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Density matrix renormalization group for 19-vertex model

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Abstract

We embody the density matrix renormalization group method for the 19-vertex model on a square lattice; the 19-vertex model is regarded to be equivalent to the XY model for small interaction. The transfer matrix of the 19-vertex model is classified by the total number of arrows incoming into one layer of the lattice. By using this property, we reduce the dimension of the transfer matrix appearing in the density matrix renormalization group method and obtain a very nice value of the conformal anomaly which are consistent with the value at the Berezinskii–Kosterlitz–Thouless transition point.

Keywords: Kosterlitz–Thouless; Renormalization group

The density matrix renormalization group method (DMRG) has been used for evaluation of eigenvalues of Hamiltonian matrix for a one-dimensional quantum system [1, 2]. This method enables us to increase the system size, within a fixed matrix size, which can be handled by recent computer resources such as the memory size and the CPU speed. Recently, the DMRG method has been applied to classical spin systems [3].

In both the cases of quantum Hamiltonian and classical transfer matrix, spin variables have a discrete degree of freedom. Therefore, one can handle matrices with finite dimension in both the cases. On the other hand, a spin system with continuous degree of freedom such as a classical XY model provides a transfer matrix whose dimension is infinite. Fortunately, it is known that the XY model on a square lattice Λ is translated into a 19-vertex model for which the transfer matrix can be described by discrete variables expressing by arrow variables [4]. The purpose of the present study is to embody the DMRG method for the 19-vertex model. We show how to reduce the dimension of matrices, which are diagonalized in the DMRG method, by using the ice rule of the 19-vertex model [5]. We obtain a very nice value of conformal anomaly, $c = 1.006(1)$, at the Berezinskii–Kosterlitz–Thouless (BKT) transition point by use of our method.

The 19 kinds of arrow configurations, i.e. vertices, are subjected to an ice rule which is generalized to include no arrow on a bond. A vertex weight $W(v_i)$ depends on the kind of the vertex $v_i \in \{1, 2, \dots, 19\}$ at a site i. The value of v_i is determined by a configuration of four arrows as

$$v_i = v_i(\alpha_i, \beta_i, \gamma_i, \delta_i), \tag{1}$$

where α_i, β_i, γ_i and δ_i denote arrows on bonds surrounding the site i. Let us express an up and a right arrow by $+1$, a down and a left arrow by -1, no arrow by 0. The ice rule is expressed using this expression as follows:

$$\alpha_i - \beta_i - \gamma_i + \delta_i = 0. \tag{2}$$

Using the weights $W(v_i)$ for the 19 vertices, we can describe the partition function Z as follows:

$$Z = \sum_{\{v_i\}}' \prod_{i \in \Lambda} W(v_i), \tag{3}$$

where the summation is taken over all permitted configurations of the vertices on the lattice Λ.

Because there exists the ice rule for the 19-vertex model, the whole transfer matrix becomes a block diagonal which is classified by the number of arrows incoming to one layer of the lattice. Using this, we can reduce the amount of calculation in the DMRG method.

We now apply the infinite DMRG method to the 19-vertex model [1, 2]. In addition to the vertex weight $W(v_i)$, we use a renormalized weight $W^{(r)}(v_i^{(r)})$, where

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$v_i^{(r)}$ means a renormalized vertex and r the number of a renormalization. As an initial value, $W^{(0)}(v_i^{(0)})$ is equal to $W(v_i)$. The transfer matrix for a given N, the number of arrows, is composed as

$$T_N^{(r)}(\eta_1, \beta_2, \eta_3, \beta_4 | \xi_1, \delta_2, \xi_3, \delta_4)$$
$$= \sum_{\alpha_1, \ldots, \alpha_4} W^{(r)}(v_1^{(r)}) W(v_2) W^{(r)}(v_3^{(r)}) W(v_4), \tag{4}$$

where the number N is obtained by

$$N = N_r(\eta_1) + \beta_2 + N_r(\eta_3) + \beta_4$$
$$= N_r(\xi_1) + \delta_2 + N_r(\xi_3) + \delta_4. \tag{5}$$

Here the number of arrows included in the renormalized vertex ξ_i is denoted by $N_r(\xi_i)$ and $N_0(\xi_i) = \delta_i$. We denote an eigenvector of this transfer matrix by $\psi_{N,k}^{(r)}(\eta_1, \beta_2, \eta_3, \beta_4)$ which corresponds to the kth eigenvalue. As the infinite DMRG method, we construct the density matrix $\hat{\rho}_{N_r(\eta_1)+\beta_2,k}^{(r)}$ as follows:

$$\rho_{N_r(\eta_1)+\beta_2,k}^{(r)}(\eta_1, \beta_2 | \xi_1, \delta_2)$$
$$\equiv \sum_{\eta_3, \beta_4} \psi_{N,k}^{(r)}(\eta_1, \beta_2, \eta_3, \beta_4) \psi_{N,k}^{(r)}(\xi_1, \delta_2, \eta_3, \beta_4). \tag{6}$$

Notice that the density matrix is labelled by $N_r(\eta_1) + \beta_2$ not by N. From Eq. (5), since $\eta_3 = \xi_3$ and $\beta_4 = \delta_4$, we have $N_r(\eta_1) + \beta_2 = N_r(\xi_1) + \delta_2$, and hence, the density matrix has a block diagonal structure classified by the total number of arrows for the half-system. In order to construct the renormalized vertex weight $W^{(r+1)}$, we diagonalize the density matrix $\hat{\rho}_{N_r(\eta_1)+\beta_2,1}$ and obtain its eigenvectors $V_{N_r(\eta_1)+\beta_2,\eta_1'}$. In the present study, we use the eigenvector of the transfer matrix for the largest eigenvalue $k = 1$. The renormalized vertex state is labelled by η_1' which means that $V_{N_r(\eta_1)+\beta_2,\eta_1'}$ corresponds to the η_1'th eigenvalue of $\hat{\rho}_{N_r(\eta_1)+\beta_2,1}$.

We determine the upper limit l of η_1' as follows:

$$l \equiv \begin{cases} 3^{r+2} & (3^{r+2} < m), \\ m & (3^{r+2} \geqslant m), \end{cases} \tag{7}$$

where m is the number of states of the density matrix which we take into account. The last step of the DMRG for the 19-vertex model is a construction of the renormalized weight for the vertex as follows:

$$W^{(r+1)}(v_1^{(r+1)}(\alpha_1, \eta_1', \alpha_3, \xi_1'))$$
$$= \sum_{\alpha_2, \eta_1, \beta_2, \xi_1, \delta_2} V_{N_r(\eta_1'),\eta_1'}(\eta_1, \beta_2) W^{(r)}(v_1^{(r)}(\alpha_1, \eta_1, \alpha_2, \xi_1))$$
$$\times W(v_2(\alpha_2, \beta_2, \alpha_3, \delta_2)) V_{N_r(\xi_1'),\xi_1'}(\xi_1, \delta_2). \tag{8}$$

The total number of arrows included in the renormalized vertex becomes

$$N_r(\eta_1') = N_r(\xi_1') = N_r(\xi_1) + \delta_2. \tag{9}$$

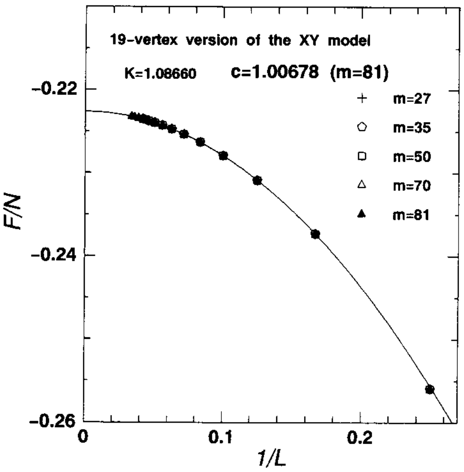

Fig. 1. The size dependence of the free energy.

We need $W^{(r+1)}(v_3^{(r+1)}(\alpha_3, \eta_3', \alpha_4, \xi_3'))$ to return to the first step of the DMRG method, but we do not need to calculate it. Since in our method the system has a symmetry of translation, we can use $W^{(r+1)}(v_1^{(r+1)})$ as $W^{(r+1)}(v_3^{(r+1)})$. Then we return to the first step represented by Eq. (4) to iterate the DMRG procedure. By iterating the above DMRG procedure we can make the system size L increase systematically. The advantage of the present method is that the dimension of the transfer matrix decreases dramatically by considering the conservation law of the number of arrows in this DMRG method. For example, in the case of $N = 0$, $m = 35$ and $L = 12$, the dimension of matrix is dramatically reduced from 531 441 down to 1545.

The value of the conformal anomaly has to be 1 at the BKT transition point. The result at $K = 1.0866$ is shown in Fig. 1. The value of interaction is equal to that estimated as the critical value in Ref. [4]. We obtain $c = 1.006(1)$ which is consistent with the value at the BKT point.

As summary, we have succeeded to reduce the matrix dimension to be diagonalized in the DMRG method for the 19-vertex model by using the ice rule. A very nice value of the conformal anomaly is obtained at BKT transition point by the present approach.

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