



## Letter to the Editor

# An extended approach for computing the critical properties in the two- and three-dimensional lattices within the effective-field renormalization group method

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## ABSTRACT

In this letter we employ the effective-field renormalization group (EFRG) to study the Ising model with nearest neighbors to obtain the reduced critical temperature and exponents  $\nu$  for bi- and three-dimensional lattices by increasing cluster scheme by extending recent works. The technique follows up the same strategy of the mean field renormalization group (MFRG) by introducing an alternative way for constructing classical effective-field equations of state takes on rigorous Ising spin identities.

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## 1. Introduction

As it is well known, exact results from statistical mechanics models are not so ubiquitous, the Onsager solution of the two-dimensional Ising model being the most useful and important example. For this reason, analytical approaches, as well as computationally exact approximations, have been developed in order to treat more complicated systems. Among these approaches, and beyond mean field theory, is the effective-field theory (EFT) based on the Callen–Suzuki relation [1–3].

In recent years, the effective-field renormalization group (EFRG) method has been applied to the study of critical phenomena in classical and quantum spin models (e.g., transverse Ising model, Blume–Capel and the model of ferromagnetic and antiferromagnetic Heisenberg quantum and the model of Heisenberg frustrated and others) that display first- and second-order phase transitions and tricritical points in our phase diagram and has provided useful qualitative and quantitative insights into critical behavior of these systems [3,4].

In this work we presented a new unified effective-field renormalization group framework (EFRG) by extending previous works on Ising systems. The scheme, which follows up the same strategy of the mean field renormalization group (MFRG) approach [5], is based on an alternative way for constructing classical effective-field equations of state by employing rigorous Ising spin identities as starting point. The method is achieved by

treating the effects of the surrounding spins of each of the cluster through a convenient differential operator expansion technique introduced in the literature by Honmura and Kaneyoshi [6] by taking all relevant self-spin correlations into account by including the contribution of the set of spins which are chosen in a systematic way by increasing the number of clusters spins on the honeycomb, square and cubic lattice.

The outline of the remainder of the paper is as follows: in Section 2, the model is introduced and the expressions for studying the critical properties in the Ising systems are briefly given, since the essential approach and discussions have been given in [7] for the case when the one and two spin clusters are taken into account. In Section 3, the results and conclusions are presented.

## 2. The model and calculation

Our prototype Hamiltonian is the Ising model described by

$$\beta\mathcal{H} = \sum_{\langle i,j \rangle} K_{ij} \sigma_i \sigma_j, \quad (1)$$

where  $K_{ij} = J_{ij}/k_B T$  is the nearest-neighbor coupling and  $\sigma_i, \sigma_j = \pm 1$ . By employing the identity of Callen–Suzuki [1,2] and the partial-trace technique, one may generate a sequence of exact spin identities, through the relation

$$m_N = \frac{1}{N} \frac{\text{tr}_\sigma \left( \sum_{k=1}^N \sigma_k e^{-\beta\mathcal{H}} \right)}{\text{tr}_\sigma e^{-\beta\mathcal{H}}}. \quad (2)$$

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By using the differential operator Eq. (2) can be written as

$$m_N = \left\langle \prod_{j_1} e^{K_{1j_1} D_{x_1}} \dots \prod_{j_n} e^{K_{nj_n} D_{x_n}} \right\rangle g(x_1, \dots, x_n) |_{x_1, \dots, x_n=0}, \quad (3)$$

where  $D_{x_p} = \partial/\partial x_p$ , and  $g(x_1, \dots, x_n)$  is obtained from present formalism and defined here by

$$g(x_1, \dots, x_n) = N^{-1} \times \frac{g_{num}(\sum K \sigma_i \sigma_j, \prod \sigma_p x_p)}{g_{den}(\sum K \sigma_i \sigma_j, \prod \sigma_p x_p)}, \quad (4)$$

with

$$g_{num} = \sum \sigma_k H_1 \left( \sum K \sigma_i \sigma_j, \sigma_1 x_1, \dots, \sigma_n x_n \right),$$

$$g_{den} = H_2 \left( \sum K \sigma_i \sigma_j, \sigma_1 x_1, \dots, \sigma_n x_n \right),$$

where  $x_p$  ( $p = 1, \dots, n$ ) represents the variables associated to the nearest neighbors from considered cluster and  $H_s$  ( $s = 1, 2$ ) denotes a new function of the variables. In EFT, we use the identity of the Van der Waerden,

$$\exp(a\sigma) = \cosh a + \sigma \sinh a,$$

valid for  $\sigma = \pm 1$  and the multispin correlation functions are decoupled as

$$\langle \sigma_i \sigma_{j'} \dots \sigma_l \sigma_{p'} \rangle \simeq \langle \sigma_i \rangle \langle \sigma_{j'} \rangle \dots \langle \sigma_l \rangle \langle \sigma_{p'} \rangle,$$

with  $i' \neq j' \neq \dots \neq l' \neq p'$ . Therefore, using the same procedure of Refs. [3,4,8] the average magnetization is given by

$$m_N = A(m, K) = \sum A_n m^n. \quad (5)$$

Here we shall consider two finite clusters with  $N'$  and  $N$  (with  $N' > N$ ) spins each of them characterized by coupling constants  $K'$  and  $K$ , respectively. Expanding Eq. (5) to the first order in these parameters and assuming translation invariance, we then get

$$m_{N'}(m', K') = A_{N'}(K') m' + O(m_{N'}^3),$$

$$m_N(m, K) = A_N(K) m + O(m_N^3), \quad (6)$$

with  $N' < N$ , where the coefficients  $A_{N',N}$  are very long functions of  $K$  and  $K'$  after performing configuration averaging. Accordingly, we will not express the actual expressions of them. By imposing the same scaling relation between the magnetization per spin, one finds the following recursion relation:

$$A_{N'}(K') = A_N(K), \quad (7)$$

which is independent of the rescaling factor. Eqs. (6) are valid for any kind of lattice structure. The solution of Eq. (7) can be obtained numerically by considering the nontrivial fixed point  $K' = K = K_c$  for honeycomb (hb), square (sq) and simple cubic (sc) lattices. On the other hand, in this context, the critical exponent  $\nu$

is given by [7,9]

$$-\nu d = \frac{\ln\left(\frac{N'}{N}\right)}{\ln\left(\frac{\partial K'}{\partial K}\right)} \Big|_{K_c}, \quad (8)$$

where  $d$  is the dimensionality.

### 3. Results and discussions

The values obtained by EFRG for reduced critical temperature,  $K_c^{-1}(N', N)$ , for various arrangements of pairs of clusters have distinct values of  $K_c^{-1}(6, 4) = 1.553$  (hb),  $K_c^{-1}(9, 6) = 2.572$  (sq) and  $K_c^{-1}(8, 4) = 4.739$  (sc). These values can be compared with the exact results and the result obtained from the series expansion method, namely  $K_c^{-1} = 1.519$  (hb),  $K_c^{-1} = 2.269$  (sq) and  $K_c^{-1} = 4.511$  (sc) [10]. On the other hand, the  $K_c^{-1}(2, 1)$  values in [7] have been given by  $K_c^{-1}(2, 1) = 2.794$  (sq) and  $K_c^{-1}(2, 1) = 2.083$  (Ka), in contrast with the unique value of 2.885 predicted by mean-field RG in [11]. Thus, the present results are refined into the reasonable direction by increasing the number of cluster spins on these lattices, when they are compared to the result of square lattice in [7]. The values of critical exponent,  $\nu$ , are  $\nu(10, 8) = 0.993[1.00]$  (hb),  $\nu(9, 6) = 0.969[1.00]$  (sq) and  $\nu(8, 4) = 0.906[0.629]$  (sc). Here, the exact and series results are quoted in brackets.

In conclusion, we observe that EFRG formalism allowed to find values substantially improved both for the critical temperature and for the critical exponent. These results are satisfactory in comparison with other schemes and support the belief that the results provided by the EFRG method can be given qualitative, and, to a certain extent, quantitative confidence.

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