

Computation of domains of analyticity for the dissipative standard map in the limit of small dissipation

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HIGHLIGHTS

- We study numerically the domain of analyticity of invariant circles.
- Computations suggest that the domain's boundary may have a self-similar structure.
- We conjecture Gevrey regularity with respect to the small dissipation parameter.

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ABSTRACT

Conformally symplectic systems include mechanical systems with a friction proportional to the velocity. Geometrically, these systems transform a symplectic form into a multiple of itself making the systems dissipative or expanding. In the present work we consider the limit of small dissipation. The example we study is a family of conformally symplectic standard maps of the cylinder for which the conformal factor, $b(\varepsilon)$, is a function of a small complex parameter, ε .

We assume that for $\varepsilon = 0$ the map preserves the symplectic form and the dependence on ε is cubic, i.e., $b(\varepsilon) = 1 - \varepsilon^3$. We compute perturbative expansions formally in ε and use them to estimate the shape of the domains of analyticity of invariant circles as functions of ε . We also give evidence that the functions might belong to a Gevrey class at $\varepsilon = 0$. We also perform numerical continuation of the solutions as they pass through the boundary of the domain to illustrate that the monodromy of the solutions is trivial. The numerical computations we perform support conjectures on the shape of the domains of analyticity.

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1. Introduction

We study the limit of small dissipation/expansion of a family of conformally symplectic standard maps. Since conformally symplectic systems include Hamiltonian systems when one adds a small friction term that is proportional to the velocity, then this kind of systems is relevant in applications [1]. In particular, we approximate the shape of domains of analyticity of invariant circles of a family of conformally symplectic standard maps of the cylinder, $\mathcal{M} = \mathbb{S}^1 \times \mathbb{R}$, depending on a small parameter, ε , that vanishes as the conformal factor tends to one.

The present work addresses some rigorous results and conjectures in [2] from numerical point of view. For instance, it is remarkable that invariant circles which are attractors/repellers in the dissipative/expanding case [3], converge in the limit of small

dissipation to invariant circles in the symplectic case. It was noted in [2] that the small divisors depend on the complex parameter ε and give rise to regions where the functions parameterizing the circles cannot be analytic with respect to ε but miss by very little. A conjecture in [2] states that the tori are analytic in a domain in the complex ε plane that is obtained by taking from a ball centered at zero, a sequence of small balls with centers along smooth curves passing through the origin. The radii of the excluded balls decrease faster than any power of the distance of the centers of the balls to the origin, see Fig. 1. In fact, it was rigorously proved in [2] that this domain is a lower bound. The main objective of the present work is to illustrate the results in one example, provide numerical evidence and indications of new results. Our computations indicate that there are singularities which cluster around several points at which one does not expect the functions to be analytic.

A common method to compute invariant circles of a map of the cylinder $f_\varepsilon : \mathcal{M} \rightarrow \mathcal{M}$, is by computing a parameterization $K_\varepsilon : \mathbb{S}^1 \rightarrow \mathcal{M}$ of the invariant circle which satisfies an invariance

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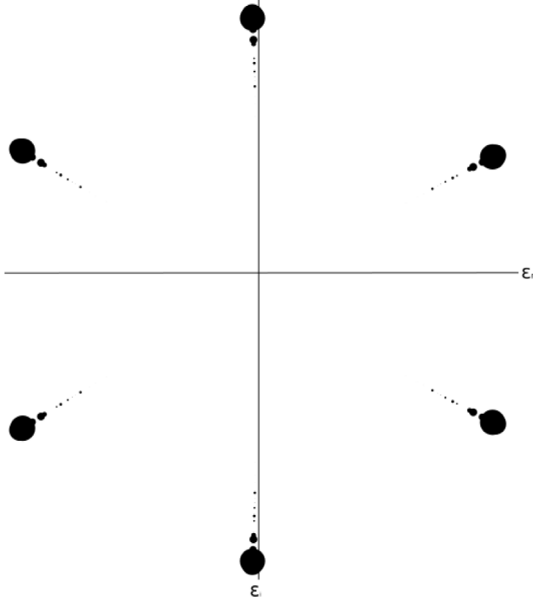


Fig. 1. Domain of analyticity according to [2].

equation. The invariance equation is

$$f_\varepsilon \circ K_\varepsilon = K_\varepsilon \circ T_\omega$$

with $T_\omega(\theta) = (\theta + \omega)$. The invariance equation states that the dynamics on the invariant circle are conjugated to a rigid rotation of the circle by an irrational number ω . One of the advantages of working with the dissipative standard map is that the parameterization function K_ε can be written in terms of a periodic function, $u_\varepsilon : \mathbb{S}^1 \rightarrow \mathbb{R}$, as it is explained in Section 2.1. The method we use to find the singularities is to approximate the conjugacy function $u_\varepsilon(\theta)$ by means of a Lindstedt series expansion in ε . The Lindstedt method produces polynomials in ε of high order,

$$u_\varepsilon^{\leq N}(\theta) = \sum_{n=0}^N u_n(\theta) \varepsilon^n$$

with $N \approx 10^3$, whose coefficients $u_n : \mathbb{S}^1 \rightarrow \mathbb{C}$ are periodic functions. We then use the Lindstedt series of the conjugacies to obtain Padé rational functions whose poles are expected to approximate the poles of the original function u_ε . Padé extrapolation methods of Lindstedt series have been widely used by several authors [4–8] in the symplectic case. Since the Padé extrapolation method is based on approximating an analytic function with a rational function, the computation of poles is done by approximating the roots of the denominator of the Padé function. The denominator is a polynomial that can be of very high degree, and computing its roots depends heavily on numerical precision. Since the computations are very sensitive to precision, we perform them using $\approx 10^3$ digits which allows us to compute singularities for values of ε that are at a distance ≈ 0.3 from $\varepsilon = 0$ in the complex plane. We expect that higher precision together with higher order degree series, would allow us to compute poles that are closer to the origin. However, the method already allows us to have an approximation of the boundary of the domain in regions at which one does not expect the functions to be analytic, as it was predicted by the conjecture in [2], even when the singularities are not very close to $\varepsilon = 0$. For this reason it is very hard to notice that the functions that we are computing are not analytic. We also find conjectures on the rate of growth of the terms of the Lindstedt series.

We note that the shapes of the domains that we present here are remarkably different from what one sees in the symplectic case, see [6–9]. This is partly due to the fact that in the symplectic case or in the dissipative case, the small divisors do not depend on the conformal factor $b(\varepsilon)$ which in our case is a function of ε .

Some explorations of the shape of the analyticity domains in the dissipative standard map have been performed using the parameterization method in [10], that is very similar to the one described in Section 3.3. In [10], it is noticed that the breakdown of invariant tori in the conservative and the dissipative case are similar when the conformal factor b is a constant, [11]. A different behavior in the breakdown of invariant tori involving bundle collapse is observed in the dissipative standard map in [12]. Explorations of the shape of the domains of analyticity in ε in the conservative case with the use of the parameterization method appear in [9,13].

2. Preliminaries

We consider the dissipative standard map defined on the cylinder $\mathcal{M} = \mathbb{S}^1 \times \mathbb{R}$ given by $f_\varepsilon(x_n, y_n) = (x_{n+1}, y_{n+1})$ and

$$\begin{aligned} y_{n+1} &= b_\varepsilon y_n + c_\varepsilon + \varepsilon V'(x_n) \\ x_{n+1} &= x_n + y_{n+1}, \end{aligned} \quad (2.1)$$

where $y_n \in \mathbb{R}$, $x_n \in \mathbb{S}^1$, $\varepsilon \in \mathbb{R}$, and $V'(x) = \frac{1}{2\pi} \sin(2\pi x)$. Here we consider the case when the dissipative parameter, b_ε , is given by $b_\varepsilon = b(\varepsilon) = 1 - \varepsilon^3$, and the drift parameter $c_\varepsilon = c(\varepsilon)$ is a function that depends on the small parameter ε . Note that adding a dissipation to the system is a very singular perturbation and could lead to the creation of attractors/repellers without quasi periodic motions. For that reason, one has to consider this external parameter, c_ε . The dissipative parameter b_ε coincides with the Jacobian of the function. We note that adding an odd power of epsilon to the b_ε term is the physically relevant choice. For this work, we choose to include a third power since it is the first non-trivial odd case. We note that the Jacobian is the rate of dissipation/expansion of the map (2.1), this rate will be dependent of the parameter ε . In particular, the case $\varepsilon = 0$ coincides with the zero dissipation limit. We must emphasize that we tie the parameter ε to the nonlinearity of the map since in this case $\varepsilon = 0$ also coincides with the integrable case and that will hugely simplify our computation. In particular, by doing this we make sure that the symplectic case, which will be the zero-th order of our series, will be trivial.

In fact, it is discussed in [2] that (2.1) is conformally symplectic. If $\Omega = dy \wedge dx$ is the standard symplectic form of the cylinder, the map f_ε satisfies that

$$f_\varepsilon^* \Omega = b_\varepsilon \Omega. \quad (2.2)$$

For certain values of c_ε , we know that maps of the form (2.2) have analytic invariant circles corresponding to quasi-periodic orbits with Diophantine rotation numbers, ω . The Lindstedt series analysis in Section 3.1 determines that one condition for the mapping (2.1) to admit an invariant circle is that $c_\varepsilon = \omega \varepsilon^3 + \mathcal{O}(\varepsilon^4)$. In the following, we discuss the properties that the rotation number should satisfy so that one can have quasi-periodic orbits parameterized by a function.

2.1. Quasi-periodic orbits

We consider a frequency ω that satisfies the Diophantine condition,

$$|\omega q - p| \geq \nu |q|^{-\tau}, \quad p \in \mathbb{Z}, \quad q \in \mathbb{Z} \setminus \{0\} \quad (2.3)$$

where $\nu \in \mathbb{R}^+$ and $\tau \in \mathbb{R}$ with $\tau \geq 1$.

Quasi periodic orbits of the dissipative standard map (2.1) are found using a parametric representation of the variable $x_n \in \mathbb{S}^1$ as

$$x_n = \theta_n + u_\varepsilon(\theta_n), \quad \theta \in \mathbb{S}^1, \quad (2.4)$$

where $u_\varepsilon : \mathbb{S}^1 \rightarrow \mathbb{R}$ is a 1-periodic function. We assume that the variable θ_n varies linearly as $\theta_{n+1} = \theta_n + \omega$ where ω is the rotation frequency.

It follows from Eq. (2.1) that

$$x_{n+1} - (1 + b_\varepsilon)x_n + b_\varepsilon x_{n-1} - c_\varepsilon + \varepsilon V'(x_n) = 0. \quad (2.5)$$

We look for quasi periodic solutions by finding u_ε and $c_\varepsilon = c(\varepsilon)$ such that

$$E_{c_\varepsilon}[u_\varepsilon] \equiv u_\varepsilon(\theta + \omega) - (1 + b_\varepsilon)u_\varepsilon(\theta) + b_\varepsilon u_\varepsilon(\theta - \omega) + (1 - b_\varepsilon)\omega - c_\varepsilon + \varepsilon V'(\theta + u_\varepsilon(\theta)) = 0. \quad (2.6)$$

We remark that the nature of the two unknowns is different since $u_\varepsilon(\theta)$ is a smooth complex 1-periodic function of $\theta \in \mathbb{S}^1$ depending on the complex parameter ε and c_ε is a complex number depending on ε . The conjecture in [2], states that ε is a complex parameter whose range lays in a complex domain that is obtained by taking out from a neighborhood of $\varepsilon = 0$, points inside balls with centers along smooth curves passing through the origin. In [2] there is also a rigorous proof that the domain described in the conjecture is a lower bound that approximates the exact domain of analyticity.

It is clear that once we find a pair $(u_\varepsilon, c_\varepsilon)$ satisfying (2.6), we can recover the embedding of the quasi-periodic orbit by the parameterization $K_\varepsilon : \mathbb{S}^1 \rightarrow \mathcal{M}$,

$$K_\varepsilon(\theta) = \begin{pmatrix} \theta + u_\varepsilon(\theta) \\ \omega + u_\varepsilon(\theta) - u_\varepsilon(\theta - \omega) \end{pmatrix}. \quad (2.7)$$

3. Methods for computing solutions

We will use two different methods for finding the solution pair $(u_\varepsilon, c_\varepsilon)$ of (2.6). The first method is based on a Lindstedt series approximation of the solutions written as formal power series of the small parameter ε . In our case the small parameter ε will account both for the size of the perturbation and the distance of the conformal factor to the symplectic case. This method produces approximate solutions in the sense that if

$$u_\varepsilon^{\leq N}(\theta) = \sum_{k=0}^N u_k(\theta)\varepsilon^k \quad \text{and} \quad c_\varepsilon^{\leq N}(\varepsilon) = \sum_{k=0}^N c_k\varepsilon^k \quad (3.1)$$

are polynomials in ε , we say that (3.1) is an approximate solution of order N whenever $\|E_{c_\varepsilon^{\leq N}(\varepsilon)}[u_\varepsilon^{\leq N}]\| = O(\varepsilon^{N+1})$, where E is the functional defined in (2.6) and $\|\cdot\|$ is the supremum norm over all $\theta \in \mathbb{S}^1$. The Lindstedt series method that we describe in Section 3.1 provides a way to construct an approximate solution of any given order $N \in \mathbb{N}$.

In Section 3.3, we include an algorithm to find the solution $(u_\varepsilon, c_\varepsilon)$ by means of a Newton method. The method starts from an approximate solution pair (u_a, c_a) so that the norm of $E_{c_a}[u_a]$ is small and provides a correction (v, δ) by imposing that the new solution $(u_a + v, c_a + \delta)$ satisfies the functional equation $E_{c_a+\delta}[u_a + v]$ up to first order in (v, δ) . This method can be shown to converge using scales of Banach spaces.

3.1. Lindstedt series

The Lindstedt series method consists of performing a formal series expansion in a small parameter ε . According to (2.6), and the fact that $b(\varepsilon) = 1 - \varepsilon^3$, we look for a solution, $(u_\varepsilon, c_\varepsilon)$, of

$$u_\varepsilon(\theta + \omega) - (2 - \varepsilon^3)u_\varepsilon(\theta) + (1 - \varepsilon^3)u_\varepsilon(\theta - \omega) + \varepsilon^3\omega - c(\varepsilon) = -\varepsilon V'(\theta + u_\varepsilon(\theta)) \quad (3.2)$$

as a power series expansion. That is, we look for solutions

$$u_\varepsilon(\theta) = \sum_{k=0}^{\infty} u_k(\theta)\varepsilon^k \quad \text{and} \quad c(\varepsilon) = \sum_{k=0}^{\infty} c_k\varepsilon^k, \quad (3.3)$$

where each u_n is a function from \mathbb{S}^1 to \mathbb{C} and each $c_n \in \mathbb{C}$.

This solution can be computed by equating powers of ε in (3.2). Taking the Taylor expansion at $\varepsilon = 0$

$$-\varepsilon V'(\theta + u_\varepsilon(\theta)) = \sum_{k=0}^{\infty} S_k(\theta)\varepsilon^k \quad (3.4)$$

and substituting (3.3) into (3.2), we have that

$$\begin{aligned} & \sum_{k=0}^{\infty} u_k(\theta + \omega)\varepsilon^k - (2 - \varepsilon^3) \sum_{k=0}^{\infty} u_k(\theta)\varepsilon^k + (1 - \varepsilon^3) \sum_{k=0}^{\infty} u_k(\theta - \omega)\varepsilon^k \\ & \times \sum_{k=0}^{\infty} u_k(\theta - \omega)\varepsilon^k + \varepsilon^3\omega - \sum_{k=0}^{\infty} c_k\varepsilon^k = \sum_{k=0}^{\infty} S_k(\theta)\varepsilon^k. \end{aligned} \quad (3.5)$$

Remark 1. When $V'(\theta) = \frac{1}{2\pi} \sin(2\pi\theta)$, or a trigonometric polynomial, the $S_k(\theta)$'s can be computed very efficiently in terms of the $u_k(\theta)$'s. Following [14,15] and denoting $\mathcal{S}(\theta, \varepsilon) = \sin(2\pi(\theta + u_\varepsilon(\theta)))$, $\mathcal{C}(\theta, \varepsilon) = \cos(2\pi(\theta + u_\varepsilon(\theta)))$, the coefficients of the series expansions $\mathcal{S}(\theta, \varepsilon) = \sum_{k=0}^{\infty} \mathcal{S}_k(\theta)\varepsilon^k$ and $\mathcal{C}(\theta, \varepsilon) = \sum_{k=0}^{\infty} \mathcal{C}_k(\theta)\varepsilon^k$ are given by the following recurrence relations,

$$(N+1)\mathcal{S}_{N+1}(\theta) = 2\pi \sum_{m=0}^N (m+1)\mathcal{C}_{N-m}(\theta)u_{m+1}(\theta) \quad (3.6)$$

$$(N+1)\mathcal{C}_{N+1}(\theta) = -2\pi \sum_{m=0}^N (m+1)\mathcal{S}_{N-m}(\theta)u_{m+1}(\theta).$$

Thus, $S_k(\theta) = -\mathcal{S}_{k-1}(\theta)$ for $k \geq 1$ and $S_0 = 0$, by (3.4).

Defining the operator

$$L_\omega \varphi(\theta) := \varphi(\theta + \omega) - 2\varphi(\theta) + \varphi(\theta - \omega) \quad (3.7)$$

Eq. (3.5) can be rewritten as

$$\begin{aligned} \sum_{k=1}^{\infty} S_k(\theta)\varepsilon^k &= \sum_{k=0}^2 (L_\omega u_k(\theta) - c_k)\varepsilon^k + (L_\omega u_3(\theta) - c_3 + u_0(\theta) \\ & \quad - u_0(\theta - \omega) + \omega)\varepsilon^3 \\ & \quad + \sum_{k=4}^{\infty} (L_\omega u_k(\theta) - c_k + u_{k-3}(\theta) - u_{k-3}(\theta - \omega))\varepsilon^k, \end{aligned} \quad (3.8)$$

Some properties of the operator L_ω are summarized in the following Lemma. See [16] for details about the proof.

Lemma 2. Let $\eta : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ be a continuous function such that $\int_0^1 \eta(\theta)d\theta = 0$. If ω is Diophantine as in (2.3), then there exists a solution, $\varphi(\theta)$, to the equation

$$L_\omega \varphi(\theta) = \eta(\theta) \quad (3.9)$$

such that $\int_0^1 \varphi(\theta)d\theta = 0$. In fact, the solution is given by

$$\varphi(\theta) = \sum_{\ell \in \mathbb{Z} \setminus \{0\}} \frac{\hat{\eta}_\ell}{2(\cos(2\pi\ell\omega) - 1)} e^{2\pi i \ell \theta},$$

where $\hat{\eta}_\ell$ are the Fourier coefficients of $\eta(\theta)$.

The Lindstedt process is as follows. Matching the coefficients of the same order in (3.8) we obtain the following relations to different orders of ε . The zero-th order term tells us that the coefficients at order zero in ε have to be trivial. The equations are

$$L_\omega u_0(\theta) - c_0 = 0. \quad (3.10)$$

Choosing $c_0 = 0$, then $u_0 \equiv 0$ is the solution given by Lemma 2. This construction is analogous to the zero-th order term in the symplectic case.

Remark 3. This method has been used in [4–8,15,16] for the symplectic case. That is, making the same process for the standard map, $(x_{n+1}, y_{n+1}) = (x_n + y_{n+1}, y_n + \varepsilon V'(x_n))$, gives the following equation to all orders ε^n ,

$$L_\omega u_k(\theta) = S_k(\theta) \quad k \geq 0. \quad (3.11)$$

Moreover, $\int_0^1 S_k(\theta) d\theta = 0$ for all $k \geq 0$. This is a consequence of the symplectic structure and the fact that $S_k(\theta)$ depends on the previously computed $u_0(\theta), u_1(\theta), \dots, u_{k-1}(\theta), S_0(\theta), S_1(\theta), \dots, S_{k-1}(\theta)$ (see Remark 1).

The first and second orders in ε are also analogous to the symplectic case. For this reason the first two coefficients of c_ε will be trivial.

$$L_\omega u_k(\theta) - c_k = S_k(\theta), \quad k = 1, 2. \quad (3.12)$$

Choosing $c_1 = 0 = c_2$ the equations are reduced to the non dissipative case and, by Remark 3, the right hand side has zero average. Therefore, we can find solutions $u_1(\theta), u_2(\theta)$.

The third order in ε is the first one that is different from the conservative case.

$$L_\omega u_3(\theta) - c_3 + \omega = S_3(\theta). \quad (3.13)$$

Here we notice that the drift parameter starts playing a rôle in the existence of invariant tori. Taking $c_3 = \omega$, Eq. (3.13) becomes the same equation as in the symplectic case. Since $S_3(\theta)$ has zero average we find $u_3(\theta)$.

The equations for orders higher than 3 are remarkably different since we have a counter term coming from the previously computed orders. Namely,

$$L_\omega u_k(\theta) = S_k(\theta) - u_{k-3}(\theta) + u_{k-3}(\theta - \omega) + c_k, \quad k \geq 4. \quad (3.14)$$

Notice that, by construction, $\int_0^1 u_{k-3}(\theta - \omega) d\theta = \int_0^1 u_{k-3}(\theta) d\theta = 0$ (see Lemma 2). Now, taking

$$c_k = - \int_0^1 S_k(\theta) d\theta, \quad (3.15)$$

we can find $u_k(\theta)$ solving (3.14) for all $k \geq 4$.

We have proved the following proposition which is a particular case of part A) of Theorem 12 in [2].

Proposition 4. For any $N \in \mathbb{N}$, the procedure presented above allows to find an approximate solution,

$$u_\varepsilon^{\leq N}(\theta) = \sum_{k=0}^N u_k(\theta) \varepsilon^k \quad \text{and} \quad c^{\leq N}(\varepsilon) = \sum_{k=0}^N c_k \varepsilon^k, \quad (3.16)$$

such that

$$\|E_{c^{\leq N}(\varepsilon)}[u_\varepsilon^{\leq N}]\| = O(\varepsilon^{N+1})$$

where E is the functional defined in (2.6).

3.2. Padé extrapolation

The domain of analyticity for the solution of (2.6) can be approximated by implementing a Padé method in which we use the approximate solutions obtained by the Lindstedt series constructed in Section 3.1.

The Padé method is quite standard and is presented in several places in the literature. Here, we follow the exposition in [17]. A Padé approximant of order $[p/q]$ of a function $g(\varepsilon) = \sum_{i=0}^{\infty} g_i \varepsilon^i$ is a rational function, $P(\varepsilon)/Q(\varepsilon)$, which agrees with u to the highest possible order in ε .

That is,

$$g(\varepsilon) - \frac{P(\varepsilon)}{Q(\varepsilon)} = O(\varepsilon^{p+q+1}). \quad (3.17)$$

where $P(\varepsilon)$ and $Q(\varepsilon)$ are polynomials of degrees p and q respectively, $Q(0) = 1$.

The existence of the polynomials P and Q can be obtained by noticing that (3.17) is equivalent to

$$g(\varepsilon)Q(\varepsilon) = P(\varepsilon) + O(\varepsilon^{p+q+1})$$

and, then, considering $P(\varepsilon) = \sum_{i=0}^p P_i \varepsilon^i$ and $Q(\varepsilon) = \sum_{i=0}^q Q_i \varepsilon^i$ the coefficients of the polynomials can be found by solving the following systems of equations

$$\begin{aligned} g_i + \sum_{j=1}^i g_{i-j} Q_j &= P_i \quad 0 \leq i \leq p \\ g_i + \sum_{j=1}^q g_{i-j} Q_j &= 0 \quad p < i \leq p+q. \end{aligned} \quad (3.18)$$

The second equation of (3.18) gives the Q_j 's, and then we can find the P_j 's by substituting in the first equation. Then, the boundary of the domain of analyticity of a function can be approximated by the zeros of Q in the $[p/q]$ Padé approximant.

There are a number of implementations of the Padé methods that are used in a quite standard manner. In the present work we use the implementations included in Version 2.9.0 of GP/PARI, [18].

3.3. Newton's method

In this section we summarize an iterative scheme in scales of Banach spaces that can be very well adapted to perform numerical computations. The scheme is based on a Newton iteration starting from approximate solutions to Eq. (2.6). We briefly describe the scheme here since details of schemes of this kind and numerical implementations have been described already in the literature [2,10,12,19], and the reader can refer to these works for more details.

We start from an approximate solution (u_a, c_a) of Eq. (2.6). Namely, we have a solution so that $\|E_{c_a}[u_a]\|$ is small enough. The approximate solution could be obtained by several means. One possibility is starting from the integrable case (for ε close to zero) and performing continuation or from a Lindstedt series expansions like the ones obtained in Section 3.1. We remark that in the dissipative standard map we are studying, $\varepsilon = 0$ is the point where the map becomes symplectic. Since we use methods for conformally symplectic systems we actually start the continuation from values of ε that are not equal to zero but small.

The Newton algorithm consists of adding a correction (v, δ) to the approximate solution so that supremum norm of (2.6) evaluated in the function plus the corrections, $\|E_{c_a+\delta}[u_a + v]\|$,

is of the order of the square of the norm of (2.6) evaluated at the approximate solution,

$$\|E_{c_a+\delta}[u_a + v]\| \leq C\|E_{c_a}[u_a]\|^2.$$

One obtains the correction by solving the linearized equation of $E_{c_a+\delta}[u_a + v]$ for (v, δ) around the approximate solution, (u_a, c_a) .

In this case, the equation we have to solve is

$$D_u E_{c_a}[u_a]v - \delta = -E_{c_a}[u_a] \quad (3.19)$$

which involves unbounded operators in Banach spaces (namely $D_u E_{c_a}[u_a]v$) that are actually bounded if one considers that the operators map into Banach spaces of less regularity. It is a standard observation in Nash–Moser theory [20,21], that to set up a converging iterative Newton scheme it is not necessary to find an exact inverse of the operator $D_u E_{c_a}[u_a]$, but finding an exact inverse of an approximate operator will suffice.

In our case, we will not solve Eq. (3.19) directly but will solve a modified equation obtained by adding a term that is quadratic in the error. We remark that since a Newton method is a quadratic scheme solving a linear equation then adding a quadratic term also gives a quadratic scheme that solves the same linear equation. One obtains an approximate Newton equation by subtracting the term $v D_u E_{c_a}[u_a]h'$, which is quadratic in the error. The modified Newton equation is,

$$h' D_u E_{c_a}[u_a]v - v D_u E_{c_a}[u_a]h' = -h'(E_{c_a}[u_a] - \delta), \quad (3.20)$$

with $h'(\theta) = 1 + \frac{\partial u(\theta)}{\partial \theta}$. The l.h.s. of Eq. (3.20), factorizes into a sequence of operators that are easier to solve numerically, as it is noted in Lemma 5.

This method has been used in several works [16,22,23]. Here we only make a reference to the justification in [10], where the reader can refer to for details.

Let the operators $\mathcal{D}_-, \mathcal{D}_+^b$ be defined by

$$\begin{aligned} \mathcal{D}_- f(\theta) &= f(\theta - \omega) - f(\theta) \\ \mathcal{D}_+^b f(\theta) &= f(\theta + \omega) - b f(\theta). \end{aligned} \quad (3.21)$$

A small remark is that (3.21) are operators that are diagonal in Fourier space. In the following lemma, we write the modified Newton as a sequence of operators that are either diagonal in Real or Fourier space.

Lemma 5. *The modified Newton equation in (3.20) with $E_{c_a}[u]$ defined in (2.6) is equivalent to*

$$\mathcal{D}_+^b [-h'(\theta)h'(\theta - \omega)\mathcal{D}_-[(h')^{-1}(\theta)v(\theta)]] = -h'(E_{c_a}[u_a](\theta) - \delta). \quad (3.22)$$

Remark 6. One notices that the operators involved in the l.h.s. of Eq. (3.22) only involve differentiation, multiplication, division, shifting the arguments of functions, and solving the difference equations with constant coefficients in (3.21). All these operations can be implemented very efficiently using the computer. For instance if we discretize the periodic functions using n uniformly distributed points and we use a Fast Fourier Transform method, the modified Newton step equation can actually be solved in $O(n \log n)$ operations.

The factorization in Eq. (3.22) suggests an algorithm that is used to solve the modified Newton equation.

Algorithm 7.

- (i) Find two functions φ and v solving the equations

$$\mathcal{D}_+^b \varphi(\theta) = -h' E_{c_a}[u] \quad (3.23)$$

and

$$\mathcal{D}_+^b v(\theta) = -h'(\theta). \quad (3.24)$$

Notice that if $\varphi(\theta)$ and $v(\theta)$ are solutions of (3.23) and (3.24), respectively, then the equation $\mathcal{D}_+^b(\varphi(\theta) - \delta v(\theta)) = -h'(\theta)(E_{c_a}[u_0](\theta) - \delta)$ holds for any $\delta \in \mathbb{C}$. This will allow us to choose a complex number δ so that the average of $\frac{\varphi(\theta) - \delta v(\theta)}{h'(\theta)h'(\theta - \omega)}$ vanishes.

- (ii) Choose $\delta \in \mathbb{C}$ such that

$$\int_{\mathbb{T}} \frac{\varphi(\theta) - \delta v(\theta)}{h'(\theta)h'(\theta - \omega)} d\theta = 0.$$

- (iii) Obtain w from the solution of the constant coefficient difference equation

$$\mathcal{D}_- w(\theta) = \frac{\varphi(\theta) - \delta v(\theta)}{-h'(\theta)h'(\theta - \omega)}. \quad (3.25)$$

Notice that after choosing a δ in step (ii) so that the right hand side has zero average we can always find a periodic function w solving (3.25) when the r.h.s. is smooth enough.

- (iv) Construct $v(\theta) = h'(\theta)w(\theta)$ and obtain the improved solution (\tilde{u}, \tilde{c}) defined as

$$\tilde{u}(\theta) = u_a(\theta) + v(\theta), \quad \tilde{c} = c_a + \delta.$$

The observation in Remark 6 is that the operators in (3.22) are very efficiently implementable with the use of a computer either in Real or in Fourier space. This efficiency comes from the fact that all the operations involved in the four steps of Algorithm 7 are multiplications, additions and integrals of periodic functions that take only $O(n)$ operations in Real space; and differentiation, shifts and solving cohomology equations with constant coefficients, that take only $O(n)$ operations in Fourier space. Therefore, the most expensive operation in Algorithm 7 is transforming from Real to Fourier space and back. This can be done in $O(n \log n)$ operations by means of a Fast Fourier Transform.

Remark 8. We note that the algorithm is guaranteed to converge inside the boundaries of the analyticity domain. Indeed, in [9] was rigorously justified that the algorithm only fails to converge as the continuation reaches the boundary of analyticity. Therefore, the continuation method can also be used to assess the bounds on the domain of ε .

4. Numerical results

In this section we present the results of implementing the methods described in Section 3. All the computations were done using the golden ratio, $\omega = \frac{\sqrt{5}-1}{2}$, which satisfies (2.3) [16].

4.1. Lindstedt expansions

The construction of Lindstedt series in Section 3.1 was implemented as a numerical algorithm. The statement of Proposition 4 tells us that given any $N \in \mathbb{N}$, the outcome of the method is the pair of polynomials of degree N in (3.16). The observation of Lemma 2 is that the operator L_ω defined in Eq. (3.7) is diagonal in Fourier Space and Eq. (3.9) can be solved for ϕ if we allow to obtain functions with less regularity than the right hand side, η . We find the solution numerically by transforming to Fourier space and solving for the u_k 's from expressions (3.12) to (3.14). At every order of the process we obtain the c_k 's as a byproduct of imposing the condition that every order should have zero average.

The Lindstedt series expansions are used to obtain an approximate solution to the functional equation in (2.6) at some high order. Indeed, we discovered that with our implementations it is

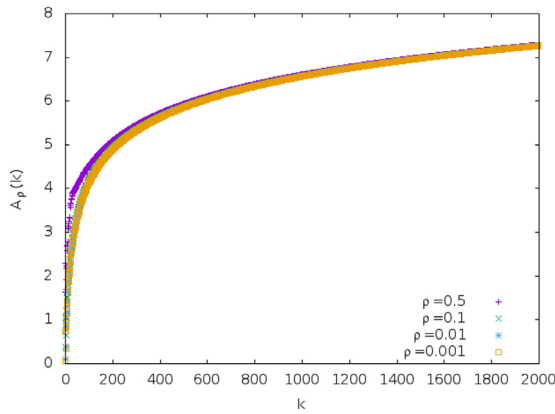


Fig. 2. Analytic norms of the coefficients of the Lindstedt expansion plotted as in expression (4.1).

very hard to notice that the functions are not analytic. Namely, the singularities that exist close to the point $\varepsilon = 0$ are very hard to detect so we have no evidence that the radius of convergence of the series is exactly zero in the complex plane. Thus, if the solution belongs to a Gevrey class then the Gevrey exponent would be very close to one.

We approximated several norms of the coefficients, $u_k(\theta)$, to have an indication of how far the functions are from being analytic. First, we use the norm on the complex strip of size $\rho > 0$, i.e., $\theta \in \mathbb{S}_\rho^1$ if $|\text{Im}(\theta)| < \rho$. Let $f : \mathbb{S}_\rho^1 \rightarrow \mathbb{S}_\rho^1$ be a function of \mathbb{S}_ρ^1 then the norm we use is

$$\|f\|_\rho = \sum_{\ell \in \mathbb{Z}} |\hat{f}_\ell|^2 e^{2\pi|\ell|\rho}$$

where \hat{f}_ℓ are the Fourier coefficients of f .

We say that the function $f(\varepsilon)$ belongs to the Gevrey class G^σ with respect to the norm $\|\cdot\|_B$ at $\varepsilon = 0$ whenever

$$\frac{1}{k!} \|\partial_\varepsilon^k f(\varepsilon)\|_B \leq CR^k k^{\sigma k},$$

for $\varepsilon = 0$, [24].

Since we want to check if the function $u_\varepsilon(\theta)$ belongs to a Gevrey class at $\varepsilon = 0$ with the analytic norms it is convenient to compute the following expressions as functions of k ,

$$A_\rho(k) \equiv \frac{1}{k} \log \|u_k(\theta)\|_\rho, \quad (4.1)$$

and then approximate the constant σ .

The expressions (4.1) as functions of k for the coefficients of the approximate solution are shown in Fig. 2.

We also used Sobolev norms defined for a real number $r > 0$ by the L^2 -norm of the r th derivative with respect to θ ,

$$\|f\|_r = \|\partial_\theta^r f\|_{L^2}.$$

Notice that when $r = 0$, $\|f\|_0$ corresponds to the L^2 norm of u . The Sobolev norms can also be written in terms of Fourier coefficients as follows,

$$\|f\|_r = \left(\sum_{k \in \mathbb{Z}} (2\pi k)^{2r} |\hat{f}_k|^2 \right)^{1/2}.$$

As in the case for analytic norms we define the following expressions for the Sobolev norms,

$$H^r(k) \equiv \frac{1}{k} \log \|u_k(\theta)\|_r. \quad (4.2)$$

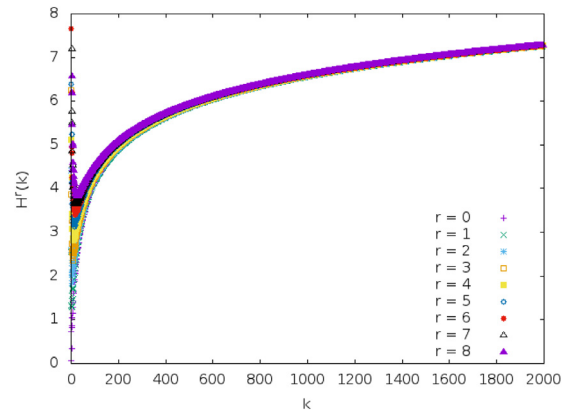


Fig. 3. Sobolev norms of the coefficients of the Lindstedt expansion plotted as functions of the order of ε .

Table 1

Numerical fit of analytic norms in expression (4.1) for different values of ρ .

	$A_\rho(k) = \log(a) + c \log(k + b)$		
	a	b	c
$\rho = 0.5$	0.719467892188978	27.3937734029272	1.00001766652951
$\rho = 0.1$	0.719927523693294	-6.67612303717059	0.999991308925563
$\rho = 0.01$	0.719826182759749	-12.4920865977342	0.999998383397402
$\rho = 0.001$	0.719813322136991	-13.0568139120788	0.999999331056825

Table 2

Numerical fit of analytic norms in expression (4.2) for different exponents, r .

	$H^r(k) = \log(a) + c \log(k + b)$		
	a	b	c
$r = 0$	0.71981186150290	-13.11936925724	0.99999943903933
$r = 1$	0.71990246696872	-8.455098498275	0.99999292602721
$r = 2$	0.71995774061454	-3.699094566292	0.99998926987818
$r = 3$	0.71997659277593	1.151414934192	0.99998826853277
$r = 4$	0.71995790042650	6.099307051805	0.99998972777892
$r = 5$	0.71990050891201	11.14758050573	0.99999346098676
$r = 6$	0.71980323117743	16.29938193670	0.9999928901585

We include the values of $H^r(k)$ for the coefficients of the approximate solution and several values of r in Fig. 3.

In both cases, the behavior of the norms coefficients $\|u_k(\theta)\|_B$ with respect to k seem to belong to Gevrey classes. In Tables 1 and 2 we include the fit of the plots in Figs. 2 and 3.

The numerical results in Tables 1 and 2 lead us to think that the solutions that we approximate are functions that are very hard to distinguish from analytic functions by just examining the truncated series expansion. One of the rigorous results in [2] states that the functions that satisfy Eq. (2.6) fail to be analytic since there is no ball around $\varepsilon = 0$ where the formal power series converges. Therefore, we conjecture that the solutions belong to a Gevrey class with an exponent that is very close to the analytic class.

Conjecture 9. The parameterization u_ε belongs to a Gevrey class, G^σ , as a function of ε . The index, σ , is close to 1.

4.2. Approximation of poles of the Lindstedt series

Here we include the poles of the Lindstedt polynomial found with the Padé method. In Fig. 4, we show the poles of the series approximated by means of the Padé method. It is well known that the Padé method computations are very sensitive to precision, see [17], so we have implemented the computations with extended precision using the software gp/Pari [18]. We

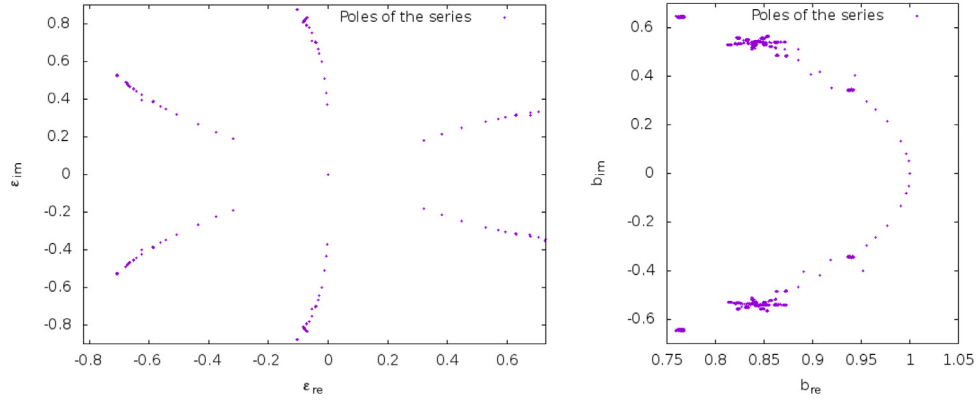


Fig. 4. Points which are simultaneously poles of Padé approximants of degree [475,475] and [500,500]. The implementation was done with 1000 digits. Left panel: Poles in the complex plane $\varepsilon \in \mathbb{C}$. Right panel: Poles evaluated in the function $b(\varepsilon) = 1 - \varepsilon^3$, with $\varepsilon \in \mathbb{C}$.

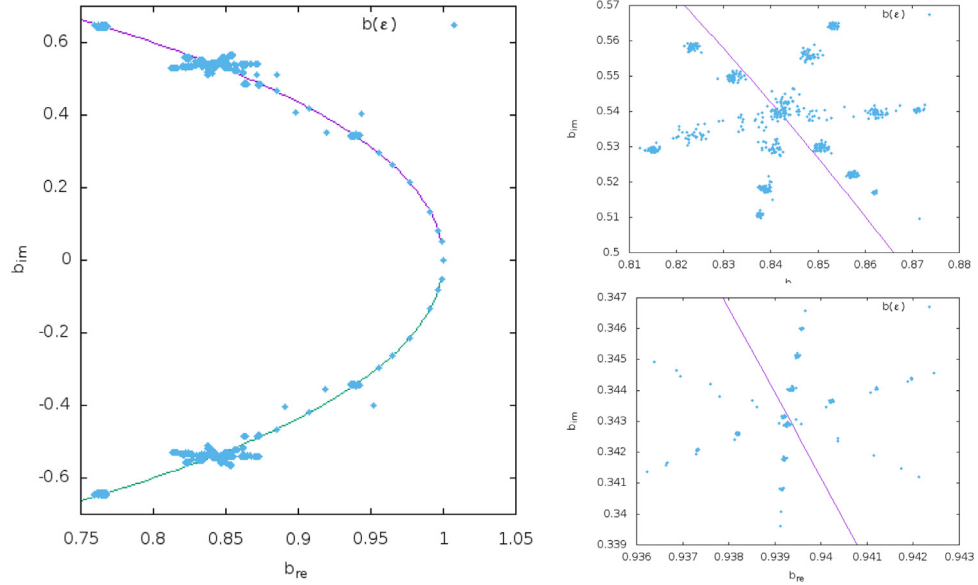


Fig. 5. The poles compared to the unit circle. Left panel: Evaluation of the poles of the series by the function $b(\varepsilon) = 1 - \varepsilon^3$. Right panels: Two zoomed in versions of the set.

show the values of the poles in the ε complex plane, and the complex values of the function $b(\varepsilon) = 1 - \varepsilon^3$. Our computations suggest that the boundary of the analyticity domain has a more complicated structure than what was predicted in [2], compare the left panel of Fig. 4 with Fig. 1. Fig. 5, contains the comparison of the values of the function $b(\varepsilon)$ with the unit circle. We also include zoomed in versions of the values of $b(\varepsilon)$ in Fig. 5.

4.3. Newton method

We used Newton's method and continuation to explore the monodromy of the solutions in the domains. A rigorous result in [2] states that the solutions defined in the domain of analyticity in ε have trivial monodromy. We verified this fact numerically by performing continuation of the solutions $(u_\varepsilon, c_\varepsilon)$ around the poles that were previously approximated using the Padé series method described in Section 4.2.

We used the approximated poles as centers of circular paths in ε over which we performed continuation while solving the invariance equation (2.6) using Algorithm 7. Once the continuation achieves a complete turn around a chosen pole, one verifies that the solution always arrives to the same starting point. This is an effect of the monodromy of the functions being trivial (see Fig. 7 and Table 3).

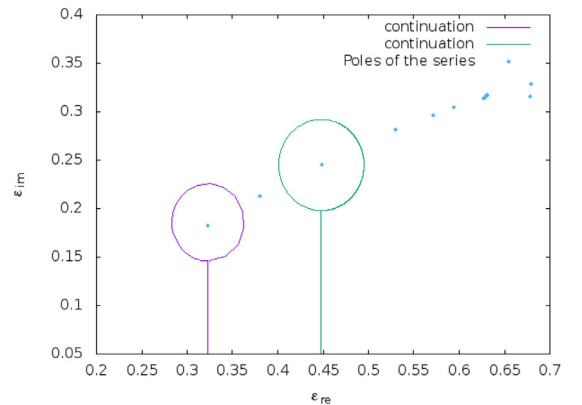


Fig. 6. Poles of the series and two different continuations done with the Newton algorithm. The continuation is done around the pole in order to illustrate that the monodromy is trivial.

We present several instances of the functions for different parameter values along a circle winding around a pole in Fig. 7. The path we used to surround the pole is presented in Fig. 6.

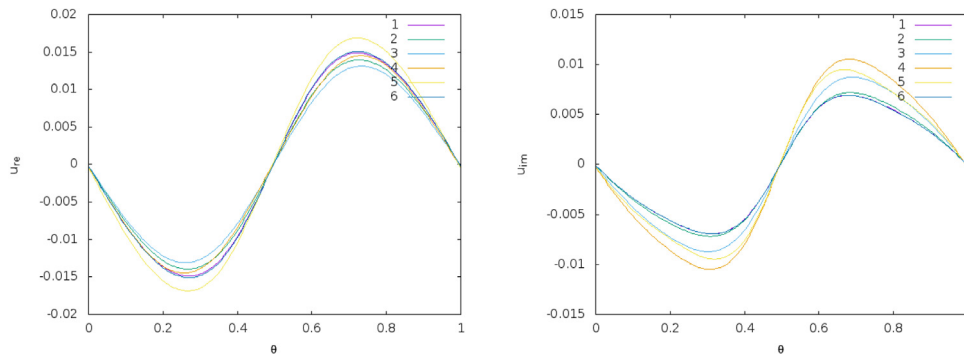


Fig. 7. Real and imaginary parts of different instances of a continuation by the Newton algorithm including the initial and final functions. One observes that there is no monodromy after a full turn around the pole.

Table 3

Values of ε and $c(\varepsilon)$ for different instances taken from the small circle in Fig. 6.

Instance	ε	$c(\varepsilon)$
1	$0.3202966 + i0.1460915$	$0.01994937 - i0.06774761$
2	$0.3008391 + i0.1527000$	$0.009976542 - i0.06136120$
3	$0.2830167 + i0.1871540$	$-0.01146081 - i0.06221038$
4	$0.3122423 + i0.2245263$	$-0.02718298 - i0.08804174$
5	$0.3613448 + i0.1973876$	$0.007928831 - i0.1127768$
6	$0.3242691 + i0.1460201$	$0.02157160 - i0.06953580$

The continuation was performed using FFTW3, [25], with the libquadmath library, [26]. The radii of the continuation paths were chosen so that the path did not come very close to the poles. Indeed, when the continuation comes close to a pole our implementation of the Newton method becomes degenerate in the sense that one needs to compute quotients of very small quantities. The reason is that when solving Eqs. (3.23) and (3.24), the divisors depending on ε , are below machine precision close to the pole and dividing over those quantities leads to large numerical errors.

4.4. Conclusion

In the present work we have explored numerically the rigorous results and conjectures in [2] in one concrete example. We have focused on the dissipative standard map (2.1) with a dissipation that vanishes whenever a small parameter ε is equal to zero. In [2] a Lindstedt series in ε is constructed up to any order explicitly and then the series is used as an approximate solution to prove the existence of invariant tori. The zero-th order term of the Lindstedt series is the symplectic case. In the present work, we have linked the parameter ε to the nonlinearity of the map. In this way, when $\varepsilon = 0$, we have that the symplectic case is the integrable map. This has allowed us to have the explicit symplectic case with any precision desired.

To perform our computations we have used the extended precision software gp/Pari [18]. We believe that it might be possible to perform similar computations to ours but starting from a non trivial symplectic case. One of the difficulties of performing that computation is that one first needs to compute a symplectic torus that has the desired precision and then write down the series expansion with the computation as a zero-th order term.

There are three conclusions that are derived from our numerical results. The first result is about the regularity of the parameterizations of the invariant circles as functions of the parameter ε . We have computed the Lindstedt series numerically and then measured the size of the series terms. This procedure allows us to conjecture that the parameterizations of the invariant circles belong to a Gevrey Class [24] with an exponent that is very close to 1, see Conjecture 9

The second numerical result was obtained by computing the poles of the series with a Padé Method, this method has been used to approximate the boundary of the domains of analyticity. The poles of the series lie close to the boundary of the domain of analyticity conjectured in [2]. The conjecture in [2] was that the boundary of the domain is constituted by a sequence of balls with centers along smooth curves passing through the origin whose radii decrease exponentially fast as they approach to $\varepsilon = 0$ in the complex plane, see Fig. 1. We discovered that computing the boundary close to $\varepsilon = 0$ with the method we used is indeed very difficult since it would require a very high numerical precision. However, we found that the boundary we computed is very similar to what was conjectured in [2], further away from $\varepsilon = 0$. Our numerical computations also suggest that the boundary of the domains could have a more complex structure since the computed poles seem to be aligned along lines in a somehow self-similar structure, see Fig. 5. We think that this phenomenon is worth investigating in a future work.

The third numerical result was obtained by implementing a Quasi-Newton Method to solve the invariance equation (2.6). With the Quasi-Newton Method, we were able to verify the result in [2] stating that the monodromy of the functions is trivial. Indeed, as we continued the invariant circles with respect to the parameter ε through the boundary of the domain and came back to the starting point, we noticed that the functions return to the same starting function. The result of the continuation is shown in Figs. 6 and 7.

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