



Centrality of K_2 -functor revisited

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ABSTRACT

We prove that $\text{St}(n, A)$ is a crossed module over $\text{GL}(n, A)$ under a local stable rank condition on an algebra A over a commutative ring. Our proof uses only elementary localization techniques in terms of pro-groups and stability results for K_1 and K_2 . We also prove similar result for the Steinberg group associated with any sufficiently isotropic general linear group constructed by a quasi-finite algebra.

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1. Introduction

In [4] W. van der Kallen proved that the Steinberg group $\text{St}(n, K)$ is a crossed module over $\text{GL}(n, K)$ for any commutative ring K and $n \geq 4$. Together with the fact that $\text{St}(n, A)$ is centrally closed for any associative ring A and $n \geq 5$ this gives explicit description of the Schur multiplier of the elementary group $E(n, K)$ for $n \geq 5$.

Later M.S. Tulenbaev extended this result for all finite K -algebras, see [12]. Both these proofs use the so-called “another presentation” of the Steinberg group in terms of arbitrary transvections instead of the elementary ones. Recently in [6,7,10] S. Sinchuk and A. Lavrenov proved centrality of K_2 -functors for Chevalley groups of types C_l , D_l , and E_l using similar methods and the linear case. In 1998, A. Bak and G. Tang announced centrality for the even unitary case, but their proof was not published.

In the local case much more is known. From surjective stability for K_2 (see [2]) it follows that $K_2(n, A)$ is central if $n \geq \text{sr}(A) + 2$, where $\text{sr}(A)$ is the stable rank of A . In particular, $K_2(n, A)$ is central for all

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$n \geq 3$ if A is semi-local. In [11] A. Stavrova proved centrality for all isotropic reductive groups over local rings (with isotropic rank at least 2).

In [9] V. Petrov and A. Stavrova defined isotropic reductive groups and proved that the elementary subgroup is normal in such groups. For non-commutative rings we may similarly define an isotropic general linear group as $\mathrm{GL}(R) = R^*$ if R contains a complete family of full orthogonal idempotents. These groups are more general than the usual matrix groups with $R = M(n, A)$, but they still have elementary subgroups and Steinberg groups. It seems that it is easy to generalize some fundamental results about matrix groups to the isotropic case: the elementary subgroup is normal in $\mathrm{GL}(R)$, there is a description of subgroups of $\mathrm{GL}(R)$ normalized by $E(R)$ in terms of ideals of R , and the Steinberg group $\mathrm{St}(R)$ is universally closed. In the matrix case we have $\mathrm{GL}(R) = \mathrm{GL}(n, A)$, $E(R) = E(n, A)$, $\mathrm{St}(R) = \mathrm{St}(n, A)$, and the results may be found in [3]. Usual proofs of these facts do not really use the whole structure of a matrix ring.

On the other hand, it is unclear even how to formulate “another presentation” in the isotropic case for non-commutative rings. In this paper we give a proof that the Steinberg group is indeed a crossed module. Our method uses an action of the local general linear group $\mathrm{GL}(S^{-1}R)$ on the Steinberg pro-group $\mathrm{St}(R)^{(\infty)}$, where S is some commutative multiplicative subset. This pro-group is the formal projective limit of “homotopes” $\mathrm{St}(R)^{(s)}$ for all $s \in S$. Our main result is

Theorem. *Let K be a commutative ring, R be a unital K -algebra with a complete family of full orthogonal idempotents $e_1, \dots, e_n$ for $n \geq 4$. Suppose that either $R = M(n, A)$ for an algebra A with $\mathrm{lsr}(A) \leq n - 2$ (and $e_i = e_{ii}$ are the matrix units) or R is quasi-finite. Then there is unique action of $\mathrm{GL}(R)$ on $\mathrm{St}(R)$ making $\mathrm{st}: \mathrm{St}(R) \rightarrow \mathrm{GL}(R)$ a crossed module.*

We also prove similar result for semi-local R and $n \geq 3$.

In section 2 we recall the definition of Steinberg groups and show how they are related under changes of the family of idempotents such as $e_1, \dots, e_n \mapsto e_1, \dots, e_{n-2}, e_{n-1} + e_n$. In section 3 we construct the pro-ring $R^{(\infty)}$ of homotopes of R and in section 4 we do the same for the Steinberg group. Section 5 contains a construction of the action of $\mathrm{GL}(S^{-1}R)$ on the Steinberg pro-group. In the last section we prove the main theorem.

2. Steinberg groups

We denote the category of sets by **Set** and the category of groups by **Grp**. If $\mathcal{C}$ is a category, then $\mathcal{C}(X, Y)$ is the set of morphisms from X to Y , $\mathcal{C}(X)$ is the set of endomorphisms of X . Compositions are written from the right to the left, i.e. as $\mathcal{C}(Y, Z) \times \mathcal{C}(X, Y) \rightarrow \mathcal{C}(X, Z)$. We write $X \in \mathcal{C}$ if X is an object of a category $\mathcal{C}$.

For group operations we use the conventions ${}^xy = xyx^{-1}$, $x^y = y^{-1}xy$, and $[x, y] = xyx^{-1}y^{-1}$. We need the commutator identities $[xy, z] = {}^x[y, z][x, z]$, $[x, yz] = [x, y]{}^y[x, z]$, ${}^x[y, z] = [{}^xy, {}^xz]$, ${}^xy = [x, y]y$, and $[x, y]^{-1} = [y, x]$.

An action of a group G on a group H is also denoted by ${}^gh$ for $g \in G$ and $h \in H$. If G and H are subgroups of some group, then $[G, H] = \langle [g, h] \mid g \in G, h \in H \rangle$ is their commutant and ${}^GH = [G, H]H = \langle {}^gh \mid g \in G, h \in H \rangle$. It is well-known that H normalizes $[G, H]$.

A precrossed module is a group homomorphism $d: H \rightarrow G$ with an action of G on H by automorphisms such that $d({}^gh) = {}^gd(h)$ for $g \in G$ and $h \in H$. A crossed module is a precrossed module with the property $d({}^{d(h)}h') = {}^hh'$ for $h, h' \in H$. For every crossed module the kernel of d is a central subgroup of H and the image is a normal subgroup of G .

Every ring in this paper is associative, but not necessarily unital. An element x in a non-unital ring R is called quasi-invertible if it is invertible under the monoidal operation $x \circ y = xy + x + y$. The group of

these elements is denoted by R° . If R has an identity, then R° is isomorphic to the group R^* of invertible elements by $x \mapsto x + 1$.

A unital ring R is called semi-simple if it is a finite product of matrix rings over division rings (of finite sizes). A unital ring R is called semi-local if $R/J(R)$ is semi-simple, where $J(R)$ is the Jacobson radical of R . A unital algebra R over a commutative ring K is called finite if it is a finitely generated K -module and quasi-finite if it is a direct limit of finite algebras. Base changes of finite and quasi-finite algebras are finite and quasi-finite respectively. Any finite algebra over a local commutative ring is semi-local. All these properties are preserved under Morita equivalence.

From now on fix a unital ring R with a complete family $e_1, \dots, e_n$ of full orthogonal idempotents for some $n \geq 1$. In other words, $e_i^2 = e_i$, $e_i e_j = 0$ for $i \neq j$, $\sum_{i=1}^n e_i = 1$, and $R = Re_i R$ for any i . We think about R as a matrix ring. Its general linear group is the group of invertible elements, $\text{GL}(R) = R^*$. We use the notation $R_{ij} = e_i R e_j$.

For all distinct $i, j \in \{1, \dots, n\}$ there are natural group homomorphisms $t_{ij}: R_{ij} \rightarrow \text{GL}(R)$, $x \mapsto 1 + x$. The elements $t_{ij}(a)$ are called elementary transvections. For example, if $R = M(n, A)$ is a matrix ring and $e_i = e_{ii}$ are the matrix units, then $t_{ij}(a)$ are ordinary elementary transvections. In any case they satisfy the Steinberg relations

- (St1) $t_{ij}(a)t_{ij}(b) = t_{ij}(a+b)$;
- (St2) $[t_{ij}(a), t_{kl}(b)] = 1$ for $j \neq k$ and $i \neq l$;
- (St3) $[t_{ij}(a), t_{jk}(b)] = t_{ik}(ab)$ for $i \neq k$.

A Steinberg group $\text{St}(R)$ is the abstract group generated by symbols $x_{ij}(a)$ for all distinct $i, j \in \{1, \dots, n\}$ and all $a \in R_{ij}$. The relations on these symbols are the Steinberg relations for x_{ij} instead of t_{ij} . Clearly, $x_{ij}: R_{ij} \rightarrow \text{St}(R)$ is a group homomorphism for all $i \neq j$, its image is denoted by $x_{ij}(\ast)$ and called a root subgroup. There is a homomorphism $\text{st}: \text{St}(R) \rightarrow \text{GL}(R)$, $x_{ij}(a) \mapsto t_{ij}(a)$. The image of st is called an elementary group and denoted by $E(R)$, it is the group generated by all elementary transvections. The kernel of st is denoted by $K_2(R)$, and the cokernel by $K_1(R)$ (it is the set of cosets in general). Since the homomorphisms t_{ij} are injective, it follows that the root subgroups $x_{ij}(\ast)$ are isomorphic to R_{ij} .

A diagonal group is the group $D(R) = \{g \in \text{GL}(R) \mid e_i g e_j = 0 \text{ for } i \neq j\}$ (it is indeed closed under inversion). This group decomposes as a direct product of $d_i(\ast)$ for all $1 \leq i \leq n$, where $d_i(a) = 1 + a - e_n$ for $a \in (R_{ii})^*$ and $d_i(\ast)$ is the group of all $d_i(a)$. The diagonal group acts on the Steinberg group by

- $d_i(a)x_{jk}(b) = x_{jk}(b)$ for $k \neq i \neq j$;
- $d_i(a)x_{ij}(b) = x_{ij}(ab)$;
- $d_i(a)x_{ji}(b) = x_{ji}(ba^{-1})$;

where a^{-1} is the inverse of a in the ring R_{ii} . The homomorphism st preserves the action of $D(R)$.

If $R = M(n, A)$ is the matrix algebra and e_i are the matrix units, then the groups defined so far coincide with the usual $\text{GL}(n, A)$, $\text{St}(n, A)$, $E(n, A)$, $K_1(n, A)$ (when it is a group), $K_2(n, A)$, and $D(n, A)$.

Consider the root system Φ of type A_{n-1} . It consists of so-called roots $e_j - e_i$ in the vector space $\mathbb{R}^n$, where $i, j \in \{1, \dots, n\}$ are distinct elements. Usually Φ is considered as a subset of its linear span $\mathbb{R}\Phi$ of dimension $n - 1$. For every root $\alpha = e_j - e_i$ we associate the root subgroup $x_\alpha(\ast) = x_{ij}(\ast)$. Then the Steinberg relations imply that

$$[x_\alpha(\ast), x_\beta(\ast)] \leq \prod_{\substack{p\alpha+q\beta \in \Phi \\ p, q > 0}} x_{p\alpha+q\beta}(\ast)$$

for all non-antiparallel roots α and β . Here the right hand side is a nilpotent subgroup of the Steinberg group (in our case even abelian). The Weyl group of the root system Φ is the group S_n , it acts on Φ by permutations of coordinates. This group also acts by permutations on the family of idempotents $e_1, \dots, e_n$, but not on the Steinberg group in general (since R is not a matrix ring). Note that the Weyl group acts transitively on Φ .

Now let $\alpha \in \Phi$ be a root. The image of the set $\Phi \setminus \{-\alpha, \alpha\}$ in the factor-space $\mathbb{R}\Phi/\mathbb{R}\alpha$ is denoted by Φ/α . The fibers of the map $\Phi \setminus \{-\alpha, \alpha\} \rightarrow \Phi/\alpha, \beta \mapsto [\beta]$ are precisely the so-called α -series of roots. We claim that there is unique dot product on the factor-space $\mathbb{R}\Phi/\mathbb{R}\alpha$ making Φ/α a root system of type A_{n-2} . Indeed, without loss of generality $\alpha = e_n - e_{n-1}$. Then $\mathbb{R}^n/\mathbb{R}\alpha$ is isomorphic to $\mathbb{R}^{n-1}$ under the map $\sum_i p_i e_i \mapsto \sum_{i < n-1} p_i e_i + (p_{n-1} + p_n)e_{n-1}$, so Φ/α maps to the standard root system of type A_{n-2} . The dot product is unique by general theory of root systems.

For any root $\alpha \in \Phi$ the root system Φ/α is actually a root system corresponding to the smaller family of idempotents in R . Without loss of generality, $\alpha = e_n - e_{n-1}$. Denote the ordinary Steinberg group by $\text{St}(R, \Phi)$ and the ordinary diagonal group by $D(R, \Phi)$. The Steinberg group associated to the family of idempotents $e_1, \dots, e_{n-2}, e_\infty = e_{n-1} + e_n$ is denoted by $\text{St}(R, \Phi/\alpha)$, and similarly to the diagonal group. There is a well-defined map $F_\alpha: \text{St}(R, \Phi/\alpha) \rightarrow \text{St}(R, \Phi)$ given by

- $x_{ij}(a) \mapsto x_{ij}(a)$ for $i, j < n-1$;
- $x_{i\infty}(a) \mapsto x_{i, n-1}(ae_{n-1})x_{in}(ae_n)$ for $i < n-1$;
- $x_{\infty j}(a) \mapsto x_{n-1, j}(e_{n-1}a)x_{nj}(e_na)$ for $j < n-1$.

In other words, every root subgroup $x_{[\beta]}(*) \leq \text{St}(R, \Phi/\alpha)$ for $\beta \in \Phi \setminus \{-\alpha, \alpha\}$ isomorphically maps onto $\prod_{\beta+p\alpha \in \Phi} x_{\beta+p\alpha}(*) \leq \text{St}(R, \Phi)$, where the product is taken over the α -series of β . Note that $t_\alpha(*) = \text{st}(x_\alpha(*))$ lies in the new diagonal group $D(R, \Phi/\alpha)$, i.e. it nicely acts on $\text{St}(R, \Phi/\alpha)$. When we need an iterated factor $(\Phi/\alpha)/[\beta]$ of the root system, we denote it by $\Phi/\{\alpha, \beta\}$ (of course, it is canonically isomorphic to $\Phi/\{\beta, \alpha\}$, they have the same corresponding families of idempotents in R). Also the root systems Φ/α and $\Phi/(-\alpha)$ are canonically isomorphic, so we may identify F_α with $F_{-\alpha}$. Later we prove that F_α is often an isomorphism.

Actually, our construction of Φ/α and a group $\text{St}(R, \Phi/\alpha)$ is a very particular case of a general notion of relative root systems from [9], where the same is done for elementary subgroups of isotropic reductive groups.

3. Pro-objects

Recall a construction of the pro-completion of a given category $\mathcal{C}$ from [5], section 6.1. A small category $\mathcal{I}$ is called filtered if it is non-empty, for any two objects i, j there is a diagram $i \rightarrow j \leftarrow k$ in $\mathcal{I}$, and every pair of parallel morphisms $i \rightrightarrows j$ is equalized by some morphism $j \rightarrow k$ in $\mathcal{I}$. A pro-object in $\mathcal{C}$ is a contravariant functor $X^{(\infty)}$ from a filtered category $\mathcal{I}_X$ to $\mathcal{C}$. Objects of $\mathcal{I}_X$ are called indices of $X^{(\infty)}$. We write $X^{(i)}$ for the values of $X^{(\infty)}$ on indices i and omit the values of $X^{(\infty)}$ on morphisms $i \rightarrow j$ in our formulas if the morphism $i \rightarrow j$ is clear from the context (for example, if j is a sufficiently large index). So a pro-object $X^{(\infty)}$ is the formal projective limit of $X^{(i)}$. We use the notation with upper indices since our pro-objects consist of homotopes of various algebraic objects.

The category of pro-objects is denoted by $\text{Pro}(\mathcal{C})$. By definition,

$$\text{Pro}(\mathcal{C})(X^{(\infty)}, Y^{(\infty)}) = \varprojlim_{j \in \mathcal{I}_Y} \varinjlim_{i \in \mathcal{I}_X} \mathcal{C}(X^{(i)}, Y^{(j)}).$$

There is a more explicit description of morphisms in $\text{Pro}(\mathcal{C})$. We say that a pre-morphism $f: X^{(\infty)} \rightarrow Y^{(\infty)}$ consists of a function f^* from the set of indices of Y to the set of indices of X , and of morphisms

$f^{(i)}: X^{(f^*(i))} \rightarrow Y^{(i)}$ for all $i \in \mathcal{I}_Y$ such that for any morphism $i \rightarrow j$ in $\mathcal{I}_Y$ there exists a sufficiently large index $k \in \mathcal{I}_X$ making the composition $X^{(k)} \rightarrow X^{(f^*(i))} \rightarrow Y^{(i)}$ equal to $X^{(k)} \rightarrow X^{(f^*(j))} \rightarrow Y^{(j)} \rightarrow Y^{(i)}$ (here f^* is not a functor between index categories). A composition of pre-morphisms $f: X^{(\infty)} \rightarrow Y^{(\infty)}$ and $g: Y^{(\infty)} \rightarrow Z^{(\infty)}$ is the pre-morphism $g \circ f$, where $(g \circ f)^*(i) = f^*(g^*(i))$ and $(g \circ f)^{(i)} = g^{(i)} \circ f^{(g^*(i))}$. Two parallel pre-morphisms $f, g: X^{(\infty)} \rightarrow Y^{(\infty)}$ are called equivalent if for every $i \in \mathcal{I}_Y$ there exists a sufficiently large index $j \in \mathcal{I}_X$ making the composition $X^{(j)} \rightarrow X^{(f^*(i))} \rightarrow Y^{(i)}$ equal to $X^{(j)} \rightarrow X^{(g^*(i))} \rightarrow Y^{(i)}$. Finally, a morphism $X^{(\infty)} \rightarrow Y^{(\infty)}$ is an equivalence class of pre-morphisms. Note that equivalence is preserved under composition.

The category $\mathcal{C}$ embeds into $\text{Pro}(\mathcal{C})$ (i.e. there is a fully faithful functor between them), because we may consider every object from $\mathcal{C}$ as a pro-object with a single index and a single morphism between indices. So $\text{Pro}(\mathcal{C})(X, Y) \cong \mathcal{C}(X, Y)$, $\text{Pro}(\mathcal{C})(X^{(\infty)}, Y) \cong \varinjlim_{i \in \mathcal{I}_X} \mathcal{C}(X^{(i)}, Y)$, and $\text{Pro}(\mathcal{C})(X, Y^{(\infty)}) \cong \varprojlim_{i \in \mathcal{I}_Y} \mathcal{C}(X, Y^{(i)})$. Also $X^{(\infty)}$ is the projective limit of $X^{(i)}$ in the category $\text{Pro}(\mathcal{C})$.

The category of pro-sets $\text{Pro}(\mathbf{Set})$ has all finite limits by [5], proposition 6.1.18. If $X^{(\infty)}$ and $Y^{(\infty)}$ are pro-sets with the same index category $\mathcal{I}$, then we may construct their product $Z^{(\infty)}$ as follows. The index category of $Z^{(\infty)}$ is $\mathcal{I}$, $Z^{(i)} = X^{(i)} \times Y^{(i)}$, the projection $Z^{(\infty)} \rightarrow X^{(\infty)}$ is given by the pre-morphism $\pi_X^{(i)}(i) = i$, $\pi_X^{(i)}(x^{(i)}, y^{(i)}) = x^{(i)}$ for $x^{(i)} \in X^{(i)}$ and $y^{(i)} \in Y^{(i)}$, and similarly for the projection $Z^{(\infty)} \rightarrow Y^{(\infty)}$. Also the diagonal morphism $X^{(\infty)} \rightarrow X^{(\infty)} \times X^{(\infty)}$ is given by the pre-morphism $\Delta^*(i) = i$, $\Delta^{(i)}(x^{(i)}) = (x^{(i)}, x^{(i)})$ for $x^{(i)} \in X^{(i)}$.

Hence we may consider algebraic objects in $\text{Pro}(\mathbf{Set})$ such as rings and groups. Any algebraic formula (say, the commutator or a polynomial with integer coefficients) defines a morphism in $\text{Pro}(\mathbf{Set})$ from a product of algebraic objects to an algebraic object. If $a^{(\infty)}$ is a variable in such a formula, then $a^{(\infty)} \in X^{(\infty)}$ means that $X^{(\infty)}$ is the domain of $a^{(\infty)}$.

The category of pro-groups $\text{Pro}(\mathbf{Grp})$ embeds into $\text{Pro}(\mathbf{Set})$. Every pro-group is a group object in $\text{Pro}(\mathbf{Set})$. It is easy to see that a morphism $f \in \text{Pro}(\mathbf{Set})(G^{(\infty)}, H^{(\infty)})$ between pro-groups comes from $\text{Pro}(\mathbf{Grp})$ if and only if it is a morphism of group objects.

Now let us return to the Steinberg groups. From now on suppose that R is a K -algebra for some commutative unital ring K and fix a multiplicative subset $S \subseteq K$. Then there is the localization map $\Psi: R \rightarrow S^{-1}R$, where $S^{-1}R$ is an $S^{-1}K$ -algebra with a complete family of full orthogonal idempotents. We construct a filtered category $\mathcal{S}$. Its objects are the elements of S , its morphisms $s \rightarrow s'$ are all $s'' \in S$ such that $ss'' = s'$, composition and the identity morphisms are obvious. By default, our pro-sets have $\mathcal{S}$ as the category of indices.

Recall the definition of homotopes from non-associative algebra. The s -homotope of R is the non-unital ring $R^{(s)} = \{a^{(s)} \mid s \in R\}$ with the addition $a^{(s)} + b^{(s)} = (a + b)^{(s)}$ and the multiplication $a^{(s)}b^{(s)} = (asb)^{(s)}$ for $a, b \in R$. We need only the case $s \in S$, so s commutes with R . Then $R^{(s)}$ is even a K -algebra with $ka^{(s)} = (ka)^{(s)}$ for $k \in K$ and $a \in R$. If $s, s' \in S$, then there is a homomorphism of K -algebras $R^{(ss')} \rightarrow R^{(s')}$, $a^{(ss')} \mapsto (as)^{(s')}$. There is a canonical decomposition $R^{(s)} = \bigoplus_{i,j=1}^n R_{ij}^{(s)}$, where $R_{ij}^{(s)} = \{a^{(s)} \mid a \in R_{ij}\}$.

Let $R^{(\infty)}$ be the formal projective limit of $R^{(s)}$ for $s \in \mathcal{S}$, it is a non-unital ring object in $\text{Pro}(\mathbf{Set})$. Clearly, $R^{(\infty)} = \bigoplus_{i,j=1}^n R_{ij}^{(\infty)}$, where $R_{ij}^{(\infty)}$ is the formal projective limit of $R_{ij}^{(s)}$. It is convenient to sometimes write the elements of $R_{ij}^{(s)} \times R_{jk}^{(s)}$ as $a^{(s)} \otimes b^{(s)}$ and the variables with the domain $R_{ij}^{(\infty)} \times R_{jk}^{(\infty)}$ as $a^{(\infty)} \otimes b^{(\infty)}$, so the multiplication morphisms are $R_{ij}^{(\infty)} \times R_{jk}^{(\infty)} \rightarrow R_{ik}^{(\infty)}$, $a^{(\infty)} \otimes b^{(\infty)} \mapsto a^{(\infty)}b^{(\infty)}$ (this is actually a notation of the morphism, not the definition).

Note that if $e_1, \dots, e_{n-2}, e_\infty = e_{n-1} + e_n$ is another family of idempotents, then $R_{i\infty}^{(\infty)} \cong R_{i,n-1}^{(\infty)} \oplus R_{in}^{(\infty)}$ for $i < n-1$, $R_{\infty j}^{(\infty)} \cong R_{n-1,j}^{(\infty)} \oplus R_{nj}^{(\infty)}$ for $j < n-1$, and $R_{\infty\infty}^{(\infty)} \cong R_{n-1,n-1}^{(\infty)} \oplus R_{n-1,n}^{(\infty)} \oplus R_{n,n-1}^{(\infty)} \oplus R_{nn}^{(\infty)}$.

Recall that a morphism $f \in \mathcal{C}(X, Y)$ is a split epimorphism if it admits a section. Split epimorphisms are preserved under pullbacks. The following lemmas essentially say that $R_{ik}^{(\infty)} \cong R_{ij}^{(\infty)} \otimes_{R_{jj}^{(\infty)}} R_{jk}^{(\infty)}$ as pro-groups, though we do not define tensor products of abelian pro-groups.

Lemma 1. Let $i, j, k \in \{1, \dots, n\}$ be indices. Then there is a positive integer N such that the morphism

$$m_N: \prod_{p=1}^N (R_{ij}^{(\infty)} \times R_{jk}^{(\infty)}) \rightarrow R_{ik}^{(\infty)}, \quad (a_p^{(\infty)} \otimes b_p^{(\infty)})_{p=1}^N \mapsto \sum_{p=1}^N a_p^{(\infty)} b_p^{(\infty)}$$

is a split epimorphism of pro-sets.

Proof. Let $1 = \sum_{p=1}^N x_p e_j y_p$ for some $x_p, y_p \in R$, such a decomposition exists since e_j is full. Consider a pre-morphism of pro-sets

$$u: R_{ik}^{(\infty)} \rightarrow \prod_{p=1}^N (R_{ij}^{(\infty)} \times R_{jk}^{(\infty)}), \quad u^*(s) = s^2, \quad u^{(s)}: c^{(s^2)} \mapsto ((e_i x_p)^{(s)} \otimes (y_p c)^{(s)})_{p=1}^N.$$

Clearly, this is indeed a pre-morphism. Moreover,

$$m_N^{(s)}(u^{(s)}(c^{(s^2)})) = \sum_{p=1}^N (e_i x_p)^{(s)} (y_p c)^{(s)} = (sc)^{(s)}. \quad \square$$

Lemma 2. Let $i, j, k \in \{1, \dots, n\}$ be indices, $G^{(\infty)}$ be a pro-group, and $g: R_{ij}^{(\infty)} \times R_{jk}^{(\infty)} \rightarrow G^{(\infty)}$ be a morphism of pro-sets. Then there is a morphism $f: R_{ik}^{(\infty)} \rightarrow G^{(\infty)}$ of pro-groups such that

$$g(a^{(\infty)} \otimes b^{(\infty)}) = f(a^{(\infty)} b^{(\infty)})$$

if and only if g satisfies the identities

- $[g(a_1^{(\infty)} \otimes b_1^{(\infty)}), g(a_2^{(\infty)} \otimes b_2^{(\infty)})] = 1$;
- $g((a_1^{(\infty)} + a_2^{(\infty)}) \otimes b^{(\infty)}) = g(a_1^{(\infty)} \otimes b^{(\infty)}) g(a_2^{(\infty)} \otimes b^{(\infty)})$;
- $g(a^{(\infty)} \otimes (b_1^{(\infty)} + b_2^{(\infty)})) = g(a^{(\infty)} \otimes b_1^{(\infty)}) g(a^{(\infty)} \otimes b_2^{(\infty)})$;
- $g(a^{(\infty)} b^{(\infty)} \otimes c^{(\infty)}) = g(a^{(\infty)} \otimes b^{(\infty)} c^{(\infty)})$ for $a^{(\infty)} \in R_{ij}^{(\infty)}$, $b^{(\infty)} \in R_{jj}^{(\infty)}$, $c^{(\infty)} \in R_{jk}^{(\infty)}$.

Proof. The necessity of the identities is clear. Since such f is unique by Lemma 1, it suffices to consider a group G instead of a pro-group. Let $1 = \sum_{p=1}^N x_p e_j y_p$ for some $x_p, y_p \in R$. Take a morphism g satisfying the identities. The morphism g is given by a map $g': R_{ij}^{(s)} \times R_{jk}^{(s)} \rightarrow G$ for sufficiently large $s \in S$ satisfying the first three identities and the identity

$$g'((asb)^{(s)} \otimes c^{(s)}) = g'(a^{(s)} \otimes (bsc)^{(s)})$$

for all $a \in R_{ij}$, $b \in R_{jj}$, $c \in R_{jk}$. Consider the map $f': R_{ik}^{(s^2)} \rightarrow G$ given by

$$f'(c^{(s^2)}) = \prod_p g'((e_i x_p)^{(s)} \otimes (y_p c)^{(s)}),$$

it is a homomorphism by the first and the third identities. For all $a \in R_{ij}$ and $b \in R_{jk}$ we have

$$\begin{aligned} f'(a^{(s^2)} b^{(s^2)}) &= \prod_p g'((e_i x)_p^{(s)} \otimes (s^2 y_p a b)^{(s)}) \\ &= \prod_p g'((e_i x_p s y_p a)^{(s)} \otimes (sb)^{(s)}) \\ &= g'(a^{(s^2)} \otimes b^{(s^2)}) \end{aligned}$$

by the second and the last identities. It is clear that f' gives the required morphism f of pro-groups. $\square$

4. Homotopes of Steinberg groups

In this section we construct a certain pro-group $\text{St}(R)^{(\infty)}$ and prove its basic properties. The “homotopes” of $\text{St}(R)$ are the groups $\text{St}(R)^{(s)}$ parameterized by $s \in S$. They are generated by symbols $x_{ij}^{(s)}(a)$ for $i \neq j$, $a \in R_{ij}$ with the modified Steinberg relations

1. $x_{ij}^{(s)}(a)x_{ij}^{(s)}(b) = x_{ij}^{(s)}(a+b)$;
2. $[x_{ij}^{(s)}(a), x_{kl}^{(s)}(b)] = 1$ for $j \neq k$ and $i \neq l$;
3. $[x_{ij}^{(s)}(a), x_{jk}^{(s)}(b)] = x_{ik}^{(s)}(asb)$ for $i \neq k$.

The structure homomorphisms are given by $x^{(ss')}(a) \mapsto x^{(s)}(s'a)$, and a pro-group $\text{St}(R)^{(\infty)}$ is the formal projective limit of $\text{St}(R)^{(s)}$ for $s \in S$. There are pre-morphisms $x_{ij}: R_{ij}^{(\infty)} \rightarrow \text{St}(R)^{(\infty)}$ of pro-groups with $x_{ij}^*(s) = s$ and $x_{ij}^{(s)}(a^{(s)}) = x_{ij}^{(s)}(a)$. There is also a pre-morphism $\text{st}: \text{St}(R)^{(\infty)} \rightarrow \text{GL}(R)^{(\infty)}$ of pro-groups, where $\text{GL}(R)^{(s)} = (R^{(s)})^\circ$ (so $\text{GL}(R)^{(1)}$ is only isomorphic to $\text{GL}(R)$), $\text{st}^*(s) = s$, $\text{st}^{(s)}(x_{ij}^{(s)}(a)) = a^{(s)}$. It follows that all maps $x_{ij}^{(s)}: R_{ij} \rightarrow \text{St}(R)^{(s)}$ are injective.

Lemma 3. *Let $G^{(\infty)}$ be a pro-group. Then every morphism $f: \text{St}(R)^{(\infty)} \rightarrow G^{(\infty)}$ of pro-groups is uniquely determined by its compositions with all $x_{ij}: R_{ij}^{(\infty)} \rightarrow \text{St}(R)^{(\infty)}$. Morphisms $g_{ij}: R_{ij}^{(\infty)} \rightarrow G^{(\infty)}$ of pro-groups for $i \neq j$ are obtained in this way if and only if they satisfy (St1)–(St3) in $\text{Pro}(\mathbf{Set})$.*

Proof. This follows directly from the definitions. Here we need that $G^{(\infty)}$ is actually a pro-group, not just a group object in $\text{Pro}(\mathbf{Set})$, and there is only a finite number of “generators” x_{ij} . $\square$

Recall that Φ is the root system of type A_{n-1} , it parameterizes “generators” x_α of the pro-group $\text{St}(R, \Phi)^{(\infty)} = \text{St}(R)^{(\infty)}$. For every root $\alpha \in \Phi$ the pro-group $\text{St}(R, \Phi/\alpha)^{(\infty)}$ is defined using the smaller family of idempotents. Now $F_\alpha: \text{St}(R, \Phi/\alpha)^{(\infty)} \rightarrow \text{St}(R, \Phi)^{(\infty)}$ is a pre-morphism of pro-groups with $F_\alpha^*(s) = s$. We are ready to prove that F_α is an epimorphism for $n \geq 3$ and an isomorphism for $n \geq 4$ in $\text{Pro}(\mathbf{Grp})$. If $S = \{1\}$, then this means that $F_\alpha: \text{St}(R, \Phi/\alpha) \rightarrow \text{St}(R, \Phi)$ is a surjection for $n \geq 3$ and a bijection for $n \geq 4$. The proof also shows that the group $\text{St}(R, \Phi)$ is perfect for $n \geq 3$.

Lemma 4. *If $n \geq 3$ and $\alpha \in \Phi$ is a root, then F_α is an epimorphism in $\text{Pro}(\mathbf{Grp})$.*

Proof. It suffices to show that for every abstract group G two pre-morphisms $f_1, f_2: \text{St}(R, \Phi)^{(\infty)} \rightarrow G$ are equivalent if their compositions with F_α are equivalent. By Lemma 3 we have to prove that $f_i \circ x_\beta$ is equivalent to $f_2 \circ x_\beta$ for all $\beta \in \Phi$. This is clear if $\beta \neq \pm\alpha$ by definition of F_α . Since we may change the sign of α and apply an element from the Weyl group, it remains to check the case $\beta = \alpha = e_n - e_{n-1}$. Note that the index 1 is different from n and $n-1$. We have

$$x_{n-1,n}(a^{(\infty)}b^{(\infty)}) = [x_{n-1,1}(a^{(\infty)}), x_{1n}(b^{(\infty)})] = F_\alpha([x_{\infty 1}(a^{(\infty)}), x_{1\infty}(b^{(\infty)})]),$$

where $e_\infty = e_{n-1} + e_n$ is the new idempotent, $a^{(\infty)} \in R_{n-1,1}^{(\infty)}$, $b^{(\infty)} \in R_{1n}^{(\infty)}$. The claim now follows by Lemma 1. $\square$

Proposition 1. *If $n \geq 4$ and $\alpha \in \Phi$ is a root, then F_α is an isomorphism in $\text{Pro}(\mathbf{Grp})$.*

Proof. We construct morphisms $\tilde{x}_{ij}: R_{ij}^{(\infty)} \rightarrow \text{St}(R, \Phi/\alpha)^{(\infty)}$ of pro-groups for all distinct $i, j \in \{1, \dots, n\}$ and show that they satisfy the Steinberg relations. Without loss of generality, $\alpha = e_n - e_{n-1}$. We denote the “generators” of $\text{St}(R, \Phi)$ and $\text{St}(R, \Phi/\alpha)$ by x_{ij} , in the first case $i, j \in \{1, \dots, n\}$, and in the second case $i, j \in \{1, \dots, n-2, \infty\}$ (where $e_\infty = e_{n-1} + e_n$).

Let

- $\tilde{x}_{ij}(a^{(\infty)}) = x_{ij}(a^{(\infty)})$ for $i, j < n-1$;
- $\tilde{x}_{ij}(a^{(\infty)}) = x_{i\infty}(a^{(\infty)})$ for $i < n-1$ and $j \in \{n-1, n\}$;
- $\tilde{x}_{ij}(a^{(\infty)}) = x_{\infty j}(a^{(\infty)})$ for $i \in \{n-1, n\}$ and $j < n-1$;
- $\tilde{x}_{n-1,n}^i(a^{(\infty)} \otimes b^{(\infty)}) = [\tilde{x}_{n-1,i}(a^{(\infty)}), \tilde{x}_{in}(b^{(\infty)})]$ for $i < n-1$, $a^{(\infty)} \in R_{n-1,i}^{(\infty)}$, $b^{(\infty)} \in R_{in}^{(\infty)}$.

So we have $\tilde{x}_\beta$ for all $\beta \neq \pm\alpha$. They clearly satisfy the Steinberg relations not involving $\pm\alpha$. Since no Steinberg relation involves both α and $-\alpha$, it suffices to construct $\tilde{x}_\alpha$ and prove the relations with α . The idea is to take $\tilde{x}_\alpha(a^{(\infty)}b^{(\infty)}) = \tilde{x}_{n-1,n}^i(a^{(\infty)} \otimes b^{(\infty)})$ for $a^{(\infty)} \in R_{n-1,i}^{(\infty)}$ and $b^{(\infty)} \in R_{in}^{(\infty)}$, and show that this is well-defined using Lemma 2. Below we use already proved Steinberg relations and the commutator identities without further mention.

If $j \neq i, n$ and $k \neq i, n-1$, then

$$\begin{aligned} \tilde{x}_{jk}(a^{(\infty)})\tilde{x}_{n-1,n}^i(b^{(\infty)} \otimes c^{(\infty)}) &= [\tilde{x}_{jk}(a^{(\infty)})\tilde{x}_{n-1,i}(b^{(\infty)}), \tilde{x}_{jk}(a^{(\infty)})\tilde{x}_{in}(c^{(\infty)})] \\ &= [\tilde{x}_{n-1,i}(b^{(\infty)}), \tilde{x}_{in}(c^{(\infty)})] = \tilde{x}_{n-1,n}^i(b^{(\infty)} \otimes c^{(\infty)}), \end{aligned}$$

i.e. $\tilde{x}_{jk}$ commutes with $\tilde{x}_{n-1,n}^i$. The “generators” $\tilde{x}_{ji}$, $\tilde{x}_{ik}$ for $j, k \notin \{i, n-1, n\}$ also commute with $\tilde{x}_{n-1,n}^i$ because

$$\begin{aligned} \tilde{x}_{ji}(a^{(\infty)})\tilde{x}_{n-1,n}^i(b^{(\infty)} \otimes c^{(\infty)}) &= [\tilde{x}_{n-1,i}(b^{(\infty)}), \tilde{x}_{jn}(a^{(\infty)}c^{(\infty)})\tilde{x}_{in}(c^{(\infty)})] \\ &= \tilde{x}_{n-1,n}^i(b^{(\infty)} \otimes c^{(\infty)}) \end{aligned}$$

and

$$\begin{aligned} \tilde{x}_{ik}(w^{(\infty)})\tilde{x}_{n-1,n}^i(u^{(\infty)} \otimes v^{(\infty)}) &= [\tilde{x}_{n-1,k}(u^{(\infty)}w^{(\infty)})\tilde{x}_{n-1,i}(u^{(\infty)}), \tilde{x}_{in}(v^{(\infty)})] \\ &= \tilde{x}_{n-1,n}^i(u^{(\infty)} \otimes v^{(\infty)}). \end{aligned}$$

Since $n \geq 4$, there is an index j distinct from $i, n-1, n$. It follows that $\tilde{x}_{n-1,i}(a^{(\infty)}b^{(\infty)})$ and $\tilde{x}_{in}(c^{(\infty)}d^{(\infty)})$ commute with $\tilde{x}_{n-1,n}^i$ for $a \in R_{n-1,j}^{(\infty)}$, $b \in R_{ji}^{(\infty)}$, $c \in R_{ij}^{(\infty)}$, $d \in R_{jn}^{(\infty)}$. By Lemma 1, $\tilde{x}_{n-1,n}^i$ commutes with $\tilde{x}_{n-1,i}$ and $\tilde{x}_{in}$. Hence we have all cases of (St2).

Now let us show that $\tilde{x}_{n-1,n}^i(a^{(\infty)} \otimes b^{(\infty)})$ is biadditive. Indeed,

$$\begin{aligned} \tilde{x}_{n-1,n}^i((a_1^{(\infty)} + a_2^{(\infty)}) \otimes b^{(\infty)}) &= [\tilde{x}_{n-1,i}(a_1^{(\infty)})\tilde{x}_{n-1,i}(a_2^{(\infty)}), \tilde{x}_{in}(b^{(\infty)})] \\ &\quad \cdot \tilde{x}_{n-1,n}^i(a_1^{(\infty)}b^{(\infty)})\tilde{x}_{n-1,n}^i(a_2^{(\infty)}b^{(\infty)}), \\ \tilde{x}_{n-1,n}^i(a^{(\infty)} \otimes (b_1^{(\infty)} + b_2^{(\infty)})) &= [\tilde{x}_{n-1,i}(a^{(\infty)}), \tilde{x}_{in}(b_1^{(\infty)})\tilde{x}_{in}(b_2^{(\infty)})] \\ &\quad \cdot \tilde{x}_{n-1,n}^i(a^{(\infty)}b_1^{(\infty)})\tilde{x}_{n-1,n}^i(a^{(\infty)}b_2^{(\infty)}). \end{aligned}$$

If i, j are distinct indices from $\{1, \dots, n-2\}$, then for $a^{(\infty)} \in R_{n-1,i}^{(\infty)}$, $b^{(\infty)} \in R_{ij}^{(\infty)}$, $c^{(\infty)} \in R_{jn}^{(\infty)}$ we have

$$\tilde{x}_{n-1,n}^i(a^{(\infty)} \otimes b^{(\infty)}c^{(\infty)}) = \tilde{x}_{n-1,i}(a^{(\infty)})[\tilde{x}_{ij}(b^{(\infty)}), \tilde{x}_{jn}(c^{(\infty)})]\tilde{x}_{in}(-b^{(\infty)}c^{(\infty)})$$

$$\begin{aligned}
&= [\tilde{x}_{n-1,j}(a^{(\infty)}b^{(\infty)})\tilde{x}_{ij}(b^{(\infty)}), \tilde{x}_{jn}(c^{(\infty)})] \\
&\cdot \tilde{x}_{in}(-b^{(\infty)}c^{(\infty)}) \\
&= \tilde{x}_{ij}(b^{(\infty)})\tilde{x}_{n-1,n}^j(a^{(\infty)}b^{(\infty)} \otimes c^{(\infty)}) \\
&= \tilde{x}_{n-1,n}^j(a^{(\infty)}b^{(\infty)} \otimes c^{(\infty)}).
\end{aligned}$$

It follows that $\tilde{x}_{n-1,n}^i(a^{(\infty)} \otimes b^{(\infty)}c^{(\infty)}d^{(\infty)}) = \tilde{x}_{n-1,n}^i(a^{(\infty)}b^{(\infty)}c^{(\infty)} \otimes d^{(\infty)})$ for $a^{(\infty)} \in R_{n-1,i}^{(\infty)}$, $b^{(\infty)} \in R_{ij}^{(\infty)}$, $c^{(\infty)} \in R_{ji}^{(\infty)}$, and $d^{(\infty)} \in R_{in}^{(\infty)}$. By Lemma 1, the last identity from the statement of Lemma 2 holds. From (St2) it follows that $\tilde{x}_{n-1,n}^i$ commutes with $\tilde{x}_{n-1,n}^j$ for not necessarily distinct $i, j < n-1$, so we may apply Lemma 2. The morphism $\tilde{x}_{n-1,n}: R_{n-1,n}^{(\infty)} \rightarrow \text{St}(R, \Phi/\alpha)$ of pro-groups is well-defined by

$$\tilde{x}_{n-1,n}(a^{(\infty)}b^{(\infty)}) = \tilde{x}_{n-1,n}^1(a^{(\infty)} \otimes b^{(\infty)})$$

for $a^{(\infty)} \in R_{n-1,1}^{(\infty)}$ and $b^{(\infty)} \in R_{1n}^{(\infty)}$, it satisfies (St1). By the relation between $\tilde{x}_{n-1,n}^i$ and $\tilde{x}_{n-1,n}^j$ we have (St3) with α in the right hand side.

It remains to prove (St3) with α in the left hand side. If i and j are different indices less than $n-1$, then

$$\begin{aligned}
\tilde{x}_{i,n-1}(a^{(\infty)})\tilde{x}_{n-1,n}^j(b^{(\infty)} \otimes c^{(\infty)}) &= [\tilde{x}_{ij}(a^{(\infty)}b^{(\infty)})\tilde{x}_{n-1,j}(b^{(\infty)}), \tilde{x}_{jn}(c^{(\infty)})] \\
&= \tilde{x}_{n-1,n}^j(b^{(\infty)} \otimes c^{(\infty)})\tilde{x}_{in}(a^{(\infty)}b^{(\infty)}c^{(\infty)})
\end{aligned}$$

and

$$\begin{aligned}
\tilde{x}_{ni}(w^{(\infty)})\tilde{x}_{n-1,n}^j(u^{(\infty)} \otimes v^{(\infty)}) &= [\tilde{x}_{n-1,j}(u^{(\infty)}), \tilde{x}_{ji}(-v^{(\infty)}w^{(\infty)})\tilde{x}_{jn}(v^{(\infty)})] \\
&= \tilde{x}_{n-1,i}(-u^{(\infty)}v^{(\infty)}w^{(\infty)})\tilde{x}_{n-1,n}^j(u^{(\infty)} \otimes v^{(\infty)}),
\end{aligned}$$

hence $[\tilde{x}_{i,n-1}(a^{(\infty)}), \tilde{x}_{n-1,n}(b^{(\infty)})] = \tilde{x}_{in}(a^{(\infty)}b^{(\infty)})$ and $[\tilde{x}_{n-1,n}(c^{(\infty)}), \tilde{x}_{ni}(d^{(\infty)})] = \tilde{x}_{n-1,i}(c^{(\infty)}d^{(\infty)})$ by Lemma 1. Here we also use that $n \geq 4$.

Now we have a morphism $G_\alpha: \text{St}(R, \Phi)^{(\infty)} \rightarrow \text{St}(R, \Phi/\alpha)^{(\infty)}$ of pro-groups such that $G_\alpha \circ x_{ij} = \tilde{x}_{ij}$ (by Lemma 3). It is clear from the construction that $G_\alpha \circ F_\alpha$ is the identity. Hence $F_\alpha \circ G_\alpha$ is also the identity by Lemma 4. $\square$

5. Localization

First we show that the semi-direct product $\text{St}(S^{-1}R) \rtimes \text{D}(S^{-1}R)$ acts on the pro-group $\text{St}(R)^{(\infty)}$. The next lemma shows that $\text{GL}(S^{-1}R)^*$ acts on the pro-group $\text{GL}(R)^{(\infty)}$ (and on the ring pro-set $R^{(\infty)}$) by inner automorphisms.

Lemma 5. *For every $\frac{a}{s} \in S^{-1}R$ the families of maps*

$$\begin{aligned}
R^{(ss')} &\rightarrow R^{(s')}, \quad b^{(ss')} \mapsto (ab)^{(s')} \\
R^{(ss')} &\rightarrow R^{(s')}, \quad b^{(ss')} \mapsto (ba)^{(s')}
\end{aligned}$$

give well-defined endomorphisms $b^{(\infty)} \mapsto \frac{a}{s}b^{(\infty)}$ and $b^{(\infty)} \mapsto b^{(\infty)}\frac{a}{s}$ of the pro-group $R^{(\infty)}$. They are additive on $\frac{a}{s}$ and satisfy the identities

$$\begin{aligned}
(uv)a^{(\infty)} &= u(va^{(\infty)}) & (ua^{(\infty)})b^{(\infty)} &= u(a^{(\infty)}b^{(\infty)}) \\
(ua^{(\infty)})v &= u(a^{(\infty)}v) & (a^{(\infty)}u)b^{(\infty)} &= a^{(\infty)}(ub^{(\infty)}) \\
(a^{(\infty)}u)v &= a^{(\infty)}(uv) & (a^{(\infty)}b^{(\infty)})u &= a^{(\infty)}(b^{(\infty)}u)
\end{aligned}$$

in $\text{Pro}(\mathbf{Set})$ for $u, v \in S^{-1}R$ and $a^{(\infty)}, b^{(\infty)} \in R^{(\infty)}$. Also $a^{(\infty)} \mapsto e_i a^{(\infty)} e_j$ is the projection $R^{(\infty)} \rightarrow R_{ij}^{(\infty)}$. For $k \in K$ the morphism $a^{(\infty)} \mapsto ka^{(\infty)} = a^{(\infty)}k$ is the ordinary multiplication in every homotope.

Proof. Clear. $\square$

Proposition 2. Suppose that either $n \geq 4$ or $n = 3$ and $S = 1$. Then there is an action of $\text{St}(S^{-1}R) \rtimes \text{D}(S^{-1}R)$ on $\text{St}(R)^{(\infty)}$ by automorphisms such that $\text{st}: \text{St}(R)^{(\infty)} \rightarrow \text{GL}(R)^{(\infty)}$ is equivariant. It is given by

- $d_i(u)x_{jk}(a^{(\infty)}) = x_{jk}(a^{(\infty)})$ for $k \neq i \neq j$;
- $d_i(u)x_{ij}(a^{(\infty)}) = x_{ij}(ua^{(\infty)})$;
- $d_i(u)x_{ji}(a^{(\infty)}) = x_{ji}(a^{(\infty)}u^{-1})$;
- $x_{ij}(v)x_{kl}(a^{(\infty)}) = x_{kl}(a^{(\infty)})$ for $j \neq k$ and $l \neq i$;
- $x_{ij}(v)x_{jk}(a^{(\infty)}) = x_{ik}(va^{(\infty)})x_{jk}(a^{(\infty)})$ for $i \neq k$;
- $x_{ij}(v)x_{ki}(a^{(\infty)}) = x_{kj}(a^{(\infty)}v)x_{ki}(a^{(\infty)})$ for $j \neq k$.

Proof. For every $u \in S^{-1}R$ the first three formulas give a well-defined endomorphism of the pro-group $\text{St}(R)^{(\infty)}$ by Lemmas 3 and 5. The properties

$$d_i(uv)(g^{(\infty)}) = d_i(u)(d_i(v)(g^{(\infty)}))$$

and

$$d_i(u)(d_j(w)(g^{(\infty)})) = d_j(w)(d_i(u)(g^{(\infty)}))$$

for all distinct i, j follow from the same lemmas. Now note that $\text{D}(S^{-1}R)$ is generated by all $(S^{-1}R_{ii})^*$ with the relations $d_i(uv) = d_i(u)d_i(v)$ and $d_i(u)d_j(w) = d_j(w)d_i(u)$ as an abstract group. If $S = 1$, then $\text{St}(S^{-1}R) = \text{St}(R) \cong \text{St}(R)^{(\infty)}$, so the remaining automorphisms are already given and satisfy all identities. They are also uniquely determined by Lemma 4.

From now on assume that $n \geq 4$. The automorphism $g^{(\infty)} \mapsto x_\alpha(v)(g^{(\infty)})$ is well-defined on $\text{St}(R, \Phi/\alpha)^{(\infty)}$, because then the element $x_\alpha(v)$ may be considered as its image in $\text{D}(S^{-1}R, \Phi/\alpha)$. Hence by Proposition 1 this is also an automorphism of $\text{St}(R, \Phi)^{(\infty)}$. We still have to show that $x_\alpha(v)$ commutes with the “generator” x_α of $\text{St}(R, \Phi)^{(\infty)}$. Without loss of generality, let $\alpha = e_{n-1} - e_n$. Other relations imply that $x_{n-1,n}(v)$ commutes with $x_{n-1,n}(a^{(\infty)}b^{(\infty)})$ for $a^{(\infty)} \in R_{n-1,1}^{(\infty)}$ and $b^{(\infty)} \in R_{1n}^{(\infty)}$. Hence it also commutes with the whole $x_{n-1,n}(c^{(\infty)})$ by Lemma 1.

By Lemma 5, the relations between $d_i(u)$ and $x_{jk}(v)$ hold for these automorphisms, as well as (St1). It remains to prove that the automorphisms induced by $x_{ij}(v)$ satisfy the Steinberg relations (St2) and (St3). If α and β are non-parallel roots, then the relation between $g^{(\infty)} \mapsto x_\alpha(v)(g^{(\infty)})$ and $g^{(\infty)} \mapsto x_\beta(w)(g^{(\infty)})$ holds on the pro-group $\text{St}(R, \Phi/\{\alpha, \beta\})^{(\infty)}$, since it holds for the images of $x_\alpha(v)$ and $x_\beta(w)$ in $\text{D}(S^{-1}R, \Phi/\{\alpha, \beta\})$. Then the relation holds on the whole pro-group $\text{St}(R, \Phi)^{(\infty)}$ by Lemma 4. It is clear from the definition that $\text{st}: \text{St}(R)^{(\infty)} \rightarrow \text{GL}(R)^{(\infty)}$ is equivariant. $\square$

For the next lemma we need maximal unipotent subgroups of the Steinberg group. Let at the moment R be arbitrary unital ring with a complete family of full orthogonal idempotents. The standard maximal unipotent subgroups of $\text{St}(R)$ are $U^+ = U^+(\Phi) = \langle x_{ij}(a) \mid i < j \rangle$ and $U^- = U^-(\Phi) = \langle x_{ij}(a) \mid i > j \rangle$.

The Steinberg relations imply that st maps U^+ and U^- isomorphically onto the groups of upper and lower unitriangular elements of $\text{GL}(R)$. If $\alpha = e_{i+1} - e_i$ is a simple root, then the subgroups $U^+(\Phi/\alpha)$ and $U^-(\Phi/\alpha)$ are well-defined and we have decompositions

$$U^+(\Phi) = U^+(\Phi/\alpha) \rtimes x_\alpha(*), \quad U^-(\Phi) = U^-(\Phi/\alpha) \rtimes x_{-\alpha}(*).$$

Moreover, both $U^+(\Phi/\alpha)$ and $U^-(\Phi/\alpha)$ are normalized by $x_\alpha(*)$ and $x_{-\alpha}(*)$.

Lemma 6. *Let R be a semi-local ring with a complete family of full orthogonal idempotents $e_1, \dots, e_n$. Then there is Gauss decomposition $\text{GL}(R) = \text{st}(U^+) \text{st}(U^-) \text{st}(U^+) \text{D}(R)$, and $\text{GL}(R)$ is the factor-group of $\text{St}(R) \rtimes \text{D}(R)$ by the relations*

$$\prod_{k=1}^N x_{i,i+1}(a_k) x_{i+1,i}(b_k) = d_i(u) d_{i+1}(v)$$

whenever the images of both sides coincide in $\text{GL}(R)$ (it suffices to take $N = 3$ for all $i \in \{1, \dots, n-1\}$).

Proof. Let us prove Gauss decomposition for $n = 2$ (if $n = 1$, there is nothing to prove). Take $g \in \text{GL}(R)$. We claim that there is $a \in R_{12}$ such that $e_2 g t_{12}(a) e_2 \in (R_{22})^*$. Without loss of generality, we may take the factor-ring of R by its Jacobson radical and consider only one of the simple factors, i.e. we may assume that R is a matrix ring over a division ring of size m . Moreover, we may assume that $e_1 = e'_{11} + \dots + e'_{tt}$ and $e_2 = e'_{t+1,t+1} + \dots + e'_{mm}$, where e'_{ij} are the matrix units. Now the matrix $e_2 g$ has rank $m - t$, so we may add first t columns of $e_2 g$ to the last $m - t$ columns with some coefficients making the rank of $e_2 g t_{12}(a) e_2$ equal to $m - t$ (the matrix a describes the coefficients of elementary transformations). Take such an a . Then there is $b \in R_{21}$ such that $e_2 g t_{12}(a) t_{21}(b) e_1 = 0$, hence $g t_{12}(a) t_{21}(b) \in t_{12}(*) \text{D}(R)$. The case of arbitrary n follows by easy induction if we pass from Φ to $\Phi/(e_{i+1} - e_i)$ and then apply the case $n = 2$ for $(e_i + e_{i+1})R(e_i + e_{i+1})$.

If $(g, h) \in \text{St}(R) \rtimes \text{D}(R)$ maps to $1 \in \text{GL}(R)$ and g lies in $U^+ U^- U^+$, then it is easy to see that the factor from U^- is trivial. Hence $g = 1$ and $h = 1$. It remains to prove that every element from $\text{St}(R) \rtimes \text{D}(R)$ is congruent to an element (g, h) with $g \in U^+ U^- U^+$ modulo the relations. Let G be the set of such elements. Then G is closed under multiplication by U^+ and $\text{D}(R)$ from the right, it also contains 1. It suffices to show that $G x_{i+1,i}(*) \subseteq G$ because $\text{St}(R)$ is generated by U^+ and all $x_{i+1,i}(*)$ (this easily follows from (St3) and Lemma 1). After passing from Φ to $\Phi/(e_{i+1} - e_i)$ it remains to show that

$$t_{i,i+1}(*) t_{i+1,i}(*) t_{i,i+1}(*) t_{i+1,i}(*) \subseteq t_{i,i+1}(*) t_{i+1,i}(*) t_{i,i+1}(*) d_i(*) d_{i+1}(*),$$

modulo the relations. But this follows from Gauss decomposition for $(e_i + e_{i+1})R(e_i + e_{i+1})$. $\square$

Recall that a sequence $a_1, \dots, a_n$ in a unital ring A is called left unimodular if there are $b_1, \dots, b_n \in A$ such that $\sum_{i=1}^n b_i a_i = 1$. The ring A satisfies $\text{sr}(A) \leq n - 1$ if for every left unimodular sequence $a_1, \dots, a_n$ there are elements $c_1, \dots, c_{n-1} \in A$ such that the shorter sequence $a_1 + c_1 a_n, \dots, a_{n-1} + c_{n-1} a_n$ is also unimodular. We say that a unital K -algebra satisfies $\text{lsr}(A) \leq n - 1$ if $\text{sr}(A_{\mathfrak{m}}) \leq n - 1$ for every maximal ideal $\mathfrak{m} \leq K$. For example, if A is a quasi-finite K -algebra, then $\text{lsr}(A) \leq 1$.

Proposition 3. *Let R be a unital K -algebra with a complete family of full orthogonal idempotents $e_1, \dots, e_n$, $S \subseteq K$ be a multiplicative subset. Suppose that either $n \geq 4$ or $n = 3$ and $S = 1$. Also suppose that either $R = \text{M}(n, A)$ for an algebra A with $\text{sr}(S^{-1}A) \leq n - 2$ (where $e_i = e_{ii}$ are the matrix units) or $S^{-1}R$ is semi-local. Then $\text{GL}(S^{-1}R)$ acts on $\text{St}(R)^{(\infty)}$, the morphism st is equivariant, and this action is consistent with the action of $\text{St}(S^{-1}R) \rtimes \text{D}(S^{-1}R)$.*

Proof. Let $\alpha_i = e_{i+1} - e_i$ be the simple roots of Φ and consider the root system $\Phi' = \Phi / \{\alpha_2, \dots, \alpha_{n-2}\}$ corresponding to the family of idempotents $e_1, e_{2'} = e_2 + \dots + e_{n-1}, e_n$. By Proposition 2 the group

$$G = \text{St}(S^{-1}R, \Phi') \rtimes \text{D}(S^{-1}R, \Phi')$$

acts on $\text{St}(R)^{(\infty)}$ and st is equivariant. Note that $G \rightarrow \text{GL}(S^{-1}R)$ is surjective: in the matrix case this follows from surjective stability of K_1 (see [1], theorem 4.2), and in the semi-local case this follows from Gauss decomposition. It remains to show that every generator of the kernel of $G \rightarrow \text{GL}(S^{-1}R)$ acts trivially.

We claim that it is possible to choose the generators to be of type

$$\left(\prod_{k=1}^N x_{[\alpha_i]}(a_k) x_{[-\alpha_i]}(b_k), h \right)$$

for $i = 1$ and $i = n - 1$. In the semi-local case this follows from Lemma 6. In the matrix case we may consider only the generators of type $(g, d_1(u) d_{2'}(v))$ and $(g, d_{2'}(u) d_n(v))$ for $g \in \text{St}(S^{-1}R, \Phi')$ by surjective stability of K_1 . But then g is a product of $x_{\pm[\alpha_1]}$ or $x_{\pm[\alpha_{n-1}]}$ by injective stability of K_1 ([13]) and surjective stability of K_2 ([2]). Finally, the proposition follows from Lemma 4 applied to $\Phi'/[\alpha_1]$ and $\Phi'/[\alpha_{n-1}]$. $\square$

6. Steinberg crossed module

We are ready to prove the main results. Recall the remark before Lemma 4 that $\text{St}(R)$ is perfect for any unital ring R with a complete family of full orthogonal idempotents $e_1, \dots, e_n$ if $n \geq 3$.

Theorem 1. *Let R be a semi-local unital ring with a complete family of full orthogonal idempotents $e_1, \dots, e_n$ with $n \geq 3$. Then there is unique action of $\text{GL}(R)$ on $\text{St}(R)$ making $\text{st}: \text{St}(R) \rightarrow \text{GL}(R)$ a crossed module, it is consistent with the action of $\text{St}(R) \rtimes \text{D}(R)$.*

Proof. Indeed, by Proposition 3 (applied to $K = \mathbb{Z}$ and $S = \{1\}$) there is an action of $\text{GL}(R)$ on $\text{St}(R)$ such that st is equivariant and this action is consistent with the conjugacy action of $\text{St}(R)$ on itself. Hence, in particular, $\text{St}(R)$ is a central perfect extension of $E(R)$. By abstract group theory it follows that for every $g \in \text{GL}(R)$ there is at most one automorphism $h \mapsto {}^g h$ of $\text{St}(R)$ making st g -equivariant. $\square$

Theorem 2. *Let K be a commutative ring, R be a unital K -algebra with a complete family of full orthogonal idempotents $e_1, \dots, e_n$ for $n \geq 4$. Suppose that either $R = M(n, A)$ for an algebra A with $\text{lsr}(A) \leq n - 2$ (and $e_i = e_{ii}$ are the matrix units) or R is quasi-finite. Then there is unique action of $\text{GL}(R)$ on $\text{St}(R)$ making $\text{st}: \text{St}(R) \rightarrow \text{GL}(R)$ a crossed module, it is consistent with the action of $\text{St}(R) \rtimes \text{D}(R)$.*

Proof. Firstly, we show any $g \in K_2(R)$ lies in the center of $\text{St}(R)$. Fix two indices $i \neq j$ and consider the ideal

$$\mathfrak{a} = \{k \in K \mid [g, x_{ij}(kR_{ij})] = 1\} \leq K.$$

It suffices to show that $\mathfrak{a}$ is not contained in any maximal ideal $\mathfrak{m} \leq K$. By Proposition 3 the element $\Psi(g)$ trivially acts on the pro-group $\text{St}(R)^{(\infty)}$, where the multiplicative subset is $S = K \setminus \mathfrak{m}$ and Ψ is the localization map (the quasi-finite case easily reduces to the case of a finite K -algebra R , so after localization it becomes a semi-local ring). This means that $\mathfrak{a}$ contains an element from $K \setminus \mathfrak{m}$.

Secondly, we show that $E(R)$ is normal in $\text{GL}(R)$ (this result is already known at least for matrix rings). Fix $g \in \text{GL}(R)$ and indices $i \neq j$. Consider the ideal

$$\mathfrak{b} = \{k \in K \mid {}^g t_{ij}(kR_{ij}) \in E(R)\} \leq K.$$

We have to prove that $\mathfrak{b}$ is not contained in any maximal ideal $\mathfrak{m} \leq K$. By Proposition 3 the element $\Psi(g)$ acts on the pro-group $\text{St}(R)^{(\infty)}$ such that st is $\Psi(g)$ -equivariant, where Ψ is the localization map with respect to the multiplicative subset $S = K \setminus \mathfrak{m}$. Hence $\mathfrak{b}$ contains an element from $K \setminus \mathfrak{m}$.

Thirdly, we have to construct the action of $\text{GL}(R)$ on $\text{St}(R)$. Fix $g \in \text{GL}(R)$. There is at most one endomorphism $h \mapsto {}^g h$ of $\text{St}(R)$ such that st is g -equivariant, because the extension $\text{St}(R) \rightarrow E(R)$ is central and perfect. If we show its existence, then ${}^{gg'}h = {}^g(g'h)$ follows by uniqueness. Let $Y_{ij}(a) = \text{st}^{-1}({}^g t_{ij}(a))$ for $a \in R_{ij}$, these objects are certain cosets of $\text{St}(R)$ by $K_2(R)$. They satisfy the Steinberg relations as follows:

- St1. $Y_{ij}(a)Y_{ij}(b) = Y_{ij}(a+b)$;
 St2. $[Y_{ij}(a), Y_{kl}(b)] \subseteq K_2(R)$ for $j \neq k$ and $i \neq l$;
 St3. $[Y_{ij}(a), Y_{jk}(b)] \subseteq Y_{ik}(ab)$ for $i \neq k$.

Since $\text{St}(R) \rightarrow E(R)$ is a central extension, all commutators $[Y_{ij}(a), Y_{kl}(b)]$ are one-element sets. We identify them with their elements. Let us show that actually $[Y_{ij}(a), Y_{kl}(b)] = 1$ for $j \neq k$ and $i \neq l$. Fix such indices i, j, k, l and an element $a \in R_{ij}$. Consider the ideal

$$\mathfrak{c} = \{k \in K \mid [Y_{ij}(a), Y_{kl}(kR_{kl})] = 1\} \leq K.$$

We show that $\mathfrak{c}$ is not contained in any maximal ideal $\mathfrak{m} \leq K$. By Proposition 3, the element $\Psi({}^g t_{ij}(a))\Psi(g) = \Psi(g)\Psi(t_{ij}(a))$ acts on $\text{St}(R)^{(\infty)}$, where $\Psi(t_{ij}(a))$ acts as in Proposition 2 (again Ψ is the localization with respect to $S = K \setminus \mathfrak{m}$). It follows that $[Y_{ij}(a), Y_{kl}(kR_{kl})] = 1$ for some $k \in K \setminus \mathfrak{c}$.

Now let $y_{ik}^j(a \otimes b) = [Y_{ij}(a), Y_{jk}(b)] \in Y_{ik}(ab)$ for $a \in R_{ij}$ and $b \in R_{jk}$ if i, j, k are distinct. The elements $y_{ik}^j(a \otimes b) \in \text{St}(R)$ are biadditive:

$$\begin{aligned} y_{ik}^j((a_1 + a_2) \otimes b) &= [Y_{ij}(a_1)Y_{ij}(a_2), Y_{jk}(b)] \\ &= [Y_{ij}(a_1), Y_{jk}(b)][Y_{ij}(a_2), Y_{jk}(b)] \\ &= y_{ik}^j(a_1 \otimes b)y_{ik}^j(a_2 \otimes b) \end{aligned}$$

and

$$\begin{aligned} y_{ik}^j(a \otimes (b_1 + b_2)) &= [Y_{ij}(a), Y_{jk}(b_1)Y_{jk}(b_2)] \\ &= [Y_{ij}(a), Y_{jk}(b_1)][Y_{ij}(a), Y_{jk}(b_2)] \\ &= y_{ik}^j(a \otimes b_1)y_{ik}^j(a \otimes b_2). \end{aligned}$$

Also for different i, j, k, l and for all $a \in R_{ij}$, $b \in R_{jk}$, $c \in R_{kl}$ we have

$$\begin{aligned} y_{il}^j(a \otimes bc) &= {}^{Y_{ij}(a)}[Y_{jk}(b), Y_{kl}(c)]Y_{jl}(-bc) \\ &= [Y_{ik}(ab)Y_{jk}(b), Y_{kl}(c)]Y_{jl}(-bc) \\ &= {}^{Y_{jk}(b)}y_{il}^k(ab \otimes c) = y_{il}^k(ab \otimes c). \end{aligned}$$

Since $n \geq 4$, it follows that $y_{ik}^j(ab \otimes c) = y_{ik}^j(a \otimes bc)$ for $a \in R_{ij}$, $b \in R_{jk}$, $c \in R_{kl}$ since there is an index l different from i, j, k (see Lemma 1 in the case $S = \{1\}$). For all $i \neq k$ choose an index j different from i, k and define a homomorphism $y_{ik}: R_{ik} \rightarrow \text{St}(R)$ by $y_{ik}(ab) = y_{ik}^j(a \otimes b)$ for $a \in R_{ij}$ and $b \in R_{jk}$ using Lemma 2 in the case $S = \{1\}$. The relations (St1) and (St2) for y_{ij} are trivial, the relation (St3) follows

again by Lemma 1 in the case $S = \{1\}$. Let ${}^gx_{ij}(a) = y_{ij}(a)$, this is the endomorphism of $\text{St}(R)$ making st g -invariant.

Finally, note that the action of $\text{GL}(R)$ on $\text{St}(R)$ is consistent with the action of $\text{St}(R) \rtimes \text{D}(R)$ by uniqueness. Hence st is a crossed module. $\square$

It is known that $\text{St}(R)$ is centrally closed for any unital ring R with a complete family of full orthogonal idempotents $e_1, \dots, e_n$ for $n \geq 5$. The proof in the matrix case is written, for example, in [8], theorem 5.10 and it actually works in the general case. Hence under the assumptions of Theorem 2 the group $\text{St}(R)$ is the universal central extension of $\text{E}(R)$ for $n \geq 5$.

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