



Construction of Grothendieck categories with enough compressible objects using colored quivers



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ABSTRACT

We introduce a new method to construct a Grothendieck category from a given colored quiver. This is a variant of the construction used to prove that every partially ordered set arises as the atom spectrum of a Grothendieck category. Using the new method, we prove that for every finite partially ordered set, there exists a locally noetherian Grothendieck category such that every nonzero object contains a compressible subobject and its atom spectrum is isomorphic to the given partially ordered set.

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1. Introduction

The *atom spectrum* of a Grothendieck category is a generalization of the set of prime ideals of a commutative ring. The idea to associate a space to a Grothendieck category can be found in [2], where the *Gabriel spectrum* was defined to be the set of isomorphism classes of indecomposable injective objects. If R is a commutative noetherian ring, Matlis' result ensures that there is a canonical bijection between the Gabriel spectrum of $\text{Mod } R$ and the prime spectrum $\text{Spec } R$, where $\text{Mod } R$ is the category of all R -modules. The atom spectrum is a variant of the Gabriel spectrum, which is defined so that there is a canonical bijection between the atom spectrum of $\text{Mod } R$ and $\text{Spec } R$ for an arbitrary commutative ring R . The idea of atom spectrum was given by Storrer [14] to reformulate Goldman's work [3] on primary decomposition of modules over noncommutative rings. The topological structure on the atom spectrum (or the Gabriel spectrum) has

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been used to classify localizing subcategories of the given Grothendieck category (see [2, Proposition VI.2.4], [5, Theorem 3.8], [12, Corollary 4.3], [7, Theorem 5.5], and [8, Theorem 7.8]).

In this paper, we will focus on the partial order on the atom spectrum $\text{ASpec } \mathcal{G}$ of a Grothendieck category \mathcal{G} . The partial order is naturally defined as the *specialization order* with respect to the topology on $\text{ASpec } \mathcal{G}$, and for a commutative ring R , the partial order on $\text{ASpec}(\text{Mod } R)$ is identified with the inclusion of prime ideals via the canonical bijection $\text{ASpec}(\text{Mod } R) \xrightarrow{\sim} \text{Spec } R$.

The possible partial order structure of the prime spectrum of a commutative ring was completely determined by Hochster [6, Proposition 10] and Speed [13, Corollary 1]: A partially ordered set is isomorphic to $\text{Spec } R$ for some commutative ring R if and only if it can be written as the inverse limit of finite partially ordered sets in the category of all partially ordered sets. It should be noticed that this property implies several well-known properties of $\text{Spec } R$ such as the fact that every prime ideal is contained in some maximal (prime) ideal. When we only consider commutative noetherian rings, the partial order structure of $\text{Spec } R$ is much more restrictive. It was shown by de Souza Doering and Lequain [1, Theorem B] that a *finite* partially ordered set P is isomorphic to $\text{Spec } R$ for some commutative noetherian ring R if and only if there is no chain of the form $x < y < z$ in P . Thus such a ring R should have dimension at most one.

The analogous problems for Grothendieck categories have quite different answers. We showed in [9, Theorem 1.2 (2)] that *every* partially ordered set arises as the atom spectrum of a Grothendieck category. As an analog to the problem for commutative noetherian rings, we also proved that *every* finite partially ordered set arises as the atom spectrum of some locally noetherian Grothendieck category ([9, Theorem 1.4 (2)]). The main idea of these results was to construct a Grothendieck category from a given *colored quiver*.

One of the significant properties of the category $\text{Mod } R$ for a commutative noetherian ring R , among all locally noetherian Grothendieck categories, is that it has enough compressible objects. We say that a Grothendieck category \mathcal{G} has *enough compressible objects* if every nonzero object in \mathcal{G} contains a compressible subobject (see Definition 2.9 for the definition of a compressible object). This holds for $\text{Mod } R$ since every nonzero R -module contains a submodule isomorphic to R/\mathfrak{p} for some $\mathfrak{p} \in \text{Spec } R$, which is compressible. The category of modules over a right noetherian ring is a locally noetherian Grothendieck category, but it does not necessarily have enough compressible objects (Remark 2.10). So it is natural to ask whether having enough compressible objects restricts the partial order structure of the atom spectrum.

The aim of this paper is to show that the answer to this question is *no*. We provide a new method to construct a Grothendieck category from a colored quiver and prove the following result:

Theorem 1.1 (Theorem 4.14). *Let P be a finite partially ordered set. Then there exists a locally noetherian Grothendieck category \mathcal{G} satisfying the following properties:*

- (1) $\text{ASpec } \mathcal{G}$ is isomorphic to P as a partially ordered set.
- (2) \mathcal{G} has enough compressible objects.

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2. Preliminaries

In this section, we collect some basic materials on Grothendieck categories and their atom spectra.

Convention 2.1. A *direct sum* (resp. a *direct product*) in a Grothendieck category means a possibly infinite coproduct (resp. product) whose index set is set-theoretically small.

Definition 2.2. Let \mathcal{G} be a Grothendieck category.

- (1) A *weakly closed subcategory* (also called a *prelocalizing subcategory*) of \mathcal{G} is a full subcategory closed under subobjects, quotient, and direct sums.
- (2) A *closed subcategory* of \mathcal{G} is a weakly closed subcategory that is also closed under direct products.

Remark 2.3. For a ring R , we denote by $\text{Mod } R$ the category of all (set-theoretically small) right R -modules. This is a Grothendieck category. For a two-sided ideal I of R , the category $\text{Mod}(R/I)$ is canonically identified with the closed subcategory

$$\{ M \in \text{Mod } R \mid MI = 0 \}$$

of $\text{Mod } R$. It is well known that all closed subcategories of $\text{Mod } R$ is of this form (see, for example, [8, Theorem 11.3]).

The notion of atoms in the category of modules over a ring was introduced by Storrer [14] in order to reformulate the theory of primary decomposition due to Goldman [3]. The definition below is a generalized version, which was given in [7] to deal with arbitrary abelian categories.

Definition 2.4. Let \mathcal{G} be a Grothendieck category.

- (1) A nonzero object H in \mathcal{G} is called *monoform* if for every nonzero subobject L of H , the only subobject of H that is isomorphic to some subobject of H/L is the zero subobject.
- (2) We say that two monoform objects H_1 and H_2 are *atom-equivalent* if H_1 has a nonzero subobject that is isomorphic to some subobject of H_2 .
- (3) The *atom spectrum* of \mathcal{G} is defined to be

$$\text{ASpec } \mathcal{G} := \frac{\{\text{monoform objects in } \mathcal{G}\}}{\text{atom equivalence}}.$$

The equivalence class of a monoform object H is denoted by \overline{H} . Each element of $\text{ASpec } \mathcal{G}$ is called an *atom* in \mathcal{G} .

The atom spectrum has a structure of a topological space and a partially ordered set:

Definition 2.5. Let \mathcal{G} be a Grothendieck category.

- (1) For each object M in \mathcal{G} , we define the *atom support* of M to be

$$\text{ASupp } M := \{ \overline{H} \in \text{ASpec } \mathcal{G} \mid H \text{ is a monoform subquotient of } M \}.$$

- (2) We define the *localizing topology* on $\text{ASpec } \mathcal{G}$ as follows: A subset Φ of $\text{ASpec } \mathcal{G}$ is open if and only if $\Phi = \text{ASupp } M$ for some $M \in \mathcal{G}$.
- (3) We define a partial order \leq on $\text{ASpec } \mathcal{G}$, called the *specialization order*, as follows: For $\alpha, \beta \in \text{ASpec } \mathcal{G}$, $\alpha \leq \beta$ if and only if α belongs to the closure of $\{\beta\}$ with respect to the localizing topology.

The localizing topology satisfies the axioms of a topology (see [9, Proposition 3.2]). The atom spectrum endowed with the localizing topology is a Kolmogorov space, and this implies that the binary relation \leq satisfies the axioms of a partial order (see [9, section 4]).

Remark 2.6. For a commutative ring R , there is a bijection $\text{Spec } R \xrightarrow{\sim} \text{ASpec}(\text{Mod } R)$ given by $\mathfrak{p} \mapsto \overline{R/\mathfrak{p}}$ as shown in [14, p. 631]. This is an isomorphism of partially ordered sets ([9, Proposition 4.3]). A subset of $\text{ASpec}(\text{Mod } R)$ is open with respect to the localizing topology if and only if the corresponding subset of $\text{Spec } R$ is closed under specialization, that is, it is upward-closed with respect to the inclusion of prime ideals ([7, Proposition 7.2 (2)]). Thus the localizing topology is different from the Zariski topology.

For a Grothendieck category \mathcal{G} , every open subset of $\text{ASpec } \mathcal{G}$ is upward-closed with respect to the specialization order, but the converse does not hold in general (see [10, Remark 2.11]).

Remark 2.7. For a locally noetherian Grothendieck category \mathcal{G} , there is a canonical bijection between the atom spectrum of \mathcal{G} and the *Gabriel spectrum* of \mathcal{G} , which consists of all isomorphism classes of indecomposable injective objects in \mathcal{G} . See [10, Proposition 2.3] for the definitions of the bijection and the corresponding partial order on the Gabriel spectrum.

Remark 2.8. If \mathcal{X} is a weakly closed subcategory of a Grothendieck category \mathcal{G} , then there is a canonical injective continuous map $\text{ASpec } \mathcal{X} \hookrightarrow \text{ASpec } \mathcal{G}$ defined by $\overline{H} \mapsto \overline{H}$. This map is also a homomorphism of partially ordered sets, and induces a homeomorphism from $\text{ASpec } \mathcal{X}$ to an open subset of $\text{ASpec } \mathcal{G}$, which was denoted by $\text{ASupp } \mathcal{X}$ in [9, Proposition 5.12]. We identify $\text{ASpec } \mathcal{X}$ with its image in $\text{ASpec } \mathcal{G}$.

Definition 2.9. Let \mathcal{G} be a Grothendieck category. An object H in \mathcal{G} is called *compressible* if each nonzero subobject of H contains a subobject isomorphic to H .

Remark 2.10. In a locally noetherian Grothendieck category, every compressible object is monoform (see [11, Proposition 2.12 (2)]). On the other hand, a monoform object does not necessarily have a compressible object. Indeed, Goodearl [4] gave an example of a left and right noetherian ring R that admits a monoform right R -module with no compressible submodule.

Remark 2.11. Let \mathcal{G} be a Grothendieck category with enough compressible objects. Then each atom α in \mathcal{G} is represented by some compressible object H , and $\text{ASupp } H$ is the *smallest* open subset of $\text{ASpec } \mathcal{G}$ among those containing α (see [9, Definition 3.1 and Proposition 3.2]). This implies that the intersection of an arbitrary family of open subsets of $\text{ASpec } \mathcal{G}$ is again an open subset. Such a topological space is called an *Alexandroff space*.

As a consequence, the topology of $\text{ASpec } \mathcal{G}$ can be recovered from its partial order as follows: a subset of $\text{ASpec } \mathcal{G}$ is open if and only if it is upward-closed with respect to the partial order (see, for example, [9, Proposition 4.1]). In particular, the Grothendieck categories that we will construct in Theorem 4.14 have this property.

3. Grothendieck categories associated to colored quivers

We fix a field K . For a given colored quiver Γ , we will associate a K -algebra F_Γ , an F_Γ -module M_Γ , and a Grothendieck category \mathcal{G}_Γ .

For a set C , we denote by C^* the free (multiplicative) monoid on C . The identity element is denoted by 1, and any other element of C^* is of the form $\mathbf{c} = c_1 c_2 \cdots c_l$, where $l \geq 1$ and $c_1, \dots, c_l \in C$.

Definition 3.1.

- (1) A *colored quiver* is a sextuple $\Gamma = (\Gamma_0, \Gamma_1, C, s, t, u)$, where Γ_0 , Γ_1 , and C are sets, and

$$s: \Gamma_1 \rightarrow \Gamma_0, \quad t: \Gamma_1 \rightarrow \Gamma_0, \quad \text{and} \quad u: \Gamma_1 \rightarrow C$$

are maps. The elements of Γ_0 , Γ_1 , and C are called *vertices*, *arrows*, and *colors* in Γ , respectively. For each arrow r in Γ , the images $s(r)$, $t(r)$, and $u(r)$ are called the *source*, *target*, and *color* of r , respectively.

We often write $\Gamma = (\Gamma_0, \Gamma_1, C)$ by omitting s , t , and u .

- (2) Let $\Gamma = (\Gamma_0, \Gamma_1, C)$ be a colored quiver. A *path* of length $l \geq 1$ in Γ is a sequence $\mathbf{r} = r_1 r_2 \cdots r_l$ of l arrows in Γ such that $t(r_i) = s(r_{i+1})$ for all $i = 1, \dots, l-1$. A *path* of length 0 is a symbol e_v , where v is a vertex in Γ .

For each path $\mathbf{r} = r_1 r_2 \cdots r_l$ in Γ , the *source* and *target* of \mathbf{r} are defined to be

$$s(\mathbf{r}) := s(r_1) \quad \text{and} \quad t(\mathbf{r}) := t(r_l),$$

respectively. The *sequence of colors* of \mathbf{r} is

$$u(\mathbf{r}) := u(r_1)u(r_2) \cdots u(r_l) \in C^*.$$

For $\mathbf{r} = e_v$, let $s(\mathbf{r}) = t(\mathbf{r}) = v$ and $u(\mathbf{r}) = 1 \in C^*$.

Notation 3.2. An arrow r in a colored quiver is visualized as

$$s(r) \xrightarrow{u(r)} t(r) .$$

Thus, for example, a path $\mathbf{r} = r_1 r_2$ of length 2 can be written as

$$s(r_1) \xrightarrow{u(r_1)} v \xrightarrow{u(r_2)} t(r_2) ,$$

where $v := t(r_1) = s(r_2)$.

Definition 3.3. Let C be a set.

- (1) We define the K -algebra $K\langle\langle C \rangle\rangle$ as follows: As a K -vector space,

$$K\langle\langle C \rangle\rangle = \prod_{\mathbf{c} \in C^*} K\mathbf{c},$$

and each element is written as a formal sum $\sum_{\mathbf{c} \in C^*} \lambda_{\mathbf{c}} \mathbf{c}$ with $\lambda_{\mathbf{c}} \in K$. The multiplication is defined as

$$\left(\sum_{\mathbf{c} \in C^*} \lambda_{\mathbf{c}} \mathbf{c} \right) \left(\sum_{\mathbf{c}' \in C^*} \lambda'_{\mathbf{c}'} \mathbf{c}' \right) = \sum_{\mathbf{c} \in C^*} \left(\sum_{\mathbf{c}_1, \mathbf{c}_2} \lambda_{\mathbf{c}_1} \lambda'_{\mathbf{c}_2} \right) \mathbf{c},$$

where \mathbf{c}_1 and \mathbf{c}_2 run over all pairs of elements of C^* with $\mathbf{c}_1 \mathbf{c}_2 = \mathbf{c}$. Note that there are only finitely many such pairs for each \mathbf{c} .

- (2) For each $f = \sum_{\mathbf{c} \in C^*} \lambda_{\mathbf{c}} \mathbf{c} \in K\langle\langle C \rangle\rangle$, define its *support* to be

$$\text{supp } f := \{ \mathbf{c} \in C^* \mid \lambda_{\mathbf{c}} \neq 0 \}.$$

Definition 3.4. Let $\Gamma = (\Gamma_0, \Gamma_1, C)$ be a colored quiver.

- (1) For each subset $B \subset C^*$ and each vertex v in Γ , define the set $P_v(B)$ to be the set of all paths \mathbf{r} in Γ such that $t(\mathbf{r}) = v$ and $u(\mathbf{r}) \in B$.
- (2) We say that an element $f \in K\langle\langle C \rangle\rangle$ is *admissible* if $P_v(\text{supp } f)$ is a finite set for all vertices v in Γ .
- (3) Denote by F_Γ the set of all admissible elements of $K\langle\langle C \rangle\rangle$.

Lemma 3.5. Let $\Gamma = (\Gamma_0, \Gamma_1, C)$ be a colored quiver. Then F_Γ is a K -subalgebra of $K\langle\langle C \rangle\rangle$.

Proof. For each B and v , we have

$$P_v(B) = \bigcup_{\mathbf{c} \in B} P_v(\{\mathbf{c}\}).$$

Thus the inclusions

$$\text{supp}(\lambda f) \subset \text{supp } f \quad \text{and} \quad \text{supp}(f + g) \subset \text{supp } f \cup \text{supp } g,$$

where $f, g \in K\langle\langle C \rangle\rangle$ and $\lambda \in K$, imply that F_Γ is a K -subspace of $K\langle\langle C \rangle\rangle$. In order to conclude the claim, it suffices to show that fg is admissible whenever $f, g \in K\langle\langle C \rangle\rangle$ are admissible.

Let v be a vertex in Γ . Since $P_v(\text{supp } g)$ is finite, we can write $P_v(\text{supp } g) = \{\mathbf{r}_1, \dots, \mathbf{r}_n\}$. Take arbitrary $\mathbf{r} \in P_v(\text{supp } fg)$. Since $u(\mathbf{r}) \in \text{supp } fg$, there exist paths \mathbf{r}' and \mathbf{r}'' in Γ such that $\mathbf{r} = \mathbf{r}'\mathbf{r}''$, $u(\mathbf{r}') \in \text{supp } f$, and $u(\mathbf{r}'') \in \text{supp } g$. Here $\mathbf{r}'' \in P_v(\text{supp } g)$, so that $\mathbf{r}'' = \mathbf{r}_i$ for some i , and $\mathbf{r}' \in P_{s(\mathbf{r}_i)}(\text{supp } f)$. We have shown that every path in $P_v(\text{supp } fg)$ is of the form $\mathbf{r}'\mathbf{r}''$, where \mathbf{r}' belongs to the finite set $P_{s(\mathbf{r}_i)}(\text{supp } f)$ for some $i = 1, \dots, n$ and \mathbf{r}'' belongs to the finite set $P_v(\text{supp } g)$. Therefore $P_v(\text{supp } fg)$ is a finite set. fg is admissible. \square

Definition 3.6. Let $\Gamma = (\Gamma_0, \Gamma_1, C)$ be a colored quiver. We define the right F_Γ -module M_Γ as follows: As a K -vector space,

$$M_\Gamma = \prod_{v \in \Gamma_0} Kx_v,$$

where Kx_v is a one-dimensional space generated by x_v . Each element of M_Γ is written as a formal sum $\sum_{v \in \Gamma_0} \mu_v x_v$. For each

$$f = \sum_{\mathbf{c} \in C^*} \lambda_{\mathbf{c}} \mathbf{c} \in F_\Gamma \quad \text{and} \quad y = \sum_{v \in \Gamma_0} \mu_v x_v \in M_\Gamma,$$

define

$$yf := \sum_{\mathbf{r}} \lambda_{u(\mathbf{r})} \mu_{s(\mathbf{r})} x_{t(\mathbf{r})},$$

where \mathbf{r} runs over all paths in Γ . The sum makes sense because, for each vertex v in Γ , there are only finitely many paths \mathbf{r} satisfying $t(\mathbf{r}) = v$ and $\lambda_{u(\mathbf{r})} \neq 0$ by the admissibility of f . It is easy to verify that M_Γ is actually a right F_Γ -module.

For each $y = \sum_{v \in \Gamma_0} \mu_v x_v \in M_\Gamma$, define its *support* to be

$$\text{supp } y := \{v \in \Gamma_0 \mid \mu_v \neq 0\}.$$

Definition 3.7. Let $\Gamma = (\Gamma_0, \Gamma_1, C)$ be a (set-theoretically small) colored quiver. We define \mathcal{G}_Γ to be the smallest weakly closed subcategory of $\text{Mod } F_\Gamma$ containing M_Γ .

Example 3.8. If $\Gamma = (\Gamma_0, \Gamma_1, C)$ is the colored quiver

$$\begin{array}{c} v \\ \curvearrowright \\ c \end{array}$$

with $C = \{c\}$, then $K\langle\langle C \rangle\rangle$ is the (commutative) formal power series ring $K[[c]]$ and F_Γ is the polynomial ring $K[c]$. The $K[c]$ -module M_Γ is isomorphic to the simple module $K[c]/(c-1)K[c]$. Thus $\mathcal{G}_\Gamma = \text{Mod}(K[c]/(c-1)K[c]) \cong \text{Mod } K$.

4. Construction of Grothendieck categories

As in the previous section, let K be a field. We denote by \mathbb{N} the set of nonnegative integers.

We will introduce an operation to obtain a new colored quiver from a given one, which allows us to construct a colored quiver $\tilde{\Gamma}$ such that the Grothendieck category $\mathcal{G}_{\tilde{\Gamma}}$ satisfies the desired properties in Theorem 1.1.

Definition 4.1. Let $\Gamma = (\Gamma_0, \Gamma_1, C)$ be a colored quiver. Define a new colored quiver $\tilde{\Gamma} = (\tilde{\Gamma}_0, \tilde{\Gamma}_1, \tilde{C})$ as follows:

- $\tilde{\Gamma}_0 = \mathbb{N} \times \Gamma_0$.
- $\tilde{\Gamma}_1 = (\mathbb{N} \times \Gamma_1) \amalg \{r_{v,w}^i \mid i \in \mathbb{N}, v, w \in \Gamma_0\}$, where \amalg denotes the disjoint union of sets and $r_{v,w}^i$ are pairwise distinct symbols.
- $\tilde{C} = C \amalg \{c_{v,w} \mid v, w \in \Gamma_0\}$, where $c_{v,w}$ are pairwise distinct symbols.
- For each $(i, r) \in \mathbb{N} \times \Gamma_1$,

$$s(i, r) = (i, s(r)), \quad t(i, r) = (i, t(r)), \quad \text{and} \quad u(i, r) = u(r).$$

- For each $r_{v,w}^i$,

$$s(r_{v,w}^i) = (i, v), \quad t(r_{v,w}^i) = (i+1, w), \quad \text{and} \quad u(r_{v,w}^i) = c_{v,w}.$$

Remark 4.2. Definition 4.1 is actually a special case of [9, Definition 7.17], but it is different from the operation observed in [9, Proposition 7.22], which played a crucial role in the proof of [9, Theorem 7.23].

Indeed, if we set Ω to be the colored quiver

$$\omega_0 \xrightarrow{\xi} \omega_1 \xrightarrow{\xi} \dots$$

with the set of colors $\Xi = \{\xi\}$ and all Γ^{ω_i} to be Γ in [9, Definition 7.17], then the output $\Omega(\{\Gamma^\omega\}_{\omega \in \Omega_0})$ is the $\tilde{\Gamma}$ in Definition 4.1 (up to change of symbols).

On the other hand, in order to obtain the colored quiver Γ in [9, Proposition 7.22], we have to set Ω to be

$$\omega_0 \xrightarrow{\xi_0} \omega_1 \xrightarrow{\xi_1} \dots$$

with the set of colors $\Xi = \{\xi_i \mid i \in \mathbb{N}\}$, where ξ_i are pairwise distinct, and each Γ^{ω_i} to be Γ^i .

Example 4.3. If $\Gamma = (\Gamma_0, \Gamma_1, C)$ is the colored quiver

$$\begin{array}{c} v \\ \downarrow c \\ w \end{array}$$

with $C = \{c\}$, then $\tilde{\Gamma}$ is

$$\begin{array}{ccccccc} (0, v) & \xrightarrow{c_{v,v}} & (1, v) & \xrightarrow{c_{v,v}} & (2, v) & \xrightarrow{c_{v,v}} & \cdots \\ & \searrow c_{v,w} & \nearrow c_{w,v} & & \searrow c_{v,w} & \nearrow c_{w,v} & \\ & c & & & c & & \\ & \nearrow c_{w,v} & \searrow c_{v,w} & & \nearrow c_{w,v} & \searrow c_{v,w} & \\ (0, w) & \xrightarrow{c_{w,w}} & (1, w) & \xrightarrow{c_{w,w}} & (2, w) & \xrightarrow{c_{w,w}} & \cdots \end{array}$$

with $\tilde{C} = \{c, c_{v,v}, c_{v,w}, c_{w,v}, c_{w,w}\}$.

Example 4.4. If Γ is the colored quiver with only one vertex v , no arrows, and no colors, then $\tilde{\Gamma}$ is

$$(0, v) \xrightarrow{c_{v,v}} (1, v) \xrightarrow{c_{v,v}} (2, v) \xrightarrow{c_{v,v}} \cdots$$

with $\tilde{C} = \{c_{v,v}\}$. Thus $K\langle\langle\tilde{C}\rangle\rangle$ is the formal power series ring $K[[c_{v,v}]]$ with a single variable $c_{v,v}$, and $M_\Gamma = F_\Gamma = K\langle\langle\tilde{C}\rangle\rangle = K[[c_{v,v}]]$. Therefore $\mathcal{G}_\Gamma = \text{Mod } K[[c_{v,v}]]$.

Remark 4.5. In the setting of Definition 4.1, $K\langle\langle C \rangle\rangle$ is a K -subalgebra of $K\langle\langle\tilde{C}\rangle\rangle$ since $C \subset \tilde{C}$. Write the inclusion map as $\nu: K\langle\langle C \rangle\rangle \hookrightarrow K\langle\langle\tilde{C}\rangle\rangle$. We can define a surjective ring homomorphism $\pi: K\langle\langle\tilde{C}\rangle\rangle \twoheadrightarrow K\langle\langle C \rangle\rangle$ by

$$\sum_{\mathbf{c} \in \tilde{C}^*} \lambda_{\mathbf{c}} \mathbf{c} \mapsto \sum_{\mathbf{c} \in C^*} \lambda_{\mathbf{c}} \mathbf{c},$$

which satisfies $\pi \circ \nu = \text{id}$. It can be verified that ν and π induce K -algebra homomorphisms between F_Γ and $F_{\tilde{\Gamma}}$. The kernel of the induced surjective homomorphism $F_{\tilde{\Gamma}} \twoheadrightarrow F_\Gamma$ is

$$I_\Gamma := \left\{ \sum_{\mathbf{c} \in \tilde{C}^*} \lambda_{\mathbf{c}} \mathbf{c} \in F_{\tilde{\Gamma}} \mid \lambda_{\mathbf{c}} = 0 \text{ for all } \mathbf{c} \in C^* \right\}.$$

Therefore

$$\text{Mod } F_\Gamma \xrightarrow{\sim} \text{Mod } \frac{F_{\tilde{\Gamma}}}{I_\Gamma} = \{ N \in \text{Mod } F_{\tilde{\Gamma}} \mid NI_\Gamma = 0 \}.$$

We regard $\text{Mod } F_\Gamma$ as a closed subcategory of $\text{Mod } F_{\tilde{\Gamma}}$ in this way.

Definition 4.6. Let $\Gamma = (\Gamma_0, \Gamma_1, C)$ be a colored quiver and let $\tilde{\Gamma} = (\tilde{\Gamma}_0, \tilde{\Gamma}_1, \tilde{C})$ as in Definition 4.1. For each $i \in \mathbb{N}$, define the $F_{\tilde{\Gamma}}$ -submodule $M_{\geq i} \subset M_{\tilde{\Gamma}}$ by

$$M_{\geq i} = \left\{ y \in M_{\tilde{\Gamma}} \mid \text{supp } y \subset \bigcup_{j \geq i} (\{j\} \times \Gamma_0) \right\}.$$

Define the K -subspace $M_i \subset M_{\tilde{F}}$ by

$$M_i = \{y \in M_{\tilde{F}} \mid \text{supp } y \subset (\{i\} \times \Gamma_0)\}.$$

Remark 4.7. In the setting of Definition 4.6, there is an isomorphism $M_{\tilde{F}} \xrightarrow{\sim} M_{\geq i}$ of $F_{\tilde{F}}$ -modules given by

$$\sum_{j \in \mathbb{N}} \sum_{v \in \Gamma_0} \lambda_{(j,v)} x_{(j,v)} \mapsto \sum_{j \in \mathbb{N}} \sum_{v \in \Gamma_0} \lambda_{(j,v)} x_{(j+i,v)}.$$

We identify the K -vector space M_i with the right $F_{\tilde{F}}$ -module $M_{\geq i}/M_{\geq i+1}$ via the composite

$$M_i \hookrightarrow M_{\geq i} \twoheadrightarrow M_{\geq i}/M_{\geq i+1}.$$

Note that $M_i = M_{\geq i}/M_{\geq i+1}$ belongs to the closed subcategory $\text{Mod } F_{\tilde{F}}$ of $\text{Mod } F_{\tilde{F}}$. We can define an isomorphism $M_{\tilde{F}} \xrightarrow{\sim} M_i$ of $F_{\tilde{F}}$ -modules (also of $F_{\tilde{F}}$ -modules) by

$$\sum_{v \in \Gamma_0} \lambda_v x_v \mapsto \sum_{v \in \Gamma_0} \lambda_v x_{(i,v)}.$$

This means that $M_{\tilde{F}}$ is isomorphic to a subquotient of $M_{\tilde{F}}$ in $\text{Mod } F_{\tilde{F}}$. Consequently, $\mathcal{G}_{\tilde{F}}$ is a weakly closed subcategory of $\mathcal{G}_{\tilde{F}}$.

The following lemmas will be used to show some properties of the Grothendieck category $\mathcal{G}_{\tilde{F}}$:

Lemma 4.8. Let $\Gamma = (\Gamma_0, \Gamma_1, C)$ be a colored quiver and let $\tilde{\Gamma} = (\tilde{\Gamma}_0, \tilde{\Gamma}_1, \tilde{C})$ as in Definition 4.1. Let $y \in M_{\tilde{F}}$, and take $i \in \mathbb{N}$ such that

$$(\text{supp } y) \cap (\{i\} \times \Gamma_0) \neq \emptyset.$$

Then $M_{\geq i+1} \subset yF_{\tilde{F}}$.

Proof. We can assume $y \in M_{\geq i}$ by replacing i by the smallest one without loss of generality. Write

$$y = \sum_{j \geq i} \sum_{v \in \Gamma_0} \lambda_{(j,v)} x_{(j,v)}.$$

By the assumption, there exists $u \in \Gamma_0$ such that $\lambda_{(i,u)} \neq 0$. By replacing y by $\lambda_{(i,u)}^{-1} y$, we may assume $\lambda_{(i,u)} = 1$.

Let $z \in M_{\geq i+1}$. For each integer $d \geq 1$, define $z_d \in M_{\geq i+d}$ and $f_d \in F_{\tilde{F}}$ inductively as follows: Let $z_1 := z$. When $z_d \in M_{\geq i+d}$ is defined, write

$$z_d = \sum_{j \geq i+d} \sum_{v \in \Gamma_0} \mu_{(j,v)} x_{(j,v)},$$

and define

$$f_d := \sum_{v \in \Gamma_0} \mu_{(i+d,v)} c_{(u,u)}^{d-1} c_{(u,v)}.$$

Let $z_{d+1} := z_d - yf_d$. Since

$$\begin{aligned}
yf_d &= \sum_{j \geq i} \sum_{v \in \Gamma_0} \lambda_{(j,u)} \mu_{(i+d,v)} x_{(j+d,v)} = \sum_{j \geq i+d} \sum_{v \in \Gamma_0} \lambda_{(j-d,u)} \mu_{(i+d,v)} x_{(j,v)} \\
&= \sum_{v \in \Gamma_0} \mu_{(i+d,v)} x_{(i+d,v)} + \sum_{j \geq i+d+1} \sum_{v \in \Gamma_0} \lambda_{(j-d,u)} \mu_{(i+d,v)} x_{(j,v)},
\end{aligned}$$

we have $z_{d+1} \in M_{\geq i+d+1}$. It follows that $f := \sum_{d \geq 1} f_d \in F_{\tilde{\Gamma}}$, and

$$z - yf = z_1 - \sum_{d \geq 1} yf_d = 0.$$

Thus $z = yf \in yF_{\tilde{\Gamma}}$. \square

Lemma 4.9. *Let $\Gamma = (\Gamma_0, \Gamma_1, C)$ be a colored quiver and let $\tilde{\Gamma} = (\tilde{\Gamma}_0, \tilde{\Gamma}_1, \tilde{C})$ as in Definition 4.1. For each nonzero $F_{\tilde{\Gamma}}$ -submodule $L \subset M_{\tilde{\Gamma}}$, there exists $i \in \mathbb{N}$ such that*

$$L = (L \cap M_i) \oplus M_{\geq i+1}$$

as a K -vector space.

Proof. Let i be the smallest number satisfying

$$(\text{supp } y) \cap (\{i\} \times \Gamma_0) \neq \emptyset$$

for some $y \in L$. Then by Lemma 4.8,

$$M_{\geq i+1} \subset yF_{\tilde{\Gamma}} \subset L \subset M_{\geq i}.$$

In particular, $(L \cap M_i) \oplus M_{\geq i+1} \subset L$.

Each $z \in L$ can be written as $z = z' + z''$, where $z' \in M_i$ and $z'' \in M_{\geq i+1}$. There exists $f \in F_{\tilde{\Gamma}}$ such that $yf = z''$. Hence $z' = z - yf \in L \cap M_i$. This shows that $L \subset (L \cap M_i) \oplus M_{\geq i+1}$. \square

Proposition 4.10. *Let $\Gamma = (\Gamma_0, \Gamma_1, C)$ be a colored quiver with $\Gamma_0 \neq \emptyset$ and let $\tilde{\Gamma} = (\tilde{\Gamma}_0, \tilde{\Gamma}_1, \tilde{C})$ as in Definition 4.1. Suppose that Γ satisfies the following conditions:*

- (a) M_{Γ} is a noetherian object in \mathcal{G}_{Γ} .
- (b) \mathcal{G}_{Γ} has enough compressible objects, that is, every nonzero object in \mathcal{G}_{Γ} has a compressible subobject.

Then the following hold:

- (1) $M_{\tilde{\Gamma}}$ is a compressible noetherian object in $\mathcal{G}_{\tilde{\Gamma}}$. Consequently, $\mathcal{G}_{\tilde{\Gamma}}$ is a locally noetherian Grothendieck category.
- (2) $\mathcal{G}_{\tilde{\Gamma}}$ has enough compressible objects.
- (3) As a partially ordered set,

$$\text{ASpec } \mathcal{G}_{\tilde{\Gamma}} = \text{ASpec } \mathcal{G}_{\Gamma} \cup \{\overline{M_{\tilde{\Gamma}}}\},$$

where $\overline{M_{\tilde{\Gamma}}}$ is smaller than any element in $\text{ASpec } \mathcal{G}_{\Gamma}$.

Proof. (1) Every nonzero submodule L of $M_{\tilde{\Gamma}}$ contains some $M_{\geq i+1}$ as a subobject by Lemma 4.9. Since $M_{\geq i+1}$ is isomorphic to $M_{\tilde{\Gamma}}$ by Remark 4.7, $M_{\tilde{\Gamma}}$ is compressible. Since we have the filtration

$$M_{\tilde{F}} = M_{\geq 0} \supset M_{\geq 1} \supset \cdots,$$

it suffices to show that $M_{\geq i}/M_{\geq i+1}$ is a noetherian for each $i \geq 0$. Again by Remark 4.7, this is isomorphic to M_{Γ} . Since M_{Γ} is a noetherian F_{Γ} -module by the assumption, it is also noetherian as a $M_{\tilde{F}}$ -module.

As explained in the paragraph before Remark 7.9 in [9], it follows that $\mathcal{G}_{\tilde{F}}$ is locally noetherian.

(2) Since $\mathcal{G}_{\tilde{F}}$ is locally noetherian, every nonzero object contains a monoform subobject ([7, Theorem 2.9]). As in the paragraph after Remark 7.8 in [9], $\text{ASupp } M_{\tilde{F}} = \text{ASpec } \mathcal{G}_{\tilde{F}}$. Thus each monoform object contains a subobject that is isomorphic to a monoform object of the form L'/L , where $L \subset L'$ are subobjects of $M_{\tilde{F}}$.

If $L = 0$, then L' contains some $M_{\geq i+1}$ as a subobject, which is compressible since $M_{\geq i+1} \cong M_{\tilde{F}}$.

If $L \neq 0$, then $L = (L \cap M_i) \oplus M_{\geq i+1}$ and $L \cap M_i \subsetneq M_i$ for some i , and hence L'/L contains the submodule

$$\frac{(L' \cap M_i) \oplus M_{\geq i+1}}{(L \cap M_i) \oplus M_{\geq i+1}} \cong \frac{L' \cap M_i}{L \cap M_i},$$

which is a nonzero subquotient of $M_i \cong M_{\Gamma} \in \mathcal{G}_{\Gamma}$. Hence it contains a compressible subobject by the assumption.

(3) The proof of (2) shows in particular that every monoform object in $\mathcal{G}_{\tilde{F}}$ contains either a submodule isomorphic to $M_{\tilde{F}}$ or a compressible (thus monoform) subobject in \mathcal{G}_{Γ} . Hence the equality in the claim follows. Since $M_{\tilde{F}}$ is a compressible object and its atom support is $\text{ASpec } \mathcal{G}_{\tilde{F}}$, the atom $\overline{M_{\tilde{F}}}$ is smallest by [9, Proposition 4.2]. Since no nonzero subobject of $M_{\tilde{F}}$ belongs to \mathcal{G}_{Γ} , the atom $\overline{M_{\tilde{F}}}$ does not belong to $\text{ASpec } \mathcal{G}_{\Gamma}$. \square

We also use the disjoint union of colored quivers:

Definition 4.11 ([9, Definition 7.17]). Let $\{\Gamma^i\}_{i \in I}$ be a family of colored quivers, where $\Gamma^i = (\Gamma_0^i, \Gamma_1^i, C^i)$. Its *disjoint union* is defined to be

$$\coprod_{i \in I} \Gamma^i = \left(\coprod_{i \in I} \Gamma_0^i, \coprod_{i \in I} \Gamma_1^i, \bigcup_{i \in I} C^i \right),$$

where s , t , and u are defined to be those induced from the colored quivers Γ^i .

Remark 4.12. In the setting of Definition 4.11, each $\text{Mod } \Gamma^i$ is regarded as a closed subcategory of $\text{Mod } \Gamma$, where $\Gamma := \coprod_{i \in I} \Gamma^i$, in the same way as Remark 4.5. Thus each \mathcal{G}_{Γ^i} is a weakly closed subcategory of \mathcal{G}_{Γ} .

Proposition 4.13. Let $\{\Gamma^i\}_{i=1}^n$ be a finite family of colored quivers and let $\Gamma := \coprod_{i=1}^n \Gamma^i$. Suppose that each Γ_i satisfies the following conditions:

- (a) M_{Γ^i} is a noetherian object in \mathcal{G}_{Γ^i} .
- (b) \mathcal{G}_{Γ^i} has enough compressible objects.

Then the following hold:

- (1) M_{Γ} is a noetherian object in \mathcal{G}_{Γ} . Consequently, \mathcal{G}_{Γ} is a locally noetherian Grothendieck category.
- (2) \mathcal{G}_{Γ} has enough compressible objects.
- (3) As a partially ordered set,

$$\text{ASpec } \mathcal{G}_{\Gamma} = \bigcup_{i=1}^n \text{ASpec } \mathcal{G}_{\Gamma^i}.$$

Proof. These can be shown similarly to Proposition 4.10, using $M_\Gamma = M_{\Gamma^1} \oplus \cdots \oplus M_{\Gamma^n}$. \square

We are ready to prove the main result of this paper.

Theorem 4.14. *Let P be a finite partially ordered set. Then there exists a colored quiver Γ satisfying the following properties:*

- (1) M_Γ is a noetherian object in \mathcal{G}_Γ . Consequently, \mathcal{G}_Γ is a locally noetherian Grothendieck category.
- (2) \mathcal{G} has enough compressible objects.
- (3) $\text{ASpec } \mathcal{G}_\Gamma$ is isomorphic to P as a partially ordered set.

Proof. For each $p \in P$, we associate a colored quiver Γ^p inductively as follows:

- If p is a maximal element, then $\Gamma^p = (\Gamma_0^p, \Gamma_1^p, C^p)$ is

$$\begin{array}{c} v_p \\ \curvearrowright \\ c_p \end{array}$$

with $C^p = \{c_p\}$, where c_p is a symbol that is not introduced in the definition of any other Γ^q .

- If p is not a maximal element, then let

$$\Omega^p := \coprod_{\substack{q \in P \\ p < q}} \Gamma^q$$

and define $\Gamma^p := \widetilde{\Omega}^p$ using Definition 4.1, where the symbols $c_{v,w}$ used in the definition of $\widetilde{\Omega}^p$ are chosen so that they are not introduced in the definition of any other Γ^q .

The colored quiver Γ is defined by

$$\Gamma := \coprod_{p \in P} \Gamma^p.$$

Then all claims follow from Example 3.8, Proposition 4.10, and Proposition 4.13. \square

References

- [1] Ada Maria de Souza Doering, Yves Lequain, The gluing of maximal ideals—spectrum of a Noetherian ring—going up and going down in polynomial rings, *Trans. Am. Math. Soc.* 260 (2) (1980) 583–593, MR 574801.
- [2] Pierre Gabriel, Des catégories abéliennes, *Bull. Soc. Math. Fr.* 90 (1962) 323–448, MR 0232821.
- [3] Oscar Goldman, Rings and modules of quotients, *J. Algebra* 13 (1969) 10–47, MR 0245608.
- [4] K.R. Goodearl, Incompressible critical modules, *Commun. Algebra* 8 (19) (1980) 1845–1851, MR 588447.
- [5] Ivo Herzog, The Ziegler spectrum of a locally coherent Grothendieck category, *Proc. Lond. Math. Soc.* (3) 74 (3) (1997) 503–558, MR 1434441.
- [6] M. Hochster, Prime ideal structure in commutative rings, *Trans. Am. Math. Soc.* 142 (1969) 43–60, MR 0251026.
- [7] Ryo Kanda, Classifying Serre subcategories via atom spectrum, *Adv. Math.* 231 (3–4) (2012) 1572–1588, MR 2964615.
- [8] Ryo Kanda, Classification of categorical subspaces of locally Noetherian schemes, *Doc. Math.* 20 (2015) 1403–1465, MR 3452186.
- [9] Ryo Kanda, Specialization orders on atom spectra of Grothendieck categories, *J. Pure Appl. Algebra* 219 (11) (2015) 4907–4952, MR 3351569.
- [10] Ryo Kanda, Integrality of noetherian Grothendieck categories, *arXiv:1711.06946v1*, 46 pp.
- [11] Ryo Kanda, Finiteness of the number of minimal atoms in Grothendieck categories, *J. Algebra* 527 (2019) 182–195, MR 3922832.
- [12] Henning Krause, The spectrum of a locally coherent category, *J. Pure Appl. Algebra* 114 (3) (1997) 259–271, MR 1426488.

- [13] T.P. Speed, On the order of prime ideals, *Algebra Univers.* 2 (1972) 85–87, MR 0306061.
- [14] Hans H. Storrer, On Goldman's primary decomposition, in: *Lectures on Rings and Modules (Tulane Univ. Ring and Operator Theory Year, 1970–1971, Vol. I)*, in: *Lecture Notes in Math.*, vol. 246, Springer, Berlin, 1972, pp. 617–661, MR 0360717.