^\top m + a \geq 0} c_m x^{F^\top m + a} \in S_\alpha, \tag{2.5}$$

which is *homogenization* with respect to α. To see the analogy with the classical notion of homogenization, note that the action of G on $\mathbb{C}^k$ induces an action of G on S by $(g \cdot f)(x) = f(g^{-1} \cdot x)$ for $g \in G, f \in S$. If $f \in S_\alpha$, it is the image of some Laurent polynomial under (2.5) and we can write

$$(g \cdot f)(x) = \sum_{F^\top m + a \geq 0} c_m (g^{-1} \cdot x)^{F^\top m + a} = g^{-a} f(x) \tag{2.6}$$

since by the definition of the reductive group $g^{F^\top m} = 1$. This shows that the number g^{-a} does not depend on the representative divisor D we choose for $\alpha \in \mathrm{Cl}(X_\Sigma)$. It therefore makes sense to write $g^{-\alpha} = g^{-(F^\top m + a)}$. Equation (2.6) shows that the homogeneous components $S_\alpha \subset S$ with respect to the grading (2.3) are the eigenspaces of the action of G on S and that

$$V_{X_\Sigma}(f) = \{p \in X_\Sigma : f(x) = 0 \text{ for some } x \in \pi^{-1}(p)\} \subset X_\Sigma \tag{2.7}$$



$$F = [u_1\ u_2\ u_3\ u_4] = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & -1 \end{bmatrix}$$

Fig. 1. Fan and matrix of primitive ray generators of the Hirzebruch surface $\mathscr{H}_2$.

is well defined if f is homogeneous. An ideal $I \subset S$ is called homogeneous if it is generated by homogeneous polynomials, and it is straightforward to extend (2.7) to define $V_{X_\Sigma}(I)$. The ring S equipped with the grading (2.3) and the irrelevant ideal (2.2) is called the *total coordinate ring*, *homogeneous coordinate ring* or *Cox ring* of X_Σ.

Example 2.2. The complete fans Σ we will encounter in this paper are normal fans of full dimensional lattice polytopes [37, §2.3]. If

$$P = \{m \in M_\mathbb{R} \mid \langle u_i, m \rangle \geq -a_i, i = 1, \dots, k\}$$

is the minimal facet representation of a full dimensional lattice polytope $P \subset M_\mathbb{R}$, then its normal fan Σ_P defines a toric variety X_{Σ_P}, which we will often denote by X for simplicity of notation. There are bijective correspondences between rays in Σ_P, facets of P, torus invariant prime divisors in X and indeterminates in the Cox ring. The matrix F contains the primitive inward pointing facet normals of P. For example, the toric variety of the standard n-simplex is $\mathbb{P}^n$.

Example 2.3. As a running example, we will consider the problem of finding the intersections of two curves on the Hirzebruch surface $\mathscr{H}_2$. The associated fan Σ and the matrix F of ray generators are shown in Fig. 1. The Cox ring $S = \mathbb{C}[x_1, x_2, x_3, x_4]$ is graded by $\mathrm{Cl}(\mathscr{H}_2) \simeq \mathbb{Z}^4/\mathrm{im}F^\top \simeq \mathbb{Z}^2$, with $\deg(x^b) = \deg(x_1^{b_1}x_2^{b_2}x_3^{b_3}x_4^{b_4}) = (b_1 - 2b_2 + b_3, b_2 + b_4)$. The reductive group and exceptional set are given by $G = \{(\lambda, \mu, \lambda, \lambda^2\mu) \mid (\lambda, \mu) \in (\mathbb{C}^*)^2\} \subset (\mathbb{C}^*)^4$ and $Z = V(x_1, x_3) \cup V(x_2, x_4) \subset \mathbb{C}^4$ respectively. Since $\mathscr{H}_2$ is smooth, it is simplicial (in the notation from above $U = \mathscr{H}_2$) and $\mathrm{Pic}(\mathscr{H}_2) = \mathrm{Cl}(\mathscr{H}_2)$.

3. Problem setup

In this section, we give a detailed description of the problem considered in this paper and we discuss our assumptions. We start from n given Laurent polynomials $\hat{f}_1, \dots, \hat{f}_n \in \mathbb{C}[M]$ (that is, we consider *square* systems). Denote

$$\hat{f}_j = \sum c_{m,j} \chi^m$$

and let $P_j \subset M_\mathbb{R}$ be the Newton polytope of $\hat{f}_j$: $P_j = \mathrm{Conv}(m \in M \mid c_{m,j} \neq 0) \subset M_\mathbb{R}$. Let $P = P_1 + \ldots + P_n$ be the Minkowski sum of these polytopes. We assume that P is full-dimensional and we let $X = X_{\Sigma_P}$ be the complete normal toric variety corresponding to its normal fan. To each P_j, we associate a basepoint free[1] Cartier divisor D_{P_j} on X, given by

[1] For $\alpha = [D] \in \mathrm{Pic}(X)$, we say that $p \in X$ is a *basepoint* of $S_\alpha \simeq \Gamma(X, \mathscr{O}_X(D))$ if every global section of the associated line bundle $\mathscr{O}_X(D)$ vanishes at p. The divisor D and its associated degree $\alpha \in \mathrm{Pic}(X)$ are called *basepoint free* if S_α has no basepoints.

$$D_{P_j} = \sum_{i=1}^{k} a_{j,i} D_i, \qquad a_{j,i} = -\min_{m \in P_j} \langle u_i, m \rangle$$

and we denote $a_j = (a_{j,1}, \dots, a_{j,k}) \in \mathbb{Z}^k, [D_{P_j}] = \alpha_j \in \operatorname{Pic}(X)$. For this construction, $D_{P_i+P_j} = D_{P_i} + D_{P_j}$ and for $\mathcal{J} \subset \{1, \dots, n\}$, $P_{\mathcal{J}} = \sum_{j \in \mathcal{J}} P_j$ we have

$$\Gamma(X, \mathscr{O}_X(\sum_{j \in \mathcal{J}} D_{P_j})) = \Gamma(X, \mathscr{O}_X(D_{P_{\mathcal{J}}})) = \bigoplus_{m \in P_{\mathcal{J}} \cap M} \mathbb{C} \cdot \chi^m. \tag{3.1}$$

By definition, $m \in P_j \cap M$ if and only if $F^\top m + a_j \geq 0$, so we have

$$\hat{f}_j = \sum_{m \in P_j \cap M} c_{m,j} \chi^m \in \Gamma(X, \mathscr{O}_X(D_{P_j})). \tag{3.2}$$

Homogenizing with respect to α_j according to (2.5) gives (see [40])

$$\hat{f}_j \mapsto f_j = \sum_{m \in P_j \cap M} c_{m,j} x^{F^\top m + a_j} \in S_{\alpha_j}.$$

Equation (3.2) shows that $\hat{f}_j$ is a global section of the line bundle given by $\mathscr{O}_X(D_{P_j})$ [37, Chapter 6]. Its divisor of zeroes is the effective divisor $\operatorname{div}(\hat{f}_j) + D_{P_j}$, whose support is exactly $V_X(f_j)$. This construction gives a homogeneous ideal $I = \langle f_1, \dots, f_n \rangle \subset S$. We will make the following assumptions on I.

Assumption 1. $V_X(I)$ is zero-dimensional. We denote $V_X(I) = \{\zeta_1, \dots, \zeta_\delta\} \subset X$.

Assumption 2. $V_X(I) \subset U \subset X$, where U is the 'simplicial part' of X as in Subsection 2.2.

Assumption 3. I defines a reduced subscheme of $U \subset X$. That is, all points ζ_i are 'simple roots' of I.

It is clear that when $n = 2$, Assumption 2 can be dropped. For $n = 3$, U is the complement of finitely many points in X: one point for each vertex of P corresponding to a non-simplicial, full dimensional cone of Σ_P. It follows that we can drop Assumption 2 also for $n = 3$, since 'face systems' corresponding to vertices do not contribute any solutions (see for instance the appendix in [27]). For $n > 3$, Assumption 2 can be dropped if X is simplicial. We will comment on Assumption 3 in Section 4 (Remark 4.1).

In order to say something more about the number δ in Assumption 1, we recall the definition of mixed volume. The n-dimensional mixed volume of a collection of n polytopes $P_1, \dots, P_n$ in $M_\mathbb{R} \simeq \mathbb{R}^n$, denoted $\operatorname{MV}(P_1, \dots, P_n)$, is the coefficient of the monomial $\lambda_1 \lambda_2 \cdots \lambda_n$ in $\operatorname{Vol}_n(\sum_{i=1}^n \lambda_i P_i)$. A formula for the mixed volume that will be useful is (see [41,42])

$$\operatorname{MV}(P_1, \dots, P_n) = \sum_{\ell=0}^{n} (-1)^{n-\ell} \sum_{\substack{\mathcal{J} \subset \{1,\dots,n\} \\ |\mathcal{J}| = \ell}} |(P_0 + P_{\mathcal{J}}) \cap M|, \tag{3.3}$$

for any lattice polytope $P_0 \subset \mathbb{R}^n$ corresponding to a basepoint free divisor D_{P_0}. The following important theorem was named after Bernstein, Khovanskii and Kushnirenko and tells us what the number δ is.

Theorem 3.1 *(BKK Theorem). Let $I = \langle f_1, \dots, f_n \rangle \subset S$ be a homogeneous ideal constructed as above. If I defines $\delta < \infty$ points on X, counting multiplicities, then δ is given by* $\operatorname{MV}(P_1, \dots, P_n)$*. For generic choices of the coefficients of the f_i, the number of roots in $T_X \simeq T_N = (\mathbb{C}^*)^n$ is exactly equal to* $\operatorname{MV}(P_1, \dots, P_n)$ *and they all have multiplicity one.*

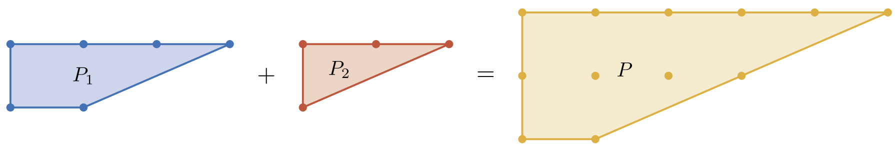


Fig. 2. Newton polytopes involved in Example 3.1.

Proof. See [38, §5.5]. For sketches of the proof we refer to [9,43]. Other proofs can be found in Bernstein's original paper [44] and in [27]. □

Theorem 3.1 is a generalization of Bézout's theorem for projective space. Motivated by this result, for the rest of this article $\delta = \mathrm{MV}(P_1, \dots, P_n)$. We can represent each $\zeta_j \in V_X(I)$ by a set of homogeneous coordinates $z_j = (z_{j1}, \dots, z_{jk}) \in \mathbb{C}^k \setminus Z$. Let $\pi^{-1}(\zeta_j) = G \cdot z_j \subset \mathbb{C}^k \setminus Z$ be the corresponding $(k-n)$-dimensional closed G-orbit and let $\overline{G \cdot z_j}$ be the closure in $\mathbb{C}^k$. It follows from our assumptions that

$$V(I) \setminus Z = G \cdot z_1 \cup \dots \cup G \cdot z_\delta \qquad \text{and} \qquad V(I) = \overline{G \cdot z_1} \cup \dots \cup \overline{G \cdot z_\delta} \cup Z',$$

with $Z' \subset Z$ a closed subvariety. We define $J = I(\overline{G \cdot z_1} \cup \dots \cup \overline{G \cdot z_\delta})$ to be the ideal of the union of orbit closures, which is radical and saturated with respect to the irrelevant ideal K. The ideal J is the one investigated in [42] (in the simplicial case). It is clear that $I \subset J$. In some special cases where Z is very small, the ideals I and J coincide. This happens for instance for $X = \mathbb{P}^n$ or for any weighted projective space $X = \mathbb{P}(w_0, \dots, w_n)$.

Example 3.1. Let us consider the polynomials

$$\hat{f}_1 = 1 + t_1 + t_2 + t_1 t_2 + t_1^2 t_2 + t_1^3 t_2,$$
$$\hat{f}_2 = 1 + t_2 + t_1 t_2 + t_1^2 t_2.$$

We think of $\hat{f}_1, \hat{f}_2$ as elements of $\mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}] \simeq \mathbb{C}[M]$ with $M = \mathbb{Z}^2$ the character lattice of $T_N = (\mathbb{C}^*)^2$. The polytopes P_1, P_2 and P are shown in Fig. 2. Note that the normal fan Σ_P of P is the fan of Fig. 1, so the toric variety associated to this system is $X = X_{\Sigma_P} = \mathscr{H}_2$. We identify $\mathrm{Cl}(X)$ with $\mathbb{Z}^2$ as in Example 2.3. It is easy to check that $\alpha_1 = [D_{P_1}] = [D_3 + D_4] = (1,1) \in \mathrm{Cl}(X)$ and $\alpha_2 = [D_{P_2}] = [D_4] = (0,1) \in \mathrm{Cl}(X)$. This gives the following homogeneous polynomials in the Cox ring $S = \mathbb{C}[x_1, \dots, x_4]$:

$$f_1 = x_3 x_4 + x_1 x_4 + x_2 x_3^3 + x_1 x_2 x_3^2 + x_1^2 x_2 x_3 + x_1^3 x_2,$$
$$f_2 = x_4 + x_2 x_3^2 + x_1 x_2 x_3 + x_1^2 x_2.$$

The mixed volume is $\delta = \mathrm{MV}(P_1, P_2) = 3$. To see that the ideal $I = \langle f_1, f_2 \rangle$ satisfies our assumptions, we compute its primary decomposition.[2]

$$I = \langle x_1 + x_3, x_2 x_3^2 + x_4 \rangle \cap \langle x_1, x_2 x_3^2 + x_4 \rangle \cap \langle x_3, x_1^2 x_2 + x_4 \rangle \cap \langle x_2, x_4 \rangle$$

which gives the decomposition of the associated variety $V(I) = \overline{G \cdot z_1} \cup \overline{G \cdot z_2} \cup \overline{G \cdot z_3} \cup Z'$ with orbit representatives $z_1 = (-1, -1, 1, 1), z_2 = (0, -1, 1, 1), z_3 = (1, -1, 0, 1)$ and $Z' = V(x_2, x_4) \subset Z$. This shows that I defines the expected number of simple, isolated points on $X = \mathscr{H}_2$. The first solution $\zeta_1 = \pi(z_1) \in T_N$ lies in the torus, the others satisfy $\zeta_2 = \pi(z_2) \in D_1$, $\zeta_3 = \pi(z_3) \in D_3$. The ideal J in this example is the intersection of the first three primary components of I. We find $J = \langle x_1^2 x_3 + x_1 x_3^2, f_2 \rangle$.

[2] We used Macaulay2 to perform the symbolic computations in this example [45].

4. Multigraded regularity and homogeneous Lagrange polynomials

The regularity of a graded module measures its complexity (for instance, in terms of the degree of minimal generators). The notion of regularity has been studied in a multigraded context. The general situation is treated in [46]. The zero-dimensional case is further investigated in [42] and some more results in a multiprojective setting can be found in [25,47]. In our case, the regularity (as defined below) of the ideal I in Section 3 will determine in which graded piece S_α of the Cox ring S we can work to define our multiplication maps in Section 5. The 'larger' this graded piece (i.e. the larger the dimension of S_α as a $\mathbb{C}$-vector space), the larger the matrices involved in the presented algorithm in Section 6. We will define homogeneous Lagrange polynomials and show how they are related to multigraded regularity. As in Subsection 2.1, these Lagrange polynomials and their dual basis will have a nice interpretation as eigenvectors of multiplication maps. For $\alpha \in \mathrm{Cl}(X)$, we denote $n_\alpha = \dim_{\mathbb{C}}(S_\alpha)$. Since X is complete, $n_\alpha < \infty, \forall \alpha \in \mathrm{Cl}(X)$ [37, Proposition 4.3.8]. The ideals $I, J \subset S$ are as defined in Section 3. In particular, I satisfies Assumptions 1-3. For $\alpha \in \mathrm{Cl}(X)$, let $S_\alpha = \bigoplus_{i=1}^{n_\alpha} \mathbb{C} \cdot x^{b_i}, b_i \in \mathbb{N}^k$ and consider the map

$$\Phi_\alpha : \mathbb{C}^k \setminus Z \dashrightarrow \mathbb{P}^{n_\alpha - 1} \simeq \mathbb{P}(S_\alpha^\vee) \simeq \mathbb{P}(\Gamma(X, \mathscr{O}_X(D))^\vee) : (x_1, \ldots, x_k) \mapsto (x^{b_1}, \ldots, x^{b_{n_\alpha}}).$$

Note that Φ_α may have basepoints (hence the dashed arrow) and it is constant on G-orbits. We will say that $\alpha \in \mathrm{Cl}(X)$ is basepoint free if Φ_α has no basepoints (this extends the definition for basepoint free $\alpha \in \mathrm{Pic}(X)$ to the class group). We say that $\zeta \in U \subset X$ is a basepoint of S_α if $\pi^{-1}(\zeta)$ are basepoints of Φ_α. The following lemma is straightforward and we omit the proof.

Lemma 4.1. *Let $\alpha = [D] \in \mathrm{Cl}(X)$ be such that no ζ_j is a basepoint of S_α. For generic $h \in S_\alpha$, we have $\zeta_j \notin V_X(h), j = 1, \ldots, \delta$.*

Note that in particular, the condition of Lemma 4.1 is always satisfied for basepoint free α. The grading on S defines a grading on the quotient S/I: $(S/I)_\alpha = S_\alpha / I_\alpha$. It follows from Lemma 4.1 that for any $\alpha = [D] \in \mathrm{Cl}(X)$ such that no ζ_j is a basepoint of S_α, the following $\mathbb{C}$-linear map is well defined for generic $h \in S_\alpha$:

$$\psi_\alpha : (S/I)_\alpha \to \mathbb{C}^\delta : f + I_\alpha \mapsto \left(\frac{f}{h}(z_1), \ldots, \frac{f}{h}(z_\delta) \right). \tag{4.1}$$

We fix such a generic $h \in S_\alpha$. Note that the definition of ψ_α does not depend on the choice of representative z_j of $G \cdot z_j$. We will now investigate for which $\alpha \in \mathrm{Cl}(X)$ the map ψ_α defines coordinates on $(S/I)_\alpha$, that is, for which α it is an isomorphism (note that this is independent of the choice of h satisfying $\zeta_j \notin V_X(h)$). It is clear that for this to happen, we need $\dim_{\mathbb{C}}((S/I)_\alpha) = \delta$. The dimension of the graded parts of S/I is given by the multigraded analog of the Hilbert function [42].

Definition 4.1 *(Hilbert function).* For a homogeneous ideal I in the Cox ring S of X, the Hilbert function of I is given by $\mathrm{HF}_I : \mathrm{Cl}(X) \to \mathbb{N} : \alpha \mapsto \dim_{\mathbb{C}}((S/I)_\alpha)$.

We note that in [42], the Hilbert function of the scheme $V_X(I)$ is equal to HF_J as defined above. In order to state a necessary and sufficient condition for surjectivity of ψ_α, we will introduce a homogeneous analog of the Lagrange polynomials introduced in Subsection 2.1.

Definition 4.2 *(homogeneous Lagrange polynomials).* Let $\alpha \in \mathrm{Cl}(X)$ be such that no ζ_j is a basepoint of S_α and let $h \in S_\alpha$ be such that $\zeta_j \notin V_X(h), j = 1, \ldots, \delta$. A set of elements $\ell_1, \ldots, \ell_\delta \in S_\alpha$ is called a set of homogeneous Lagrange polynomials of degree α with respect to h if for $j = 1, \ldots, \delta$,

1. $\zeta_i \in V_X(\ell_j), i \neq j$,
2. $\zeta_j \in V_X(h - \ell_j)$.

In terms of the homogeneous coordinates z_j, a set of homogeneous Lagrange polynomials satisfies $\ell_j(z_i) = 0, i \neq j$ and $\ell_j(z_j) = h(z_j), j = 1, \ldots, \delta$.

Remark 4.1. Let $\ell_j, j = 1, \ldots, \delta$ be a set of homogeneous Lagrange polynomials of degree α with respect to h. The cosets $\ell_j + I_\alpha \in (S/I)_\alpha$ are a dual basis for the evaluation functionals $v_j \in (S/I)_\alpha^\vee$ given by $v_j : (S/I)_\alpha \to \mathbb{C} : f + I_\alpha \mapsto (f/h)(z_j)$. If I defines points with multiplicities (the case of 'fat points', violating Assumption 3), a starting point would be to extend this set of evaluation functionals to a basis of $(S/I)_\alpha^\vee$, using analogs of differentiation operators. It is known that the theory for the affine root finding problem (Subsection 2.1) extends nicely in this way; see for instance [9, Chapter 4, Proposition 2.7], [2, Section 4.3] or [48]. We leave this for future research.

In what follows, we use the same function h to define ψ_α and a set of homogeneous Lagrange polynomials.

Proposition 4.1. *Let $\alpha \in \mathrm{Cl}(X)$ be such that no ζ_j is a basepoint of S_α. Then*

1. *ψ_α is injective if and only if $I_\alpha = J_\alpha$. In this case* $\mathrm{HF}_I(\alpha) \leq \delta$,
2. *ψ_α is surjective if and only if there exists a set of homogeneous Lagrange polynomials of degree α. In this case* $\mathrm{HF}_I(\alpha) \geq \delta$.

Proof. Let $f, h \in S_\alpha$ such that $\zeta_j \notin V_X(h), j = 1, \ldots, \delta$. If ψ_α is injective, then $f \in J_\alpha \Rightarrow \psi_\alpha(f + I_\alpha) = 0 \Rightarrow f \in I_\alpha$. So $J_\alpha \subset I_\alpha$ and the other inclusion is trivial. Conversely, if $I_\alpha = J_\alpha$, then $\psi_\alpha(f + I_\alpha) = 0 \Rightarrow f \in J_\alpha \Rightarrow f \in I_\alpha$, so ψ_α is injective. The corresponding statement about HF_I follows easily.
If ψ_α is surjective, take $\ell_j \in \psi_\alpha^{-1}(e_j)$. Conversely, if $\ell_j, j = 1, \ldots, \delta$ is a set of homogeneous Lagrange polynomials of degree α, $\psi_\alpha(\ell_j + I_\alpha) = e_j$ and ψ_α is surjective. Again, the statement about HF_I follows easily. □

Corollary 4.1. *If $\alpha \in \mathrm{Pic}(X)$ is ample*[3] *and I is radical, then ψ_α is injective.*

Proof. In this case $I = I(\overline{G \cdot z_1} \cup \cdots \cup \overline{G \cdot z_\delta} \cup Z')$ by the Nullstellensatz. Take $f \in J_\alpha$. Since any polynomial in S_α for α ample vanishes on Z ($S_\alpha \subset K$, see e.g. [49]), f vanishes on $Z' \subset Z$. Therefore $f \in I_\alpha$ and $J_\alpha \subset I_\alpha \subset J_\alpha$. Now apply Proposition 4.1. □

The following proposition shows that the existence of homogeneous Lagrange polynomials of degree $\alpha \in \mathrm{Cl}(X)$ is equivalent to the fact that the points $\Phi_\alpha(z_j)$ span a linear space of dimension $\delta - 1$ in $\mathbb{P}^{n_\alpha - 1}$. Let $p_j \in \mathbb{C}^{n_\alpha}$ be a set of homogeneous coordinates (in the standard sense) of $\Phi_\alpha(z_j) \in \mathbb{P}^{n_\alpha - 1}$ and define the matrix $L_\alpha = [p_1 \ \cdots \ p_\delta] \in \mathbb{C}^{n_\alpha \times \delta}$.

Proposition 4.2. *Let $\alpha \in \mathrm{Cl}(X)$ be such that no ζ_j is a basepoint of S_α. There exists a set of Lagrange polynomials of degree α if and only if L_α has rank δ.*

Proof. The rank of L_α is δ if and only if there exists a left inverse matrix $L_\alpha^\dagger \in \mathbb{C}^{\delta \times n_\alpha}$ such that $L_\alpha^\dagger L_\alpha = \mathrm{id}_\delta$ is the $\delta \times \delta$ identity matrix. We will show that this is equivalent to the existence of a set of homogeneous

[3] A divisor D and its degree $\alpha = [D]$ are called *very ample* if D is basepoint free and $X \to \mathbb{P}(\Gamma(X, \mathscr{O}_X(D))^\vee)$ is a closed embedding. If kD (or $k\alpha$) is very ample for some $k \geq 1$, then D (or α) is called *ample*. See [37, Chapter 6] for definitions and properties.

Lagrange polynomials of degree α. Suppose that $L_\alpha^\dagger$ exists. The rows of $L_\alpha^\dagger$ should be interpreted as elements of S_α represented in the basis $\{x^{b_1}, \dots, x^{b_{n_\alpha}}\}$. The columns of L_α are elements of $S_\alpha^\vee$ represented in the dual basis. Let the j-th row of $L_\alpha^\dagger$ correspond to $\tilde{\ell}_j \in S_\alpha$. It is clear from $L_\alpha^\dagger L_\alpha = \mathrm{id}_\delta$ that

$$\langle \tilde{\ell}_j, p_i \rangle = \tilde{\ell}_j(z_i) = \begin{cases} 1 & i = j, \\ 0 & \text{otherwise}. \end{cases}$$

By Lemma 4.1, there is $h \in S_\alpha$ such that $h(z_j) \neq 0, j = 1, \dots, \delta$. Then $\ell_j = h(z_j)\tilde{\ell}_j, j = 1, \dots, \delta$ are a set of homogeneous Lagrange polynomials. Conversely, if a set of homogeneous Lagrange polynomials exists, construct a matrix $\tilde{L}_\alpha^\dagger$ by plugging the coefficients of ℓ_j into the j-th row. Then there is $h \in S_\alpha$ such that $\tilde{L}_\alpha^\dagger L_\alpha = \mathrm{diag}(h(z_1), \dots, h(z_\delta))$ is an invertible diagonal matrix. The left inverse is $L_\alpha^\dagger = \mathrm{diag}(h(z_1), \dots, h(z_\delta))^{-1} \tilde{L}_\alpha^\dagger$. □

Based on these results, we make the following definition.

Definition 4.3 *(Regularity).* The regularity $\mathrm{Reg}(I) \subset \mathrm{Cl}(X)$ of I is the subset of degrees $\alpha \in \mathrm{Cl}(X)$ for which no ζ_j is a basepoint of S_α and the following equivalent conditions are satisfied:

1. ψ_α is an isomorphism,
2. $\mathrm{HF}_I(\alpha) = \delta$ and $I_\alpha = J_\alpha$,
3. $\mathrm{HF}_I(\alpha) = \delta$ and there exists a set of homogeneous Lagrange polynomials of degree α,
4. $I_\alpha = J_\alpha$ and there exists a set of homogeneous Lagrange polynomials of degree α.

Theorem 4.1. *If $\alpha \in \mathrm{Reg}(I)$, $\alpha_0 \in \mathrm{Cl}(X)_+$ is such that no ζ_j is a basepoint of S_{α_0} and $\mathrm{HF}_I(\alpha + \alpha_0) = \delta$, then $\alpha + \alpha_0 \in \mathrm{Reg}(I)$.*

Proof. Let $\ell_j, j = 1, \dots, \delta$ be a set of homogeneous Lagrange polynomials of degree α w.r.t. $h \in S_\alpha$. It is easy to verify that for generic $h_0 \in S_{\alpha_0}$, $h_0\ell_j, j = 1, \dots, \delta$ is a set of homogeneous Lagrange polynomials of degree $\alpha + \alpha_0$ w.r.t. hh_0. □

If $\alpha \in \mathrm{Pic}(X)$ is basepoint free and $\mathrm{HF}_I(\alpha) = \delta$, then to show that $\alpha \in \mathrm{Reg}(I)$, by Proposition 4.2 it suffices to show that L_α is of rank δ. If α is 'large enough' (the associated polytope has enough lattice points), this seems reasonable to expect. Alternatively, by Proposition 4.1 it suffices to show that $I_\alpha = J_\alpha$. Based on experimental evidence we propose the following conjecture.

Conjecture 1. Let $I = \langle f_1, \dots, f_n \rangle \subset S$ be a homogeneous ideal obtained as in Section 3 such that $V_X(I)$ is a zero-dimensional subscheme of $U \subset X$. Let $\alpha_i = \deg(f_i) \in \mathrm{Pic}(X)$ be the basepoint free degrees of the generators. Then $\alpha_0 + \alpha_1 + \dots + \alpha_n \in \mathrm{Reg}(I)$ for all $\alpha_0 \in \mathrm{Cl}(X)_+$ such that no ζ_j is a basepoint of S_{α_0}.

In the rest of this section, we prove some weaker results to support Conjecture 1 and we continue our running example by investigating the regularity.

We consider the question for which $\alpha \in \mathrm{Cl}(X)$ we have $\mathrm{HF}_I(\alpha) = \delta$. The following theorem generalizes Theorem 3.16 in [42] in the case where Z is small enough.

Theorem 4.2. *Let $I = \langle f_1, \dots, f_n \rangle \subset S$ be a homogeneous ideal obtained as in Section 3 such that $V_X(I)$ is a zero-dimensional subscheme of $U \subset X$. Let $\alpha_i = \deg(f_i) \in \mathrm{Pic}(X)$ be the basepoint free degrees of the generators. If $\mathrm{codim}(Z) \geq n$ then for all basepoint free $\alpha_0 \in \mathrm{Pic}(X)_+$, $\mathrm{HF}_I(\alpha_0 + \alpha_1 + \dots + \alpha_n) = \delta$.*

Proof. Consider the Koszul complex

$$0 \to S(-\sum_{i=1}^{n} \alpha_i) \to \bigoplus_{\substack{\mathcal{J} \subset \{1,\dots,n\} \\ |\mathcal{J}|=n-1}} S(-\alpha_{\mathcal{J}}) \to \cdots \to \bigoplus_{\substack{\mathcal{J} \subset \{1,\dots,n\} \\ |\mathcal{J}|=2}} S(-\alpha_{\mathcal{J}}) \to \bigoplus_{i=1}^{n} S(-\alpha_i) \to S$$

where $\alpha_{\mathcal{J}} = \sum_{i \in \mathcal{J}} \alpha_i$ and $S(-\alpha)$ is the Cox ring with twisted grading: $S(-\alpha)_\beta = S(\beta - \alpha)$. Since the orbit closures $\overline{G \cdot z_j}$ have dimension $k - n$ and by assumption $\dim(Z) \leq k - n$, the f_i form a regular sequence in S. Hence the Koszul complex is exact. Restricting to the degree $\alpha = \alpha_0 + \alpha_1 + \ldots + \alpha_n$ part we get

$$0 \to S(\alpha_0) \to \bigoplus_{\substack{\mathcal{J} \subset \{1,\dots,n\} \\ |\mathcal{J}|=n-1}} S(\alpha - \alpha_{\mathcal{J}}) \to \cdots \to \bigoplus_{\substack{\mathcal{J} \subset \{1,\dots,n\} \\ |\mathcal{J}|=2}} S(\alpha - \alpha_{\mathcal{J}}) \to \bigoplus_{i=1}^{n} S(\alpha - \alpha_i) \to S_\alpha.$$

Since α_0 is basepoint free, it corresponds to a polytope P_0 and we have by (2.4) and (3.1)

$$\dim_{\mathbb{C}}(S_{\alpha_0 + \alpha_{\mathcal{J}}}) = |(P_0 + P_{\mathcal{J}}) \cap M|$$

with $P_{\mathcal{J}} = \sum_{i \in \mathcal{J}} P_i$ for any subset $\mathcal{J} \subset \{0, \dots, n\}$. Counting dimensions we get

$$\dim_{\mathbb{C}}((S/I)_\alpha) = \sum_{\ell=0}^{n} (-1)^{n-\ell} \sum_{\substack{\mathcal{J} \subset \{1,\dots,n\} \\ |\mathcal{J}|=\ell}} |(P_0 + P_{\mathcal{J}}) \cap M|,$$

and the right hand side is the formula (3.3) for the mixed volume $\delta = \mathrm{MV}(P_1, \dots, P_n)$. □

Note that the conditions of Theorem 4.2 are satisfied by all toric surfaces ($n = 2$). Here is an analogous result for the case where the system is 'unmixed' (in some sense) and the corresponding polytope is normal.

Theorem 4.3. *Let $I = \langle f_1, \dots, f_n \rangle \subset S$ be a homogeneous ideal obtained as in Section 3 such that $V_X(I)$ is a zero-dimensional subscheme of X. Let $\alpha_i = \deg(f_i) \in \mathrm{Pic}(X)$ be the basepoint free degrees of the generators. If there is a basepoint free degree $\alpha_\star \in \mathrm{Pic}(X)$ corresponding to a normal polytope, such that $\alpha_i = t_i \alpha_\star$ for positive integers t_i, then $\mathrm{HF}_I(t\alpha_\star) = \delta$ for $t \geq \sum_{i=1}^{n} t_i$.*

Proof. The assumption on α_i implies that $P_i = t_i P_\star + m_i$ for a normal polytope $P_\star$, lattice points m_i and positive integers t_i. We can assume without loss of generality that $m_i = 0, i = 1, \dots, n$. We consider the embedding $X_{\mathscr{A}} \subset \mathbb{P}^{|\mathscr{A}|-1}$ of X where $\mathscr{A} = P_\star \cap M$. More precisely, $X_{\mathscr{A}}$ is the image of $\Phi_{\alpha_\star}$ [37, Proposition 5.4.7]. Let $u_m, m \in \mathscr{A}$ be homogeneous coordinates on $\mathbb{P}^{n_{\alpha_\star}-1} = \mathbb{P}^{|\mathscr{A}|-1}$. The toric ideal of $X_{\mathscr{A}}$ is denoted $I_{\mathscr{A}} \subset \mathbb{C}[u_m, m \in \mathscr{A}]$ and the $\mathbb{Z}$-graded coordinate ring of $X_{\mathscr{A}}$ is $\mathbb{C}[X_{\mathscr{A}}] = \mathbb{C}[u_m, m \in \mathscr{A}]/I_{\mathscr{A}}$. By [37, Theorem 5.4.8], we have $S_{\alpha_i} \simeq \mathbb{C}[X_{\mathscr{A}}]_{t_i}$ and $f_i \in S_{\alpha_i}$ corresponds to an element $h_i + I_{\mathscr{A}} \in \mathbb{C}[X_{\mathscr{A}}]_{t_i}$. We define the homogeneous ideal $I' = \langle h_1 + I_{\mathscr{A}}, \dots, h_n + I_{\mathscr{A}} \rangle \subset \mathbb{C}[X_{\mathscr{A}}]$. By assumption, I' defines a 0-dimensional subscheme of $X_{\mathscr{A}}$, so $h_1 + I_{\mathscr{A}}, \dots, h_n + I_{\mathscr{A}}$ is a regular sequence in $\mathbb{C}[X_{\mathscr{A}}]$. The ring $\mathbb{C}[X_{\mathscr{A}}]$ is arithmetically Cohen-Macaulay [37, Exercise 9.2.8], so the corresponding Koszul complex

$$0 \to K_n \to K_{n-1} \to \cdots \to K_2 \to K_1 \to \mathbb{C}[X_{\mathscr{A}}] \quad \text{with} \quad K_t = \bigoplus_{\substack{\mathcal{J} \subset \{1,\dots,n\} \\ |\mathcal{J}|=t}} \mathbb{C}[X_{\mathscr{A}}](-\sum_{i \in \mathcal{J}} t_i)$$

is exact. Since $P_\star$ is a normal polytope, we have $\dim_{\mathbb{C}}(\mathbb{C}[X_{\mathscr{A}}]_t) = |tP_\star \cap M|$. Counting dimensions and using the same formula as before for $\delta = \mathrm{MV}(P_1, \dots, P_n) = \mathrm{MV}(P_\star, \dots, P_\star)$ we find that $\dim_{\mathbb{C}}((\mathbb{C}[X_{\mathscr{A}}]/I')_t) = \delta$

for $t \geq \sum_{i=1}^{n} t_i$. Combining this with $(\mathbb{C}[X_{\mathscr{A}}]/I')_t \simeq (S/I)_{t\alpha_\star}$ (see [37, Theorem 5.4.8]) we get the desired result. $\square$

We note that in the case where X is a product of projective spaces, stronger bounds than those of Theorem 4.2 and Theorem 4.3 are known [25].

Example 4.1. We continue Example 3.1. The polytope $P = P_1 + P_2$ (shown in Fig. 2) has 12 lattice points. Therefore $n_\alpha = 12$, with $\alpha = [D_P] \in \mathrm{Pic}(X)$. Since $\delta = 3$, L_α is a 12×3 matrix. Its rows are indexed by the monomials spanning S_α, and its columns by the representatives z_j. The transpose is given by

$$L_\alpha^\top = \begin{array}{c} \begin{array}{cccccccccccc} x_3x_4^2 & x_1x_4^2 & x_2x_3^3x_4 & x_1x_2x_3^2x_4 & x_1^2x_2x_3x_4 & x_1^3x_2x_4 & x_2^2x_3^5 & x_1x_2^2x_3^4 & x_1^2x_2^2x_3^3 & x_1^3x_2^2x_3^2 & x_1^4x_2^2x_3 & x_1^5x_2^2 \end{array} \\ \left[\begin{array}{cccccccccccc} 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \end{array} \begin{array}{c} \\ z_1 \\ z_2 \\ z_3 \end{array}.$$

Consider $h = 39(x_3x_4^2 - x_1x_4^2) \in S_\alpha$ and note that $h(z_j) \neq 0$ for all j. A set of homogeneous Lagrange polynomials w.r.t. h is given by

$$\frac{2\,\tilde{L}_\alpha^\dagger}{13} = \begin{array}{c} \begin{array}{cccccccccccc} x_3x_4^2 & x_1x_4^2 & x_2x_3^3x_4 & x_1x_2x_3^2x_4 & x_1^2x_2x_3x_4 & x_1^3x_2x_4 & x_2^2x_3^5 & x_1x_2^2x_3^4 & x_1^2x_2^2x_3^3 & x_1^3x_2^2x_3^2 & x_1^4x_2^2x_3 & x_1^5x_2^2 \end{array} \\ \left[\begin{array}{cccccccccccc} 0 & 0 & 0 & 2 & -2 & 0 & 0 & -2 & 2 & -2 & 2 & 0 \\ 2 & 0 & -2 & -1 & 1 & 0 & 2 & 1 & -1 & 1 & -1 & 0 \\ 0 & 2 & 0 & 1 & -1 & -2 & 0 & -1 & 1 & -1 & 1 & 2 \end{array}\right] \end{array} \begin{array}{c} \\ \ell_1 \\ \ell_2 \\ \ell_3 \end{array},$$

which is related to the pseudo inverse of L_α by

$$L_\alpha^\dagger = \mathrm{diag}(h(z_1), h(z_2), h(z_3))^{-1}\tilde{L}_\alpha^\dagger = \mathrm{diag}(1/78, 1/39, 1/39)\tilde{L}_\alpha^\dagger.$$

To check that $I_\alpha = J_\alpha$ we compute $\mathrm{HF}_I(\alpha) = \mathrm{HF}_J(\alpha) = 3$. Hence we have $\alpha \in \mathrm{Reg}(I)$. In fact, in this example I is radical and α is ample, so $I_\alpha = J_\alpha$ follows from Corollary 4.1.

5. A toric eigenvalue, eigenvector theorem

In this section, we will work with multiplication maps between graded pieces of S/I. Again, I is a homogeneous ideal in S obtained as in Section 3 satisfying Assumptions 1-3. For $\alpha, \alpha_0 \in \mathrm{Cl}(X)_+$, a homogeneous element $g \in S_{\alpha_0}$ defines a linear map

$$M_g : (S/I)_\alpha \to (S/I)_{\alpha+\alpha_0} : f + I_\alpha \mapsto gf + I_{\alpha+\alpha_0}$$

representing 'multiplication with g'. Just as in the affine case, these multiplication maps will be the key ingredient to formulate our root finding problem as a linear algebra problem. We state a toric version of the eigenvalue, eigenvector theorem and show how the eigenvalues can be used to recover homogeneous coordinates of the solutions and equations for the corresponding G-orbits. Our main result uses the following Lemma.

Lemma 5.1. *Let $\alpha, \alpha_0 \in \mathrm{Cl}(X)_+$ be such that $\alpha, \alpha + \alpha_0 \in \mathrm{Reg}(I)$ and no ζ_j is a basepoint of S_{α_0}. Then for generic $h_0 \in S_{\alpha_0}$, $M_{h_0} : (S/I)_\alpha \to (S/I)_{\alpha+\alpha_0} : f + I_\alpha \mapsto h_0 f + I_{\alpha+\alpha_0}$ is an isomorphism of vector spaces.*

Proof. Let ψ_α be given as in (4.1) for some $h \in S_\alpha$. We can take $hh_0 \in S_{\alpha+\alpha_0}$ to define $\psi_{\alpha+\alpha_0}$. Then $\psi_{\alpha+\alpha_0} \circ M_{h_0} = \psi_\alpha$ shows that M_{h_0} is invertible. □

Theorem 5.1 *(Toric eigenvalue, eigenvector theorem). Let $\alpha, \alpha_0 \in \mathrm{Cl}(X)_+$ be such that $\alpha, \alpha+\alpha_0 \in \mathrm{Reg}(I)$ and no ζ_j is a basepoint of S_{α_0}. Then for any $g \in S_{\alpha_0}$, $M_{g/h_0} = M_{h_0}^{-1} \circ M_g : (S/I)_\alpha \to (S/I)_\alpha$ has eigenpairs*

$$\left(\frac{g}{h_0}(z_j), \ell_j + I_\alpha\right), \qquad \left(v_j, \frac{g}{h_0}(z_j)\right), \quad j = 1, \dots, \delta,$$

where the $\ell_j + I_\alpha$ are cosets of homogeneous Lagrange polynomials of degree α and the v_j are the dual basis of $(S/I)_\alpha^\vee$.

Proof. The map M_{h_0} is an isomorphism by Lemma 5.1. We define ψ_α, $\psi_{\alpha+\alpha_0}$ as in (4.1) with $h \in S_\alpha$, $hh_0 \in S_{\alpha+\alpha_0}$ respectively. A straightforward computation shows that $\psi_{\alpha+\alpha_0} \circ M_{h_0}(\ell_j + I_\alpha) = e_j$. Analogously, we have $\psi_{\alpha+\alpha_0} \circ M_g(\ell_j + I_\alpha) = \frac{g}{h_0}(z_j) e_j$. It follows that $h_0(z_j) M_g(\ell_j + I_\alpha) = g(z_j) M_{h_0}(\ell_j + I_\alpha)$, and therefore

$$M_{g/h_0}(\ell_j + I_\alpha) = \frac{g}{h_0}(z_j)(\ell_j + I_\alpha),$$

which proves the statement about the right eigenpairs, since the $\ell_j + I_\alpha$ are linearly independent. For the statement about the left eigenpairs, note that for any $f \in S_\alpha$

$$v_j \circ M_{g/h_0}(f + I_\alpha) = v_j \circ M_{h_0}^{-1}(gf + I_{\alpha+\alpha_0})$$

and since M_{h_0} is an isomorphism, there is $\tilde{f} \in S_\alpha$ such that $gf - h_0\tilde{f} \in I_{\alpha+\alpha_0}$. Therefore, for each $z_j \in V(I)$ we have

$$\frac{gf - h_0\tilde{f}}{h_0 h}(z_j) = 0 \Rightarrow \frac{\tilde{f}}{h}(z_j) = \frac{g}{h_0}(z_j)\frac{f}{h}(z_j)$$

and thus, since $M_{h_0}^{-1}(gf + I_{\alpha+\alpha_0}) = \tilde{f} + I_\alpha$, we have

$$v_j \circ M_{g/h_0}(f + I_\alpha) = v_j(\tilde{f} + I_\alpha) = \frac{g}{h_0}(z_j) v_j(f + I_\alpha).$$

The v_j are linearly independent, so this concludes the proof. □

Let $S_{\alpha_0} = \bigoplus_{i=1}^{n_{\alpha_0}} \mathbb{C} \cdot x^{b_i}$ where $\alpha_0 \in \mathrm{Cl}(X)_+$ is such that no ζ_j is a basepoint of S_{α_0}. We now show how the eigenvalues of the $M_{x^{b_i}/h_0}$ lead directly to a set of defining equations of $G \cdot z_j, j = 1, \dots, \delta$ if α_0 is 'large enough'. For every cone $\sigma \in \Sigma_P$, we define $U_\sigma = \mathbb{C}^k \setminus V(x^{\hat{\sigma}}) = \mathrm{MaxSpec}(S_{x^{\hat{\sigma}}})$. Note that $\mathbb{C}^k \setminus Z = \bigcup_{\sigma \in \Sigma_P} U_\sigma$. Let D_{α_0} be a representative divisor: $\alpha_0 = [D_{\alpha_0}] = [\sum_{i=1}^k a_{0,i} D_i]$. Let $P_0 \subset M_\mathbb{R}$ be the polytope $\{m \in M_\mathbb{R} \mid F^\top m + a_0 \geq 0\}$. If $\alpha_0 \in \mathrm{Pic}(X)$, then for every $\sigma \in \Sigma_P$ there is $m_\sigma \in P_0 \cap M$ such that

$$\langle u_i, m_\sigma \rangle + a_{0,i} = 0, \quad \forall \rho_i \in \sigma(1), \tag{5.1}$$

see for instance [39, Lemma 3.4] or [37, Theorem 4.2.8]. If D_{α_0} is not Cartier, such an m_σ does not exist for every cone $\sigma \in \Sigma_P$. We will denote the subset of cones for which $m_\sigma \in P_0$ satisfying (5.1) exists by $\widetilde{\Sigma}_P \subset \Sigma_P$. This set is nonempty since $\{0\} \in \widetilde{\Sigma}_P$. We write $P_0 \cap M = \{m_1, \dots, m_{n_{\alpha_0}}\}$, $b_i = F^\top m_i + a_0$ and $b_\sigma = F^\top m_\sigma + a_0$. For all $\sigma \in \widetilde{\Sigma}_P$ we denote $P_0 \cap M - m_\sigma = \{m_1 - m_\sigma, \dots, m_{n_{\alpha_0}} - m_\sigma\}$ (note that

$0 \in P_0 \cap M - m_\sigma$) and $\sigma^\vee = \{m \in M_\mathbb{R} \mid \langle u, m\rangle \geq 0, \forall u \in \sigma\}, \sigma^\perp = \{m \in M_\mathbb{R} \mid \langle u, m\rangle = 0, \forall u \in \sigma\}$. We partition $P_0 \cap M - m_\sigma$ into $\mathcal{M}_\sigma^\perp = (P_0 \cap M - m_\sigma) \cap \sigma^\perp$ and $\mathcal{M}_\sigma = (P_0 \cap M - m_\sigma) \setminus \mathcal{M}_\sigma^\perp$. The inclusion

$$\mathbb{N}\mathcal{M}_\sigma + \mathbb{Z}\mathcal{M}_\sigma^\perp = \left\{ \sum_{m \in \mathcal{M}_\sigma} c_m m + \sum_{m \in \mathcal{M}_\sigma^\perp} d_m m \mid c_m \in \mathbb{N}, d_m \in \mathbb{Z} \right\} \subset \sigma^\vee \cap M$$

is clear. In what follows, we will show that if equality holds for some simplicial $\sigma \in \widetilde{\Sigma}_P$ with $z_j \in U_\sigma$, then α_0 is 'large enough' to recover equations for $G \cdot z_j$ from the eigenvalues of the $M_{x^{b_i}/h_0}$.

Theorem 5.2. *Let $z \in U_\sigma$ for a simplicial cone $\sigma \in \widetilde{\Sigma}_P$ such that $\pi(z)$ is not a basepoint of S_{α_0}. Take $h_0 \in S_{\alpha_0}$ such that $h_0(z) \neq 0$ and let $\lambda_i = z^{b_i}/h_0(z), i = 1, \dots, n_{\alpha_0}$. If $\sigma^\vee \cap M = \mathbb{N}\mathcal{M}_\sigma + \mathbb{Z}\mathcal{M}_\sigma^\perp$, then $G \cdot z \subset U_\sigma$ is the subvariety defined by the ideal*

$$\left\langle x^{b_i - b_\sigma} - \lambda_i \frac{h_0(x)}{x^{b_\sigma}} \mid i = 1, \dots, n_{\alpha_0} \right\rangle \subset S_{x^{\hat{\sigma}}}.$$

To prove Theorem 5.2, we need the following auxiliary lemma.

Lemma 5.2. *Let $\sigma \in \widetilde{\Sigma}_P$ be a simplicial cone. For any point $z \in U_\sigma$, the orbit $G \cdot z$ is the subvariety defined by*

$$G \cdot z = \{x \in U_\sigma \mid x^{F^\top m} - z^{F^\top m}, m \in \sigma^\vee \cap M\} \subset U_\sigma.$$

If $\sigma^\vee \cap M = \mathbb{N}\{m_1, \dots, m_\kappa\} + \mathbb{Z}\{m_{\kappa+1}, \dots, m_s\}$, then

$$\{x \in U_\sigma \mid x^{F^\top m} - z^{F^\top m}, m \in \sigma^\vee \cap M\} = \{x \in U_\sigma \mid x^{F^\top m_i} - z^{F^\top m_i}, i = 1, \dots, s\}.$$

Proof. Note that $x^{F^\top m} - z^{F^\top m} \in S_{x^{\hat{\sigma}}}, \forall m \in \sigma^\vee \cap M$ and $m_{\kappa+1}, \dots, m_s \in \sigma^\perp \cap M$. The first statement is shown in the proof of Theorem 2.1 in [39]. For the second statement, the inclusion '$\subset$' is obvious. To show the opposite inclusion, take $m \in \sigma^\vee \cap M$ and write $m = c_1 m_1 + \ldots + c_s m_s$ with $c_1, \dots, c_\kappa \in \mathbb{N}$, $c_{\kappa+1}, \dots, c_s \in \mathbb{Z}$. Then

$$x^{F^\top m} = \prod_{i=1}^{\kappa} (x^{F^\top m_i})^{c_i} \prod_{j=\kappa+1}^{s} (x^{F^\top m_j})^{c_j}$$

and if $x^{F^\top m_i} = z^{F^\top m_i}, i = 1, \dots, s$, it follows that $x^{F^\top m} = z^{F^\top m}$. $\square$

Proof of Theorem 5.2. It follows from Lemma 5.2 that $G \cdot z$ is the variety of

$$\left\langle x^{F^\top(m_i - m_\sigma)} - z^{F^\top(m_i - m_\sigma)} \mid i = 1, \dots, n_{\alpha_0} \right\rangle = \left\langle x^{b_i - b_\sigma} - z^{b_i - b_\sigma} \mid i = 1, \dots, n_{\alpha_0} \right\rangle.$$

Write $h_0(x) = \sum_{i=1}^{n_{\alpha_0}} c_i x^{b_i}, c_i \in \mathbb{C}$. It is easy to check that

$$\left(\begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix} - \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_{n_{\alpha_0}} \end{bmatrix} \begin{bmatrix} c_1 & \dots & c_{n_{\alpha_0}} \end{bmatrix} \right) \begin{bmatrix} x^{b_1 - b_\sigma} - z^{b_1 - b_\sigma} \\ \vdots \\ x^{b_{n_{\alpha_0}} - b_\sigma} - z^{b_{n_{\alpha_0}} - b_\sigma} \end{bmatrix} = \begin{bmatrix} x^{b_1 - b_\sigma} - \lambda_1 \frac{h_0(x)}{x^{b_\sigma}} \\ \vdots \\ x^{b_{n_{\alpha_0}} - b_\sigma} - \lambda_{n_{\alpha_0}} \frac{h_0(x)}{x^{b_\sigma}} \end{bmatrix}$$

and for generic c_i, the matrix on the left is invertible (it's invertible for $c_i = 0$, so the determinant is a nonzero polynomial in the c_i). $\square$

Theorem 5.3. *Let $z \in U_\sigma$ with $\sigma \in \widetilde{\Sigma}_P$ simplicial be such that $\pi(z)$ is not a basepoint of S_{α_0} and $\sigma^\vee \cap M = \mathbb{N}\mathcal{M}_\sigma + \mathbb{Z}\mathcal{M}_\sigma^\perp$. For generic $h_0 \in S_{\alpha_0}$ satisfying $h_0(z) \neq 0$, the variety*

$$Y_z = V\left(x^{b_i} - \frac{z^{b_i}}{h_0(z)}, i = 1, \dots, n_{\alpha_0}\right) \subset \mathbb{C}^k$$

is nonempty and $Y_z \subset G \cdot z$.

The proof of Theorem 5.3 uses the following lemma.

Lemma 5.3. *If $\alpha_0 \in \mathrm{Cl}(X)_+$ is such that $\sigma^\vee \cap M = \mathbb{N}\mathcal{M}_\sigma + \mathbb{Z}\mathcal{M}_\sigma^\perp$ for some $\sigma \in \widetilde{\Sigma}_P$, then α_0 is not a torsion element of $\mathrm{Cl}(X)$.*

Proof. Suppose $u\alpha_0 = 0$ for some $u > 0$. Then $F^\top m + ua_0 = 0$ for some $m \in M$, and therefore $F^\top(m/u) + a_0 = 0$. Since Σ_P is complete, this means that $P_0 = \{m/u\}$ and P_0 either has 1 lattice point (if $m/u \in M$, in which case $\alpha_0 = 0$), or it has none. Since $\alpha_0 \in \mathrm{Cl}(X)_+$, we can assume $0 \in P_0$ and this must be the only lattice point in P_0. Then $\sigma^\vee \cap M = \mathbb{N}\mathcal{M}_\sigma + \mathbb{Z}\mathcal{M}_\sigma^\perp = \{0\}$. But $\sigma^\vee$ has dimension n because σ is strongly convex ([37, Proposition 1.2.12]), so this is a contradiction. □

Proof of Theorem 5.3. Since α_0 is not a torsion element of $\mathrm{Cl}(X)$ (Lemma 5.3), we have the exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathrm{Cl}(X) \longrightarrow \mathrm{Cl}(X)/(\mathbb{Z} \cdot \alpha_0) \longrightarrow 0$$

where $\mathbb{Z} \to \mathrm{Cl}(X)$ sends $u \mapsto u\alpha_0 \in \mathrm{Cl}(X)$. Taking $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{C}^*)$ shows that $G \to \mathbb{C}^* : g \mapsto g^{\alpha_0}$ is surjective (because $\mathbb{C}^*$ is divisible). Therefore we can find $g \in G$ such that $g^{\alpha_0} = h_0(z)^{-1}$ and thus $h_0(g \cdot z) = 1$. Every $x \in Y_z$ satisfies $x^{b_i} - (g \cdot z)^{b_i} = 0, i = 1, \dots, n_{\alpha_0}$: this follows from $(g \cdot z)^{b_i} = z^{b_i}/h_0(z)$. In particular, $x^{b_\sigma} = (g \cdot z)^{b_\sigma} \neq 0$ ($z \in U_\sigma$ and hence $g \cdot z \in U_\sigma$ since U_σ is G-invariant) and therefore x satisfies $x^{b_i - b_\sigma} = (g \cdot z)^{b_i - b_\sigma}, i = 1, \dots n_{\alpha_0}$. By Lemma 5.2 it follows that $g \cdot z \in Y_z \subset G \cdot z$. □

Recall that we took α_0 such that no ζ_j is a basepoint of S_{α_0}. We conclude this section with the following immediate corollary of Theorems 5.2 and 5.3.

Corollary 5.1. *Let $\lambda_{ij} = z_j^{b_i}/h_0(z_j)$ be the j-th eigenvalue of the i-th multiplication map $M_{x^{b_i}/h_0}$. For $j = 1, \dots, \delta$, assume that $z_j \in U_{\sigma_j}$ for a simplicial cone $\sigma_j \in \widetilde{\Sigma}_P$ satisfying $\sigma_j^\vee \cap M = \mathbb{N}\mathcal{M}_{\sigma_j} + \mathbb{Z}\mathcal{M}_{\sigma_j}^\perp$. The ideal*

$$\left\langle x^{b_i - b_{\sigma_j}} - \lambda_{ij} \frac{h_0(x)}{x^{b_{\sigma_j}}} \mid i = 1, \dots, n_{\alpha_0} \right\rangle \subset S_{x^{\hat{\sigma}_j}}$$

defines the orbit $G \cdot z_j \subset U_{\sigma_j}$, and for any point $z_j' \in V(x^{b_i} - \lambda_{ij}, i = 1, \dots, n_{\alpha_0}) \subset U_{\sigma_j}$, we have $\pi(z_j') = \zeta_j$.

Corollary 5.1 implies that we can find homogeneous coordinates of the solutions from the eigenvalues λ_{ij} by solving a system of binomial equations if P_0 'has enough lattice points'. Concretely, for every point z_j there has to be a cone $\sigma_j \in \widetilde{\Sigma}_P$ such that $z_j \in U_{\sigma_j}$ and $\sigma_j^\vee \cap M = \mathbb{N}\mathcal{M}_{\sigma_j} + \mathbb{Z}\mathcal{M}_{\sigma_j}^\perp$. Note that if all solutions are in the torus, then $z_j \in U_\sigma$ for $\sigma = \{0\} \in \widetilde{\Sigma}_P$ and this condition translates to the fact that $\mathbb{Z}(P_0 \cap M - m) = M$ for some $m \in P_0 \cap M$. If P_0 is very ample, then $\alpha_0 \in \mathrm{Pic}(X)$, so $\widetilde{\Sigma}_P = \Sigma_P$ and $\sigma^\vee \cap M = \mathbb{N}\mathcal{M}_\sigma + \mathbb{Z}\mathcal{M}_\sigma^\perp$ holds for all $\sigma \in \Sigma_P$ [37, Proposition 1.3.16]. We will elaborate on how to solve this system of binomial equations in the next section.

6. Algorithm

In this section we present an eigenvalue algorithm for computing homogeneous coordinates of the points in $V_X(I)$, where I is an ideal satisfying Assumptions 1-3. As in Theorem 5.1, let $\alpha, \alpha_0 \in \mathrm{Cl}(X)_+$ be such that $\alpha, \alpha + \alpha_0 \in \mathrm{Reg}(I)$ and no ζ_j is a basepoint of S_{α_0}. In practice, we will take $\alpha = \alpha_1 + \cdots + \alpha_n$ where $\alpha_i = \deg(f_i)$ (by Conjecture 1) and α_0 'large enough' to recover all solutions (Corollary 5.1). We denote

$$S_{\alpha_0} = \bigoplus_{i=1}^{n_{\alpha_0}} \mathbb{C} \cdot x^{b_i}.$$

We have that $\mathrm{HF}_I(\alpha) = \mathrm{HF}_I(\alpha + \alpha_0) = \delta$. Given a generic $h_0 \in S_{\alpha_0}$ and a surjective linear map $N : S_{\alpha+\alpha_0} \to \mathbb{C}^\delta$ with $\ker N = I_{\alpha+\alpha_0}$, we define

$$N_{h_0} : S_\alpha \to \mathbb{C}^\delta : f \mapsto N(h_0 f)$$

and assume that N_{h_0} is surjective as well. Such a map N can be computed directly from the input equations. We will come back to this later. Let $N^* : B \to \mathbb{C}^\delta$ be the restriction of N_{h_0} to a subspace $B \subset S_\alpha$ of dimension δ such that N^* is invertible, and let

$$N_i : B \to \mathbb{C}^\delta : f \mapsto N(x^{b_i} f), \qquad i = 1, \ldots, n_{\alpha_0}.$$

Theorem 6.1. *The map $\nu : B \simeq (S/I)_\alpha : g \mapsto g + I_\alpha$ is an isomorphism of vector spaces and the linear maps $(N^*)^{-1} \circ N_i : B \to B$ satisfy $\nu \circ (N^*)^{-1} \circ N_i = M_{x^{b_i}/h_0} \circ \nu$ where $M_{x^{b_i}/h_0}$ are the maps from Theorem 5.1.*

Proof. By Lemma 5.1, $h_0 f \in I_{\alpha+\alpha_0}$ if and only if $f \in I_\alpha$. Therefore $\ker N_{h_0} = I_\alpha$. The first statement follows from $S_\alpha = B \oplus \ker N_{h_0}$. Since $\ker N = I_{\alpha+\alpha_0}$, N is well defined mod $I_{\alpha+\alpha_0}$. We define

$$\tilde{N} : (S/I)_{\alpha+\alpha_0} \to \mathbb{C}^\delta : f + I_{\alpha+\alpha_0} \mapsto N(f).$$

Since $\tilde{N}$ is a surjective linear map between δ-dimensional vector spaces, it is invertible. For $g \in B$, $N^*(g) = N(h_0 g) = (\tilde{N} \circ M_{h_0})(g + I_\alpha)$ so $\nu \circ (N^*)^{-1} = (\tilde{N} \circ M_{h_0})^{-1}$. Analogously we find $N_i(g) = (\tilde{N} \circ M_{x^{b_i}})(g + I_\alpha)$. The theorem follows from $(\nu \circ (N^*)^{-1} \circ N_i)(g) = ((\tilde{N} \circ M_{h_0})^{-1} \circ (\tilde{N} \circ M_{x^{b_i}}))(g + I_\alpha) = (M_{h_0}^{-1} \circ M_{x^{b_i}} \circ \nu)(g)$. □

Theorem 6.1 tells us that, identifying B with $(S/I)_\alpha$, the homogeneous multiplication operators are given by $(N^*)^{-1} \circ N_i$. After fixing a basis $\mathcal{B}$ for B the multiplication operators are commuting $\delta \times \delta$ matrices and we can compute their simultaneous diagonalization to find the values $\lambda_{ij} = z_j^{b_i}/h_0(z_j)$.

We now show how the map N can be computed from the input equations. Our strategy is based on techniques for computing Truncated Normal Forms (TNFs), as introduced in [18]. We use the notation $V = S_{\alpha+\alpha_0}$, $V_i = S_{\alpha+\alpha_0-\alpha_i}$ and by the *Resultant map* $\mathrm{Res} : V_1 \times \cdots \times V_n \to V$ we mean the linear map

$$(q_1, \ldots, q_n) \mapsto q_1 f_1 + \ldots + q_n f_n.$$

When represented in matrix form, using monomial bases for the vector spaces involved, this map looks a lot like the resultant matrices coming from Macaulay and toric resultants [9, Chapters 3 and 7]. Since $\mathrm{im}\mathrm{Res} = I_{\alpha+\alpha_0}$, the cokernel map of Res is a map $N : V \to \mathbb{C}^\delta \simeq V/\mathrm{im}\mathrm{Res}$ with the properties we need.

The next step is to find the homogeneous coordinates of $V_X(I)$ from the λ_{ij}. Suppose that $z_j \in U_{\sigma_j}$ for $\sigma_j \in \widetilde{\Sigma}_P$ and that α_0 is such that $\sigma_j^\vee \cap M = \mathbb{N}\mathcal{M}_{\sigma_j} + \mathbb{Z}\mathcal{M}_{\sigma_j}^\perp$. By Corollary 5.1, it remains to compute one point on the variety $V(x^{b_i} - \lambda_{ij}, i = 1, \ldots, n_{\alpha_0})$ for $j = 1, \ldots, \delta$. If $\zeta_j \in T_X$, we can do this efficiently using only linear algebra as follows. Let $A = [b_1 \ \cdots \ b_{n_{\alpha_0}}] \in \mathbb{Z}^{k \times n_{\alpha_0}}$ be the matrix of exponents and compute

Algorithm 1 Computes the Cox coordinates of the points defined by $I = \langle f_1, \dots, f_n \rangle$.

1: Res $\leftarrow$ Matrix of the resultant map $V_1 \times \cdots \times V_n \to V$
2: $N \leftarrow$ Matrix of the cokernel $V \to \mathbb{C}^\delta$ of Res
3: $h_0 \leftarrow$ Generic element of S_{α_0}
4: Construct a matrix of N_{h_0}
5: Find $B \subset S_\alpha$ such that $(N_{h_0})_{|B}$ is invertible
6: $N^* \leftarrow (N_{h_0})_{|B}$
7: Construct a matrix of N_i, $1 \le i \le n_{\alpha_0}$
8: **for** $i = 1, \dots, n_{\alpha_0}$ **do**
9: $\quad M_{x^{b_i}/h_0} \leftarrow (N^*)^{-1} N_i$
10: **end for**
11: Compute $\lambda_{ij}, 1 \le i \le n_{\alpha_0}, 1 \le j \le \delta$ by sim. diag. of the $M_{x^{b_i}/h_0}$
12: **for** $j = 1, \dots, \delta$ **do**
13: $\quad \tilde{J}_j \leftarrow \left\langle x^{b_i} - \lambda_{ij}, 1 \le i \le n_{\alpha_0} \right\rangle \subset S$
14: $\quad$ **if** $(\min_i |\lambda_{ij}|)/(\sum_{i=1}^{n_{\alpha_0}} |\lambda_{ij}|^2)^{1/2} > \mathrm{tol}$ **then**
15: $\qquad$ Find one point $z_j \in \mathbb{C}^k$ on $V(\tilde{J}_j)$ using SNF
16: $\quad$ **else**
17: $\qquad$ Find one point $z_j \in \mathbb{C}^k$ on $V(\tilde{J}_j)$ using Newton iteration
18: $\quad$ **end if**
19: **end for**
20: **return** $z_1, \dots, z_\delta$

its Smith normal form: $UAV = S$ with U, V unimodular and $S = [\mathrm{diag}(m_1, \dots, m_r, 0, \dots, 0)\ 0] \in \mathbb{Z}^{k \times n_{\alpha_0}}$, where $m_i | m_{i+1}$. We make the substitution of variables $x_\ell = y_1^{U_{1\ell}} \cdots y_k^{U_{k\ell}}$ to obtain the equivalent system of equations given by $y^{Ub_i} = \lambda_{ij}$. Applying the invertible transformation given by the matrix V, this simplifies to

$$y_\ell^{m_\ell} = \prod_{i=1}^{n_{\alpha_0}} \lambda_{ij}^{V_{i\ell}},\ \ell = 1, \dots, r \quad \text{and} \quad 1 = \prod_{i=1}^{n_{\alpha_0}} \lambda_{ij}^{V_{i\ell}},\ r < \ell \le k.$$

This imposes no conditions on $y_\ell, \ell > r$, so we can put $y_\ell = 1, \ell > r$. Taking the logarithm then shows that

$$\log y = [\log y_1 \ \cdots \ \log y_k] = [w \ 0_{k-r}]$$

where $w = [\log \lambda_{1j} \ \cdots \log \lambda_{n_{\alpha_0} j}][V_{:,1} \ \cdots \ V_{:,r}]\mathrm{diag}(1/m_1, \dots, 1/m_r)$ and 0_{k-r} is a row vector of length $k - r$ with zero entries. To find the homogeneous coordinates, we only need to invert our change of coordinates and the logarithm:

$$\log x = [\log x_1 \ \cdots \ \log x_k] = \log y\ U, \qquad x_\ell = e^{\log x_\ell}, \ell = 1, \dots, k.$$

Taking the logarithm has some advantages for the implementation: it reduces all computations to some matrix multiplications and it may prevent overflow. When ζ_j is not in the torus, some of the λ_{ij} may be zero. In this case, to compute a point on $V(x^{b_i} - \lambda_{ij}, i = 1, \dots, n_{\alpha_0})$, we may use a simple Newton iteration, for instance. In the *nearly* degenerate situation, where λ_{ij} is close to zero for some i, the approach above suffers from rounding errors. We take this into account by using the Smith normal form technique when $(\min_i |\lambda_{ij}|)/(\sum_{i=1}^{n_{\alpha_0}} |\lambda_{ij}|^2)^{1/2} > \mathrm{tol}$, where $|\cdot|$ denotes the modulus and tol is a predefined tolerance. This leads to Algorithm 1.

In line 5 of the algorithm, the choice of the subspace B is important for the numerical stability. A good choice is using QR factorization with optimal column pivoting as in [17,18] which results in a basis for $(S/I)_\alpha$ consisting of monomials in S. An alternative is using the singular value decomposition, in which case B is the orthogonal complement of I_α in S_α [50, Section 3]. We use the SVD for the experiments in this article.

Algorithm 1 requires some computations involving polytopes. If one is interested in solving many systems with the same structure, it is advantageous to do these computations in an 'offline' phase. The 'online' algorithm then takes a basis of S_{α_0}, S_α and $S_{\alpha+\alpha_0}$, a facet representation of P and P_0 and the mixed volume $\delta = \mathrm{MV}(P_1, \dots, P_n)$ as inputs. The 'offline' version of the algorithm computes all this information from the input equations, and generates an α_0 such that $\mathbb{Z}(P_0 \cap M - m) = M$. This is enough to find (at least) all solutions in the torus by Corollary 5.1.

To retrieve the coordinates in $(\mathbb{C}^*)^n$ of toric solutions from their homogeneous coordinates, we use the map (2.1).

Remark 6.1. We conclude this Section with a remark on the complexity of Algorithm 1 as compared to the TNF algorithm of [18]. The first step in both algorithms is to compute the cokernel of a resultant map Res. Since for both algorithms the monomials indexing the vector space V in the definition of Res are the lattice points contained in a slightly enlarged version of the polytope $P = P_1 + \dots + P_n$, this step takes roughly the same computation time for both algorithms. Even though the Cox ring has dimension $k > n$, the dimensions of its graded pieces correspond to the lattice points contained in n-dimensional polytopes. This is an important observation, because for larger problems, the computation of the cokernel of Res is the most expensive step of the algorithm. Next, both algorithms compute the multiplication matrices from this cokernel. This is more expensive for the algorithm in this paper: there are more multiplication maps. Another important difference is that for the TNF algorithm, the eigenvalues of the multiplication maps immediately give the coordinates of the solutions, whereas Algorithm 1 processes these eigenvalues to find the homogeneous coordinates (line 12-19). We conclude that Algorithm 1 is computationally more expensive overall. This should be considered the price that is payed for being more robust in nearly degenerate situations, which is the main objective in this paper. However, the increase of complexity is not dramatic: systems with thousands of solutions can be solved within reasonable time (see Subsection 7.3), and there is certainly room for performance optimization in the current Matlab implementation, which is tested in the next Section.

7. Examples

Algorithm 1 is implemented in Matlab. Polymake is used for computations involving polytopes [51], except for the mixed volume, which is computed using PHCpack [7]. In this section, we test the implementation on several examples and compare the results with those of some other polynomial system solvers. All computations are done in double precision arithmetic on an 8 GB RAM machine with an intel Core I7-6820HQ CPU working at 2.70 GHz. To measure the quality of an approximate solution, we compute the *residual* as defined in [17, Section 7] as a measure for the relative backward error. In double precision arithmetic, a residual of order 10^{-16} is the best one can hope for. The goal of the experiments is to show that Algorithm 1 meets our objectives: it finds *all* solutions with *good accuracy* within reasonable time. In particular, it does so for (nearly) degenerate systems with solutions on or near the exceptional divisors of X that cannot be solved by other state of the art solvers.

7.1. Points on $\mathscr{H}_2$

We finish our running example by using Algorithm 1 to compute homogeneous coordinates of the solutions of the system defined in Example 2.3. We use tol $= 10^{-12}$, $\alpha = \alpha_1 + \alpha_2$. For $\alpha_0 = \alpha_2$, Algorithm 1 finds three solutions. All three residuals are of order 10^{-16}.

To illustrate the results, we use the *moment map*

$$\mu : \mathbb{C}^k \setminus Z \to P : x \mapsto \frac{1}{\sum_{m \in P \cap M} |x^{F^\top m + a}|} \sum_{m \in P \cap M} |x^{F^\top m + a}| m,$$

where $|\cdot|$ denotes the modulus. The map μ is constant on G-orbits and takes a point $x \in \mathbb{C}^k \setminus Z$ to a convex combination of the lattice points of P. It has the property that torus invariant prime divisors are

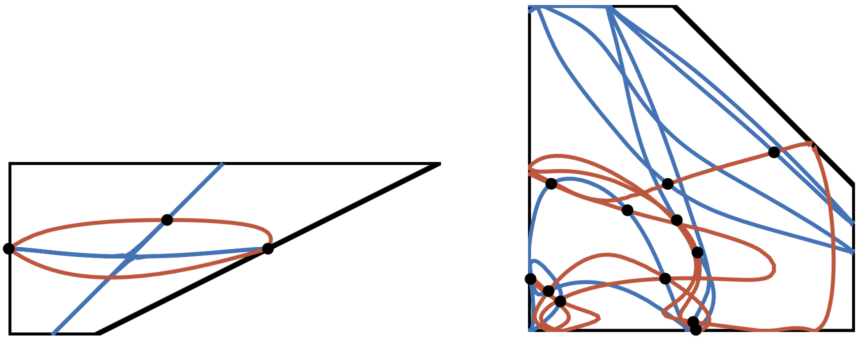
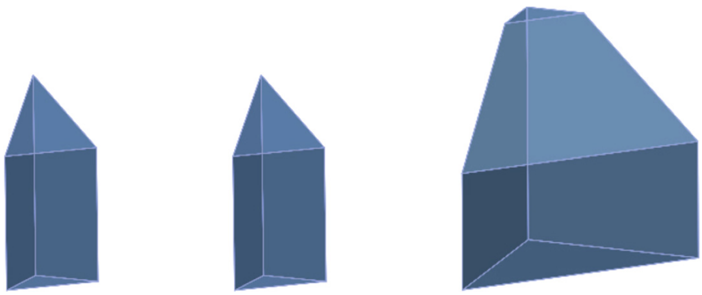


Fig. 3. Left: images in P of the real part of $V(f_1)$ (—) and $V(f_2)$ (—) from Example 2.3 under the moment map μ. The images of the computed real solutions are shown as black dots (●). Right: same picture for a different system. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)



(a) P_1 (b) P_2 (c) P_3

Fig. 4. Newton polytopes of the equations of the eight point radial distortion problem.

sent to their corresponding facets and $(\mathbb{C}^*)^k$ is sent to the interior of P. More information can be found in [38, Section 4.2] and [52, Section 2]. Fig. 3 shows that two of the computed solutions lie on divisors and one is in the torus. The image under μ of all of the solutions must lie on an intersection of the images of $V(f_1) \setminus Z, V(f_2) \setminus Z$ (but not all intersections correspond to solutions). As an illustration, we have included the same picture for a system with more solutions in the right part of the same figure. The polytopes for this system are $P_1 = [0, 4] \times [0, 4]$ and $P_2 = 5\Delta_2$ where Δ_2 is the standard simplex. There are $\delta = 40$ solutions, 12 of them are real.

7.2. A problem from computer vision

One of the so-called ‘minimal problems’ in computer vision is the problem of estimating radial distortion from eight point correspondences in two images. In [53], Kukelova and Pajdla propose a formulation of this problem as a system of 3 polynomial equations in 3 unknowns. The Newton polytopes are visualized in Fig. 4. The mixed volume is $\delta = \mathrm{MV}(P_1, P_2, P_3) = 17$ and the matrix of facet normals is

$$F = \begin{bmatrix} 0 & -1 & -1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & -1 \end{bmatrix},$$

so the Cox ring S has dimension 6. We assign random real coefficients drawn from a standard normal distribution to all lattice points in the polytopes and solve the system using Algorithm 1. We first run the offline version, which generates the polytope P_0. In this case, P_0 is the standard simplex. All 17 solutions are found with a residual of order 10^{-16} within ± 0.1 s (using the online version of the algorithm). To show the robustness of Algorithm 1 in the nearly degenerate case, i.e. the case where there are solutions on or near the torus invariant prime divisors, we perform the following experiment. Consider the lattice points

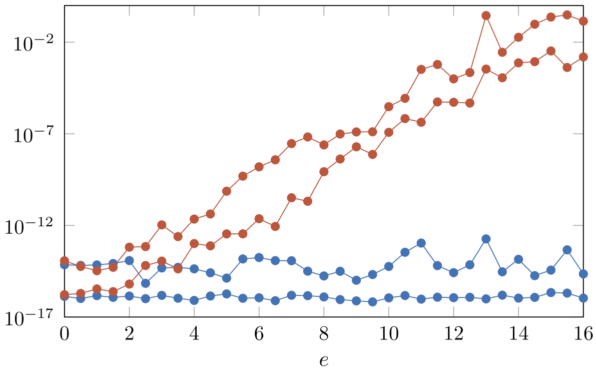


Fig. 5. Minimal and maximal residual for different values of the parameter e for the parametrized eight point radial distortion problem, for Algorithm 1 (—●—) and the toric TNF algorithm (—●—).

$$\mathscr{F}_3 = \{m \in P_1 \cap M \mid \langle u_3, m \rangle + 3 = 0\}, \qquad \mathscr{G}_3 = (P_1 \cap M) \setminus \mathscr{F}_3.$$

The points in $\mathscr{F}_3$ are the lattice points on the facet of P_1 corresponding to $u_3 = (-1, -1, -1)$. Set

$$\hat{g}_i = \sum_{m \in \mathscr{F}_3} c_{m,i} \chi^m + \sum_{m \in \mathscr{G}_3} c_{m,i} \chi^m, \qquad i = 1, 2$$

with $c_{m,i}$ real numbers drawn from a standard normal distribution. Now let $\hat{f}_1 = \hat{g}_1$ and

$$\hat{f}_2(e) = \sum_{m \in \mathscr{F}_3} (10^{-e} c_{m,2} + (1 - 10^{-e}) c_{m,1}) \chi^m + \sum_{m \in \mathscr{G}_3} c_{m,2} \chi^m, e \in [0, \infty).$$

The equation $\hat{f}_2 = 0$ is parametrized by the real parameter e. The third equation $\hat{f}_3 = 0$ is chosen randomly. When $e = 0$, $\hat{f}_2 = \hat{g}_2$ and the system is generic, as before. When $e \to \infty$, the part of $\hat{f}_2$ corresponding to $\mathscr{F}_3$ converges to the part of $\hat{f}_1$ corresponding to $\mathscr{F}_3$, meaning that there will be solutions 'at infinity' on the divisor D_3. We solve the system for $e = 0, 1/2, 1, 3/2, \ldots, 16$ and compute both the maximal residual $r_{\max}$ and the minimal residual $r_{\min}$ for the 17 solutions found by Algorithm 1 with tol $= 10^{-4}$ and the solutions found by the toric version of the Truncated Normal Form (TNF) algorithm [18]. The TNF solver computes the multiplication matrices for the input equations (in the classical sense) using heuristically 'the best possible basis' from a numerical point of view. The numerical results in [18,50] motivate the choice of this method as a reference. The result of the experiment is shown in Fig. 5. Note that not only the residuals of the solutions approaching the divisor deteriorate for the TNF algorithm. Accuracy is lost on *all* solutions. The reason is that even for the 'best' basis selected by this algorithm, the computation of the classical multiplication matrices is ill-conditioned because the system is nearly degenerate. Looking at the computed Cox coordinates, we see that for three of the solutions, the coordinate x_3 goes to zero as e increases, so 3 out of 17 solutions approach the divisor D_3.

One can perform the same experiment for any other facet of P_1. However, in order to find the solutions on the divisors, the polytope P_0 must be large enough and it might not be sufficient that its lattice points generate the lattice (Corollary 5.1). Repeating the same experiment, but this time using $\mathscr{F}_2$ instead of $\mathscr{F}_3$, the solutions in the torus are still found with good accuracy by Algorithm 1. Accuracy is lost on the solutions approaching D_2. The reason is that the standard simplex does not 'show' this facet. Using $P_0 = \mathrm{Conv}((0,0,0),(1,0,0),(0,1,0),(0,0,1),(1,0,1),(0,1,1),(0,0,2))$ we find homogeneous coordinates of all solutions.

Table 1
Results for generic systems with mixed supports.

n	NZ	$d_{\max}$	δ	k	n_{α_0}	OFFLINE			ONLINE		
						t	D_{mean}	$D_{\max}$	t	D_{mean}	$D_{\max}$
2	20	10	144	12	3	1.9e+1	15	14	2.0e-1	15	14
2	20	20	505	14	4	2.4e+1	14	12	1.9e+0	14	11
2	20	30	1268	15	3	5.8e+1	14	12	1.9e+1	14	12
2	20	40	2390	16	3	2.6e+2	14	11	1.4e+2	14	13
2	20	50	3275	16	3	3.7e+2	14	12	2.3e+2	14	11
2	20	60	4469	12	3	7.8e+2	11	7	5.2e+2	11	8
2	40	30	1522	15	3	9.5e+1	14	11	3.4e+1	14	10
2	60	30	1670	15	4	1.2e+2	14	12	5.3e+1	14	12
2	200	30	1672	10	3	1.1e+2	15	10	6.0e+1	15	9
3	5	3	18	21	4	2.2e+1	14	12	1.1e-1	15	13
3	5	5	136	36	4	3.9e+1	14	9	6.3e-1	14	13
3	10	5	190	60	5	3.5e+1	15	7	2.1e+0	15	11
3	10	7	592	63	5	1.3e+2	14	10	3.2e+1	15	7
4	5	3	81	106	6	6.9e+1	14	11	3.7e+1	14	11

7.3. Generic problems

To give an idea of the computation time and the type of systems Algorithm 1 can handle, we perform the following experiment. Consider the parameters n, NZ, $d_{\max} \in \mathbb{N} \setminus \{0\}$. For $j = 1, \dots, n$ we generate a set $\mathscr{A}_j \subset \mathbb{Z}^n$ of NZ lattice points by selecting NZ points in $\mathbb{N}^n$ with coordinates drawn uniformly from $\{0, 1, \dots, d_{\max}\}$ and shifting these points by substracting the first point from all other points. Then for each $m \in \mathscr{A}_j$ we generate a random real number $c_{m,j}$ drawn from a standard normal distribution and we set

$$\hat{f}_j = \sum_{m \in \mathscr{A}_j} c_{m,j} \chi^m.$$

If two or more points $m \in \mathscr{A}_j$ coincide, we add the $c_{m,j}$ together, so NZ is an upper bound for the number of terms in $\hat{f}_j$. We use Algorithm 1 to compute the Cox coordinates of the solutions of the resulting system and their image under (2.1). In Table 1 we report the number of solutions δ, the dimension k of the Cox ring, the number n_{α_0} for the automatically generated α_0, and, for both the offline and the online solver, the maximal residual $r_{\max}$, the geometric mean of the residuals of all solutions r_{mean} and the computation time t (in seconds). The residuals are represented by $D_{\text{mean}} = \lceil -\log_{10} r_{\text{mean}} \rceil$ and $D_{\max} = \lceil -\log_{10} r_{\max} \rceil$. It follows from Bernstein's second theorem [44,27] that solutions on divisors can only occur if the involved polytopes have common tropisms corresponding to positive dimensional faces. An important case in which this may happen is the unmixed case in which all input polytopes are equal. We repeat the experiment, but this time we keep the supports $\mathscr{A} = \mathscr{A}_1 = \ldots = \mathscr{A}_n$ fixed. Table 2 shows some results. Of course, for this type of systems, the dimension of the Cox ring (or, equivalently, the number of facets of the Minkowski sum of the input polytopes) is lower and the system of binomial equations from Corollary 5.1 is easier to solve.

7.4. Comparison with homotopy methods

As discussed in the introduction, homotopy continuation methods provide very successful numerical solvers for systems of small degrees in large numbers of variables. Algebraic methods prove to be more robust in the case of high degrees and small dimensions, see for instance the numerical experiments in [18]. In this sense, these two important classes of numerical solvers are complementary to each other. As an illustration, we repeat the mixed experiment from Subsection 7.3 for three challenging 2-dimensional systems and compare the results with two homotopy implementations that are considered state of the art: Bertini (v1.6) [6] and PHCpack (v2.4.64) [7]. For both these solvers, we use standard double precision settings and

Table 2
Results for generic systems with unmixed supports.

n	NZ	$d_{\max}$	δ	k	n_{α_0}	OFFLINE			ONLINE		
						t	D_{mean}	$D_{\max}$	t	D_{mean}	$D_{\max}$
2	20	60	3638	7	3	5.8e+2	13	11	3.8e+2	13	10
3	10	10	834	14	6	3.5e+2	13	12	1.9e+2	13	12
4	6	3	15	7	8	3.3e+1	15	15	8.4e-1	15	14
4	6	4	28	6	11	4.3e+1	14	13	5.4e+0	15	14
4	6	5	216	9	7	5.7e+2	12	11	2.7e+2	12	11
4	6	6	339	8	6	1.5e+3	6	4	2.0e+3	6	5
5	6	3	10	6	8	7.5e+1	15	14	1.0e+1	15	15

Table 3
Results for generic systems using Algorithm 1 and the homotopy packages PHCpack and Bertini.

n	NZ	$d_{\max}$	δ	Algorithm 1					PHCpack		Bertini	
				k	n_{α_0}	t_{OFFLINE}	t_{ONLINE}	$\hat{\delta}$	t	$\hat{\delta}$	t	$\hat{\delta}$
2	20	20	622	14	3	4.4e+1	2.8e+0	622	1.7e+0	597	2.2e+1	605
2	200	30	1700	14	3	1.5e+2	7.1e+1	1700	1.3e+1	1671	4.9e+2	1119
2	800	40	3117	9	3	3.5e+2	2.3e+2	3117	7.7e+1	3055	7.6e+3	2832

the backward errors of the computed solutions are of the order of the machine precision because these solvers intrinsically use Newton refinement. The results are reported in Table 3. For each solver, the number $\hat{\delta}$ is the number of correctly computed solutions (with residual $< 10^{-9}$). Note that both homotopy solvers miss some solutions for all these problems. PHCpack is very efficient for this type of generic problems because it implements polyhedral homotopies [27,28]. This means in practice that exactly δ paths are tracked. Bertini tracks 1258, 3135 and 6320 paths for the first, second and third problem respectively. This experiment shows that even for generic systems, for large δ and small n the state of the art homotopy algorithms do not find all solutions. The method introduced in this paper aims at solving (nearly) degenerate, non-generic systems. In practice, this often means that there are 'large solutions'. To show that such situations cause trouble for homotopy methods, even for small δ, we consider the experiment of Subsection 7.2. Solving the system for $e = 4.5$ using Algorithm 1 we find three solutions whose coordinates have a modulus of order 10^4. PHCpack and Bertini both find only 14 solutions (the homotopy solvers give up on the paths converging to the 'large solutions').

8. Conclusion

We have presented a toric eigenvalue, eigenvector theorem that allows to compute homogeneous coordinates of solutions of systems of Laurent polynomial equations (satisfying the assumptions in Section 3) on a natural toric compactification X of $(\mathbb{C}^*)^n$. This results in a numerical linear algebra based algorithm that proves to be robust in the case of (nearly) degenerate systems with solutions on the torus invariant prime divisors. The algorithm is particularly successful for small dimensions n and large degrees. It relies on a conjecture related to the regularity of I (Conjecture 1), which is checked numerically to be true in all of the presented experiments and supported by some weaker results in Section 4.

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