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The refined inviscid stability condition and cellular instability of viscous shock waves

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ABSTRACT

Combining the work of Serre and Zumbrun, Benzoni-Gavage, Serre, and Zumbrun, and Texier and Zumbrun, we propose as a mechanism for the onset of cellular instability of viscous shock and detonation waves in a finite-cross-section duct, the violation of the refined planar stability condition of Zumbrun–Serre, a viscous correction of the inviscid planar stability condition of Majda. More precisely, we show for a model problem involving flow in a rectangular duct with artificial periodic boundary conditions that transition to multidimensional instability through violation of the refined stability condition of planar viscous shock waves on the whole space generically implies for a duct of sufficiently large cross-section, a cascade of Hopf bifurcations involving more and more complicated cellular instabilities. The refined condition is numerically calculable as described by Benzoni-Gavage–Serre–Zumbrun.

1. Introduction

It is well known both experimentally and numerically [1–9] that shock and detonation waves propagating in a finite cross-section duct can exhibit time-oscillatory or "cellular" instabilities, in which the initially nearly planar shock takes on nontrivial transverse geometry. Majda et al. [10–12] have studied the onset of such instabilities by weakly nonlinear optics expansion of the associated planar inviscid shock in the whole space. More recently, Kasimov–Stewart [9] and Texier–Zumbrun [13–15] have studied these instabilities as Hopf bifurcations of flow in a finite-cross-section duct, associated with passage across the imaginary axis of eigenvalues of the linearized operator about the wave.

In this paper, combining the analyses of [16,15,17], we make an explicit connection between stability of planar shocks on the whole space, and Hopf bifurcation in a finite cross-section duct, by a mechanism different from that investigated by Majda et al. Specifically, we point out that violation of the *refined stability condition* of [18,19,16], a viscous correction of the inviscid planar stability condition of Majda [20–22], is generically associated with Hopf bifurcation in a finite cross-section duct corresponding to the observed cellular instability, for cross-section M sufficiently large. Indeed, we show more, that this is associated with a cascade of bifurcations to higher and higher wave numbers and more and more complicated solutions, with features on finer and finer length/time scales.

1.1. Equations and assumptions

Consider a planar viscous shock solution

$$u(x,t)=\bar u(x_1-st) \tag{1.1}$$

of a two-dimensional system of viscous conservation laws

$$u_t+\sum f^j(u)_{x_j}=\Delta_x u,\quad u\in\mathbb{R}^n,\ x\in\mathbb{R}^2,\ t\in\mathbb{R}^+ \tag{1.2}$$

on the whole space. This may be viewed alternatively as a planar traveling-wave solution of (1.2) on an infinite channel

$$\mathcal{C}:=\{x:\ (x_1,x_2)\in\mathbb{R}^1\times[-M,M]\}$$

under periodic boundary conditions

$$u(x_1,M)=u(x_1,-M). \tag{1.3}$$

We take this as a simplified mathematical model for compressible flow in a duct, in which we have neglected boundary-layer phenomena along the wall $\partial\Omega$ in order to isolate the oscillatory phenomena of our main interest.

Following [13], consider a one-parameter family of standing planar viscous shock solutions $\bar u^\varepsilon(x_1)$ of a smoothly-varying family of conservation laws

$$u_t=\mathcal{F}(\varepsilon,u):=\Delta_x u-\sum_{j=1}^{2}F^j(\varepsilon,u)_{x_j},\quad u\in\mathbb{R}^n \tag{1.4}$$

in a fixed channel $\mathcal{C}$, with periodic boundary conditions (typically, shifts $\sum F^j(\varepsilon,u)_{x_j}:=\sum f^j(u)_{x_j}-s(\varepsilon)u_{x_1}$ of a single equation (1.2)

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written in coordinates $x_1 \to x_1 - s(\varepsilon)t$ moving with traveling-wave solutions of varying speeds $s(\varepsilon)$), with linearized operators $L(\varepsilon) := \partial\mathcal{F}/\partial u|_{u=\bar u^\varepsilon}$. Profiles $\bar u^\varepsilon$ satisfy the standing-wave ODE

$$u' = F^1(\varepsilon, u) - F^1(\varepsilon, u_-). \tag{1.5}$$

Let

$$A^1_\pm(\varepsilon) := \lim_{z\to\pm\infty} F^1_u(\varepsilon, \bar u^\varepsilon). \tag{1.6}$$

Following [19,13,17], we make the assumptions:

(H0) $F^j \in C^k$, $k \geq 2$.

(H1) $\sigma(A^1_\pm(\varepsilon))$ real, distinct, and nonzero, and $\sigma(\sum \xi_j A^j_\pm(\varepsilon))$ real and semisimple for $\xi \in \mathbb{R}^d$.

For most of our results, we require also:

(H2) Considered as connecting orbits of (1.5), $\bar u^\varepsilon$ are transverse and unique up to translation, with dimensions of the stable subpace $S(A^1_+)$ and the unstable subspace $U(A^1_-)$ summing for each ε to $n + 1$.

(H3) $\det(r^-_1, \dots, r^-_{c-1}, r^+_{c+1}, \dots, r^+_n, u_+ - u_-) \neq 0$, where $r^-_1, \dots, r^-_{c-1}$ are eigenvectors of A^1_- associated with negative eigenvalues and $r^-_{c+1}, \dots, r^-_n$ are eigenvectors of A^1_+ associated with positive eigenvalues.

Hypothesis (H2) asserts in particular that $\bar u^\varepsilon$ is of standard *Lax type*, meaning that the axial hyperbolic convection matrices $A^1_+(\varepsilon)$ and $A^1_-(\varepsilon)$ at plus and minus spatial infinity have, respectively, $n - c$ positive and $c - 1$ negative real eigenvalues for $1 \leq c \leq n$, where c is the characteristic family associated with the shock: in other words, there are precisely $n - 1$ outgoing hyperbolic characteristics in the far field. Hypothesis (H3) may be recognized as the *Liu–Majda condition* corresponding to one-dimensional stability of the associated inviscid shock. In the present, viscous, context, this, together with transverality, (H2), plays the role of a spectral nondegeneracy condition corresponding in a generalized sense [23,19] to simplicity of the embedded zero eigenvalue associated with eigenfunction $\partial_{x_1}\bar u$ and translational invariance.

1.2. Stability conditions

Our first set of results, generalizing the one-dimensional analysis of [17], characterize stability/instability of waves $\bar u^\varepsilon$ in terms of the spectrum of the linearized operator $L(\varepsilon)$. Fixing ε, we suppress the parameter ε. We start with the routine observation that the semilinear parabolic equation (1.2) has a center-stable manifold about the equilibrium solution $\bar u$.

Proposition 1.1. *Under assumptions* (H0)–(H1)*, there exists in an H^2 neighborhood of the set of translates of $\bar u$ a codimension-p translation invariant C^k (with respect to H^2) center stable manifold $\mathcal{M}_{cs}$, tangent at $\bar u$ to the center stable subspace Σ_{cs} of L, that is (locally) invariant under the forward time-evolution of* (1.2)–(1.3) *and contains all solutions that remain bounded and sufficiently close to a translate of $\bar u$ in forward time, where p is the (necessarily finite) number of unstable, i.e., positive real part, eigenvalues of L.*

Proof. By standard considerations [24,13], $L(\varepsilon)$ possesses no essential spectrum and at most a finite set of positive real part eigenvalues on $\Re\lambda > 0$. With this observation, the result follows word-for-word by the argument of [17] in the one-dimensional case, which depends only on the properties of L as a sectorial second-order elliptic operator, and on semilinearity and translation-invariance of the underlying Eq. (1.2). □

Introduce now the *nonbifurcation condition*:

(D1) L has no nonzero imaginary eigenvalues.

As discussed above, (H2)–(H3) correspond to a generalized notion of simplicity of the embedded eigenvalue $\lambda = 0$ of L. Thus, (D1) together with (H2)–(H3) correspond to the assumption that there are no additional (usual or generalized) eigenvalues on the imaginary axis other than the translational eigenvalue at $\lambda = 0$; that is, the shock is not in transition between different degrees of stability, but has stability properties that are insensitive to small variations in parameters.

Theorem 1.2. *Under* (H0)–(H3) *and* (D1)*, $\bar u$ is nonlinearly orbitally stable as a solution of* (1.2)–(1.3) *under sufficiently small perturbations in $L^1 \cap H^2$ lying on the codimension p center stable manifold $\mathcal{M}_{cs}$ of $\bar u$ and its translates, where p is the number of unstable eigenvalues of L, in the sense that, for some $\alpha(\cdot)$, all L^p,*

$$\begin{aligned}
&|u(x,t) - \bar u(x - \alpha(t))|_{L^p} \\
&\quad \leq C(1+t)^{-\frac{1}{2}(1-\frac{1}{p})}|u(x,0) - \bar u(x)|_{L^1\cap H^2}, \\
&|u(x,t) - \bar u(x - \alpha(t))|_{H^2} \leq C(1+t)^{-\frac{1}{4}}|u(x,0) - \bar u(x)|_{L^1\cap H^2}, \\
&|\dot\alpha(t)| \leq C(1+t)^{-\frac{1}{2}}|u(x,0) - \bar u(x)|_{L^1\cap H^2}, \\
&|\alpha(t)| \leq C|u(x,0) - \bar u(x)|_{L^1\cap H^2}.
\end{aligned} \tag{1.7}$$

Moreover, it is orbitally unstable with respect to small H^2 perturbations not lying in $\mathcal{M}_{cs}$, in the sense that the corresponding solution leaves a fixed-radius neighborhood of the set of translates of $\bar u$ in finite time.

Remark 1.3. Theorem 1.2 includes in passing the result that existence of unstable eigenvalues implies nonlinear instability, hence completely characterizes stability/instability of waves under the nondegeneracy condition (D1). The rates of decay (1.7) are exactly those of the one-dimensional case [17].

1.3. Bifurcation conditions

We next recall the following result from [13,14] characterizing Hopf bifurcation of $\bar u^\varepsilon$ in terms of conditions on the spectrum of $L(\varepsilon)$. Define the *Hopf bifurcation condition*:

(D2) Outside the essential spectrum of $L(\varepsilon)$, for ε and $\delta > 0$ sufficiently small, the only eigenvalues of $L(\varepsilon)$ with real part of absolute value less than δ are a crossing conjugate pair $\lambda_\pm(\varepsilon) := \gamma(\varepsilon) \pm i\tau(\varepsilon)$ of $L(\varepsilon)$, with $\gamma(0) = 0$, $\partial_\varepsilon\gamma(0) > 0$, and $\tau(0) \neq 0$.

Proposition 1.4 (*[13,14]*)**.** *Let $\bar u^\varepsilon$,* (1.4) *be a family of traveling-waves and systems satisfying assumptions* (H0)–(H3) *and* (D2)*, and $\eta > 0$ sufficiently small. Then, for $a \geq 0$ sufficiently small and $C > 0$ sufficiently large, there are C^1 functions $\varepsilon(a)$, $\varepsilon(0) = 0$, and $T^*(a)$, $T^*(0) = 2\pi/\tau(0)$, and a C^1 family of solutions $u^a(x_1, t)$ of* (1.4) *with $\varepsilon = \varepsilon(a)$, time-periodic of period $T^*(a)$, such that*

$$C^{-1}a \leq \sup_{x_1\in\mathbb{R}} e^{\eta|x_1|}\left|u^a(x,t) - \bar u^{\varepsilon(a)}(x_1)\right| \leq Ca \quad \text{for all } t \geq 0. \tag{1.8}$$

Up to fixed translations in x, t, for ε sufficiently small, these are the only nearby solutions as measured in norm $\|f\|_{X_1} := \|(1 + |x_1|)f(x)\|_{L^\infty(x)}$ that are time-periodic with period $T \in [T_0, T_1]$, for any fixed $0 < T_0 < T_1 < +\infty$. Indeed, they are the only nearby solutions of form $u^a(x,t) = \mathbf{u}^a(x - \sigma^a t, t)$ with $\mathbf{u}^a$ periodic in its second argument.

Proof. This result was established in Theorem 1.4, [13] with (1.8) replaced by

$$C^{-1}a \leq \sup_{x_1\in\mathbb{R}} (1 + |x_1|)\left|u^a(x,t) - \bar u^{\varepsilon(a)}(x_1)\right| \leq Ca \tag{1.9}$$

and under the further assumption that there are no eigenvalues of $L(\varepsilon)$ with strictly positive real part other than possibly $\lambda_\pm(\varepsilon)$. As $L(\varepsilon)$ by standard considerations [24,13] possesses at most a finite set of positive real part eigenvalues, an examination of the proof shows that the more general case follows by essentially the same argument, reducing by the Lyapunov–Schmidt reduction described in [13] to a finite-dimensional equation on the direct sum of the oscillatory eigenspace associated with $\lambda_\pm$ and the unstable eigenspace of L, then appealing to standard, finite-dimensional theory to conclude the appearance of Hopf bifurcation with bound (1.9). The stronger result of exponential localization, (1.8), may be obtained by combining the argument of [13] with the strengthened cancellation estimates of Proposition 2.5 [14]. As the distinction between (1.8) and (1.9) is not important for the present discussion, we omit the (straightforward) details. □

Remark 1.5. Together with Theorem 1.2, Proposition 1.4 implies that, under the Hopf bifurcation assumption (D2) together with the further assumption that $L(\varepsilon)$ have no strictly positive real part eigenvalues other than possibly $\lambda_\pm$, waves $\bar{u}^\varepsilon$ are linearly and nonlinearly *stable* for $\varepsilon < 0$ and *unstable* for $\varepsilon > 0$, with bifurcation/exchange of stability at $\varepsilon = 0$.

1.4. Longitudinal vs. transverse bifurcation

The analysis of [13] in fact gives slightly more information. Denote by

$$\Pi^\varepsilon f := \sum_{j=\pm} \phi_j^\varepsilon(x)\langle\tilde{\phi}_j^\varepsilon, f\rangle \tag{1.10}$$

the $L(\varepsilon)$-invariant projection onto the oscillatory eigenspace $\Sigma^\varepsilon := \mathrm{Span}\{\phi_\pm^\varepsilon\}$, where $\phi_\pm^\varepsilon$ are the eigenfunctions associated with $\lambda_\pm(\varepsilon)$. Then, we have the following result, proved but not explicitly stated in [13].

Proposition 1.6. *Under the assumptions of Proposition 1.4, also*

$$\sup_{x_1} e^{\eta|x_1|}|u^a - \bar{u} - \Pi^\varepsilon(u^a - \bar{u})| \le Ca^2 \quad \textit{for all } t \ge 0. \tag{1.11}$$

Proof. The weaker bound

$$\sup_{x_1} (1 + |x_1|)|u^a - \bar{u} - \Pi^\varepsilon(u^a - \bar{u})| \le Ca^2 \quad \text{for all } t \ge 0, \tag{1.12}$$

is established in the course of the Lyapunov reduction of [13]; see (2.17), Proposition 2.9, case $\omega \equiv 0$. The stronger version (1.11) follows by the same argument together with the strengthened cancellation estimates of Proposition 2.5 [14]. □

Bounds (1.8) and (1.11) together yield the standard finite-dimensional property that bifurcating solutions lie to quadratic order in the direction of the oscillatory eigenspace of $L(\varepsilon)$. From this, we may draw the following additional conclusions about the structure of bifurcating waves. By separation of variables, and x_2-independence of the coefficients of $L(\varepsilon)$, we have that the eigenfunctions ψ of $L(\varepsilon)$ decompose into families

$$e^{i\xi x_2}\psi(x_1), \quad \xi = \frac{\pi k}{M}, \tag{1.13}$$

associated with different integers k, where $2M$ is cross-sectional width. Thus, there are two very different cases: (i) (*longitudinal instability*) the bifurcating eigenvalues $\lambda_\pm(\varepsilon)$ are associated with wave-number $k = 0$, or (ii) (*transverse instability*) the bifurcating eigenvalues $\lambda_\pm(\varepsilon)$ are associated with wave-numbers $\pm k \ne 0$.

Corollary 1.7. *Under the assumptions of Proposition 1.4, u^a depend nontrivially on x_2 if and only if the bifurcating eigenvalues $\lambda_\pm$ are associated with transverse wave-numbers $\pm k \ne 0$.*

Proof. For $k \ne 0$, the result follows by the fact that, by (1.8) and (1.11), $\Pi(u^a - \bar{u})$ is the dominant part of $u^a - \bar{u}$, and the fact that Πf by inspection depends nontrivially on x_2 whenever $\Pi f \ne 0$. For $k = 0$, the result follows by uniqueness, and the fact that, restricted to the one-dimensional case, the same argument yields a bifurcating solution depending only on x_1. □

Bifurcation through longitudinal instability corresponds to "galloping" or "pulsating" instabilities described in detonation literature, while symmetry-breaking bifurcation through transverse instability corresponds to "cellular" instabilities introducing nontrivial transverse geometry to the structure of the propagating wave.

1.5. The refined stability condition and bifurcation

Longitudinal or "galloping" bifurcation, though almost certainly occurring for detonations (see [13,15] and references therein), has up to now not been observed for shock waves as far as we know (though we see no reason why they should not in general be possible), nor has there been proposed any specific mechanism by which this might occur. The main purpose of the present paper, as we now describe, is to point out that for transverse or "cellular" bifurcations, to the contrary, there is a simple and natural mathematical mechanism, closely related to the inviscid stability theory for shocks in the whole space, by which they can and likely do occur.

1.5.1. The inviscid stability condition

Inviscid stability analysis for shocks in the whole space centers about the *Lopatinski determinant*

$$\Delta(\tilde{\xi}, \lambda) := \begin{pmatrix} \mathcal{R}_1^- & \cdots & \mathcal{R}_{p-1}^- & \mathcal{R}_{p+1}^+ & \cdots & \mathcal{R}_n^+ & \lambda[u] + i\tilde{\xi}[f^2] \end{pmatrix}, \tag{1.14}$$

$\tilde{\xi} \in \mathbb{R}^1$, $\lambda = \gamma + i\tau \in \mathbb{C}$, $\tau > 0$, a spectral determinant whose zeroes correspond to normal modes $e^{\lambda t}e^{i\tilde{\xi}x_2}w(x_1)$ of the constant-coefficient linearized equations about the discontinuous shock solution. Here, $\{\mathcal{R}_{p+1}^+, \dots, \mathcal{R}_n^+\}$ and $\{\mathcal{R}_1^-, \dots, \mathcal{R}_{p-1}^-\}$ denote bases for the unstable/resp. stable subspaces of

$$\mathcal{A}_\pm(\tilde{\xi}, \lambda) := (\lambda I + i\tilde{\xi}df^2(u_\pm))(df^1(u_\pm))^{-1}. \tag{1.15}$$

Weak stability $|\Delta| > 0$ for $\gamma > 0$ is clearly necessary for linearized stability, while strong, or uniform stability, $|\Delta|/|(\tilde{\xi}, \lambda)| \ge c_0 > 0$, is sufficient for nonlinear stability. Between strong instability, or failure of weak stability, and strong stability, there lies a region of neutral stability corresponding to the appearance of surface waves propagating along the shock front, for which Δ is nonvanishing for $\Re\lambda > 0$ but has one or more roots $(\tilde{\xi}_0, \lambda_0)$ with $\lambda_0 = i\tau_0$ pure imaginary. This region of neutral inviscid stability typically occupies an open set in physical parameter space [20–22,25,19,26]. For details, see, e.g., [27,20–22,28–31,18,19,26,32,25,33], and references therein.

It has been suggested [10–12] that nonlinear hyperbolic evolution of surface waves in the region of neutral linear stability might explain the onset of complex behavior such as Mach stem formation/kinking of the shock. We pursue here a variant of this idea based instead on interaction between neglected viscous effects and transverse spatial scales.

1.5.2. The refined stability condition

Viscous stability analysis for shocks in the whole space centers about the *Evans function* $D(\tilde{\xi}, \lambda)$, $\tilde{\xi} \in \mathbb{R}^1$, $\lambda = \gamma + i\tau \in \mathbb{C}$, $\tau > 0$, a spectral determinant analogous to the Lopatinski determinant of the inviscid theory, whose zeroes correspond to normal modes $e^{\lambda t}e^{i\tilde\xi x_2}w(x_1)$, of the linearized equations about $\bar u$ (now variable-coefficient), or spectra of the linearized operator about the wave [34,35,18]. The main result of [18], establishing a rigorous relation between viscous and inviscid stability, was the asymptotic expansion

$$D(\tilde\xi,\lambda)=\mu\Delta(\tilde\xi,\lambda)+o(|(\tilde\xi,\lambda)|) \tag{1.16}$$

of D about the origin $(\tilde\xi,\lambda)=(0,0)$, where μ is a constant measuring transversality of $\bar u$ as a connecting orbit of the traveling-wave ODE. Equivalently, considering $D(\tilde\xi,\lambda)=D(\rho\tilde\xi_0,\rho\lambda_0)$ as a function of polar coordinates $(\rho,\tilde\xi_0,\lambda_0)$, we have

$$D|_{\rho=0}=0 \quad\text{and}\quad (\partial/\partial\rho)|_{\rho=0}D=\mu\Delta(\tilde\xi_0,\lambda_0). \tag{1.17}$$

An important consequence of (1.16) is that *weak inviscid stability*, $|\Delta|>0$, *is necessary for weak viscous stability*, $|D|>0$ (an evident necessary condition for linearized viscous stability). For, (1.16) implies that the zero set of D is tangent at the origin to the cone $\{\Delta=0\}$ (recall, (1.14), that Δ is homogeneous, degree one), hence enters $\{\tau>0\}$ if $\{\Delta=0\}$ does. Moreover, in case of *neutral inviscid stability* $\Delta(\xi_0,i\tau_0)=0$, $(\xi_0,i\tau_0)\neq(0,0)$, one may extract a further, *refined stability condition*

$$\Re\beta\ge 0 \quad\text{for}\quad \beta:=-D_{\rho\rho}/D_{\rho\lambda}|_{\rho=0} \tag{1.18}$$

necessary for weak viscous stability. For, (1.17) then implies $D_\rho|_{\rho=0}=\mu\Delta(\xi_0,i\tau_0)=0$, whence Taylor expansion of D yields that the zero level set of D is concave or convex toward $\tau>0$ according as the sign of β; see [18] for details. As discussed in [18,19], the constant β has a heuristic interpretation as an effective diffusion coefficient for surface waves moving along the front.

As shown in [18,16], the formula (1.18) is well-defined whenever Δ is analytic at $(\tilde\xi_0,i\tau)$, in which case D considered as a function of polar coordinates is analytic at $(0,\tilde\xi_0,i\tau_0)$, and $i\tau_0$ is a simple root of $\Delta(\tilde\xi_0,\cdot)$. The determinant Δ in turn is analytic at $(\tilde\xi_0,i\tau_0)$, for all except a finite set of branch singularities $\tau_0=\tilde\xi_0\eta_j$. As discussed in [16,26,32], the apparently nongeneric behavior that the family of holomorphic functions Δ^ε associated with shocks $(u_+^\varepsilon,u_-^\varepsilon)$ have roots $(\xi_0^\varepsilon,i\tau_0(\varepsilon))$ with $i\tau_0$ pure imaginary on an open set of ε is explained by the fact that, on certain components of the complement on the imaginary axis of this finite set of branch singularities, $\Delta^\varepsilon(\xi_0^\varepsilon,\cdot)$ takes the imaginary axis to itself. Thus, zeros of odd multiplicity persist on the imaginary axis, by consideration of the topological degree of Δ^ε as a map from the imaginary axis to itself.

Moreover, the same topological considerations show that a simple imaginary root of this type can only enter or leave the imaginary axis at a branch singularity of $\Delta^\varepsilon(\tilde\xi^\varepsilon,\cdot)$ or at infinity, which greatly aids in the computation of transition points for inviscid stability [16,19,26,32]. As described in [26,32,29], escape to infinity is always associated with transition to strong instability. Indeed, using real homogeneity of Δ, we may rescale by $|\lambda|$ to find in the limit as $|\lambda|\to\infty$ that $0=|\lambda_0|^{-1}\Delta(\tilde\xi_0,\lambda_0)=\Delta(\tilde\xi_0/|\lambda_0|,\lambda_0/|\lambda_0|)\to\Delta(0,i)$, which, by the complex homogeneity $\Delta(0,\lambda)\equiv\lambda\Delta(0,1)$ of the one-dimensional Lopatinski determinant $\Delta(0,\cdot)$, yields *one-dimensional instability* $\Delta(0,1)=0$. As described in [19], Section 6.2, this is associated not with surface waves, but the more dramatic phenomenon of *wave-splitting*, in which the axial structure of the front bifurcates from a single shock to a more complicated multi-wave Riemann pattern.

Example 1.8. For gas dynamics, complex symmetry, $\bar\Delta(\tilde\xi,\lambda)=\Delta(-\tilde\xi,\bar\lambda)$, and rotational invariance, $\Delta(\tilde\xi,\lambda)=\Delta(-\tilde\xi,\lambda)$, imply that

$$\Delta(\tilde\xi,i\tau)=\Delta(|\tilde\xi|^2,|\tau|^2). \tag{1.19}$$

Explicit computation [27,20,19] yields that $\Delta(\tilde\xi_0,\cdot)$ has a pair of branch points of square-root type, located at $|\tau_0|^2=|\tilde\xi_0|^2(M^2-1)$, where M is the downstream Mach number c^2/u^2, where c is sound speed and u the axial particle velocity of the shock on the downstream side, defined as the side in the direction of particle velocity. Transition from strong stability to neutral stability occurs through a pair of simple imaginary zeros entering the imaginary axis at the branch points, and transition from neutral stability to strong instability occurs through escape of these zeros to infinity, with associated one-dimensional instability/wave-splitting.

Remark 1.9. We note in passing that one-dimensional inviscid stability $\Delta^\varepsilon(0,1)\neq 0$ is equivalent to (H3) through the relation

$$\Delta^\varepsilon(0,\lambda)=\lambda\det(r_1^-,\dots,r_c^-,r_{c+1}^+,\dots,r_n^+,u_+-u_-). \tag{1.20}$$

1.5.3. Transverse bifurcation of flow in a duct

We now make an elementary observation connecting cellular bifurcation of flow in a duct to stability of shocks in the whole space: specifically, to violation of the refined stability condition. Assume for the family $\bar u^\varepsilon$ the (transition of) *stability conditions*:

(B1) For ε sufficiently small, the inviscid shock $(u_+^\varepsilon,u_-^\varepsilon)$ is weakly stable; more precisely, $\Delta^\varepsilon(1,\lambda)$ has no roots $\Re\lambda\ge 0$ except for a single simple pure imaginary root $\lambda(\varepsilon)=i\tau_*(\varepsilon)\neq 0$ lying away from the singularities of Δ^ε.

(B2) The refined stability coefficient $\beta(\varepsilon)$ defined in (1.18) satisfies $\Re\beta(0)=0$, $\partial_\varepsilon\Re\beta(0)<0$.

Lemma 1.10 (*[18,19]*). *Assuming* (H0)–(H2) *and* (B1), *for* ε, $\tilde\xi$ *sufficiently small, there exist a smooth family of roots* $(\tilde\xi,\lambda_*^\varepsilon(\tilde\xi))$ *of* $D(\tilde\xi,\lambda)$ *with*

$$\lambda_*^\varepsilon(\tilde\xi)=i\tilde\xi\tau_*(\varepsilon)-\tilde\xi^2\beta(\varepsilon)+\delta(\varepsilon)\tilde\xi^3+r(\varepsilon,\tilde\xi)\tilde\xi^4,$$
$$r\in C^1(\varepsilon,\tilde\xi). \tag{1.21}$$

Moreover, these are the unique roots of D satisfying $\Re\lambda\ge -|\tilde\xi|/C$ for some $C>0$ and $\rho=|(\tilde\xi,\lambda)|$ sufficiently small.

Proof. This follows by the Implicit Function Theorem applied to the function $\check D(\rho,\lambda_0):=\rho^{-1}D(\rho\tilde\xi_0,\rho\lambda_0)$, about the values $(\rho,\tilde\lambda_0)=(0,i\tilde\xi_0\tau_*(\varepsilon))$, where $\tilde\xi_0$ is without loss of generality held fixed, using the facts that D expressed in polar coordinates $(\rho,\tilde\xi_0,\lambda_0)$ satisfies $D|_{\rho=0}\equiv 0$ and $\partial_\rho D|_{\rho=0}\equiv\Delta$, so that D_ρ, $D_{\lambda\lambda}$, and D_λ all vanish at $(0,\tilde\xi_0,\tilde\xi_0 i\tau_*(\varepsilon))$. For details, see the proof of Theorem 3.7, [19]. □

Corollary 1.11. *Assuming* (H0)–(H2) *and* (B1)–(B2), *for* ε, $\tilde\xi$ *sufficiently small, there is a unique C^1 function $\mathcal{E}(\tilde\xi)\neq 0$, $\mathcal{E}(0)=0$, such that $\Re\lambda_*^\varepsilon(\tilde\xi)=0$ for $\varepsilon=\mathcal{E}(\tilde\xi)$. In the generic case $\Re\delta(0)\neq 0$, moreover,*

$$\mathcal{E}(\tilde\xi)\sim(\Re\delta(0)/\partial_\varepsilon\beta(0))\tilde\xi. \tag{1.22}$$

Proof. As a consequence of (1.21), we have for some smooth G

$$\Re\left(\frac{\lambda_*^\varepsilon(\tilde\xi)}{\tilde\xi^2}\right)=-\Re\beta(\varepsilon)+\tilde\xi G(\varepsilon,\tilde\xi), \tag{1.23}$$

$G:=(\delta(\varepsilon)+r(\varepsilon,\tilde\xi))$, whence the equation $0=\Re\left(\frac{\lambda_*^\varepsilon(\tilde\xi)}{\tilde\xi^2}\right)=-\Re\beta(\varepsilon)+\tilde\xi G(\tilde\xi,\varepsilon)$ has a unique root $\varepsilon=\mathcal{E}(\tilde\xi)$ by assumption (B2) and standard scalar bifurcation theory. From $G=\delta(\varepsilon)\tilde\xi+O(|\tilde\xi|^2)$, we find, in the generic case $\Re\delta(0)\neq 0$, that $\partial_{\tilde\xi}(\tilde\xi G)|_{\tilde\xi,\varepsilon=0,0}=\Re\delta(0)\neq 0$, yielding (1.22). □

Remark 1.12. As described further in [16], the above results on the refined stability condition apply also in the case of "real" or partial viscosity, in particular to the physical Navier–Stokes equations of compressible gas dynamics and MHD.

To (B1) and (B2), adjoin now the additional assumptions:

(B3) $\delta(0) \neq 0$.

(B4) At $\varepsilon = 0$, the Evans function $D(\tilde{\xi}, \lambda)$ has no roots $\tilde{\xi} \in \mathbb{R}$, $\Re\lambda \geq 0$ outside a sufficiently small ball about the origin.

Then, we have the following main result.

Theorem 1.13. *Assuming* (H0)–(H2) *and* (B1)–(B4), *for* $\varepsilon_{max} > 0$ *sufficiently small and each cross-sectional width M sufficiently large, there is a finite sequence* $0 < \varepsilon_1(M) < \cdots < \varepsilon_k(M) < \cdots \leq \varepsilon_{max}$, *with* $\varepsilon_k(M) \sim (\Re\delta(0)/\partial_\varepsilon\beta(0))\frac{\pi k}{M}$, *such that, as* ε *crosses successive* ε_k *from the left, there occur a series of transverse (i.e., "cellular") Hopf bifurcations of* $\bar{u}^\varepsilon$ *associated with wave-numbers* $\pm k$, *with successively smaller periods* $T_k(\varepsilon) \sim \tau_*(0)\frac{2M}{k}$.

Proof. By (B4), for $|\varepsilon| \leq \varepsilon_{max}$ sufficiently small, we have by continuity that there exist no roots of $D(\tilde{\xi}, \lambda)$ for $\Re\lambda \geq -1/C$, $C > 0$, outside a small ball about the origin. By Lemma 1.10, within this small ball, there are no roots other than possibly $(\tilde{\xi}, \lambda^\varepsilon(\tilde{\xi}))$ with $\Re\lambda \geq 0$: in particular, *no nonzero purely imaginary spectra are possible other than at values* $\lambda^\varepsilon(\tilde{\xi})$ for operator $L(\varepsilon)$ acting on functions on the whole space.

Considering L instead as an operator acting on functions on the channel $\mathcal{C} := \{x : (x_1, x_2) \in \mathbb{R}^1 \times [-M, M]\}$, we find by discrete Fourier transform/separation of variables that its spectra are exactly the zeros of $D(\xi_k, \lambda)$, as $\xi_k = \frac{\pi k}{M}$ runs through all integer wave-numbers k; see (1.13). Applying Corollary 1.11, and using (B3), we find, therefore, that pure imaginary eigenvalues of $L(\varepsilon)$ with $|\varepsilon| \leq \varepsilon_{max}$ sufficiently small occur precisely at values $\varepsilon = \varepsilon_k$, and consist of crossing conjugate pairs $\lambda^k_\pm(\varepsilon)$ associated with wave-numbers $\pm k$, satisfying Hopf bifurcation condition (D2) with

$$\Im\lambda^k_\pm(\varepsilon) \sim \tau_*(0)\frac{\pi k}{M}.$$

Applying Proposition 1.4, we obtain the result. □

Remark 1.14. As evidenced by decreasing periods T_k, this phenomenon of increasing-complexity solutions is completely different from the more familiar one of period-doubling.

Remark 1.15. Lemma 1.10 and Corollary 1.11 are readily generalized to the case with (B1) replaced by (B1′) For ε sufficiently small, the inviscid shock $(u^\varepsilon_+, u^\varepsilon_-)$ is weakly stable, with all pure imaginary roots simple and lying away from the singularities of Δ^ε. In this case we obtain a family of roots/crossings of the imaginary axis, one for each imaginary root of Δ^ε. In particular, for gas dynamics, due to rotational invariance (see Example 1.8), we obtain families of four crossing eigenvalues $\lambda_\pm(\varepsilon_k)$, with each of λ_+ and λ_- occurring at both wave-numbers k and $-k$. This is not a standard Hopf bifurcation, but a more complicated version with $O(2)$ symmetry, and so we cannot apply directly Theorem 1.13.

1.6. Discussion and open problems

We have presented in a simple setting a rigorous mathematical demonstration of a mechanism by which destabilization of hyperbolic surface waves arising in the inviscid shock stability problem in the whole space can, at appropriate transverse length scales, lead to Hopf bifurcation of a viscous shock in a finite-cross-section duct: specifically, destabilization of the effective transverse viscosity β investigated in [18,16], or violation of the *refined stability condition*. This appears to be a fundamentally different mechanism than the hyperbolic ones proposed by Majda et al. [10–12] via weakly nonlinear geometric optics.

We point out that as shock parameters cross the inviscid strong instability boundary, gas-dynamical shocks undergo one-dimensional instability, or *wave-splitting*, a more dramatic change in front topology than the cellular instabilities we seek to investigate. Thus, cellular instability must occur *before* the strong instability boundary is reached. Experimental observations, though not conclusive, indicate that nonetheless it occurs near the strong instability boundary [1], suggesting that it lies in the region of neutral inviscid stability as we have conjectured.

More, if the transition to cellular instability occurs at low frequencies, it *must* occur by the scenario described, or else the viscous shock would remain stable up to the point of wave-splitting. If, on the other hand, it occurs at high frequencies, then as pointed out in [16], then it necessarily involves Hopf bifurcation, by one-dimensional inviscid stability, (H2) (satisfied for typical equations of state). Thus, it would appear quite promising to search for Hopf bifurcations in the region of neutral inviscid stability, whether of the "low-frequency" type studied here or a "high-frequency" type involving unknown mechanisms. This would be a very interesting direction for numerical investigations, for example by the numerical Evans function techniques of [36–41].

Another interesting direction would be investigation of the stability coefficient β, both numerically and analytically. As pointed out in [16], this is numerically quite well-conditioned. One might also consider attempting to carry out an asymptotic analysis near the endpoints of the region of neutral stability, at which the imaginary root τ_* approaches either a branch point of Δ or else infinity.

At a technical level, an interesting open problem is to carry out a bifurcation analysis in the rotationally symmetric case, for example, for gas dynamics, in which the bifurcation associated with crossing $\lambda_\pm$ is no longer a standard Hopf bifurcation but a more complicated type involving $O(2)$ symmetry. For a description of Hopf bifurcation with $O(2)$ symmetry, see, for example, [42]. For a circular cross-section, there is besides axial translation an additional continuous group invariance of rotation in the transverse direction, leading to the possibility of "spinning" instabilities. These degenerate cases require further analysis at the level of the finite-dimensional reduced equations; however, the initial reduction to finite dimensions, as carried out in [13], is essentially the same. Other open problems are to extend to detonations, as done for the one-dimensional case in [15] and to treat also real, or partial viscosities. As discussed in [13,14], the latter problem involves interesting issues involving Lagrangian vs. Eulerian formulations.

2. Conditional stability analysis

Nonlinear stability follows quite similarly as in the one-dimensional case [17], decomposing behavior into a one-dimensional (averaged in x_2) flow driving time-exponentially damped transverse modes.[1]

Define the perturbation variable

$$v(x, t) := u(x_1 + \alpha(t), x_2, t) - \bar{u}(x_1) \tag{2.1}$$

for u a solution of (1.2)–(1.3), where α is to be specified later. Subtracting the equations for $u(x_1 + \alpha(t), x_2, t)$ and $\bar{u}(x_1)$, we

[1] Indeed, though we do not do it here, this prescription could be followed quite literally at the nonlinear level.

obtain the nonlinear perturbation equation

$$v_t - Lv = \sum_{j=1}^{2} N_j(v)_{x_j} + \partial_t \alpha(\bar{u}_{x_1} + \partial_{x_1} v), \tag{2.2}$$

where

$$L := \Delta_x - \sum_{j=1}^{2} \partial_{x_j} A^j(x), \quad A^j := df_j(\bar{u}) \tag{2.3}$$

denotes the linearized operator about $\bar{u}$ and $N_j(v) := -(f^j(\bar{u} + v) - f^j(\bar{u}) - df^j(\bar{u})v)$, where, so long as $|v|_{H^1}$ (hence $|v|_{L^\infty}$ and $|u|_{L^\infty}$) remains bounded,

$$\begin{aligned} &N^j(v) = O(|v|^2), \qquad \partial_x N^j(v) = O(|v||\partial_x v|), \\ &\partial_x^2 N^j(v) = O(|\partial_x v|^2 + |v||\partial_x^2 v|). \end{aligned} \tag{2.4}$$

2.1. Projector bounds

Let Π_u denote the eigenprojection of L onto its unstable subspace Σ_u, and $\Pi_{cs} = \mathrm{Id} - \Pi_u$ the eigenprojection onto its center stable subspace Σ_{cs}.

Lemma 2.1. *Assuming* (H0)–(H1), *there is $\tilde{\Pi}_j$ defined in (2.7) such that*

$$\Pi_j \partial_x = \partial_x \tilde{\Pi}_j \tag{2.5}$$

for $j = u,\ cs$ and, for all $1 \le p \le \infty$, $0 \le r \le 4$,

$$\begin{aligned} &|\Pi_u|_{L^p \to W^{r,p}},\ |\tilde{\Pi}_u|_{L^p \to W^{r,p}} \le C, \\ &|\tilde{\Pi}_{cs}|_{W^{r,p} \to W^{r,p}},\ \ |\tilde{\Pi}_{cs}|_{W^{r,p} \to W^{r,p}} \le C. \end{aligned} \tag{2.6}$$

Proof. Recalling that L has at most finitely many unstable eigenvalues, we find that Π_u may be expressed as

$$\Pi_u f = \sum_{j=1}^{p} \phi_j(x) \langle \tilde{\phi}_j, f \rangle,$$

where ϕ_j, $j = 1, \dots p$ are generalized right eigenfunctions of L associated with unstable eigenvalues λ_j, satisfying the generalized eigenvalue equation $(L - \lambda_j)^{r_j} \phi_j = 0$, $r_j \ge 1$, and $\tilde{\phi}_j$ are generalized left eigenfunctions. Noting that L is divergence form, and that $\lambda_j \neq 0$, we may integrate $(L - \lambda_j)^{r_j} \phi_j = 0$ over $\mathbb{R}$ to obtain $\lambda_j^{r_j} \int \phi_j \mathrm{d}x = 0$ and thus $\int \phi_j \mathrm{d}x = 0$. Noting that ϕ_j, $\tilde{\phi}_j$ and derivatives decay exponentially in x_1 by separation of variables and standard one-dimensional theory [24,23,43], we find that $\phi_j = \partial_x \Phi_j$ with Φ_j and derivatives exponentially decaying in x_1, hence

$$\tilde{\Pi}_u f = \sum_j \Phi_j \langle \partial_x \tilde{\phi}, f \rangle. \tag{2.7}$$

Estimating $|\partial_x^j \Pi_u f|_{L^p} = |\sum_j \partial_x^j \phi_j \langle \tilde{\phi}_j f \rangle|_{L^p} \le \sum_j |\partial_x^j \phi_j|_{L^p} |\tilde{\phi}_j|_{L^q} |f|_{L^p} \le C|f|_{L^p}$ for $1/p + 1/q = 1$ and similarly for $\partial_x^r \tilde{\Pi}_u f$, we obtain the claimed bounds on Π_u and $\tilde{\Pi}_u$, from which the bounds on $\Pi_{cs} = \mathrm{Id} - \Pi_u$ and $\tilde{\Pi}_{cs} = \mathrm{Id} - \tilde{\Pi}_u$ follow immediately. □

2.2. Linear estimates

Let $G_{cs}(x, t; y) := \Pi_{cs} \mathrm{e}^{Lt} \delta_y(x)$ denote the Green kernel of the linearized solution operator on the center stable subspace Σ_{cs}. Then, we have the following detailed pointwise bounds established in [13,43].

Proposition 2.2 ([13,43]). *Assuming* (H0)–(H2), (D1)–(D3), *the center stable Green function may be decomposed as $G_{cs} = E + \tilde{G}$, where*

$$E(x, t; y) = \partial_{x_1} \bar{u}(x_1) e(y_1, t), \tag{2.8}$$

$$\begin{aligned} e(y_1, t) \ = \ \sum_{a_k^- > 0} \Bigg(&\mathrm{errfn}\left(\frac{y_1 + a_k^- t}{\sqrt{4(t+1)}} \right) \\ &- \ \mathrm{errfn}\left(\frac{y_1 - a_k^- t}{\sqrt{4(t+1)}} \right) \Bigg) l_k^-(y_1) \end{aligned} \tag{2.9}$$

for $y_1 \le 0$ and symmetrically for $y_1 \ge 0$, $l_k^- \in \mathbb{R}^n$ constant, and $a_j^\pm$ are the eigenvalues of $df(u_\pm)$, and

$$\left| \int_{\mathcal{C}} \partial_x^s \tilde{G}(\cdot, t; y) f(y) \mathrm{d}y \right|_{L^p} \le C(1 + t^{-\frac{s}{2}}) t^{-\frac{1}{2}(\frac{1}{q} - \frac{1}{p})} |f|_{L^p \cap L^q}, \tag{2.10}$$

$$\left| \int_{\mathcal{C}} \partial_x^s \tilde{G}_y(\cdot, t; y) f(y) \mathrm{d}y \right|_{L^p} \le C(1 + t^{-\frac{s}{2}}) t^{-\frac{1}{2}(\frac{1}{q} - \frac{1}{p}) - \frac{1}{2}} |f|_{L^p \cap L^q}, \tag{2.11}$$

for all $t \ge 0$, $0 \le s \le 2$, some $C > 0$, for any $1 \le q \le p$ and $f \in L^q \cap L^p$.

Proof. As observed in [13], it is equivalent to establish decomposition

$$G = G_u + E + \tilde{G} \tag{2.12}$$

for the full Green function $G(x, t; y) := \mathrm{e}^{Lt} \delta_y(x)$, where

$$G_u(x, t; y) := \Pi_u \mathrm{e}^{Lt} \delta_y(x)$$

denotes the Green kernel of the linearized solution operator on Σ_u.

Using separation of variables, moreover, we may decompose $G = \sum_k G^k$, where G^k is the Green function acting on Fourier modes of wave number k, i.e., $G^k = \mathcal{F}^{-1} G \delta(\tilde{\xi} - \pi k/M) \mathcal{F}$, where $\mathcal{F}$ denotes Fourier transform in x_2, and $\tilde{\xi}$ Fourier frequency. This reduces the problem to that of deriving the asserted bounds separately on the one-dimensional Green function G^0 and on the complement $\sum_{k \neq 0} G^k$, where the difficulty, due to lack of spectral gap, is concentrated in the estimation of the one-dimensional part G^0. Here, we are using also the fact that the Fourier projection $\Pi_0 f(x) \equiv \frac{1}{2M} \int_{-M}^{M} f(y_2) \mathrm{d}y_2$ of a function in x_2 onto the subspace of zero wave number functions satisfies $|\Pi_0 f|_{L^p(x_2)} \le |f|_{L^1(x_2)} \le |f|_{L^p(x_2)}$ for any $p \ge 1$, by Hölder's inequality, so that Π_0 and its complementary projection $I - \Pi_0$ are bounded as operators on $L^p(x_2)$.

The bounds on the one-dimensional Green function G^0 have already been established in [17], Proposition 4.2, by essentially the same stationary phase estimates used in [44] in the stable case $\Pi_u = 0$; see [13,17] for further discussion. The bounds on the complement $\sum_{k \neq 0} G^k$ follow by the straightforward semigroup estimate $|\mathrm{e}^{\tilde{L}t} f|_{L^p} \le C\mathrm{e}^{-\eta t} |f|_{L^p}$, $\eta > 0$, where $\tilde{L}$ denotes the projection of L onto the intersection of its center stable subspace and the subspace of functions with transverse Fourier wave numbers $\neq 0$, which evidently has a nonzero spectral gap $\sigma(\tilde{L}) \le -2\eta < 0$ for some $\eta > 0$; see [13] for related computations. □

Corollary 2.3 ([17]). *The kernel e satisfies for all $t > 0$*

$$\begin{aligned} &|e_y(\cdot, t)|_{L^p},\ |e_t(\cdot, t)|_{L^p} \le Ct^{-\frac{1}{2}(1 - 1/p)}, \\ &|e_{ty}(\cdot, t)|_{L^p} \le Ct^{-\frac{1}{2}(1 - 1/p) - 1/2}. \end{aligned}$$

Proof. Direct computation using definition (2.9). □

2.3. Reduced equations

Recalling that $\partial_{x_1}\bar{u}$ is a stationary solution of the linearized equations $u_t = Lu$, so that $L\partial_{x_1}\bar{u} = 0$, or

$$\int_{\mathcal{C}} G(x,t;y)\bar{u}_{x_1}(y_1)dy = e^{Lt}\bar{u}_{x_1}(x_1) = \partial_{x_1}\bar{u}(x_1),$$

we have, applying Duhamel's principle to (2.2),

$$v(x,t) = \int_{\mathcal{C}} G(x,t;y)v_0(y)\,dy - \int_0^t \int_{\mathcal{C}} G_y(x,t-s;y)(N(v)+\dot{\alpha}v)(y,s)\,dy\,ds + \alpha(t)\partial_{x_1}\bar{u}(x_1).$$

Defining

$$\alpha(t) = -\int_{\mathcal{C}} e(y,t)v_0(y)\,dy + \int_0^t \int_{\mathcal{C}} e_y(y,t-s)(N(v)+\dot{\alpha}\,v)(y,s)dyds, \tag{2.13}$$

following [23,17,45,44], where e is defined as in (2.9), and recalling the decomposition $G = E + G_u + \tilde{G}$ of (2.12), we obtain the *reduced equations*

$$v(x,t) = \int_{\mathcal{C}} (G_u+\tilde{G})(x,t;y)v_0(y)\,dy - \int_0^t \int_{\mathcal{C}} (G_u+\tilde{G})_y(x,t-s;y)(N(v)+\dot{\alpha}v)(y,s)dy\,ds, \tag{2.14}$$

and, differentiating (2.13) with respect to t, and observing that $e_y(y_1,s) \to 0$ as $s \to 0$, as the difference of approaching heat kernels,

$$\dot{\alpha}(t) = -\int_{\mathcal{C}} e_t(y,t)v_0(y)\,dy + \int_0^t \int_{\mathcal{C}} e_{yt}(y,t-s)(N(v)+\dot{\alpha}v)(y,s)\,dy\,ds. \tag{2.15}$$

2.4. Nonlinear damping estimate

Proposition 2.4 ([44]). *Assuming* (H0)–(H3), *let $v_0 \in H^2$, and suppose that for $0 \le t \le T$, the H^2 norm of v remains bounded by a sufficiently small constant, for v as in (2.1) and u a solution of (1.2)–(1.3). Then, for some constants $\theta_{1,2} > 0$, for all $0 \le t \le T$,*

$$\|v(t)\|_{H^2}^2 \le Ce^{-\theta_1 t}\|v(0)\|_{H^2}^2 + C\int_0^t e^{-\theta_2(t-s)}(|v|_{L^2}^2 + |\dot{\alpha}|^2)(s)\,ds. \tag{2.16}$$

Proof. Energy estimates identical with those of the one-dimensional proof in [17], using the fact that boundary terms in x_2 are identically zero due to periodic boundary conditions. □

2.5. Proof of nonlinear stability

Decompose the nonlinear perturbation v as

$$v(x,t) = w(x,t) + z(x,t), \tag{2.17}$$

where

$$w := \Pi_{cs}v, \qquad z := \Pi_u v. \tag{2.18}$$

Applying Π_{cs} to (2.14) and recalling commutator relation (2.5), we obtain an equation

$$w(x,t) = \int_{\mathcal{C}} \tilde{G}(x,t;y)w_0(y)\,dy - \int_0^t \int_{\mathcal{C}} \tilde{G}_y(x,t-s;y)\tilde{\Pi}_{cs}(N(v)+\dot{\alpha}v)(y,s)dy\,ds \tag{2.19}$$

for the flow along the center stable manifold, parametrized by $w \in \Sigma_{cs}$.

Lemma 2.5. *Assuming* (H0)–(H1), *for v lying initially on the center stable manifold $\mathcal{M}_{cs}$,*

$$|z|_{W^{r,p}} \le C|w|_{H^2}^2 \tag{2.20}$$

for some $C > 0$, for all $1 \le p \le \infty$ and $0 \le r \le 4$, so long as $|w|_{H^2}$ remains sufficiently small.

Proof. By tangency of the center stable manifold to Σ_{cs}, we have immediately $|z|_{H^2} \le C|w|_{H^2}^2$, whence (2.20) follows by equivalence of norms for finite-dimensional vector spaces, applied to the p-dimensional subspace Σ_u. (Alternatively, we may see this by direct computation using the explicit description of $\Pi_u v$ afforded by Lemma 2.1.) □

Proof of Theorem 1.2. Recalling by Proposition 1.1 that solutions remaining for all time in a sufficiently small radius neighborhood $\mathcal{N}$ of the set of translates of $\bar{u}$ lie in the center stable manifold $\mathcal{M}_{cs}$, we obtain trivially that solutions not originating in $\mathcal{M}_{cs}$ must exit $\mathcal{N}$ in finite time, verifying the final assertion of orbital instability with respect to perturbations not in $\mathcal{M}_{cs}$.

Consider now a solution $v \in \mathcal{M}_{cs}$, or, equivalently, a solution $w \in \Sigma_{cs}$ of (2.19) with $z = \Phi_{cs}(w) \in \Sigma_u$. Define

$$\zeta(t) := \sup_{0\le s\le t}\left(|w|_{H^2}(1+s)^{\frac{1}{4}} + (|w|_{L^\infty} + |\dot{\alpha}(s)|)(1+s)^{\frac{1}{2}}\right). \tag{2.21}$$

We shall establish:

Claim. For all $t \ge 0$ for which a solution exists with ζ uniformly bounded by some fixed, sufficiently small constant, there holds

$$\zeta(t) \le C_2(E_0 + \zeta(t)^2) \quad \text{for } E_0 := |v_0|_{L^1\cap H^2}. \tag{2.22}$$

From this result, provided $E_0 < 1/4C_2^2$, we have that $\zeta(t) \le 2C_2E_0$ implies $\zeta(t) < 2C_2E_0$, and so we may conclude by continuous induction that

$$\zeta(t) < 2C_2E_0 \tag{2.23}$$

for all $t \ge 0$, from which we readily obtain the stated bounds. (By standard short-time H^s existence theory, $v \in H^2$ exists and ζ remains continuous so long as ζ remains bounded by some uniform constant, hence (2.23) is an open condition.)

Proof of Claim. By (2.6), $|w_0|_{L^1\cap H^2} = |\Pi_{cs}v_0|_{L^1\cap H^2} \le CE_0$. Likewise, by Lemma 2.5, (2.21), (2.4), and Lemma 2.1, for $0 \le s \le t$,

$$|\tilde{\Pi}_{cs}(N(v)+\dot{\alpha}v)(y,s)|_{L^2} \le C\zeta(t)^2(1+s)^{-\frac{3}{4}}. \tag{2.24}$$

Combining the latter bounds with representations (2.19) and (2.15) and applying Proposition 2.2, we obtain

$$\begin{aligned} |w(x,t)|_{L^p} &\le \left|\int_{\mathcal{C}} \tilde{G}(x,t;y)w_0(y)\,dy\right|_{L^p} + \left|\int_0^t \int_{\mathcal{C}} \tilde{G}_y(x,t-s;y)\tilde{\Pi}_{cs}(N(v)+\dot{\alpha}v)(y,s)dy\,ds\right|_{L^p} \\ &\le E_0(1+t)^{-\frac{1}{2}(1-\frac{1}{p})} + C\zeta(t)^2\int_0^t (t-s)^{-\frac{3}{4}+\frac{1}{2p}}(1+s)^{-\frac{3}{4}}dy\,ds \\ &\le C(E_0+\zeta(t)^2)(1+t)^{-\frac{1}{2}(1-\frac{1}{p})} \end{aligned} \tag{2.25}$$

and, similarly, using Hölder's inequality and applying Corollary 2.3,

$$\begin{aligned}
|\dot\alpha(t)| &\le \int_{\mathcal{C}} |e_t(y,t)||v_0(y)|\,dy \\
&\quad + \int_0^t \int_{\mathcal{C}} |e_{yt}(y,t-s)||(N(v)+\dot\alpha v)(y,s)|\,dy\,ds \\
&\le |e_t|_{L^\infty}|v_0|_{L^1} + C\zeta(t)^2 \int_0^t |e_{yt}|_{L^2}(t-s)|(N(v)+\dot\alpha v)|_{L^2}(s)ds \\
&\le E_0(1+t)^{-\frac12} + C\zeta(t)^2 \int_0^t (t-s)^{-\frac34}(1+s)^{-\frac34}ds \\
&\le C(E_0+\zeta(t)^2)(1+t)^{-\frac12}. \qquad (2.26)
\end{aligned}$$

By Lemma 2.5,

$$|z|_{H^2}(t) \le C|w|^2_{H^2}(t) \le C\zeta(t)^2. \qquad (2.27)$$

In particular, $|z|_{L^2}(t) \le C\zeta(t)^2(1+t)^{-\frac12}$. Applying Proposition 2.4 and using (2.25) and (2.26), we thus obtain

$$|w|_{H^2}(t) \le C(E_0+\zeta(t)^2)(1+t)^{-\frac14}. \qquad (2.28)$$

Combining (2.25), (2.26), and (2.28), we obtain (2.22) as claimed. As discussed earlier, from (2.22), we obtain by continuous induction (2.23), or $\zeta \le 2C_2|v_0|_{L^1\cap H^2}$, whereupon the claimed bounds on $|v|_{L^p}$ and $|v|_{H^2}$ follow by (2.25) and (2.28), and on $|\dot\alpha|$ by (2.26). Finally, a computation parallel to (2.26) (see, e.g., [44,26]) yields $|\alpha(t)| \le C(E_0+\zeta(t)^2)$, from which we obtain the last remaining bound on $|\alpha(t)|$. □

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