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Fourier space derivation of the demagnetization tensor for uniformly magnetized objects of cylindrical symmetry

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ABSTRACT

We derive an analytical solution for the demagnetization tensor of a cylinder of finite size and uniform magnetization in both Cartesian and cylindrical coordinate systems, by employing a recently developed Fourier space approach. Simple arguments are given to show that a previously published solution using this approach is incomplete, and a detailed and comprehensive derivation containing the necessary corrections is given. The corrected result is shown to be in agreement with older analytical solutions based on an electrostatic potential approach. We subsequently expand our solution to objects of general cylindrical symmetry and derive an expression for the demagnetization tensor that involves a single remaining integral over a finite domain, enabling the use of conventional numerical tools to efficiently calculate the demagnetization tensor.

1. Introduction

Demagnetization effects can be of great importance in magnetism both because the demagnetizing field is associated with an important energy contribution and also because the stray field from a magnetic material has the potential to affect its surroundings. However, demagnetizing fields inside magnetized objects are difficult to calculate unless the sample is ellipsoidal in shape, since only then the demagnetizing field is uniform [1]. This simple scenario includes the sphere, the flat plate, and the infinitely long cylinder as subsets. In other geometries an analytical solution is not so forthcoming. Though a rather unwieldy method for analytical solutions was outlined some time ago [2], a recently developed Fourier space approach [3] has extended the range of analytically solvable geometries in a much more general and elegant fashion (for examples see [4–9]). This ingenious approach has been used to calculate the demagnetization tensor of a uniformly magnetized cylinder [10], but we found the published solution to contain errors and room for a significant extension. We therefore present a comprehensive rederivation of the demagnetization tensor that closely follows the original argument while accounting for the necessary corrections (these are detailed in Appendix B). We then compare our new result to published solutions based on the electrostatic potential approach [2,11]. Subsequently, we extend the treatment to uniformly magnetized objects of general cylindrical symmetry and show that the demagnetization tensor in this case can be written in terms of a single integral over the length of the object.

2. Fourier space approach

Assuming a uniform magnetization of an arbitrarily shaped object, Beleggia and De Graef [3] showed that the position-dependent demagnetization tensor $N_{ij}(\boldsymbol{r})$ of the object can be elegantly expressed as an inverse Fourier transform of an expression involving the Fourier transform of a single function $D(\boldsymbol{r})$ called the shape function, which encodes the shape of the object, taking the value 1 if $\boldsymbol{r}$ is inside the object and 0 if $\boldsymbol{r}$ is outside. The demagnetization tensor is then given by

$$N_{ij}(\boldsymbol{r}) = \frac{1}{(2\pi)^3}\int d^3\boldsymbol{k}\,\frac{D(\boldsymbol{k})}{k^2}k_i k_j e^{i\boldsymbol{k}\cdot\boldsymbol{r}}, \tag{1}$$

where $D(\boldsymbol{k})$ is the Fourier transform of $D(\boldsymbol{r})$. Note that Eq. (1) implies that the demagnetization tensor is symmetric ($N_{ij} = N_{ji}$). Additionally, it can be shown that the trace of the tensor has to satisfy

$$\mathrm{Tr}[N_{ij}(\boldsymbol{r})] = D(\boldsymbol{r}), \tag{2}$$

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which we use in our discussion below to verify the computation of the demagnetization tensor.

In the following derivation we will closely follow the approach given in [10] for computing the demagnetization tensor. We will present a set of integrals and their solutions with comparisons to known standard integrals whenever necessary, which are reproduced in Appendix A for completeness.

Furthermore, we employ the following conversion between Cartesian and cylindrical coordinates for coordinates in real space ($\boldsymbol{r}$) and reciprocal space ($\boldsymbol{k}$):

$$\boldsymbol{r} = (x, y, z) = (r\cos\theta, r\sin\theta, z), \quad d^3\mathbf{r} = r dr\, d\theta\, dz,$$
$$\boldsymbol{k} = (k_x, k_y, k_z) = (k_\perp\cos\phi, k_\perp\sin\phi, k_z), \quad d^3\mathbf{k} = k_\perp dk_\perp\, d\phi\, dk_z. \tag{3}$$

3. Shape amplitude in cylindrical symmetry

An object of general cylindrical symmetry is shown in Fig. 1. Defining the multidimensional Heaviside Theta function

$$H_\theta(x_1, x_2, \ldots) = \begin{cases} 0, & \text{if } \textit{any} \text{ of the } x_i < 0 \\ 1, & \text{otherwise} \end{cases}, \tag{4}$$

the shape amplitude $D(\boldsymbol{k})$ of the object can be evaluated as follows:

$$D(\mathbf{k}) = \int d^3\mathbf{r}\, D(\mathbf{r}) e^{-i\mathbf{k}\cdot\mathbf{r}} = \int d^3\mathbf{r}\, H_\theta(r - r_1(z), r_2(z) - r, d - |z|) e^{-i\mathbf{k}\cdot\mathbf{r}} \tag{5}$$

$$\begin{aligned} D(\mathbf{k}) &= \int_{-d}^{d} dz\, e^{-izk_z} \int_{r_1(z)}^{r_2(z)} dr\, r \int_0^{2\pi} d\theta\, e^{-irk_\perp\cos(\theta-\phi)} \\ &= \int_{-d}^{d} dz\, e^{-izk_z} \int_{r_1(z)}^{r_2(z)} dr\, r 2\pi J_0(|rk_\perp|) \end{aligned} \tag{6}$$

$$D(\mathbf{k}) = \int_{-d}^{d} dz\, e^{-izk_z} \frac{2\pi}{k_\perp} [f_2(k_\perp, z) - f_1(k_\perp, z)], \tag{7}$$

where $f_i(k_\perp, z) = r_i(z) J_1(r_i(z) k_\perp)$. The functions J_i are the Bessel functions of the first kind and we made use of the definition of J_0 and Eq. (5.52.1) in [12]. Note that because $r \ge 0$ and $k_\perp \ge 0$ we removed the magnitude signs in the Bessel functions' arguments. For a finite cylinder we can use $r_1 \to 0$ and $r_2 \to R$ to arrive at

$$D(\mathbf{k}) \to \frac{4\pi R \sin(dk_z) J_1(Rk_\perp)}{k_\perp k_z}, \tag{8}$$

in agreement with Eq. (57) in [10]. However, it turns out to be easier to work with Eq. (7) to compute the inverse Fourier transforms contained in the expression for $N_{ij}(\mathbf{r})$.

The functional form of Eq. (7) shows that the object of general cylindrical symmetry can equivalently be described using the shape functions of two simply connected (no holes) cylindrical objects. This is depicted in Fig. 1 and represents an example of decomposing the shape function of an object into a combination of simpler shape functions and thereby "build" the object of interest. In this case, the two emerging terms in Eq. (7) are functionally identical and therefore in the following discussion we need to only restrict ourselves to one term of this form, where we will denote the function $f_i(k_\perp, z)$ simply by $f(k_\perp, z) \equiv R(z) J_1(R(z) k_\perp)$.

4. Demagnetization tensor

Combining Eqs. (1) and (7) yields

$$\begin{aligned} N_{ij}(\mathbf{r}') = \frac{1}{4\pi^2} \int_{-d}^{d} dz \int_0^\infty dk_\perp f(k_\perp, z) \\ \int_0^{2\pi} d\phi\, e^{ik_\perp r'\cos(\phi-\theta')} \int_{-\infty}^{\infty} dk_z \frac{k_i k_j}{k_\perp^2 + k_z^2} e^{ik_z(z'-z)}. \end{aligned} \tag{9}$$

Inserting the Cartesian components for k_i and k_j the six unique Cartesian tensor elements become

$$\begin{pmatrix} N_{xx} \\ N_{yy} \\ N_{zz} \\ N_{xy} \\ N_{xz} \\ N_{yz} \end{pmatrix}(\mathbf{r}') = \frac{1}{4\pi^2} \int_{-d}^{d} dz \int_0^\infty dk_\perp f(k_\perp, z) \int_0^{2\pi} d\phi\, e^{ik_\perp r'\cos(\phi-\theta')} \begin{pmatrix} \cos^2\phi \\ \sin^2\phi \\ 1 \\ \cos\phi\sin\phi \\ \cos\phi \\ \sin\phi \end{pmatrix} \int_{-\infty}^{\infty} dk_z \frac{e^{ik_z(z'-z)}}{k_\perp^2 + k_z^2} \begin{pmatrix} k_\perp^2 \\ k_\perp^2 \\ k_z^2 \\ k_\perp^2 \\ k_\perp k_z \\ k_\perp k_z \end{pmatrix}, \tag{10}$$

in agreement with Eq. (37) in [10]. We will now deal with the integrals over ϕ and k_z in turn.

4.1. Integrals over ϕ

The integrals over ϕ can be solved by employing trigonometric addition formulae and using the standard integral with equation number (3.915.2) in [12]. In addition, we utilize the fact that we are free to choose the limits of the integral range, as long as the lower and upper integral limits differ by 2π. Without loss of generality we can thus define the limits to be ψ and $\psi + 2\pi$, where we choose the constant ψ to have the value $\psi = \theta' - \pi$ in order to symmetrize the limits of the final integrals. Defining the variable $\varphi = \phi - \theta'$ the integrals can all be solved by the same procedure as follows:

$$\begin{aligned} xy: \quad & \int_\psi^{\psi+2\pi} d\phi\, e^{ik_\perp r'\cos(\phi-\theta')} \cos\phi\sin\phi \\ &= \frac{1}{2}\int_{-\pi}^{\pi} d\varphi\, e^{ik_\perp r'\cos\varphi} \sin(2\varphi + 2\theta') = -\pi\sin(2\theta') J_2(k_\perp r') \end{aligned} \tag{11}$$

$$\begin{aligned} xx: \quad & \frac{1}{2}\int_{-\pi}^{\pi} d\varphi\, e^{ik_\perp r'\cos\varphi}(1 + \cos(2\varphi + 2\theta')) \\ &= \pi(J_0(r'k_\perp) - \cos(2\theta') J_2(r'k_\perp)) \end{aligned} \tag{12}$$

$$yy:\quad \frac{1}{2}\int_{-\pi}^{\pi} \mathrm{d}\varphi\, e^{ik_\perp r'\cos\varphi}(1-\cos(2\varphi+2\theta')) \\ = \pi(J_0(r'k_\perp)+\cos(2\theta')J_2(r'k_\perp)) \tag{13}$$

$$zz:\quad \int_{-\pi}^{\pi} \mathrm{d}\varphi\, e^{ik_\perp r'\cos\varphi} = 2\pi J_0(r'k_\perp) \tag{14}$$

$$xz:\quad \int_{-\pi}^{\pi} \mathrm{d}\varphi\, e^{ik_\perp r'\cos\varphi}\cos(\varphi+\theta') = i2\pi\cos(\theta')J_1(r'k_\perp) \tag{15}$$

$$yz:\quad \int_{-\pi}^{\pi} \mathrm{d}\varphi\, e^{ik_\perp r'\cos\varphi}\sin(\varphi+\theta') = i2\pi\sin(\theta')J_1(r'k_\perp). \tag{16}$$

This leads to Eq. (10) becoming:

$$\begin{pmatrix} N_{xx} \\ N_{yy} \\ N_{zz} \\ N_{xy} \\ N_{xz} \\ N_{yz} \end{pmatrix}(\mathbf{r}') = \frac{1}{4\pi}\int_{-d}^{d}\mathrm{d}z \int_0^\infty \mathrm{d}k_\perp f(k_\perp, z) \begin{pmatrix} k_\perp^2(J_0(r'k_\perp)-\cos 2\theta' J_2(r'k_\perp)) \\ k_\perp^2(J_0(r'k_\perp)+\cos 2\theta' J_2(r'k_\perp)) \\ 2J_0(k_\perp r') \\ -k_\perp^2 \sin 2\theta' J_2(r'k_\perp) \\ i2k_\perp\cos\theta' J_1(r'k_\perp) \\ i2k_\perp \sin\theta' J_1(r'k_\perp) \end{pmatrix} \int_{-\infty}^{\infty}\mathrm{d}k_z \frac{e^{ik_z(z'-z)}}{k_\perp^2+k_z^2}\begin{pmatrix} 1 \\ 1 \\ k_z^2 \\ 1 \\ k_z \\ k_z \end{pmatrix}. \tag{17}$$

It should be noted that in all the above formulae we again used $|r'k_\perp| = r'k_\perp$ since $k_\perp \geq 0$ and $r' \geq 0$.

4.2. Integrals over k_z

For the k_z integrals the authors of [10] report the use of the standard integrals (3.723.2), (3.723.3) and (3.738.2) in [12] to arrive at their solution. Unfortunately, the convergence criteria of these standard integrals, as given in [12], are not always fulfilled in this problem. It turns out that for the integrals containing 1 and k_z in the numerator it does not make a difference, whereas for the integral containing k_z^2 an extra term appears, i.e. the authors of [10] used (3.738.2) in the case $m = 2n + 1$ (with $n=1$, $m=3$) which is outside the domain of applicability of this standard integral. We will now solve these integrals, while taking particular care with their convergence.

The first integral with $k_\perp^0 = 1$ in the numerator can be solved by using the standard integral (3.723.2) in [12], in the case that $\mathrm{Re}(k_\perp) > 0$ and $(z'-z) \geq 0$. The latter restriction can be resolved by writing $(z'-z) = -(z-z') = -|z'-z|$ for the case that $(z'-z) < 0$. For $k_\perp > 0$ the integral evaluates to

$$k_z^0:\ G_0 \equiv \int_{-\infty}^{\infty}\mathrm{d}k_z \frac{e^{ik_z(z'-z)}}{k_\perp^2+k_z^2} = \frac{\pi}{k_\perp}e^{-|z'-z|k_\perp} \\ = \frac{\pi}{k_\perp}\begin{cases} e^{-(z'-z)k_\perp}, & z'-z \geq 0 \\ e^{+(z'-z)k_\perp}, & z'-z < 0 \end{cases}. \tag{18}$$

However, the restriction of $k_\perp > 0$ requires attention now, because in the limit of $k_\perp \to 0$, which is covered by the range of the integral over $k_\perp$ as shown in Eq. (17), this result clearly diverges. To resolve this we have to go back to Eq. (17) and include all the functions that contain factors of $k_\perp$. We use the expansions of the Bessel functions for small arguments: $\lim_{x\to 0} J_\nu(x) = c_\nu x^\nu + \ldots$, from Eq. (9.1.10) in [13], where c_ν are simple constants. Thus, we can find the limit for the terms including $k_\perp$

$$\lim_{k_\perp\to 0}\frac{e^{ik_z(z'-z)}}{k_\perp^2+k_z^2}J_1(Rk_\perp)k_\perp^2(c_{ij}J_0(r'k_\perp)+d_{ij}J_2(r'k_\perp)) \\ = \lim_{k_\perp\to 0}\frac{e^{ik_z(z'-z)}}{k_\perp^2+k_z^2}(Ck_\perp^3+\mathrm{O}(k_\perp^5)) = 0, \tag{19}$$

where c_{ij}, d_{ij} and C are constants. Therefore, we conclude that in our case we can safely use the integral (18), as the case of $k_\perp = 0$ is assured to give no contribution.

The next integral with k_z in the numerator can be solved in two ways. The first, and slightly longer route, is to use the standard integral in [12] with number (3.723.3) and combine it with standard integral (3.721.1) for the special case $k_\perp = 0$. A second approach is to apply Leibniz's Theorem for the differentiation of an integral, $\frac{\mathrm{d}}{\mathrm{d}c}\int_{a(c)}^{b(c)} f(x,c)\,\mathrm{d}x = \int_{a(c)}^{b(c)}\frac{\partial}{\partial c}f(x,c)\,\mathrm{d}x + f(b,c)\frac{\mathrm{d}b}{\mathrm{d}c} - f(a,c)\frac{\mathrm{d}a}{\mathrm{d}c}$, e.g. from Eq. (3.3.7) in [13], to differentiate the earlier integral (18) with respect to $(z'-z)$ as follows:

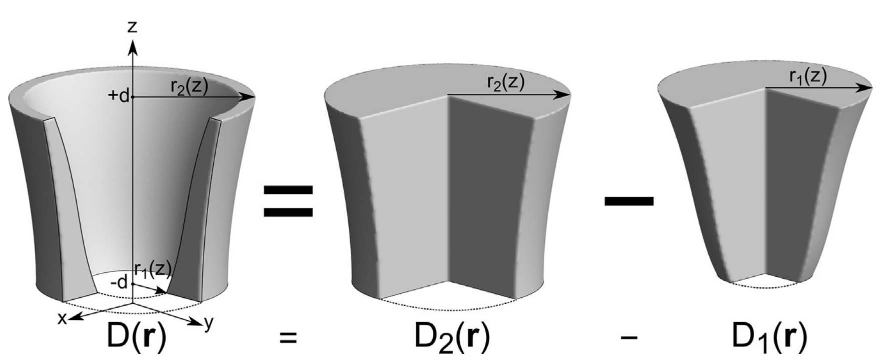

Fig. 1. Schematic of a cylindrically symmetric object and the decomposition into simpler shape functions.

$$k_z^1\!: G_1 \equiv \int_{-\infty}^{\infty} \mathrm{d}k_z \frac{k_z e^{ik_z(z'-z)}}{k_\perp^2 + k_z^2} = \frac{\mathrm{d}G_0}{i\mathrm{d}(z'-z)}$$
$$= i\pi\left(e^{-(z'-z)k_\perp} H_\theta(z'-z) - e^{(z'-z)k_\perp} H_\theta(z-z')\right), \tag{20}$$

Note that the result vanishes for $z' - z = 0$, which is easily verified due to the odd symmetry of the integral in that case. We have also written the result in terms of Heaviside Theta functions.

The final integral with k_z^2 in the numerator can now be solved by differentiating (20) again:

$$k_z^2\!: G_2 \equiv \int_{-\infty}^{\infty} \mathrm{d}k_z \frac{k_z^2 e^{ik_z(z'-z)}}{k_\perp^2 + k_z^2} = \frac{\mathrm{d}G_1}{i\mathrm{d}(z'-z)}$$
$$= \pi\left[e^{-(z'-z)k_\perp}\left(-k_\perp H_\theta(z'-z) + \frac{\mathrm{d}H_\theta(z'-z)}{\mathrm{d}(z'-z)}\right) - e^{(z'-z)k_\perp}\left(k_\perp H_\theta(z-z') - \frac{\mathrm{d}H_\theta(z-z')}{\mathrm{d}(z-z')}\right)\right]$$
$$= -\pi k_\perp\left(e^{-(z'-z)k_\perp} H_\theta(z'-z) + e^{(z'-z)k_\perp} H_\theta(z-z')\right) + 2\pi\delta(z'-z)e^{-|z'-z|k_\perp}. \tag{21}$$

In the last step we used $\mathrm{d}H_\theta(x)/\mathrm{d}x = \delta(x)$, see for example Chapter 5 in [14]. The crucial difference between the results above and those reported in [10] is the extra delta-function term that appears in (21), which we show below is essential in satisfying the condition on the trace (Eq. (2)) of the demagnetization tensor. With these integrals the expression for N_{ij} becomes

$$\begin{pmatrix} N_{xx} \\ N_{yy} \\ N_{zz} \\ N_{xy} \\ N_{xz} \\ N_{yz} \end{pmatrix}(\mathbf{r}') = \frac{1}{4}\int_{-d}^{d} \mathrm{d}z \int_0^\infty \mathrm{d}k_\perp f(k_\perp, z) k_\perp e^{-|z'-z|k_\perp} \begin{pmatrix} J_0(r'k_\perp) - \cos 2\theta' J_2(r'k_\perp) \\ J_0(r'k_\perp) + \cos 2\theta' J_2(r'k_\perp) \\ -2J_0(r'k_\perp)(1 - 2\delta(z'-z)/k_\perp) \\ -\sin 2\theta' J_2(r'k_\perp) \\ -2\cos\theta' J_1(r'k_\perp)\mathrm{sgn}(z'-z) \\ -2\sin\theta' J_1(r'k_\perp)\mathrm{sgn}(z'-z) \end{pmatrix} \tag{22}$$

$$\begin{pmatrix} N_{xx} \\ N_{yy} \\ N_{zz} \\ N_{xy} \\ N_{xz} \\ N_{yz} \end{pmatrix}(\mathbf{r}') = \int_0^\infty \mathrm{d}k_\perp \int_{-d}^{d} \mathrm{d}z\, f(k_\perp, z) e^{-|z'-z|k_\perp}\Big[k_\perp \bar{S}_{ij}(k_\perp, r', \theta')\xi_{ij}^{\mathrm{sgn}(z-z')}$$
$$+ \delta_{i3}\delta_{j3} J_0(r'k_\perp)\delta(z'-z)\Big]. \tag{23}$$

(Note we are *not* employing the Einstein summation convention here.) The following symbols have been defined:

$$\bar{S}_{ij}(k_\perp, r', \theta') \equiv \sum_{\mu=0}^{2} \bar{\alpha}_{ij}^{\mu}(\theta') J_\mu(r'k_\perp') \tag{24}$$

$$\bar{\alpha}_{ij}^{0}(\theta') \equiv \frac{1}{4}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad \bar{\alpha}_{ij}^{1}(\theta') \equiv \frac{1}{2}\begin{pmatrix} 0 & 0 & \cos\theta' \\ 0 & 0 & \sin\theta' \\ \cos\theta' & \sin\theta' & 0 \end{pmatrix},$$
$$\xi_{ij}^{\pm} \equiv \begin{pmatrix} 1 & -1 & \pm 1 \\ -1 & 1 & \pm 1 \\ \pm 1 & \pm 1 & -1 \end{pmatrix}, \quad \bar{\alpha}_{ij}^{2}(\theta') \equiv \frac{1}{4}\begin{pmatrix} -\cos 2\theta' & \sin 2\theta' & 0 \\ \sin 2\theta' & \cos 2\theta' & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{25}$$

5. Trace condition

Substituting the expression for $f(k_\perp, z) = R(z)J_1(R(z)k_\perp)$ into Eq. (23) enables the evaluation of the two remaining integrals over z and $k_\perp$. However, it is first instructive to show that Eq. (23) satisfies the trace condition (Eq. (2)). Applying $\mathrm{Tr}[aM + bN] = a\,\mathrm{Tr}[M] + b\,\mathrm{Tr}[N]$ for scalars a, b and matrices M, N gives us

$$\mathrm{Tr}[N_{ij}(\mathbf{r}')] = \sum_{\mu=0}^{2} g^{\mu}(r', z')\mathrm{Tr}[\bar{\alpha}_{ij}^{\mu}(\theta')\xi_{ij}^{\mathrm{sgn}(z-z')}]$$
$$+ \int_{-d}^{d} \mathrm{d}z\, \delta(z'-z) \int_0^\infty \mathrm{d}k_\perp\, R(z)J_1(R(z)k_\perp)J_0(k_\perp r') \tag{26}$$

$$\mathrm{Tr}[N_{ij}(\mathbf{r}')] = H_\theta(R(z) - r', d - |z'|) = D(\mathbf{r}'). \tag{27}$$

The scalar functions $g^{\mu}(r', z')$ represent the integrals involving the $\bar{S}_{ij}$ term in Eq. (23). The first term of Eq. (26) is trivially zero by the structure of the matrices defined in Eq. (25), whereas for the second term the standard integral (6.512.3[8]) from [12] has been utilized.

6. Finite cylinder case – remaining integrals

The integral over $k_\perp$ in Eq. (23) can be solved in full generality, as done in Section 8. However, for the case of a finite cylinder, for which $R(z) = R$, it turns out to be easier to first integrate over z. Substituting $f(k_\perp) = RJ_1(Rk_\perp)$ into Eq. (23) and using the result from combining Eqs. (26) and (27) leads to

$$N_{ij}(\mathbf{r}') = \delta_{i3}\delta_{j3}D(\mathbf{r}') + \bar{N}_{ij}(\mathbf{r}')$$
$$= \delta_{i3}\delta_{j3}D(\mathbf{r}') + \int_0^\infty \mathrm{d}k_\perp \int_{-d}^{d} \mathrm{d}z\, Rk_\perp J_1(Rk_\perp)\bar{S}_{ij}(k_\perp, r', \theta')$$
$$\xi_{ij}^{\mathrm{sgn}(z-z')} e^{-|z'-z|k_\perp}. \tag{28}$$

The remaining z-integral can now be evaluated in the different regions, as follows:

$$z' < -d\!: \quad \int_{-d}^{d} \mathrm{d}z\, \xi_{ij}^{+} e^{(z'-z)k_\perp} = 2\xi_{ij}^{+} e^{z'k_\perp} \frac{\sinh(dk_\perp)}{k_\perp}, \tag{29}$$

$$z' > d\!: \quad \int_{-d}^{d} \mathrm{d}z\, \xi_{ij}^{-} e^{-(z'-z)k_\perp} = 2\xi_{ij}^{-} e^{-z'k_\perp} \frac{\sinh(dk_\perp)}{k_\perp}, \tag{30}$$

$$-d < z' < d\!: \quad \int_{-d}^{z'} \mathrm{d}z\, \xi_{ij}^{-} e^{-(z'-z)k_\perp} + \int_{z'}^{d} \mathrm{d}z\, \xi_{ij}^{+} e^{(z'-z)k_\perp}$$
$$= \frac{\xi_{ij}^{-}}{k_\perp}\left(1 - e^{-(d+z')k_\perp}\right) + \frac{\xi_{ij}^{+}}{k_\perp}\left(1 - e^{-(d-z')k_\perp}\right). \tag{31}$$

Therefore, we find that

$$\bar{N}_{ij}(\mathbf{r}') = \sum_{\mu=0}^{2} \bar{\alpha}_{ij}^{\mu}(\theta') \int_0^\infty \mathrm{d}K\, J_1(K)J_\mu(\rho K) \begin{cases} \xi_{ij}^{-}[e^{-(\zeta-\tau)K} - e^{-(\zeta+\tau)K}], & \zeta > \tau \\ \xi_{ij}^{-}(1 - e^{-(\tau+\zeta)K}) + \xi_{ij}^{+}(1 - e^{-(\tau-\zeta)K}), & |\zeta| < \tau \\ \xi_{ij}^{+}[e^{(\zeta+\tau)K} - e^{(\zeta-\tau)K}], & \zeta < -\tau \end{cases}, \tag{32}$$

where we introduced a new set of coordinates scaled by the radius R of the cylinder:

$$\rho \equiv \frac{r'}{R}, \quad \zeta \equiv \frac{z'}{R}, \quad K \equiv Rk_\perp,$$
$$\rho K = r'k_\perp, \quad \tau = \frac{d}{R} = \text{aspect ratio of cylinder}. \tag{33}$$

The remaining integrals (which are of Lipschitz–Hankel type) can be written as

$$I_\mu(\rho, \alpha) \equiv \int_0^\infty \mathrm{d}K\, J_1(K) J_\mu(K\rho) e^{-\alpha K}. \tag{34}$$

Defining the quantities α_- and α_+ by (recall that $\tau = d/R > 0$)

$$\alpha_- \equiv |\zeta - \tau| = \begin{cases} \zeta - \tau, & \zeta > \tau \\ \tau - \zeta, & |\zeta| < \tau \\ \tau - \zeta, & \zeta < -\tau \end{cases}, \quad \alpha_+ \equiv |\zeta + \tau| = \begin{cases} \zeta + \tau, & \zeta > \tau \\ \zeta + \tau, & |\zeta| < \tau \\ -(\zeta + \tau), & \zeta < -\tau \end{cases}, \tag{35}$$

enables us to rewrite the demagnetization tensor in the following way:

$$N_{ij}(\rho, \theta', \zeta) = \delta_{i3}\delta_{j3} D(\rho, \theta', \zeta) + \sum_{\mu=0}^{2} \bar{\alpha}_{ij}^{\mu}(\theta') \begin{cases} \xi_{ij}^{-}(I_\mu(\rho, \alpha_-) - I_\mu(\rho, \alpha_+)), & \zeta > \tau \\ \xi_{ij}^{-}(I_\mu(\rho, 0) - I_\mu(\rho, \alpha_+)) + \xi_{ij}^{+}(I_\mu(\rho, 0) - I_\mu(\rho, \alpha_-)), & |\zeta| < \tau \\ \xi_{ij}^{+}(I_\mu(\rho, \alpha_+) - I_\mu(\rho, \alpha_-)), & \zeta < -\tau \end{cases}. \tag{36}$$

Note again that we are *not* employing the Einstein summation convention here.

6.1. Solutions to Lipschitz–Hankel Integrals I_μ

The integrals in Eq. (34) have known analytical solutions, which can be either taken from [15] or [16] (see section (2.12.38.1) in [16]). It should be noted that because we are interested in the case of $\mu = \{0, 1, 2\}$, we are in the regime where the integrals always converge except at the boundary points where $\rho = 1$ and $\alpha_\pm = 0$ (see the introduction section in [15]), which correspond to the rims of the flat cylinder surfaces.

Let us first introduce the notation used in [15] for the integrals at hand:

$$I(\mu, \nu; \lambda) = \int_0^\infty J_\mu(at) J_\nu(bt) e^{-ct} t^\lambda \, \mathrm{d}t. \tag{37}$$

6.2. Integral I_0

For this integral we identify $a=1$, $b = \rho$, $c = \alpha_\pm$ and combine Eqs. (4.7) and (3.4) in [15] to obtain

$$I_0(\rho, \alpha_\pm) = I(1, 0; 0) = \begin{cases} -\dfrac{k_\pm \alpha_\pm}{2\pi\sqrt{\rho}} K(m_\pm) - \dfrac{1}{2}\Lambda_0(\beta_\pm, \kappa_\pm) + 1, & \rho < 1 \\ -\dfrac{k_\pm \alpha_\pm}{2\pi\sqrt{\rho}} K(m_\pm) + \dfrac{1}{2}, & \rho = 1 \\ -\dfrac{k_\pm \alpha_\pm}{2\pi\sqrt{\rho}} K(m_\pm) + \dfrac{1}{2}\Lambda_0(\beta_\pm, \kappa_\pm), & \rho > 1 \end{cases}, \tag{38}$$

where we defined

$$m_\pm = k_\pm^2 = \sin^2(\kappa_\pm) = \frac{4\rho}{(\rho + 1)^2 + \alpha_\pm^2}, \tag{39}$$

$$\beta_\pm = \sin^{-1}\left(\frac{\alpha_\pm}{\sqrt{(\rho - 1)^2 + \alpha_\pm^2}} \right), \tag{40}$$

$$\Lambda_0(\beta, \kappa) = \frac{2}{\pi}(K(m)E(\beta, \pi/2 - \kappa) - (K(m) - E(m))F(\beta, \pi/2 - \kappa)). \tag{41}$$

The functions $K(m)$ and $E(m)$ are the complete elliptic integrals of the first and second kind, respectively. The (incomplete) elliptic integrals, $F(\beta, \gamma)$ and $E(\beta, \gamma)$, of the first and second kind, respectively, are used to define Heuman's Lambda Function $\Lambda_0(\beta, \kappa)$ (see Section 17.4.39 in [13]).

6.3. Integral I_1

After making the identification $a=1$, $b = \rho$, $c = \alpha_\pm$ we can combine Eqs. (4.2) and (3.4) from [15] to arrive at

$$I_1(\rho, \alpha_\pm) = I(1, 1; 0) = \frac{1}{\pi k_\pm \sqrt{\rho}}((2 - m_\pm)K(m_\pm) - 2E(m_\pm)). \tag{42}$$

6.4. Integral I_2

For this integral we need to make use of some of the recurrence relations reported in [15]. In this case we are interested in the integral $I(2, 1; 0)$ with the identification $a = \rho$, $b=1$, $c = \alpha_\pm$. Using Eqs. (8.1) and (9.5) from [15] we obtain

$$I(2, 1; 0) = \frac{b}{a} I(1, 0; 0) - \frac{c}{a} I(1, 1; 0) \tag{43}$$

Substituting from Eqs. (4.2), (4.7) and (3.4) in [15] and rearranging yields

$$I_2(\rho, \alpha_\pm) = I(2, 1; 0) = \begin{cases} \dfrac{2\alpha_\pm}{\pi k_\pm \rho^{3/2}} E(m_\pm) - \dfrac{\alpha_\pm k_\pm(\alpha_\pm^2 + \rho^2 + 2)}{2\pi\rho^{5/2}} K(m_\pm) - \dfrac{1}{2\rho^2}\Lambda_0(\beta_\pm, \kappa_\pm) + \dfrac{1}{\rho^2}, & \rho > 1 \\ \dfrac{2\alpha_\pm}{\pi k_\pm \rho^{3/2}} E(m_\pm) - \dfrac{\alpha_\pm k_\pm(\alpha_\pm^2 + \rho^2 + 2)}{2\pi\rho^{5/2}} K(m_\pm) - \dfrac{1}{2\rho^2}, & \rho = 1 \\ \dfrac{2\alpha_\pm}{\pi k_\pm \rho^{3/2}} E(m_\pm) - \dfrac{\alpha_\pm k_\pm(\alpha_\pm^2 + \rho^2 + 2)}{2\pi\rho^{5/2}} K(m_\pm) + \dfrac{1}{2\rho^2}\Lambda_0(\beta_\pm, \kappa_\pm), & \rho < 1 \end{cases}. \tag{44}$$

7. Transformation to cylindrical coordinate system

So far we have expressed the demagnetization tensor in a Cartesian coordinate system, as in Eqs. (23) and (36). The transformation matrix $\mathbf{M}$ between a Cartesian and a cylindrical coordinate system is

$$\begin{pmatrix} \hat{\boldsymbol{r}} \\ \hat{\boldsymbol{\theta}} \\ \hat{\boldsymbol{z}} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{\boldsymbol{x}} \\ \hat{\boldsymbol{y}} \\ \hat{\boldsymbol{z}} \end{pmatrix} \equiv \mathbf{M} \begin{pmatrix} \hat{\boldsymbol{x}} \\ \hat{\boldsymbol{y}} \\ \hat{\boldsymbol{z}} \end{pmatrix}. \tag{45}$$

The demagnetization tensor in the cylindrical coordinate system is then

$$N_{cyl} = \begin{pmatrix} N_{rr} & N_{r\theta} & N_{rz} \\ N_{r\theta} & N_{\theta\theta} & N_{\theta z} \\ N_{rz} & N_{\theta z} & N_{zz} \end{pmatrix} = \mathbf{M} N_{cart} \mathbf{M}^{\top} = \mathbf{M} \begin{pmatrix} N_{xx} & N_{xy} & N_{xz} \\ N_{xy} & N_{yy} & N_{yz} \\ N_{xz} & N_{yz} & N_{zz} \end{pmatrix} \mathbf{M}^{\top}, \tag{46}$$

where we used the symmetry requirement $N_{ij} = N_{ji}$. Substituting the Cartesian components from (23) and simplifying the resulting expression yields the simple and intuitive relation

$$N_{cyl} = N_{cyl}(r, z) = N_{cart}|_{\theta=0} = \left. \begin{pmatrix} N_{xx} & 0 & N_{xz} \\ 0 & N_{yy} & 0 \\ N_{xz} & 0 & N_{zz} \end{pmatrix} \right|_{\theta=0}. \tag{47}$$

This holds for the case of an object of general cylindrical symmetry. In the specific scenario of a finite cylinder, the non-zero components of the tensor in the reduced cylindrical coordinate system are

$$N_{rr}(\rho, \zeta) = \frac{1}{4}[2(I_0(\rho, 0) - I_2(\rho, 0))H_\theta(\tau - |\zeta|) + (I_0(\rho, \alpha_-) - I_2(\rho, \alpha_-))(H_\theta(\zeta - \tau) - H_\theta(\tau - \zeta)) + (I_0(\rho, \alpha_+) - I_2(\rho, \alpha_+))(H_\theta(-\tau - \zeta) - H_\theta(\tau + \zeta))] \tag{48}$$

$$N_{\theta\theta}(\rho, \zeta) = \frac{1}{4}[2(I_0(\rho, 0) + I_2(\rho, 0))H_\theta(\tau - |\zeta|) + (I_0(\rho, \alpha_-) + I_2(\rho, \alpha_-))(H_\theta(\zeta - \tau) - H_\theta(\tau - \zeta)) + (I_0(\rho, \alpha_+) + I_2(\rho, \alpha_+))(H_\theta(-\tau - \zeta) - H_\theta(\tau + \zeta))] \tag{49}$$

$$N_{zz}(\rho, \zeta) = D(\rho, \zeta) - \frac{1}{2}[2I_0(\rho, 0)H_\theta(\tau - |\zeta|) + I_0(\rho, \alpha_-)(H_\theta(\zeta - \tau) - H_\theta(\tau - \zeta)) + I_0(\rho, \alpha_+)(H_\theta(-\tau - \zeta) - H_\theta(\tau + \zeta))] \tag{50}$$

$$N_{rz}(\rho, \zeta) = \frac{1}{2}(I_1(\rho, \alpha_+) - I_1(\rho, \alpha_-)), \tag{51}$$

where expressions for $I_0(\rho, 0)$ and $I_2(\rho, 0)$ can be most easily obtained by employing equation (6.512. 3[8]) in [12]. The above results can be shown to agree with those previously published by Joseph and Schlömann [2] and Kraus [11] upon a simple translation of the coordinate system: $\zeta \to \zeta + \tau$ (equivalent to $z \to z + d$). As before, the trace condition (Eq. (2)) is satisfied with $\mathrm{Tr}[N_{cyl}(\rho, \zeta)] = D(\rho, \zeta)$.

For an excellent discussion concerning the eigenvalues and eigenvectors of the demagnetization tensor we refer the reader to Section 4.2 in [10].

8. General cylindrical case – remaining integrals

It is possible to simplify the expression for the demagnetization tensor in general cylindrical symmetry given in (23) even further. Remembering from Section 3 that we can always write N_{ij} in terms of contributions due to simply connected objects of cylindrical symmetry we only need to focus on one such simply connected object. Defining the outer radius of such an object as $R(z)$ and focussing on the non-trivial part of $N_{ij}(\mathbf{r}') = \delta_{i3}\delta_{j3}D(\mathbf{r}') + \bar{N}_{ij}(\mathbf{r}')$ we find that in the notation of Eq. (37);

$$\bar{N}_{ij}(\mathbf{r}') = \sum_{\mu=0}^{2} \bar{\alpha}_{ij}^{\mu}(\theta') \int_{-d}^{d} \mathrm{d}z\, R(z)\xi_{ij}^{\mathrm{sgn}(z-z')} \int_0^\infty \mathrm{d}k_\perp\, k_\perp J_1(R(z)k_\perp)J_\mu(r'k_\perp)e^{-|z'-z|k_\perp} \tag{52}$$

$$= \sum_{\mu=0}^{2} \bar{\alpha}_{ij}^{\mu}(\theta') \int_{-d}^{d} \mathrm{d}z\, R(z)\xi_{ij}^{\mathrm{sgn}(z-z')} I(1, \mu; 1). \tag{53}$$

Note that setting $\theta' \to 0$ yields the tensor in a cylindrical coordinate system. In the cases of $\mu = 0$ and $\mu = 1$ we can directly take the solutions for $I(1, \mu; 1)$ from Eqs. (4.8) and (4.4) in [15], respectively, whereas for $\mu = 2$ we first exploit the recurrence relation (8.2) in [15]. Defining $k^2 = 4ab/((a+b)^2 + c_\pm^2)$ with the identification $a = R(z)$, $b = r'$ and $c_\pm = \pm(z - z')$ we obtain

$$I(1, 0; 1) = \frac{k^3(a^2 - b^2 - c_\pm^2)}{8\pi a(1 - k^2)(ab)^{3/2}}E(k) + \frac{k}{2\pi a\sqrt{ab}}K(k), \tag{54}$$

$$I(1, 1; 1) = \frac{c_\pm k}{4\pi(ab)^{3/2}}\left(\frac{(2 - k^2)}{1 - k^2}E(k) - 2K(k)\right), \tag{55}$$

$$I(1, 2; 1) = \frac{2}{b}I(1, 1; 0) - I(1, 0; 1), \tag{56}$$

$$I(1, 2; 1) = \frac{8a - k^2(4a + b)}{2\pi k(ab)^{3/2}}K(k) - \frac{32a^2(1 - k^2) + k^4(a^2 - b^2 - c_\pm^2)}{8\pi(1 - k^2)ka(ab)^{3/2}}E(k). \tag{57}$$

The $I(1, \mu; 1)$ integrals converge if $c_\pm > 0$ (see the introduction section in [15]). This is true in our case, with the $c_\pm = 0$ contribution already included in the $\delta_{i3}\delta_{j3}D(\mathbf{r}')$ term. More formally we could write $\xi_{ij}^{\mathrm{sgn}(z-z')} = \xi_{ij}^{+}H_\theta(z - z') + \xi_{ij}^{-}H_\theta(z' - z)$ in (53) to emphasize this. Furthermore, the $I(1, \mu; 1)$ are finite if $k \neq 1$, which is satisfied as long as $c \neq 0$ and $a \neq b$, i.e. not on the boundary of the object.

Expression (53) is as far as the demagnetization tensor for a simply connected object of general cylindrical symmetry can be simplified without knowing its radius $R(z)$ as a function of position along the symmetry axis. The major advantage of this result is that the only remaining integral is over a finite domain, which makes it accessible to conventional numerical techniques and thereby greatly enhances its utility in practical applications.

9. Discussion and conclusions

The approach described in this paper yields analytical results that can be used to compare with numerical simulations. The cylindrical geometry is suited to the rod-like samples produced by the mirror furnace growth technique. In fact, the present study was inspired by the need to account for demagnetization fields arising in muon-spin rotation experiments on such samples [17]. Furthermore, having analytical expressions for standard simple shapes, such as boxes [2] and cylinders, allows more complex structures to be built up and hence model more complex situations such as non-uniformly magnetized systems [18].

In conclusion, we utilized a recently developed Fourier space approach to compute the demagnetization tensor of a uniformly magnetized cylinder of finite extent. Our work corrects a previously published analytical solution for this geometry and presents the steps of the rederivation in detail. Our final analytical result in a cylindrical coordinate system is in agreement with published solutions employing the rather more unwieldy electrostatic potential approach. Additionally, we have provided a solution for the case of a simply connected object of general cylindrical symmetry, which depends on only one finite integral and thus can be solved via conventional numerical methods. We have illustrated that within the Fourier space approach the demagnetization tensor of complex shapes can be obtained by combining simpler, simply connected shapes.

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Appendix A. Standard integrals and expressions

This is a compilation of the different standard integrals used here and in [10]. The numbers in brackets represent the equation numbers used in [12]:

(3.721.1): $$\int_0^\infty \frac{\sin(ax)}{x}\,\mathrm{d}x = \frac{\pi}{2}\,\mathrm{sgn}(a),\quad \text{for } a \in \mathbb{R}$$

(3.723.2): $$\int_0^\infty \frac{\cos(ax)}{\beta^2 + x^2}\,\mathrm{d}x = \frac{\pi}{2\beta}e^{-a\beta},\quad \text{for } a \geq 0,\ \mathrm{Re}(\beta) > 0$$

(3.723.3): $$\int_0^\infty \frac{x\,\sin(ax)}{\beta^2 + x^2}\,\mathrm{d}x = \frac{\pi}{2}e^{-a\beta},\quad \text{for } a > 0,\ \mathrm{Re}(\beta) > 0$$

(3.738.2): $$\int_0^\infty \frac{x^{m-1}\cos(ax)}{x^{2n} + \beta^{2n}}\,\mathrm{d}x = \frac{\pi\beta^{m-2n}}{2n}$$

$$\sum_{k=1}^{n} e^{-a\beta\sin((2k-1)\pi/2n)}$$

$$\sin\left[\frac{(2k-1)m\pi}{2n} + a\beta\cos\left(\frac{(2k-1)\pi}{2n}\right)\right],$$

$$\text{for } m = \text{odd},\ a > 0,\ |\arg(\beta)| < \frac{\pi}{2n},\ 0 < m < 2n + 1$$

(3.915.2): $$\int_0^\pi e^{i\beta\cos(x)}\cos(nx)\,\mathrm{d}x = i^n\pi J_n(\beta),\quad \text{for } n \in \mathbb{Z}$$

(5.52.1): $$\int x^{p+1}Z_p(x)\,\mathrm{d}x = x^{p+1}Z_{p+1}(x),$$

$$\text{for } Z_p = \text{a Bessel function (1st, 2nd or 3rd kind)}$$

(6.512. 3[8]): $$\int_0^\infty J_\nu(\alpha x)J_{\nu-1}(\beta x)\,\mathrm{d}x = \begin{cases} \dfrac{\beta^{\nu-1}}{\alpha^\nu}, & \beta < \alpha \\ \dfrac{1}{2\beta}, & \beta = \alpha \\ 0, & \beta > \alpha \end{cases},\quad \text{for } \mathrm{Re}(\nu) > 0.$$

The following two expressions concern the small argument expansion of Bessel functions of the first kind and Leibniz's Theorem for the differentiation of an integral. The numbers in brackets correspond to the equation numbers in [13]:

(9.1.10): $$\lim_{z\to 0} J_\nu(z) = \lim_{z\to 0}\left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{\left(\frac{-z^2}{4}\right)^k}{k!\,\Gamma(\nu + k + 1)} = c_\nu z^\nu + O(z^{\nu+2}),$$

$$\text{for } \nu = 0, 1, \ldots \text{ and } c_\nu \text{ a constant}$$

(3.3.7): $$\frac{\mathrm{d}}{\mathrm{d}c}\int_{a(c)}^{b(c)} f(x, c)\,\mathrm{d}x = \int_{a(c)}^{b(c)} \frac{\partial}{\partial c} f(x, c)\,\mathrm{d}x + f(b, c)\frac{\mathrm{d}b}{\mathrm{d}c} - f(a, c)\frac{\mathrm{d}a}{\mathrm{d}c}.$$

Appendix B. Comparison with reference [10]

B.1. Trace condition

We can compare Eq. (26) with the equivalent expression one gets from taking the trace of $N_{ij}(\mathbf{r}')$ as reported in Eq. (42) in [10]. In this case, we obtain a single term that looks exactly like the first term in Eq. (26) above. Hence, it also has to vanish due to the structure of the matrices defined in Eq. (43) in [10] and therefore lead to $\mathrm{Tr}[N_{ij}(\mathbf{r}')] = 0$ everywhere. Thus, Eq. (42) in [10] is incomplete. We conclude that the appearance of the extra delta-function in Eq. (23) is essential in ensuring that the trace of $N_{ij}(\mathbf{r}')$ is equal to the shape function $D(\mathbf{r}')$.

B.2. Finite cylinder result: Eq. (36)

The resulting expression for the demagnetization tensor of the finite cylinder in Eq. (36) differs from its counterpart in Eq. (48) in [10], which we will show now to be inconsistent with the previous result in Eq. (42) in [10]. Thus, if we take Eq. (48) in [10] and compute its trace we find that, in the notation of [10], the $\alpha_{ij}^{(0)}$ term gives the only non-vanishing contributions

$$\mathrm{Tr}[N_{ij}(\rho, \theta', \zeta)] = H_{\tau\zeta} I_0(\rho, 0) - s_{\zeta\tau} I_0(\rho, \alpha_-) - I_0(\rho, \alpha_+). \tag{B.1}$$

We note that this result is neither consistent with the trace condition $\mathrm{Tr}[N_{ij}(\mathbf{r})] = D(\mathbf{r})$ nor the result one obtains when taking the trace of the earlier expression (42) in [10] (see Appendix B.1). We conclude that the rearranging in [10] from Eqs. (42) to (48) cannot be correct.

B.3. Additional corrections

The matrix $\tilde{\alpha}_{ij}^2(\theta')$ in Eq. (25) differs from its counterpart in [10] by a factor of 2 in the xy-element, due to the integral (11) being calculated incorrectly in [10].

The three solutions to the Lipschitz–Hankel integrals I_μ reported in [10] all contain minor errors. The solutions for I_0 and I_2 in Eqs. (38) and (44), respectively, show that the subscripts of their counterparts in Eq. (50) in [10] should be swapped. The result for I_1 in Eq. (42) differs from the corresponding solution reported in [10] by a factor of 2 in front of $E(m_\pm)$.

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