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Stability of the Boltzmann equation with potential forces on torus

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ABSTRACT

In this paper, we are concerned with the stability of solutions to the Cauchy problem of the Boltzmann equation with potential forces on torus. It is shown that the natural steady state with the symmetry of origin is asymptotically stable in the Sobolev space with exponential rate in time for any initially smooth, periodic, origin symmetric small perturbation, which preserves the same total mass, momentum and mechanical energy. For the non-symmetric steady state, it is also shown that it is stable in L^1-norm for any initial data with the finite total mass, mechanical energy and entropy.

1. Introduction

The motion of the dilute gas in the presence of the potential force field is described by the Boltzmann equation:

$$\partial_t f + \xi \cdot \nabla_x f - \nabla_x \phi \cdot \nabla_\xi f = Q(f, f), \tag{1.1}$$

$$f(0, x, \xi) = f_0(x, \xi). \tag{1.2}$$

Here, the unknown $f = f(t, x, \xi) \geq 0$ stands for the spatially periodic number density of gas particles which have position $x = (x_1, x_2, x_3) \in \mathbb{T}^3 = [-\pi, \pi]^3$ and velocity $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$ at time $t \geq 0$, and initial data $f_0(x, \xi)$ is given. $-\nabla_x \phi$ is the given external force generated by the stationary potential function $\phi = \phi(x)$. The bilinear collision operator Q with hard-sphere interaction [1] is defined by

$$Q(f, g) = \int_{\mathbb{R}^3 \times S^2} (f' g'_* - f g_*) |(\xi - \xi_*) \cdot \omega| \mathrm{d}\omega \mathrm{d}\xi_*,$$

$$f = f(t, x, \xi), \qquad f' = f(t, x, \xi'), \qquad g_* = g(t, x, \xi_*), \qquad g'_* = g(t, x, \xi'_*),$$

$$\xi' = \xi - [(\xi - \xi_*) \cdot \omega]\omega, \qquad \xi'_* = \xi_* + [(\xi - \xi_*) \cdot \omega]\omega, \quad \omega \in S^2.$$

Let $\phi(x)$ be normalized such that

$$\int_{\mathbb{T}^3} \mathrm{e}^{-\phi(x)} \mathrm{d}x = 1,$$

and also the global Maxwellian

$$\mathbf{M} = \frac{1}{(2\pi)^{3/2}} \exp\left(-|\xi|^2/2\right)$$

be normalized to have zero bulk velocity and unit density and temperature. It is easy to check that (1.1) has a stationary solution $f_S = f_S(x, \xi)$ given by

$$f_S = \mathrm{e}^{-\phi(x)} \mathbf{M}.$$

The goal of this paper is to study the stability of solutions to the Cauchy problem (1.1)-(1.2) with respect to the stationary state f_S under some conditions. The first result about the asymptotical stability of solutions in the framework of L^2 space is stated as follows.

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Theorem 1.1. *Assume that*

(A1) $\phi(-x) = \phi(x), f_0(-x, -\xi) = f_0(x, \xi) \geq 0$;
(A2)

$$\iint_{\mathbb{T}^3\times\mathbb{R}^3} \left[1, \xi, \frac{1}{2}|\xi|^2 + \phi(x)\right] (f_0 - f_S)\mathrm{d}x\mathrm{d}\xi = 0;$$

(A3) ϕ *is bounded,* $\|\nabla_x\phi\|_{W^{N,\infty}}$ *is small with* $N \geq 4$.

Then, if

$$\left\|\frac{f_0 - f_S}{\sqrt{\mathbf{M}}}\right\|_{H^N}$$

is small enough, the Cauchy problem (1.1)- (1.2) *admits a unique global classical solution* $f(t, x, \xi)$, *satisfying*

$$f(t, -x, -\xi) = f(t, x, \xi) \geq 0,$$
$$\iint_{\mathbb{T}^3\times\mathbb{R}^3} \left[1, \xi, \frac{1}{2}|\xi|^2 + \phi(x)\right] (f(t) - f_S)\mathrm{d}x\mathrm{d}\xi = 0;$$

and

$$\left\|\frac{f(t) - f_S}{\sqrt{\mathbf{M}}}\right\|_{H^N}^2 + \lambda\int_0^t \left\|\frac{f(\tau) - f_S}{\sqrt{\mathbf{M}}}\right\|_{H^N_\nu}^2 d\tau \leq C\left\|\frac{f_0 - f_S}{\sqrt{\mathbf{M}}}\right\|_{H^N}^2, \tag{1.3}$$

where $H^N = H^N(T^3 \times \mathbb{R}^3)$, $H^N_\nu = H^N(\mathbb{T}^3 \times \mathbb{R}^3; \nu(\xi)\mathrm{d}x\mathrm{d}\xi)$, *and* $\nu = \nu(\xi)$ *is the collision frequency. Moreover, it holds that*

$$\left\|\frac{f(t) - f_S}{\sqrt{\mathbf{M}}}\right\|_{H^N} \leq C\left\|\frac{f_0 - f_S}{\sqrt{\mathbf{M}}}\right\|_{H^N} \mathrm{e}^{-\lambda t}, \tag{1.4}$$

for any $t \geq 0$.

The second result about the L^1-stability of solutions is stated as follows.

Theorem 1.2. *Assume that* $f(t, x, \xi) \geq 0$ *is a global solution to the Cauchy problem* (1.1)- (1.2), *satisfying the conservation of mass*

$$\iint_{\mathbb{T}^3\times\mathbb{R}^3} f(t)\mathrm{d}x\mathrm{d}\xi = \iint_{\mathbb{T}^3\times\mathbb{R}^3} f_0\mathrm{d}x\mathrm{d}\xi \tag{1.5}$$

and the natural bound

$$\sup_{t\geq 0}\iint_{\mathbb{T}^3\times\mathbb{R}^3} f(t)(\phi(x) + \frac{1}{2}|\xi|^2 + \log f(t))\mathrm{d}x\mathrm{d}\xi < \infty. \tag{1.6}$$

Then, for any $\eta > 0$, *there exists* $\delta > 0$ *such that if*

$$\iint_{\mathbb{T}^3\times\mathbb{R}^3} \left(\frac{1}{2}|\xi|^2 + \phi(x)\right)|f_0 - f_S|\mathrm{d}x\mathrm{d}\xi + \iint_{\mathbb{T}^3\times\mathbb{R}^3} (f_0\log f_0 - f_S\log f_S) < \delta, \tag{1.7}$$

then

$$\sup_{t\geq 0}\iint_{\mathbb{T}^3\times\mathbb{R}^3} |f(t) - f_S|\mathrm{d}x\mathrm{d}\xi \leq \eta. \tag{1.8}$$

Theorem 1.1 shows that the solution with the same total mass, momentum and mechanical energy as the steady state f_S is asymptotically stable with exponential rate in time under the smooth initial perturbations if the potential function ϕ and the initial data f_0 are symmetric with respect to the origin. Let us explain the condition of the origin symmetry a little more. The total conservation laws often play a key role in the study of stability of the Boltzmann equation over the bounded domain because the Poincaré inequality is able to be applied, see [2,3] and [4,5]. When there is a given external force acting on the gas, only the conservation of the total mass holds in general. For the stationary potential force, the total mechanical energy is also conservative, but it is still missing for the conservation of the total momentum. In order to recover it, we postulate the origin symmetry on the potential function. Actually, for the case of potential forces, the Boltzmann equation remains unchanged under the origin symmetric transformation, and hence by uniqueness the solution also preserves it if initial data is symmetric for the origin. On the other hand, any distribution function with the origin symmetry has the zero total momentum since it holds that

$$\begin{aligned}\iint_{\mathbb{T}^3\times\mathbb{R}^3} \xi f(t, x, \xi)\mathrm{d}x\mathrm{d}\xi &= \iint_{\mathbb{T}^3\times\mathbb{R}^3} (-\tilde{\xi})f(t, -\tilde{x}, -\tilde{\xi})\mathrm{d}\tilde{x}\mathrm{d}\tilde{\xi}\\ &= -\iint_{\mathbb{T}^3\times\mathbb{R}^3} \tilde{\xi}f(t, \tilde{x}, \tilde{\xi})\mathrm{d}\tilde{x}\mathrm{d}\tilde{\xi}.\end{aligned}$$

Thus the symmetric condition (A1) in Theorem 1.1 yields that the total momentum vanishes for any time $t \geq 0$. We also mention that an interesting kinetic model related to the Vlasov–Fokker–Planck equations was recently considered in [6] to prove the asymptotical stability of the steady state for small symmetric perturbations.

For the non-symmetric case, it is unknown whether the steady state f_S is asymptotically stable. However, in this case, Theorem 1.2 shows that the steady state f_S is stable in L^1-norm. It is straightforward to obtain the stability by using the relative entropy

$$H(f|f_S) = \iint_{\mathbb{T}^3\times\mathbb{R}^3} f \log \frac{f}{f_S} \mathrm{d}x\mathrm{d}\xi.$$

See [7,8] and the references therein for many applications of the entropy method. We remark that Theorem 1.2 holds for the general potential function and initial data, for which actually the global existence of weak solutions satisfying (1.6) is assured by DiPerna–Lions renormalized solution theory [9,10]. On the other hand, it should be pointed out that in the absence of forces, [2] provided the well-known result about the almost exponential time-decay for the relative entropy $H(f(t)|f_S)$ under the additional regularity conditions that all the moments of f are uniformly bounded in time and f is bounded in all Sobolev spaces uniformly in time. The corresponding result of [2] could be generalized to the case of the symmetric potential force and initial data. In fact, for any solution f obtained in Theorem 1.1, it holds that

$$\|f(t) - f_S\|_{L^1_{x,\xi}} \leq C \left\| \frac{f(t)-f_S}{\sqrt{\mathbf{M}}} \right\|_{L^2_{x,\xi}} \leq C_{f_0,\phi} \mathrm{e}^{-\lambda t},$$

and the additional regularity conditions are satisfied by postulating

$$\phi \in \bigcap_{N\geq 0} H^N_x, \qquad \frac{f_0 - f_S}{\sqrt{\mathbf{M}}} \in \bigcap_{N\geq 0} H^N_{x,\xi}.$$

The same issue has been considered in [3] for the linear Fokker–Planck equation:

$$\partial_t f + \xi \cdot \nabla_x f - \nabla_x \phi \cdot \nabla_\xi f = \nabla_\xi \cdot (\nabla_\xi f + f\xi).$$

Also see [11] for more details.

Here we mention some work related to this paper. In fact, even for the more complicated model like the Vlasov–Maxwell–Boltzmann system on torus, the global existence of solutions near homogeneous steady states has been proved in [4], where the time-derivative is needed in the energy functional due to the hyperbolic property of the Maxwell equations. The similar stability results on torus were given in [12] and [5] for the Boltzmann equation without forces and the Vlasov–Poisson–Boltzmann system, respectively, and later almost exponential or exponential decay was obtained in [13,14]. In this paper, we shall remove the time-derivative since it makes initial data has to take on the higher regularity and integrability than solutions. Thus, it is necessary to make the refined energy estimate. The main idea to achieve this goal, given in [15] for the global existence of perturbed solutions in $L^2_\xi(H^N_x)$, is to introduce the free energy functional in the estimates of the macroscopic part of solutions. We point out that this is in the same spirit of Kawashima's compensation function in the Fourier space [16]. The same method was recently applied in [17] to the study of the Vlasov–Poisson–Boltzmann system in the whole space for the one-species of gas.

The energy method for the Boltzmann equation in the whole space was independently developed in [18] and [19,20]. After that, there were further extensive studies of the Boltzmann and related kinetic equation such as the stability of global Maxwellians for the Vlasov–Maxwell–Boltzmann system in $\mathbb{R}^3$ by [21], the stability and convergence rate of local Maxwellians for the Boltzmann equation in the presence of external forces by [22–26], Green's functions of the Boltzmann equation by [27,28] and the stability of wave patterns for the Boltzmann equation by [29,30].

The rest of this paper is arranged as follows. In Section 2, we study the properties of the macroscopic part of solutions under the macro–micro decomposition. In Section 3, we prove Theorem 1.1 by obtaining the uniform a priori estimates. In Section 4, we prove Theorem 1.2.

Notations. We use $\langle \cdot, \cdot \rangle$ to denote the inner product in the Hilbert space $L^2(\mathbb{T}^3_x \times \mathbb{R}^3_\xi)$ or $L^2(\mathbb{T}^3_x)$ or $L^2(\mathbb{R}^3_\xi)$, and $\|\cdot\|$ to denote the corresponding L^2 norm. Sometimes we also write $\|\cdot\|_{L^2_{x,\xi}}$, $\|\cdot\|_{L^2_x}$ and $\|\cdot\|_{L^2_\xi}$ when it is needed to be precise. We also define

$$\langle u, v\rangle_\nu \equiv \langle \nu(\xi)u, v\rangle$$

for suitable functions $u = u(x, \xi)$ and $v = v(x, \xi)$ to be the weighted inner product in $L^2(\mathbb{T}^3_x \times \mathbb{R}^3_\xi)$, and use $\|\cdot\|_\nu$ for the corresponding weighted L^2 norm. For the multiple indices $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and $\beta = (\beta_1, \beta_2, \beta_3)$, as usual we denote

$$\partial^\alpha_x \partial^\beta_\xi = \partial^{\alpha_1}_{x_1}\partial^{\alpha_2}_{x_2}\partial^{\alpha_3}_{x_3}\partial^{\beta_1}_{\xi_1}\partial^{\beta_2}_{\xi_2}\partial^{\beta_3}_{\xi_3}.$$

The length of α is $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$. For simplicity, we also use ∂_i to denote ∂_{x_i} for each $i = 1, 2, 3$. In addition, C denotes a generic positive (generally large) constant and λ denotes a generic positive (generally small) constant.

2. Macro–micro decomposition

This section makes some basic preparations for the proof of Theorem 1.1. Thus we suppose that all conditions (A1)–(A3) in Theorem 1.1 hold through this section. We first derive the total conservation laws under the symmetric assumption (A1). And then, we reformulate the Cauchy problem (1.1)–(1.2) in terms of perturbations to the stationary state f_S. Notice that f_S also satisfies $f_S(-x, -\xi) = f_S(x, \xi)$, and hence perturbations preserve it. Furthermore, we shall make the macro–micro decomposition, and derive the local macroscopic conservation laws and evolutions of high-order moments of the microscopic component, where the latter implies that the macroscopic parts are dissipative.

Multiplying Eq. (1.1) by the moments 1, ξ, $|\xi|^2/2$ and taking velocity integration, one can get the local macroscopic balance laws

$$\partial_t \int_{\mathbb{R}^3} f \mathrm{d}\xi + \nabla_x \cdot \int_{\mathbb{R}^3} \xi f \mathrm{d}\xi = 0, \tag{2.1}$$

$$\partial_t \int_{\mathbb{R}^3} \xi f \mathrm{d}\xi + \nabla_x \cdot \int_{\mathbb{R}^3} \xi \otimes \xi f \mathrm{d}\xi = -\nabla_x \phi \int_{\mathbb{R}^3} f \mathrm{d}\xi, \tag{2.2}$$

$$\partial_t \int_{\mathbb{R}^3} \frac{1}{2}|\xi|^2 f \mathrm{d}\xi + \nabla_x \cdot \int_{\mathbb{R}^3} \frac{1}{2}|\xi|^2 \xi f \mathrm{d}\xi = -\nabla_x \phi \cdot \int_{\mathbb{R}^3} \xi f \mathrm{d}\xi. \tag{2.3}$$

After the further space integration, the above equations give the total balance laws:

$$\frac{\mathrm{d}}{\mathrm{d}t} \iint_{\mathbb{T}^3\times\mathbb{R}^3} f(t) \mathrm{d}x\mathrm{d}\xi = 0, \tag{2.4}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \iint_{\mathbb{T}^3\times\mathbb{R}^3} \xi f(t) \mathrm{d}x\mathrm{d}\xi = \iint_{\mathbb{T}^3\times\mathbb{R}^3} \nabla_x \phi f(t) \mathrm{d}x\mathrm{d}\xi, \tag{2.5}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \iint_{\mathbb{T}^3\times\mathbb{R}^3} (\frac{1}{2}|\xi|^2 + \phi(x)) f(t) \mathrm{d}x\mathrm{d}\xi = 0, \tag{2.6}$$

where the following integration by parts were used:

$$\begin{aligned}\iint_{\mathbb{T}^3\times\mathbb{R}^3} \nabla_x \phi \cdot \xi f \mathrm{d}x\mathrm{d}\xi &= -\int_{\mathbb{T}^3} \phi \nabla_x \cdot \int_{\mathbb{R}^3} \xi f \mathrm{d}\xi \mathrm{d}x \\ &= \int_{\mathbb{T}^3} \phi \partial_t \int_{\mathbb{R}^3} f \mathrm{d}\xi \mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t} \iint_{\mathbb{T}^3\times\mathbb{R}^3} \phi f(t) \mathrm{d}x\mathrm{d}\xi.\end{aligned}$$

Notice that if ϕ is an even function, then the form of Eq. (1.1) remains unchanged under the change of variables $(x, \xi) \to (-x, -\xi)$. Thus, the property

$$f_0(-x, -\xi) = f_0(x, \xi)$$

for initial data f_0 is preserved for the solution $f(t)$ at any positive time $t > 0$, that is

$$f(t, -x, -\xi) = f(t, x, \xi). \tag{2.7}$$

Under the above symmetry of $f(t, x, \xi)$, it holds that

$$\begin{aligned}\iint_{\mathbb{T}^3\times\mathbb{R}^3} \nabla_x \phi(x) f(t, x, \xi) \mathrm{d}x\mathrm{d}\xi &= \iint_{\mathbb{T}^3\times\mathbb{R}^3} \nabla_x \phi(-x) f(t, -x, -\xi) \mathrm{d}x\mathrm{d}\xi \\ &= -\iint_{\mathbb{T}^3\times\mathbb{R}^3} \nabla_x \phi(x) f(t, x, \xi) \mathrm{d}x\mathrm{d}\xi \\ &= 0.\end{aligned}$$

Therefore, from (2.4)–(2.6), one has the total conservation laws

$$\frac{\mathrm{d}}{\mathrm{d}t} \iint_{\mathbb{T}^3\times\mathbb{R}^3} f(t) \mathrm{d}x\mathrm{d}\xi = 0,$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \iint_{\mathbb{T}^3\times\mathbb{R}^3} \xi f(t) \mathrm{d}x\mathrm{d}\xi = 0,$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \iint_{\mathbb{T}^3\times\mathbb{R}^3} (\frac{1}{2}|\xi|^2 + \phi(x)) f(t) \mathrm{d}x\mathrm{d}\xi = 0.$$

Thus, by assuming initially

$$\iint_{\mathbb{T}^3\times\mathbb{R}^3} \left[1, \xi, \frac{1}{2}|\xi|^2 + \phi(x)\right] (f_0 - f_S) \mathrm{d}x\mathrm{d}\xi \equiv 0,$$

one has

$$\iint_{\mathbb{T}^3\times\mathbb{R}^3} \left[1, \xi, \frac{1}{2}|\xi|^2 + \phi(x)\right] (f(t) - f_S) \mathrm{d}x\mathrm{d}\xi \equiv 0, \tag{2.8}$$

which implies that any solution with initial data satisfying the total conservation laws and the symmetric condition has to converge to f_S.

Now, we formulate the above argument in the frame of the perturbation. Define the standard perturbation $u(t, x, \xi)$ to f_S as

$$f = f_S + \sqrt{\mathbf{M}} u.$$

Then u satisfies

$$\partial_t u + \xi \cdot \nabla_x u - \nabla_x \phi \cdot \nabla_\xi u + \frac{1}{2} \xi \cdot \nabla_x \phi u = \mathrm{e}^{-\phi} \mathbf{L} u + \Gamma(u, u), \tag{2.9}$$

with given initial data

$$u(0,x,\xi)=u_0(x,\xi)\equiv\frac{f_0-f_S}{\sqrt{\mathbf{M}}},\tag{2.10}$$

where the linear term $\mathbf{L}u$ and the nonlinear $\Gamma(u,u)$ are denoted by

$$\mathbf{L}u=\frac{1}{\sqrt{\mathbf{M}}}\left[Q(\mathbf{M},\sqrt{\mathbf{M}}u)+Q(\sqrt{\mathbf{M}}u,\mathbf{M})\right],$$
$$\Gamma(u,u)=\frac{1}{\sqrt{\mathbf{M}}}Q(\sqrt{\mathbf{M}}u,\sqrt{\mathbf{M}}u).$$

The total conservation laws (2.8) and the symmetric property (2.7) can be rewritten as

$$\iint_{\mathbb{T}^3\times\mathbb{R}^3}\left(1,\xi,\frac{1}{2}|\xi|^2+\phi(x)\right)\sqrt{\mathbf{M}}u(t)\mathrm{d}x\mathrm{d}\xi=0,\quad t\geq 0,\tag{2.11}$$

and

$$u(t,-x,-\xi)=u(t,x,\xi).$$

As usual, for fixed (t,x), $u(t,x,\xi)$ can be uniquely decomposed into

$$\begin{cases}u(t,x,\xi)=u_1+u_2,\\ u_1\equiv\mathbf{P}u\in\mathscr{N},\qquad u_2\equiv\{\mathbf{I}-\mathbf{P}\}u\in\mathscr{N}^{\perp},\\ \mathbf{P}u=\left\{a(t,x)+\displaystyle\sum_{i=1}^{3}b_i(t,x)\xi_i+c(t,x)|\xi|^2\right\}\sqrt{\mathbf{M}},\end{cases}\tag{2.12}$$

where u_1 is called the macroscopic part of $u(t,x,\xi)$ with coefficients $(a,b,c)=(a^u,b^u,c^u)$ for brevity, and u_2 the microscopic part of $u(t,x,\xi)$, and $\mathscr{N}$ is the null-space of $\mathbf{L}$ spanned by the collision invariants:

$$\mathscr{N}=\mathrm{span}\left\{\sqrt{\mathbf{M}};\ \xi_i\sqrt{\mathbf{M}},i=1,2,3;\ |\xi|^2\sqrt{\mathbf{M}}\right\}.$$

For later use, we first list the exact values of some moments of the standard Maxwellian $\mathbf{M}$:

$$\langle 1,\mathbf{M}\rangle=1,$$
$$\langle|\xi_i|^2,\mathbf{M}\rangle=1,\qquad\langle|\xi|^2,\mathbf{M}\rangle=3,$$
$$\langle|\xi_i|^2|\xi_j|^2,\mathbf{M}\rangle=1,\quad i\neq j,$$
$$\langle|\xi_i|^4,\mathbf{M}\rangle=3,\qquad\langle|\xi|^2|\xi_i|^2,\mathbf{M}\rangle=5,\qquad\langle|\xi|^4,\mathbf{M}\rangle=15,$$
$$\langle|\xi|^4|\xi_i|^2,\mathbf{M}\rangle=35,\quad\langle|\xi|^6,\mathbf{M}\rangle=105.$$

Then, one can compute the macroscopic quantities:

$$\int_{\mathbb{R}^3}f\mathrm{d}\xi=\mathrm{e}^{-\phi}+(a+3c),$$
$$\int_{\mathbb{R}^3}\xi f\mathrm{d}\xi=b,$$
$$\int_{\mathbb{R}^3}\frac{1}{2}|\xi|^2f\mathrm{d}\xi=\frac{3}{2}\mathrm{e}^{-\phi}+\frac{3}{2}(a+5c).$$

The total conservation laws in (2.11) imply

$$\int_{\mathbb{T}^3}(a+3c)\mathrm{d}x=0,\tag{2.13}$$

$$\int_{\mathbb{T}^3}b\mathrm{d}x=0,\tag{2.14}$$

$$\frac{3}{2}\int_{\mathbb{T}^3}(a+5c)\mathrm{d}x+\int_{\mathbb{T}^3}\phi(x)(a+3c)\mathrm{d}x=0.\tag{2.15}$$

Under the conservation (2.13) for mass, (2.15) is equivalent with

$$\int_{\mathbb{T}^3}c\mathrm{d}x=-\frac{1}{3}\int_{\mathbb{T}^3}a\mathrm{d}x=-\frac{1}{3}\int_{\mathbb{T}^3}\phi(x)(a+3c)\mathrm{d}x.\tag{2.16}$$

Though the averages of both a and c are not zero, it follows from (2.13) and (2.16) and the Poincaré inequality that they turn out to be bounded by L^2-norm of the first-order derivative of $a+3c$.

Proposition 2.1. *Let $u(t)$ satisfy the total conservation laws (2.11). Then, (2.13)–(2.15) hold true, and moreover, one has*

$$\left|\int_{\mathbb{T}^3} a\mathrm{d}x\right| + \left|\int_{\mathbb{T}^3} c\mathrm{d}x\right| \le C\|\phi\|_{L^\infty}\|\nabla_x(a+3c)\|.$$

Next, we derive the local macroscopic balance laws for (a, b, c) from (2.1)–(2.3). One can further compute the higher-order moments of f under the decomposition (2.12):

$$\begin{aligned}\int_{\mathbb{R}^3}\xi_i\xi_j f\,\mathrm{d}\xi &= \int_{\mathbb{R}^3}\xi_i\xi_j(\mathrm{e}^{-\phi(x)}\mathbf{M}+\sqrt{\mathbf{M}}u)\mathrm{d}\xi\\ &= \mathrm{e}^{-\phi(x)}\int_{\mathbb{R}^3}\xi_i\xi_j\mathbf{M}\mathrm{d}\xi + \int_{\mathbb{R}^3}\xi_i\xi_j\sqrt{\mathbf{M}}\mathbf{P}u\mathrm{d}\xi + \int_{\mathbb{R}^3}\xi_i\xi_j\sqrt{\mathbf{M}}\{\mathbf{I}-\mathbf{P}\}u\mathrm{d}\xi\\ &= \mathrm{e}^{-\phi(x)}\delta_{ij} + (a+5c)\delta_{ij} + \langle\xi_i\xi_j\sqrt{\mathbf{M}}, u_2\rangle,\end{aligned}$$

and

$$\begin{aligned}\int_{\mathbb{R}^3}\frac{1}{2}|\xi|^2\xi_i f\,\mathrm{d}\xi &= \int_{\mathbb{R}^3}\frac{1}{2}|\xi|^2\xi_i(\mathrm{e}^{-\phi(x)}\mathbf{M}+\sqrt{\mathbf{M}}u)\mathrm{d}\xi\\ &= \int_{\mathbb{R}^3}\frac{1}{2}|\xi|^2\xi_i\sqrt{\mathbf{M}}\mathbf{P}u\mathrm{d}\xi + \int_{\mathbb{R}^3}\frac{1}{2}|\xi|^2\xi_i\sqrt{\mathbf{M}}\{\mathbf{I}-\mathbf{P}\}u\mathrm{d}\xi\\ &= \frac{5}{2}b_i + \langle\frac{1}{2}|\xi|^2\xi_i\sqrt{\mathbf{M}}, u_2\rangle.\end{aligned}$$

Putting all identities into (2.1)–(2.3), one obtains the local macroscopic balance laws

$$\partial_t(a+3c) + \nabla_x\cdot b = 0, \tag{2.17}$$

$$\partial_t b + \nabla_x(a+5c) + \nabla_x\cdot\langle\xi\otimes\xi\sqrt{\mathbf{M}}, u_2\rangle = -\nabla_x\phi(a+3c), \tag{2.18}$$

$$\partial_t\left[\frac{3}{2}(a+5c)\right] + \frac{5}{2}\nabla_x\cdot b + \nabla_x\cdot\langle\frac{1}{2}|\xi|^2\xi\sqrt{\mathbf{M}}, u_2\rangle = -\nabla_x\phi\cdot b. \tag{2.19}$$

For the later use, from (2.17) for the conservation of mass, (2.19) for the balance law of energy can be rewritten as

$$\partial_t c + \frac{1}{3}\nabla_x\cdot b + \frac{1}{6}\nabla_x\cdot\langle|\xi|^2\xi\sqrt{\mathbf{M}}, u_2\rangle = -\frac{1}{3}\nabla_x\phi\cdot b.$$

Notice that (2.17)–(2.19) are the linearized Euler-type equations which are not closed due to the appearance of the microscopic part u_2. Thus, we also have to consider the evolution of higher-order moments of u_2:

$$\langle\xi\otimes\xi\sqrt{\mathbf{M}}, u_2\rangle, \qquad \langle|\xi|^2\xi\sqrt{\mathbf{M}}, u_2\rangle.$$

Eq. (2.9) can be rewritten as

$$\partial_t u_1 + \xi\cdot\nabla_x u_1 - \nabla_x\phi\cdot\nabla_\xi u_1 + \frac{1}{2}\xi\cdot\nabla_x\phi u_1 = -\partial_t u_2 + l + n, \tag{2.20}$$

where the linear term l and the nonlinear term n are denoted by

$$l = -\xi\cdot\nabla_x u_2 + \mathrm{e}^{-\phi}\mathbf{L}u + \nabla_x\phi\cdot\nabla_\xi u_2 - \frac{1}{2}\xi\cdot\nabla_x\phi u_2, \tag{2.21}$$

$$n = \Gamma(u, u). \tag{2.22}$$

One can use the representation of $u_1 = \mathbf{P}u$ in terms of (a, b, c) to further write (2.20) as

$$\begin{aligned}&\left\{\partial_t a - \sum_j b_j\partial_j\phi\right\}\sqrt{\mathbf{M}} + \sum_i\{\partial_t b_i + \partial_i a + (a-2c)\partial_i\phi\}\xi_i\sqrt{\mathbf{M}} + \sum_i\{\partial_t c + \partial_i b_i + b_i\partial_i\phi\}|\xi_i|^2\sqrt{\mathbf{M}}\\ &\quad + \sum_{i<j}\{\partial_i b_j + \partial_j b_i + b_j\partial_i\phi + b_i\partial_j\phi\}\xi_i\xi_j\sqrt{\mathbf{M}} + \sum_i\{\partial_i c + c\partial_i\phi\}|\xi|^2\xi_i\sqrt{\mathbf{M}} = -\partial_t u_2 + l + n.\end{aligned} \tag{2.23}$$

Define the high-order moment functions $A = (A_{ij})_{3\times 3}$ and $B = (B_1, B_2, B_3)$ by

$$A_{ij}(u) = \langle(\xi_i\xi_j - 1)\sqrt{\mathbf{M}}, u\rangle, \qquad B_i(u) = \langle(|\xi|^2 - 5)\xi_i\sqrt{\mathbf{M}}, u\rangle. \tag{2.24}$$

Applying $A_{ij}(\cdot)$ and $B_i(\cdot)$ to both sides of (2.23), one has

$$\partial_t[A_{ii}(u_2) + 2c] + 2\partial_i b_i + 2b_i\partial_i\phi = A_{ii}(l+n), \tag{2.25}$$

$$\partial_t A_{ij}(u_2) + \partial_i b_j + \partial_j b_i + b_j\partial_i\phi + b_i\partial_j\phi = A_{ij}(l+n), \quad i\neq j, \tag{2.26}$$

$$\partial_t B_i(u_2) + \partial_i c + c\partial_i\phi = B_i(l+n), \tag{2.27}$$

where (2.26) also holds for $i > j$ since it is symmetric for (i, j) due to the symmetry of A_{ij}. The main observation which initially came from [5] is the following

Proposition 2.2. *For fixed j, it holds that*

$$
\begin{aligned}
&-\partial_t\left[\sum_i \partial_i A_{ij}(u_2)+\frac{1}{2}\partial_j A_{jj}(u_2)\right]-\Delta_x b_j-\partial_j\partial_j b_j\\
&\quad=\sum_i \partial_i(b_j\partial_i\phi+b_i\partial_j\phi)-\sum_{i\neq j}\partial_j(b_i\partial_i\phi)+\frac{1}{2}\sum_{i\neq j}\partial_j A_{ii}(l+n)-\sum_i \partial_i A_{ij}(l+n).
\end{aligned}
\tag{2.28}
$$

Proof. In fact, it follows from (2.26) that

$$
\begin{aligned}
-\Delta_x b_j-\partial_j\partial_j b_j&=-\sum_{i\neq j}\partial_i(\partial_i b_j)-2\partial_j\partial_j b_j=-\sum_{i\neq j}\partial_i[-\partial_j b_i-b_j\partial_i\phi-b_i\partial_j\phi+A_{ij}(-\partial_t u_2+l+n)]-2\partial_j\partial_j b_j\\
&=\partial_j\left[\sum_{i\neq j}\partial_i b_i-2\partial_j b_j\right]+\sum_{i\neq j}\partial_i(b_j\partial_i\phi+b_i\partial_j\phi)-\sum_{i\neq j}\partial_i A_{ij}(-\partial_t u_2+l+n).
\end{aligned}
\tag{2.29}
$$

On the other hand, it follows from (2.25) that

$$
\begin{aligned}
&2\partial_t c+\sum_{i\neq j}\partial_i b_i=\frac{1}{2}\sum_{i\neq j}A_{ii}(-\partial_t u_2+l+n)-\sum_{i\neq j}b_i\partial_i\phi,\\
&2\partial_t c+2\partial_j b_j=A_{jj}(-\partial_t u_2+l+n)-2b_j\partial_j\phi,
\end{aligned}
$$

which by taking difference lead to

$$
\sum_{i\neq j}\partial_i b_i-2\partial_j b_j=\frac{1}{2}\sum_{i\neq j}A_{ii}(-\partial_t u_2+l+n)-A_{jj}(-\partial_t u_2+l+n)-\sum_{i\neq j}b_i\partial_i\phi+2b_j\partial_j\phi.
\tag{2.30}
$$

Putting (2.30) into (2.29) gives

$$
\begin{aligned}
-\Delta_x b_j-\partial_j\partial_j b_j&=\frac{1}{2}\sum_{i\neq j}\partial_j A_{ii}(-\partial_t u_2+l+n)-\sum_{i\neq j}\partial_i A_{ij}(-\partial_t u_2+l+n)-\partial_j A_{jj}(-\partial_t u_2+l+n)+\sum_{i\neq j}\partial_i(b_j\partial_i\phi+b_i\partial_j\phi)\\
&\quad-\sum_{i\neq j}\partial_j(b_i\partial_i\phi)+2\partial_j(b_j\partial_j\phi)\\
&=-\frac{1}{2}\sum_{i\neq j}\partial_j A_{ii}(\partial_t u_2)+\sum_i \partial_i A_{ij}(\partial_t u_2)+\frac{1}{2}\sum_{i\neq j}\partial_j A_{ii}(l+n)-\sum_i \partial_i A_{ij}(l+n)+\sum_i \partial_i(b_j\partial_i\phi+b_i\partial_j\phi)\\
&\quad-\sum_{i\neq j}\partial_j(b_i\partial_i\phi).
\end{aligned}
$$

Thus, (2.28) follows by noting that

$$
\sum_{i\neq j}\partial_j A_{ii}(\partial_t u_2)=\sum_i \partial_j A_{ii}(\partial_t u_2)-\partial_j A_{jj}(\partial_t u_2)=-\partial_j A_{jj}(\partial_t u_2).\quad\square
$$

Notice that the Euler-type equations (2.17)-(2.19) coupled with (2.25)-(2.27) about the evolution of high-order moment functions are a constant-coefficient first-order hyperbolic balance laws in the form of

$$
\mathbb{A}_0\partial_t U+\sum_k \mathbb{A}_k\partial_k U=S,
\tag{2.31}
$$

with constrains

$$
\sum_i A_{ii}=0,
\tag{2.32}
$$

where

$$
U=\begin{bmatrix} a+3c\\ b\\ c\\ (A_{ii})_{1\le i\le 3}\\ (A_{ij})_{1\le i<j\le 3}\\ (B_i)_{1\le i\le 3}\end{bmatrix},\qquad
S=\begin{bmatrix} 0\\ -(a+3c)\nabla_x\phi\\ -2\nabla_x\phi\cdot b\\ A_{ii}(l+n)-2b_i\partial_i\phi\\ A_{ij}(l+n)-(b_i\partial_j\phi+b_i\partial_j\phi)\\ B_i(l+n)-c\partial_i\phi\end{bmatrix}
$$

and $\mathbb{A}_0$, $\mathbb{A}_k$ are the constant 14×14 matrices. See [31] for the study of the dissipation of the general hyperbolic–parabolic system. We remark that the system (2.31)-(2.32) was firstly derived by Grad in [32], and later applied in [16] and [33] to study the time-decay of the linearized Boltzmann equation or Vlasov–Poisson–Boltzmann system, respectively, by using Kawashima's compensated function instead of the spectral analysis which was first completed by Ukai [34] and Nishida-Imai [35].

3. L^2 stability of solutions

In this section, we devote ourselves to the proof of Theorem 1.1, which follows from the local existence together with uniform a priori estimates as well as the standard continuum argument. Here, we skip the proof of the local existence for simplicity. To obtain the uniform a priori estimates in the framework of small initial perturbations, we make a priori assumption

$$\sup_{0\le t\le T}\|u(t)\|_{H^N}\le\epsilon \tag{3.1}$$

with $0<\epsilon\le 1$ small enough and $N\ge 4$, where $u(t)$ is the solution in H^N to the Cauchy problem (2.9)-(2.10) over $[0,T]$ for $0<T\le\infty$. Furthermore, to the end, we suppose that $\|\phi\|_{L^\infty}$ is finite and

$$\|\nabla_x\phi\|_{W^{N,\infty}}\le\epsilon_\phi \tag{3.2}$$

for $0<\epsilon_\phi\le 1$ small enough.

3.1. Microscopic dissipation

The dissipation of the microscopic part u_2 is based on the so-called Boltzmann's H-theorem. Precisely, as in [36], $\mathbf{L}$ can be decomposed into

$$\mathbf{L}=-\nu+K,$$

with

$$\nu(\xi)=\int_{\mathbb{R}^3\times S^2}|(\xi-\xi_*)\cdot\omega|\mathbf{M}(\xi_*)\mathrm{d}\omega\mathrm{d}\xi_*,$$

where $\nu(\xi)$ is the collision frequency satisfying

$$\nu_0(1+|\xi|)\le\nu(\xi)\le\frac{1}{\nu_0}(1+|\xi|)$$

for constant $\nu_0>0$, and K is a self-adjoint compact operator on L^2_ξ. Moreover, $\mathbf{L}$ is non-positive and there is a constant $\lambda>0$ such that

$$-\int_{\mathbb{R}^3}u\mathbf{L}u\,\mathrm{d}\xi\ge\lambda\int_{\mathbb{R}^3}\nu(\xi)\,|\{\mathbf{I}-\mathbf{P}\}u|^2\,\mathrm{d}\xi. \tag{3.3}$$

Besides, to handle estimates on the velocity derivative and the nonlinear term, we cite the following two lemmas.

Lemma 3.1 (*[5]*). *Let $|\beta|>0$. Then $\partial_\xi^\beta\nu(\xi)$ is uniformly bounded. And for any small $\eta>0$ there exists $C_{\beta,\eta}$ such that, for any u,*

$$\|\partial_\xi^\beta[Ku]\|^2\le\eta\sum_{|\beta'|=|\beta|}\|\partial_\xi^{\beta'}u\|^2+C_{\beta,\eta}\|u\|^2.$$

Lemma 3.2 (*[5]*).

$$|\langle\partial_\xi^\beta\Gamma(u,v),w\rangle|\le C\sum_{\beta'+\beta''=\beta}\left\{\int_{\mathbb{T}^3}\|\nu^{1/2}\partial_\xi^{\beta'}u\|_{L^2_\xi}\|\partial_\xi^{\beta''}v\|_{L^2_\xi}\|\nu^{1/2}w\|_{L^2_\xi}\mathrm{d}x+\int_{\mathbb{R}^3}\|\nu^{1/2}\partial_\xi^{\beta'}v\|_{L^2_\xi}\|\partial_\xi^{\beta''}u\|_{L^2_\xi}\|\nu^{1/2}w\|_{L^2_\xi}\mathrm{d}x\right\};$$

$$\|\langle\Gamma(u,v),w\rangle\|_{L^2_x}+\|\langle\Gamma(v,u),w\rangle\|_{L^2_x}\le C\|\nu^3w\|_{L^\infty_{x,\xi}}\|u\|_{L^\infty_x(L^2_\xi)}\|v\|.$$

The following lemma plays a key role in the nonlinear energy estimate and the time-decay rate estimate for the torus case, compared with the whole space.

Lemma 3.3. *Under conditions in Theorem 1.1, it holds that*

$$\|(a,b,c)\|+\|(a,b,c)\|_{L^1}\le C\|\nabla_x(a,b,c)\|.$$

Proof. This inequality follows from (2.13), (2.14) and (2.16), Proposition 2.1 and the Poincaré inequality. □

Now, we devote ourselves to obtaining the microscopic dissipation in the following lemma. The inequality (3.3), and Lemmas 3.1–3.3 are always used.

Lemma 3.4. *Assume that (3.1) and (3.2) hold for $0<\epsilon,\epsilon_\phi\le 1$ small enough. There are constants $\lambda>0$, C independent of ϵ,ϵ_ϕ such that for any $0\le t\le T$, one has*

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2+\lambda\|u_2\|_\nu^2\le C(\epsilon+\epsilon_\phi)\|\nabla_x(a,b,c)\|^2, \tag{3.4}$$

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{1\le|\alpha|\le N}\|\partial_x^\alpha u\|^2+\lambda\sum_{1\le|\alpha|\le N}\|\partial_x^\alpha u_2\|_\nu^2 \le{}& C(\epsilon+\epsilon_\phi)\sum_{|\alpha|\le N-1}\|\partial_x^\alpha\nabla_x(a,b,c)\|^2+C(\epsilon+\epsilon_\phi)\|u_2\|_\nu^2\\ &+C\epsilon_\phi\sum_{|\alpha|\le N-1}\|\partial_x^\alpha\nabla_\xi u_2\|^2,\end{aligned} \tag{3.5}$$

and

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{\substack{|\beta|=k\\|\alpha|+|\beta|\le N}}\|\partial_x^\alpha\partial_\xi^\beta u_2\|^2+\lambda\sum_{\substack{|\beta|=k\\|\alpha|+|\beta|\le N}}\|\partial_x^\alpha\partial_\xi^\beta u_2\|_\nu^2$$
$$\le C(\epsilon+\epsilon_\phi)\sum_{|\alpha|+|\beta|\le N}\|\partial_x^\alpha\partial_\xi^\beta u_2\|_\nu^2+C\sum_{|\alpha|\le N-k}\|\partial_x^\alpha\nabla_x(a,b,c)\|^2+C\sum_{|\alpha|\le N-k+1}\|\partial_x^\alpha u_2\|_\nu^2+C\chi_{\{2\le k\le N\}}\sum_{\substack{1\le|\beta|\le k-1\\|\alpha|+|\beta|\le N}}\|\partial_x^\alpha\partial_\xi^\beta u_2\|_\nu^2, \tag{3.6}$$

where $1\le k\le N$, and χ_D denotes the characteristic function of a set D.

Proof. Recall Eq. (2.9). The direct energy estimate gives

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2-\langle \mathrm{e}^{-\phi}\mathbf{L}u,u\rangle=\langle\Gamma(u,u),u\rangle+\left\langle-\frac{1}{2}\xi\cdot\nabla_x\phi,u^2\right\rangle=I_1+I_2,$$

where I_1, I_2 are estimated as

$$\begin{aligned}I_1&=\langle\Gamma(u,u),u_2\rangle\le C\|u\|_{L_x^\infty(L_\xi^2)}\|u\|_\nu^2\\&\le C\|u\|_{L_\xi^2(H_x^2)}(\|(a,b,c)\|^2+\|u_2\|_\nu^2)\\&\le C\epsilon(\|\nabla_x(a,b,c)\|^2+\|u_2\|_\nu^2),\end{aligned}$$

and

$$\begin{aligned}I_2&=-\frac{1}{2}\langle\xi\cdot\nabla_x\phi,u^2\rangle\\&\le\iint_{\mathbb{T}^3\times\mathbb{R}^3}|\xi|\cdot|\nabla_x\phi|(u_1^2+u_2^2)\mathrm{d}x\mathrm{d}\xi\\&\le C\|\nabla_x\phi\|_{L^\infty}(\|(a,b,c)\|^2+\|u_2\|_\nu^2)\\&\le C\epsilon_\phi(\|\nabla_x(a,b,c)\|^2+\|u_2\|_\nu^2).\end{aligned}$$

Thus (3.4) is proved.

To prove (3.5), let $1\le|\alpha|\le N$, and then the high-order space derivative estimates yield

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_x^\alpha u\|^2-\langle\mathrm{e}^{-\phi}\mathbf{L}\partial_x^\alpha u,\partial_x^\alpha u\rangle\\&=\langle\partial_x^\alpha\Gamma(u,u),\partial_x^\alpha u\rangle+\left\langle-\frac{1}{2}\partial_x^\alpha[\xi\cdot\nabla_x\phi u],\partial_x^\alpha u\right\rangle+\sum_{\alpha'<\alpha}C_{\alpha'}^\alpha\langle\partial_x^{\alpha-\alpha'}\nabla_x\phi\cdot\nabla_\xi\partial_x^{\alpha'}u,\partial_x^\alpha u\rangle+\sum_{\alpha'<\alpha}C_{\alpha'}^\alpha\langle\partial_x^{\alpha-\alpha'}\mathrm{e}^{-\phi}\mathbf{L}\partial_x^{\alpha'}u,\partial_x^\alpha u\rangle\\&=I_3+I_4+I_5+I_6.\end{aligned}$$

Firstly, I_3, I_4 are estimated as

$$\begin{aligned}I_3&=\langle\partial_x^\alpha\Gamma(u,u),\partial_x^\alpha u_2\rangle\\&=\sum_{\alpha'\le\alpha}C_{\alpha'}^\alpha\langle\Gamma(\partial_x^{\alpha-\alpha'},\partial_x^\alpha u),\partial_x^\alpha u_2\rangle\\&\le C\sum_{|\alpha'|\le N/2}\left(\|\partial_x^{\alpha'}u\|_{L_x^\infty(L_\xi^2)}\|\partial_x^{\alpha-\alpha'}u\|_\nu+\|\nu^{1/2}\partial_x^{\alpha'}u\|_{L_x^\infty(L_\xi^2)}\|\partial_x^{\alpha-\alpha'}u\|\right)\|\partial_x^\alpha u_2\|_\nu\\&\le C\epsilon\sum_{|\alpha'|\le N}(\|\partial_x^{\alpha'}u_2\|_\nu^2+\|\partial_x^{\alpha'}(a,b,c)\|^2)\\&\le C\epsilon\sum_{|\alpha'|\le N}\|\partial_x^{\alpha'}u_2\|_\nu^2+C\epsilon\sum_{|\alpha'|\le N-1}\|\partial_x^{\alpha'}\nabla_x(a,b,c)\|^2,\end{aligned}$$

and

$$\begin{aligned}I_4&=-\frac{1}{2}\sum_{\alpha'\le\alpha}C_{\alpha'}^\alpha\langle\xi\cdot\nabla_x\partial_x^{\alpha-\alpha'}\phi\partial_x^\alpha u,\partial_x^\alpha u\rangle\\&\le\sum_{|\alpha'|\ge3}\|\nabla_x\partial_x^{\alpha-\alpha'}\phi\|_{L_x^\infty}\|\partial_x^{\alpha'}u\|_\nu\|\partial_x^\alpha u\|_\nu+\sum_{|\alpha'|\le2}\|\nabla_x\partial_x^{\alpha-\alpha'}\phi\|_{L_x^2}\int_{\mathbb{R}^3}|\xi|\|\partial_x^{\alpha'}u\|_{L_x^\infty}\|\partial_x^\alpha u\|_{L_x^2}\mathrm{d}\xi\\&\le C\|\nabla_x\phi\|_{W_x^{N,\infty}}\left(\sum_{|\alpha'|\le N}\|\partial_x^{\alpha'}u_2\|_\nu^2+\sum_{|\alpha'|\le N-1}\|\partial_x^{\alpha'}\nabla_x(a,b,c)\|^2\right).\end{aligned}$$

Similarly, it holds for I_5, I_6 that

$$I_5\le C\|\nabla_x\phi\|_{W_x^{N,\infty}}\left(\sum_{|\alpha'|\le N}\|\partial_x^{\alpha'}u_2\|_\nu^2+\sum_{|\alpha'|\le N-1}\|\partial_x^{\alpha'}\nabla_x(a,b,c)\|^2\right)+C\|\nabla_x\phi\|_{W_x^{N,\infty}}\sum_{|\alpha'|\le N-1}\|\partial_x^{\alpha'}\nabla_\xi u_2\|^2,$$

and

$$I_6 \le \|\nabla_x\phi\|_{W_x^{N,\infty}} \sum_{|\alpha'|\le N} \|\partial_x^{\alpha'} u_2\|_\nu^2.$$

Collecting the above estimates and using the smallness of ϵ and ϵ_ϕ, (3.5) is proved.

To prove (3.6), applying the microscopic projection $\{\mathbf{I}-\mathbf{P}\}$ to Eq. (2.9), one has the microscopic evolution equation:

$$\begin{aligned}&\partial_t u_2 + \xi\cdot\nabla_x u_2 - \nabla_x\phi\cdot\nabla_\xi u_2 + \frac{1}{2}\xi\cdot\nabla_x\phi u_2 + \nu(\xi)u_2\\ &\quad = \mathrm{e}^{-\phi}Ku_2 + \Gamma(u,u) - \{\mathbf{I}-\mathbf{P}\}\left[\xi\cdot\nabla_x u_1 - \nabla_x\phi\cdot\nabla_\xi u_1 + \frac{1}{2}\xi\cdot\nabla_x\phi u_1\right] + \mathbf{P}\left[\xi\cdot\nabla_x u_2 - \nabla_x\phi\cdot\nabla_\xi u_2 + \frac{1}{2}\xi\cdot\nabla_x\phi u_2\right].\end{aligned}$$

Let $1\le |k|\le N$. Taking the derivative $\partial_x^\alpha\partial_\xi^\beta$ with $|\alpha|+|\beta|\le N$ and $|\beta|=k$, then the standard energy estimates as in [4,25] give (3.6) and the details are omitted for simplicity. □

3.2. Macroscopic dissipation

This section is devoted to obtain the macroscopic dissipation on the basis of the hyperbolic balance laws (2.31)-(2.32), and the derived parabolic-type equation (2.28). Throughout this subsection, we still suppose that the a priori assumption (3.1) and the smallness condition (3.2) hold.

First, recall the definitions (2.21)-(2.22). One has

Lemma 3.5. *Let $|\alpha|\le N-1$. Under (3.1) and (3.2), it holds that*

$$\|A_{ij}(\partial_x^\alpha l)\| + \|B_i(\partial_x^\alpha l)\| \le C\sum_{|\alpha'|\le N}\|\partial_x^{\alpha'}u_2\|,$$

$$\|A_{ij}(\partial_x^\alpha n)\| + \|B_i(\partial_x^\alpha n)\| \le C\epsilon\sum_{|\alpha'|\le N}\|\partial_x^{\alpha'}u_2\| + C\epsilon\sum_{|\alpha'|\le N-1}\|\partial_x^{\alpha'}\nabla_x(a,b,c)\|,$$

where the moment functions $A_{ij}(\cdot)$, $B_i(\cdot)$ are defined in (2.24).

Proof. In fact, $A_{ij}(l)$, $B_i(l)$ are in the form of

$$\int_{\mathbb{R}^3} e(\xi)l\mathrm{d}\xi = \int_{\mathbb{R}^3} e(\xi)\left[-\xi\cdot\nabla_x u_2 + \mathrm{e}^{-\phi}\mathbf{L}u + \nabla_x\phi\cdot\nabla_\xi u_2 - \frac{1}{2}\xi\cdot\nabla_x\phi u_2\right]\mathrm{d}\xi,$$

and similarly, $A_{ij}(n)$, $B_i(n)$ take the form as

$$\int_{\mathbb{R}^3} e(\xi)n\mathrm{d}\xi = \int_{\mathbb{R}^3} e(\xi)\Gamma(u,u)\mathrm{d}\xi,$$

where

$$e(\xi) = (\xi_i\xi_j - 1)\sqrt{\mathbf{M}} \quad \text{or} \quad (|\xi|^2-1)\xi_i\sqrt{\mathbf{M}}.$$

Thus, similar as in [21], the lemma is proved by taking the velocity integration by parts and using the exponential decay of $e(\xi)$, the compactness of K and Lemma 3.2. □

The following lemma gives the macroscopic dissipation. Its proof is initiated by [18], and here we shall use the idea of [15] to provide another proof by introducing the free energy functional. Lemmas 3.3 and 3.5 have been used in the proof.

Lemma 3.6. *Assume that (3.1) and (3.2) hold for $0<\epsilon,\epsilon_\phi\le 1$ small enough. There are constants $\lambda>0$, C independent of ϵ,ϵ_ϕ such that for any $0\le t\le T$, one has*

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\sum_{|\alpha|\le N-1}\left[\sum_{ij}\langle A_{ij}(\partial_x^\alpha u_2), \partial_x^\alpha(\partial_i b_j + \partial_j b_i)\rangle + \sum_j\langle A_{jj}(\partial_x^\alpha u_2), \partial_x^\alpha\partial_j b_j\rangle\right] + \lambda\sum_{|\alpha|\le N-1}\|\partial_x^\alpha\nabla_x b\|^2\\ &\quad\le C(\delta+\epsilon)\sum_{|\alpha|\le N-1}\|\partial_x^\alpha\nabla_x(a+3c,c)\|^2 + \frac{C}{\delta}\sum_{|\alpha|\le N}\|\partial_x^\alpha u_2\|^2,\end{aligned} \tag{3.7}$$

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\sum_{|\alpha|\le N-1}\sum_i\langle B_i(\partial_x^\alpha u_2), \partial_x^\alpha\partial_i c\rangle + \lambda\sum_{|\alpha|\le N-1}\|\partial_x^\alpha\nabla_x c\|^2\\ &\quad\le C(\delta+\epsilon)\sum_{|\alpha|\le N-1}\|\partial_x^\alpha\nabla_x(a+3c,b)\|^2 + \frac{C}{\delta}\sum_{|\alpha|\le N}\|\partial_x^\alpha u_2\|^2,\end{aligned} \tag{3.8}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{|\alpha|\le N-1}\langle\partial_x^\alpha\nabla_x(a+3c), \partial_x^\alpha b\rangle + \lambda\sum_{|\alpha|\le N-1}\|\partial_x^\alpha\nabla_x(a+3c)\|^2 \le C\sum_{|\alpha|\le N-1}\left(\|\partial_x^\alpha\nabla_x(b,c)\|^2 + \|\partial_x^\alpha\nabla_x u_2\|^2\right), \tag{3.9}$$

where the constant $0<\delta\le 1$ is arbitrary.

Proof. To prove (3.7), let $|\alpha| \le N-1$, and then it follows from (2.28) that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left\langle \sum_i \partial_i A_{ij}(\partial_x^\alpha u_2) + \frac{1}{2}\partial_j A_{jj}(\partial_x^\alpha u_2), -\partial_x^\alpha b_j \right\rangle + \|\partial_x^\alpha \nabla_x b_j\|^2 + \|\partial_x^\alpha \partial_j b_j\|^2 \\
&\quad = \left\langle \sum_i A_{ij}(\partial_x^\alpha \partial_i u_2) + \frac{1}{2} A_{jj}(\partial_x^\alpha \partial_j u_2), -\partial_x^\alpha \partial_t b_j \right\rangle + \sum_{ik\ell} C_{1,j}^{ik\ell} \langle A_{ik}(\partial_x^\alpha (l+n)), \partial_x^\alpha \partial_\ell b_j\rangle + \sum_{ik\ell} C_{2,j}^{ik\ell}\langle \partial_x^\alpha (b_i \partial_k \phi), \partial_x^\alpha \partial_\ell b_j\rangle \\
&\quad = I_1 + I_2 + I_3.
\end{aligned}$$

Here, I_1 is estimated as

$$\begin{aligned}
I_1 &\le \delta \|\partial_x^\alpha \partial_t b_j\|^2 + \frac{C}{\delta}\|\nabla_x \partial_x^\alpha u_2\|^2 \\
&\le C\delta \sum_{|\alpha|\le N-1} \|\partial_x^\alpha \nabla_x (a,c)\|^2 + \frac{C}{\delta}\|\nabla_x \partial_x^\alpha u_2\|^2,
\end{aligned}$$

where we used the balance law (2.18) for b. I_2, I_3 are estimated as

$$\begin{aligned}
I_2 &= \frac{1}{2}\|\partial_x^\alpha \nabla_x b_j\|^2 + C\sum_{ik}\|A_{ik}(\partial_x^\alpha (l+n))\|^2 \\
&\le \frac{1}{2}\|\partial_x^\alpha \nabla_x b_j\|^2 + C\sum_{|\alpha|\le N}\|\partial_x^\alpha u_2\|^2 + C\epsilon \sum_{|\alpha|\le N-1}\|\partial_x^\alpha \nabla_x (a,b,c)\|^2,
\end{aligned}$$

and

$$I_3 \le \epsilon_\phi \|\partial_x^\alpha \nabla_x b_j\|^2 + \frac{C}{\epsilon_\phi}\sum_{ik}\|\partial_x^\alpha (b_i \partial_k \phi)\|^2 \le C\epsilon_\phi \sum_{|\alpha|\le N-1}\|\partial_x^\alpha \nabla_x b\|^2.$$

Thus (3.7) is proved by collecting the above inequalities.

To prove (3.8), similarly let $|\alpha| \le N-1$, and then from (2.27), it follows that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\langle B_i(\partial_x^\alpha u_2), \partial_x^\alpha \partial_i c\rangle + \|\partial_x^\alpha \partial_i c\|^2 &= \langle B_i(\partial_x^\alpha \partial_i u_2), -\partial_x^\alpha \partial_t c\rangle + \langle B_i(\partial_x^\alpha (l+n)), \partial_x^\alpha \partial_i c\rangle + \langle -\partial_x^\alpha (c\partial_i \phi), \partial_x^\alpha \partial_i c\rangle \\
&= I_4 + I_5 + I_6,
\end{aligned}$$

where I_4 is estimated by the balance law (2.19) as

$$I_4 \le \delta \|\partial_x^\alpha \partial_t c\|^2 + \frac{C}{\delta}\|\partial_x^\alpha \nabla_x u_2\|^2 \le C\delta \sum_{|\alpha|\le N-1}\|\partial_x^\alpha \nabla_x b\|^2 + \frac{C}{\delta}\|\partial_x^\alpha \nabla_x u_2\|^2,$$

and I_5, I_6 are estimated as

$$\begin{aligned}
I_5 &\le \frac{1}{2}\|\partial_x^\alpha \partial_i c\|^2 + C\|B_i(\partial_x^\alpha (l+n))\|^2 \\
&\le \frac{1}{2}\|\partial_x^\alpha \partial_i c\|^2 + C\sum_{|\alpha|\le N}\|\partial_x^\alpha u_2\|^2 + C\epsilon \sum_{|\alpha|\le N-1}\|\partial_x^\alpha \nabla_x (a,b,c)\|^2,
\end{aligned}$$

and

$$I_6 \le \epsilon_\phi \|\partial_x^\alpha \partial_i c\|^2 + \frac{C}{\epsilon_\phi}\|\partial_x^\alpha (c\partial_i \phi)\|^2 \le C\epsilon_\phi \sum_{|\alpha|\le N-1}\|\partial_x^\alpha \nabla_x c\|^2.$$

Thus (3.8) follows from the above estimates.

To prove (3.9), for $|\alpha| \le N-1$, it follows from (2.18) that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\langle \partial_x^\alpha \nabla_x (a+3c), \partial_x^\alpha \cdot b\rangle + \|\partial_x^\alpha \nabla_x (a+3c)\|^2 \\
&\quad = \langle \partial_x^\alpha \nabla_x \partial_t (a+3c), \partial_x^\alpha \cdot b\rangle + \langle \partial_x^\alpha \nabla_x (a+3c), \partial_x^\alpha [-2\partial_x^\alpha c - \nabla_x \cdot A(u_2)]\rangle + \langle \partial_x^\alpha \nabla_x (a+3c), \partial_x^\alpha [-\nabla_x \phi (a+3c)]\rangle \\
&\quad = I_7 + I_8 + I_9,
\end{aligned}$$

where it holds that

$$\begin{aligned}
I_7 &= \|\partial_x^\alpha \nabla_x \cdot b\|^2, \\
I_8 &\le \frac{1}{2}\|\partial_x^\alpha \nabla_x (a+3c)\|^2 + C(\|\partial_x^\alpha \nabla_x c\|^2 + \|\partial_x^\alpha \nabla_x u_2\|^2), \\
I_9 &\le C\epsilon_\phi \sum_{|\alpha|\le N-1}\|\partial_x^\alpha \nabla_x (a+3c)\|^2.
\end{aligned}$$

Here (2.17) for the conservation of mass was used. Hence (3.9) follows. This completed the proof of Lemma 3.6. □

Corollary 3.1. *There exists the free energy $\mathcal{E}_{free}(\cdot)$ with*

$$|\mathcal{E}_{free}(u(t))| \leq C\|u\|^2_{L^2_\xi(H^N_x)}, \tag{3.10}$$

such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_{free}(u(t)) + \lambda\|(a+3c, b, c)\|^2_{H^N_x} \leq C\|u_2\|^2_{L^2_\xi(H^N_x)}. \tag{3.11}$$

Proof. Define $\mathcal{E}_{free}(u(t))$ by

$$\begin{aligned}\mathcal{E}_{free}(u(t)) = {} & M\sum_{|\alpha|\leq N-1}\sum_{ij}\langle A_{ij}(\partial^\alpha_x u_2), \partial^\alpha_x(\partial_i b_j + \partial_j b_i)\rangle + M\sum_{|\alpha|\leq N-1}\sum_{j}\langle A_{jj}(\partial^\alpha_x u_2), \partial^\alpha_x \partial_j b_j\rangle \\ & + M\sum_{|\alpha|\leq N-1}\sum_{i}\langle B_i(\partial^\alpha_x u_2), \partial^\alpha_x \partial_i c\rangle + \sum_{|\alpha|\leq N-1}\langle \partial^\alpha_x \nabla_x(a+3c), \partial^\alpha_x b\rangle\end{aligned}$$

for a properly large constant $M > 0$. From Lemma 3.6, it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_{free}(u(t)) + \lambda\|\nabla_x(a+3c, b, c)\|^2_{H^{N-1}_x} \leq \|u_2\|^2_{L^2_\xi(H^N_x)}.$$

Therefore, (3.11) follows by further using Lemma 3.3, and (3.10) holds by the definitions of $A_{ij}(\cdot)$ and $B_i(\cdot)$. This completes the proof of Corollary 3.1. □

3.3. Proof of uniform a priori estimates

Recall that under the a priori assumption (3.1) and the smallness condition (3.2), Lemma 3.4 and Corollary 3.1 hold. The linear combination of (3.4), (3.5) and (3.11) gives the dissipation of both the macroscopic part $(a+3c, b, c)$, microscopic part u_2 and their space derivatives with the small-coefficient L^2-norms of space–velocity derivatives as the remaining term. On the other hand, the linear combination of (3.6) for $1 \leq k \leq N$ gives the dissipation in L^2-norm of the space–velocity derivatives. Thus, the further linear combination leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(u(t)) + \lambda\mathcal{D}(u(t)) \leq 0,$$

where $\mathcal{E}(u(t))$, $\mathcal{D}(u(t))$ are equivalent with $\|u(t)\|^2_{H^N}$, $\|u(t)\|^2_{H^N_\nu}$, respectively. Since

$$\mathcal{E}(t) \leq C\mathcal{D}(u(t)),$$

then one has

$$\mathcal{E}(u(t)) \leq \mathcal{E}(u_0)\mathrm{e}^{-\lambda t}.$$

By the equivalence, (1.3) and (1.4) hold for any $0 \leq t \leq T$ in terms of the unperturbation $f \equiv f_S + \sqrt{\mathbf{M}}u$. Finally the obtained uniform a priori estimates together with the normal local existence as well as the continuum argument imply the global existence. This completes the proof of Theorem 1.1.

4. L^1 stability of solutions

For the non-symmetric potential function $\phi(x)$ or non-symmetric initial data $f_0(x, \xi)$, we cannot prove the nonlinear asymptotical stability even in the framework of small perturbations given in Theorem 1.1. However, for the general initial data with finite mass, mechanical energy and energy, the stationary state f_S is stable in L^1-norm. Precisely, as stated in Theorem 1.2, as long as the initial data is sufficiently close to f_S in terms of energy and entropy, then the solution remains close to f_S in L^1-norm. This nonlinear stability is applicable to the global weak solution from the DiPerna–Lions renormalized solution theory [10,9].

To prove Theorem 1.2, let us define the Kullback relative entropy $H(f|g)$ for the distribution function with respect to the function g as

$$H(f|g) = \iint_{\mathbb{T}^3\times\mathbb{R}^3} f\log\frac{f}{g}\mathrm{d}x\mathrm{d}\xi.$$

Recall the Csiszár–Kullback inequality which states that

$$\frac{1}{2}\|f-g\|^2_{L^1} \leq H(f|g)$$

whenever f and g are two probability distributions. One can apply the above inequality to the solution $f(t)$ to the Boltzmann equation (1.1) and the stationary state f_S. Notice that f_S has been normalized and $f(t)$ has the same total mass with f_S at any time t by the assumption (1.5). Thus one has

$$\|f(t) - f_S\|_{L^1} \leq \sqrt{2H(f|f_S)}.$$

On the other hand, from the entropy and energy inequalities, and the conservation of mass, it holds that

$$\frac{\mathrm{d}}{\mathrm{d}t}H(f|f_S) = \frac{\mathrm{d}}{\mathrm{d}t}\iint_{\mathbb{T}^3\times\mathbb{R}^3}\left[f(t)\log f(t) + (\phi(x) + \frac{1}{2}|\xi|^2)f(t)\right]\mathrm{d}x\mathrm{d}\xi \leq 0,$$

which further implies

$$H(f|f_S) \le H(f_0|f_S).$$

One can rewrite $H(f_0|f_S)$ as

$$H(f_0|f_S) = \iint_{\mathbb{T}^3\times\mathbb{R}^3} (f_0 \log f_0 - f_S \log f_S)\mathrm{d}x\mathrm{d}\xi + \iint_{\mathbb{T}^3\times\mathbb{R}^3} \left(\phi(x) + \frac{1}{2}|\xi|^2\right)(f_0 - f_S).$$

Therefore, for any $\eta > 0$, there exists $\delta = \eta^2/2$ such that if (1.7) holds, then one has

$$\|f(t) - f_S\|_{L^1} \le \eta.$$

Then (1.8) holds. This completes the proof of Theorem 1.2.

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