Extended symmetry analysis of an isothermal no-slip drift flux model

Stanislav Opanasenko [a,b], Alexander Bihlo [a,*], Roman O. Popovych [b,c,d], Artur Sergyeyev [d]

[a] *Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's (NL) A1C 5S7, Canada*
[b] *Institute of Mathematics of NAS of Ukraine, 3 Tereshchenkivska Str., 01024 Kyiv, Ukraine*
[c] *Fakultät für Mathematik, Universität Wien, Oskar-Morgenstern-Platz 1, 1090 Wien, Austria*
[d] *Mathematical Institute, Silesian University in Opava, Na Rybníčku 1, 746 01 Opava, Czech Republic*

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ABSTRACT

We perform extended group analysis for a system of differential equations modeling an isothermal no-slip drift flux. The maximal Lie invariance algebra of this system is proved to be infinite-dimensional. We also find the complete point symmetry group of this system, including discrete symmetries, using the megaideal-based version of the algebraic method. Optimal lists of one- and two-dimensional subalgebras of the maximal Lie invariance algebra in question are constructed and employed for obtaining reductions of the system under study. Since this system contains a subsystem of two equations that involves only two of three dependent variables, we also perform group analysis of this subsystem. The latter can be linearized by a composition of a fiber-preserving point transformation with a two-dimensional hodograph transformation to the Klein–Gordon equation. We also employ both the linearization and the generalized hodograph method for constructing the general solution of the entire system under study. We find *inter alia* genuinely generalized symmetries for this system and present the connection between them and the Lie symmetries of the subsystem we mentioned earlier. Hydrodynamic conservation laws and their generalizations are also constructed.

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1. Introduction

In the course of solving problems in physics one often faces systems of first-order quasilinear differential equations, that is, first-order systems which are linear in the first-order derivatives of the dependent variables but whose coefficients at these derivatives may in general depend on the dependent and independent variables. Such systems frequently occur in acoustics, fluid mechanics, gas and shock dynamics [1], and for the case of two independent variables have the general form

$$\sum_{j=1}^{n} A^{ij}\frac{\partial u^j}{\partial t} + \sum_{j=1}^{n} B^{ij}\frac{\partial u^j}{\partial x} + C^i = 0, \quad i = 1, \ldots, n, \tag{1}$$

where the $n \times n$ matrices $A = (A^{ij})$, $B = (B^{ij})$ and the n-component vector $C = (C^i)$ are functions of independent variables (t, x) and dependent variables $(u^1, \ldots, u^n)$ but not of the derivatives of the latter. Such systems and their natural generalizations to the case of more than two independent variables are known as (translation-noninvariant nonhomogeneous) hydrodynamic-type

systems and are a subject of intense research, see for example [2–10] and references therein.

An important class of such systems is given by evolutionary translation-invariant systems of hydrodynamic type in two independent variables, for which A is the $n \times n$ unit matrix, the vector C vanishes, and the matrix B depends only on dependent variables. For the sake of brevity in what follows we shall refer to such systems just as to the hydrodynamic-type systems.

If a hydrodynamic-type system can be diagonalized by a change $\mathfrak{r} = \mathfrak{r}(u)$ of dependent variables alone, that is, $\tilde{B}^{ij} = 0$ for the new matrix $\tilde{B}$ if $i \neq j$ and $V^i := \tilde{B}^{ii}$ are eigenvalues of the matrix B, then the new dependent variables are called *Riemann invariants* of this system, and the eigenvalues $V^1, \ldots, V^n$ are commonly referred to as the characteristic velocities of the system, cf. e.g. [11]. Note that in general hydrodynamic-type systems in more than two dependent variables are not diagonalizable. A diagonalizable hydrodynamic-type system is called genuinely nonlinear if $V_i^i \neq 0$ for all $i = 1, \ldots, n$, and linearly degenerate if all of these inequalities fail. Here and below, unless otherwise explicitly stated, the indices i, j and k run from 1 to n, and a function subscript like i denotes the derivative with respect to the ith Riemann invariant.

In the theory of hydrodynamic-type systems there exists an integrability criterion involving the generalized hodograph transformation [12,13]. This criterion states that a diagonalizable

strictly hyperbolic (strict hyperbolicity means that all characteristic velocities V^i are real and distinct) hydrodynamic-type system is integrable in the sense presented below if and only if the condition

$$\partial_i \frac{V_j^k}{V^j - V^k} = \partial_j \frac{V_i^k}{V^i - V^k}$$

holds for all $i \neq k \neq j$ (no sum over repeated indices here). Such hydrodynamic-type systems are called *semi-Hamiltonian* [12,13] (see e.g. [14, p. 60] and [15] for further details). Given a diagonal hydrodynamic-type system

$$\mathfrak{r}_t^i + V^i \mathfrak{r}_x^i = 0,$$

where $\mathfrak{r} = (\mathfrak{r}^1, \dots, \mathfrak{r}^n)$ is the tuple of its Riemann invariants and $V = (V^1, \dots, V^n)$ is the associated tuple of characteristic velocities, the generalized hodograph transformation allows one to locally represent the general solution of this system (except for the solutions for which $\mathfrak{r}_x^i = 0$ for some i) in the form

$$x - V^i(\mathfrak{r})t = W^i(\mathfrak{r}), \tag{2}$$

where $W = (W^1, \dots, W^n)$ is the general solution of the system

$$\frac{W_j^i}{W^j - W^i} = \frac{V_j^i}{V^j - V^i}, \quad i \neq j,$$

with the nondegeneracy condition $\det(V_j^i t + W_j^i) \neq 0$ (again no sum over repeated indices here), which guarantees that the ansatz (2) is locally solvable with respect to $\mathfrak{r}$.

In the rest of the present paper we shall deal with a three-component ($n = 3$) hydrodynamic-type system arising in the study of the two-phase flow phenomena. The problem in question is of great importance in physics thanks to its applications in several rapidly developing branches such as nuclear power and chemical industry [16,17]. In particular, the accurate prediction of void fraction in the sub-channel under two-phase flow is of fundamental significance.

Unfortunately, this problem is quite challenging and therefore various simplified models of two-phase flow phenomena were developed. One of these is the drift flux model introduced in [18], which allows to describe the mixing motion rather than the individual phases. It was thoroughly studied in [19–21], where several sub-models of the drift flux model were found, and, in particular, the concept of the slip function was introduced. The model with no-slip condition

$$\rho_t^1 + u\rho_x^1 + u_x\rho^1 = 0, \tag{3a}$$
$$\rho_t^2 + u\rho_x^2 + u_x\rho^2 = 0, \tag{3b}$$
$$(\rho^1 + \rho^2)(u_t + uu_x) + a^2(\rho_x^1 + \rho_x^2) = 0, \tag{3c}$$

where $u = u(t, x)$ is the common velocity in both phases, $\rho^1 = \rho^1(t, x)$ and $\rho^2 = \rho^2(t, x)$ are densities of liquids (or of liquid and gas), and a is a constant depending on both phases, was considered in [22]. In [23] an attempt at performing the group analysis for this system was made. Unfortunately, the said work contains a number of inaccuracies, including an incorrect computation of the maximal Lie invariance algebra as well as some mistakes in the classification of one-dimensional and two-dimensional subalgebras of the latter. Also, the system (3) has many nice additional properties as it is of hydrodynamic type, and the goal of the present paper is to revisit and to significantly extend the group analysis of (3).

To this end it is convenient to simplify the initial model (3) by an appropriate change of variables. First of all, using a simultaneous rescaling of x and u we can set $a = 1$ provided that a is positive, which is justified from the physical point of view. Introducing the new dependent variables $v = \ln(\rho^1 + \rho^2)$ and

$w = \rho^1/\rho^2$ instead of ρ^1 and ρ^2, we rewrite the system (3) as the system $\mathcal{S}$ which reads

$$u_t + uu_x + v_x = 0, \tag{4a}$$
$$v_t + uv_x + u_x = 0, \tag{4b}$$
$$w_t + uw_x = 0. \tag{4c}$$

The system (4) is obviously of hydrodynamic type. Furthermore, diagonalizing the matrix B of the representation (1) for the system (4) we map the system (4) to the system

$$\mathfrak{r}_t^1 + (\mathfrak{r}^1 + \mathfrak{r}^2 + 1)\mathfrak{r}_x^1 = 0, \tag{5a}$$
$$\mathfrak{r}_t^2 + (\mathfrak{r}^1 + \mathfrak{r}^2 - 1)\mathfrak{r}_x^2 = 0, \tag{5b}$$
$$\mathfrak{r}_t^3 + (\mathfrak{r}^1 + \mathfrak{r}^2)\mathfrak{r}_x^3 = 0 \tag{5c}$$

using the change of dependent variables $\mathfrak{r}^1 = \frac{1}{2}(u+v)$, $\mathfrak{r}^2 = \frac{1}{2}(u-v)$, $\mathfrak{r}^3 = w$. Thus, $\mathfrak{r}^1$, $\mathfrak{r}^2$ and $\mathfrak{r}^3$ are the Riemann invariants of the system (4), and the Riemann invariants of the initial system (3) are

$$\mathfrak{r}^1 = \frac{u + \ln(\rho^1 + \rho^2)}{2}, \quad \mathfrak{r}^2 = \frac{u - \ln(\rho^1 + \rho^2)}{2}, \quad \mathfrak{r}^3 = \frac{\rho^1}{\rho^2},$$

and hence the expressions for the initial dependent variables (u, ρ^1, ρ^2) in terms of the Riemann invariants read

$$u = \mathfrak{r}^1 + \mathfrak{r}^2, \quad \rho^1 = \frac{\mathfrak{r}^3 e^{\mathfrak{r}^1 - \mathfrak{r}^2}}{\mathfrak{r}^3 + 1}, \quad \rho^2 = \frac{e^{\mathfrak{r}^1 - \mathfrak{r}^2}}{\mathfrak{r}^3 + 1}.$$

We also readily see that the characteristic velocities of the system (5) are

$$V^1 = \mathfrak{r}^1 + \mathfrak{r}^2 + 1, \quad V^2 = \mathfrak{r}^1 + \mathfrak{r}^2 - 1, \quad V^3 = \mathfrak{r}^1 + \mathfrak{r}^2. \tag{6}$$

The system (5) is not genuinely nonlinear as $V_3^3 = 0$. It is strictly hyperbolic, diagonalizable and semi-Hamiltonian, so the generalized hodograph transformation can be applied here. Nevertheless, it also makes sense to extend the scope and to study this system within the framework of group analysis. Note that the subsystem of the first two equations of (5) coincides with the diagonalized form of the system describing one-dimensional isentropic gas flows with constant sound speed [11, Section 2.2.7, Eq. (16)].

Throughout the text we switch between the forms (4) and (5) of the system $\mathcal{S}$. It is often more convenient to use the diagonalized form for computation, although many results are more concisely expressed in terms of the variables u, v and w.

The rest of the paper is organized as follows. In Section 2 we compute the maximal Lie invariance algebra of the system (4). Pushing forward the Lie symmetry vector fields by relevant point transformations, we also present the corresponding algebras for both the initial system (3) and its diagonalized form (5). Section 3 is devoted to some background material on how to compute the complete point symmetry group, which contains both Lie symmetries and discrete point symmetries, for a system of differential equations. The actual computation of the complete point symmetry group of the system (4) is presented in Section 4. The classification of one- and two-dimensional subalgebras of the maximal Lie invariance algebra of the system (4) is presented in Section 5. Section 6 contains the results on Lie reductions based on the optimal lists of inequivalent subalgebras from the previous section. In Section 7 we perform group analysis of the essential subsystem (4a)–(4b) of the system (4). Employing the fact that this subsystem can be linearized by a two-dimensional hodograph transformation to the telegraph equation, in Section 8 we describe *all* (local) solutions of the system (4), where regular solutions are expressed in terms of solutions of the telegraph equation. In Section 9 we apply the generalized hodograph transformation for describing the general solution of the diagonalized

version (5) of the system $\mathcal{S}$. First-order generalized symmetries of the system (5) and their generalizations are computed in Section 10. Hydrodynamic conservation laws for the model under study and their generalizations are given in Section 11. In the final Section 12 we summarize the results of the paper and touch upon some avenues of future research.

2. Lie Symmetries

In order to compute the maximal Lie invariance algebra of a given system of differential equations we employ the infinitesimal method [24,25].

The infinitesimal generators of one-parameter Lie symmetry groups for the system $\mathcal{S}$ are defined as

$$Q = \tau \partial_t + \xi \partial_x + \eta \partial_u + \theta \partial_v + \zeta \partial_w,$$

where the components τ, ξ, η, θ and ζ depend on $t, x, u, v,$ and w. The infinitesimal invariance criterion requires that

$$\begin{aligned}
& Q^{(1)} \left(u_t + u u_x + v_x \right) |_{\mathcal{S}} = 0, \\
& Q^{(1)} \left(v_t + u v_x + u_x \right) |_{\mathcal{S}} = 0, \\
& Q^{(1)} \left(w_t + u w_x \right) |_{\mathcal{S}} = 0.
\end{aligned} \tag{7}$$

The first prolongation $Q^{(1)}$ of the vector field Q is given by

$$\begin{aligned}
Q^{(1)} = Q &+ \eta^{(1,0)} \partial_{u_t} + \eta^{(0,1)} \partial_{u_x} + \theta^{(1,0)} \partial_{v_t} + \theta^{(0,1)} \partial_{v_x} \\
&+ \zeta^{(1,0)} \partial_{w_t} + \zeta^{(0,1)} \partial_{w_x},
\end{aligned}$$

where the components $\eta^{(1,0)}, \eta^{(0,1)}, \theta^{(1,0)}, \theta^{(0,1)}, \zeta^{(1,0)}$ and $\zeta^{(0,1)}$ of the prolonged vector field $Q^{(1)}$ are readily derived from the general prolongation formula [24],

$$\begin{aligned}
\eta^\alpha &= D^\alpha \left(\eta - \tau u_t - \xi u_x \right) + \tau u_{\alpha + \delta_1} + \xi u_{\alpha + \delta_2}, \\
\theta^\alpha &= D^\alpha \left(\theta - \tau v_t - \xi v_x \right) + \tau v_{\alpha + \delta_1} + \xi v_{\alpha + \delta_2}, \\
\zeta^\alpha &= D^\alpha \left(\zeta - \tau w_t - \xi w_x \right) + \tau w_{\alpha + \delta_1} + \xi w_{\alpha + \delta_2}.
\end{aligned}$$

Here $\alpha = (\alpha_1, \alpha_2)$ is a multi-index, $\delta_1 = (1, 0)$, $\delta_2 = (0, 1)$, $D^\alpha = D_t^{\alpha_1} D_x^{\alpha_2}$, D_t and D_x are the total derivative operators with respect to t and x, respectively,

$$D_t = \partial_t + \sum_\alpha (u_{\alpha + \delta_1} \partial_{u_\alpha} + v_{\alpha + \delta_1} \partial_{v_\alpha} + w_{\alpha + \delta_1} \partial_{w_\alpha}),$$

$$D_x = \partial_x + \sum_\alpha (u_{\alpha + \delta_2} \partial_{u_\alpha} + v_{\alpha + \delta_2} \partial_{v_\alpha} + w_{\alpha + \delta_2} \partial_{w_\alpha})$$

with $u_\alpha = \partial^{\alpha_1 + \alpha_2} u / \partial t^{\alpha_1} \partial x^{\alpha_2}$, etc. Thus, the infinitesimal invariance criterion implies that

$$\begin{aligned}
& \eta^{(1,0)} + u \eta^{(0,1)} + \eta u_x + \theta^{(0,1)} = 0, \\
& \theta^{(1,0)} + u \theta^{(0,1)} + \eta v_x + \eta^{(0,1)} = 0, \\
& \zeta^{(1,0)} + u \zeta^{(0,1)} + \eta w_x = 0
\end{aligned}$$

when substituting

$$u_t = -u u_x - v_x, \quad v_t = -u v_x - u_x, \quad w_t = -u w_x.$$

Splitting with respect to the parametric derivatives u_x, v_x and w_x results in the system of determining equations for the components of Lie symmetry vector fields,

$$\zeta_t = \zeta_x = \zeta_u = \zeta_v = \tau_x = \tau_u = \tau_v = \tau_w = 0$$

$$\eta_u = \eta_v = \eta_w = \theta_u = \theta_v = \theta_w = 0,$$

$$\xi_x = \tau_t, \quad \xi_t = \eta, \quad \eta_t + u \eta_x + \theta_x = 0, \quad \theta_t + u \theta_x + \eta_x = 0.$$

The general solution of this system is

$$\tau = \xi^1 t + \tau^0, \quad \xi = \eta^0 t + \xi^1 x + \xi^0,$$

$$\eta = \eta^0, \quad \theta = \theta^0, \quad \zeta = \Omega(w),$$

where $\tau^0, \xi^0, \xi^1, \eta^0, \eta^1$ and θ^0 are arbitrary constants and Ω is an arbitrary smooth function of its argument. This proves the following theorem.

Theorem 1. *The maximal Lie invariance algebra $\mathfrak{g}$ of the system $\mathcal{S}$ is infinite-dimensional and spanned by the vector fields*

$$\begin{aligned}
\mathcal{D} &= t\partial_t + x\partial_x, \quad \mathcal{G} = t\partial_x + \partial_u, \quad \mathcal{P}^t = \partial_t, \quad \mathcal{P}^x = \partial_x, \\
\mathcal{P}^v &= \partial_v, \quad \mathcal{W}(\Omega) = \Omega(w)\partial_w,
\end{aligned} \tag{8}$$

where Ω runs through the set of smooth functions of w.

Remark 2. The systems (3) and (4) are related by the point transformation with t, x and u unchanged, and $\rho^1 = we^v/(w+1)$, $\rho^2 = e^v/(w+1)$. This transformation pushes forward the algebra $\mathfrak{g}$ to the maximal Lie invariance algebra $\tilde{\mathfrak{g}}$ of the initial system (3), where $\tilde{\mathfrak{g}} = \langle \tilde{\mathcal{D}}, \tilde{\mathcal{G}}, \tilde{\mathcal{P}}^t, \tilde{\mathcal{P}}^x, \tilde{\mathcal{P}}^v, \tilde{\mathcal{W}}(\tilde{\Omega}) \rangle$, where $\tilde{\mathcal{D}}, \tilde{\mathcal{G}}, \tilde{\mathcal{P}}^t, \tilde{\mathcal{P}}^x$ are formally of the same form as their counterparts in the algebra $\mathfrak{g}$,

$$\tilde{\mathcal{P}}^v = \rho^1 \partial_{\rho^1} + \rho^2 \partial_{\rho^2}, \quad \tilde{\mathcal{W}}(\tilde{\Omega}) = \rho^2 \tilde{\Omega}\left(\frac{\rho^1}{\rho^2}\right)(\partial_{\rho^1} - \partial_{\rho^2}),$$

and $\tilde{\Omega}$ runs through the set of the smooth functions of its argument. Note that the infinite-dimensional part $\langle \tilde{\mathcal{W}}(\tilde{\Omega}) \rangle$ of the algebra $\tilde{\mathfrak{g}}$ was missed in [23].

Remark 3. Likewise, the maximal Lie invariance algebra of the system (5) is spanned by the vector fields

$$\begin{aligned}
\hat{\mathcal{D}} &= t\partial_t + x\partial_x, \quad \hat{\mathcal{G}} = 2t\partial_x + \partial_{\mathfrak{r}^1} + \partial_{\mathfrak{r}^2}, \quad \hat{\mathcal{P}}^t = \partial_t, \quad \hat{\mathcal{P}}^x = \partial_x, \\
\hat{\mathcal{P}}^v &= \partial_{\mathfrak{r}^1} - \partial_{\mathfrak{r}^2}, \quad \hat{\mathcal{W}}(\Omega) = \Omega(\mathfrak{r}^3)\partial_{\mathfrak{r}^3},
\end{aligned}$$

where Ω runs through the set of smooth functions of $\mathfrak{r}^3$, and we also rescaled some of the spanning elements.

3. Methods of finding the complete point symmetry group

Using the Lie infinitesimal method for a system of differential equations $\mathcal{L}$ with subsequent generation of finite transformations allows one to construct the Lie symmetry group of $\mathcal{L}$, which consists of continuous symmetry transformations of the system $\mathcal{L}$ and is the identity component of the complete point symmetry group of this system. At the same time, discrete symmetry transformations are also of interest for applications. If the Lie symmetry group of $\mathcal{L}$ is known, then finding discrete symmetry transformations of $\mathcal{L}$ is equivalent to the construction of the complete point symmetry group of $\mathcal{L}$. There are two main methods for computing complete point symmetry groups of systems of differential equations in the literature, the direct method [26–28] and the algebraic method [29–33]. Although the latter in general gives only a part of restrictions on the form of point symmetry transformations and thus also involves computations within the framework of the direct method at the final step of the respective procedure, it is usually more convenient since, in comparison with using the direct method alone, the computations are less cumbersome.

The direct method is based on the definition of a (finite) point symmetry transformation and is the most universal one. The application of this method results in a system of PDEs known as the determining equations which are, in general, nonlinear and strongly coupled and thus quite difficult to solve. Several methods were developed to improve the direct method, and one of those improved methods is the algebraic one.

The algebraic method for finding the complete point symmetry group of a system of differential equations was suggested by Hydon [31,32]. The underlying idea is that each point symmetry transformation $\mathcal{T}$ of a system of differential equations $\mathcal{L}$ induces an automorphism of the maximal Lie invariance algebra $\mathfrak{g}$ of $\mathcal{L}$. If the algebra $\mathfrak{g}$ is finite-dimensional with $\dim \mathfrak{g} = n > 0$, then the above means that

$$\mathcal{T}_* e_j = \sum_{i=1}^n a_{ij} e_i, \quad j = 1, \dots, n,$$

where $\mathcal{T}_*$ is the pushforward of vector fields induced by $\mathcal{T}$, and $(a_{ij})_{i,j=1}^n$ is the matrix of an automorphism of $\mathfrak{g}$ in the chosen basis $(e_1, \ldots, e_n)$. Computing the automorphism group $\mathrm{Aut}(\mathfrak{g})$ of $\mathfrak{g}$, one obtains a set of restrictions on the form of the matrix $(a_{ij})_{i,j=1}^n$. Expanding the above condition for $\mathcal{T}_*$ gives, under the assumption that $\mathfrak{g} \neq \{0\}$, gives constraints for the transformation $\mathcal{T}$. These constraints are to be subsequently employed within the framework of the direct method.

Be that as it may, the computation of the entire automorphism group $\mathrm{Aut}(\mathfrak{g})$ can still be challenging, especially if the maximal Lie invariance algebra $\mathfrak{g}$ is infinite-dimensional. This is why another version of the algebraic method, which involves the notion of megaideals [34], also known as fully characteristics ideals [35], was developed in [29,30,33]. A megaideal $\mathfrak{i}$ of a Lie algebra $\mathfrak{g}$ is a subspace of $\mathfrak{g}$ that is invariant under any automorphism of $\mathfrak{g}$. Provided $\mathfrak{g}$ is finite-dimensional with $\dim \mathfrak{g} = n$ and possesses a megaideal $\mathfrak{i}$ spanned by the first k, $k < n$, basis elements, i.e., $\mathfrak{i} = \langle e_1, \ldots, e_k \rangle$, the matrix $(a_{ij})_{i,j=1}^n$ of any automorphism of $\mathfrak{g}$ is of block structure with $a_{ij} = 0$ for $i > k$ and $j \leqslant k$. In other words, the megaideal hierarchy of a finite-dimensional Lie algebra is directly related to the block structure of matrices of the automorphisms of this Lie algebra. This observation can also be applied to infinite-dimensional Lie algebras with finite-dimensional megaideals.

Simple tools for constructing megaideals were presented in [30,33,34]. First of all, both the improper subalgebras of a Lie algebra $\mathfrak{g}$ (the zero subalgebra and $\mathfrak{g}$ itself) are (improper) megaideals of $\mathfrak{g}$. Moreover, sums, intersections and Lie products of megaideals are megaideals, megaideals of megaideals of $\mathfrak{g}$ are megaideals of $\mathfrak{g}$, all elements of the derived series, the ascending and the descending central series of $\mathfrak{g}$, in particular, the center and the derivatives of $\mathfrak{g}$, as well as the radical and the nilradical are megaideals of $\mathfrak{g}$. Recall that the nth derivative $\mathfrak{g}^{(n)}$ of the Lie algebra $\mathfrak{g}$ is defined by $\mathfrak{g}^{(n)} = [\mathfrak{g}^{(n-1)}, \mathfrak{g}^{(n-1)}]$ for $n \geqslant 1$ with $\mathfrak{g}^{(0)} = \mathfrak{g}$ and the nth power $\mathfrak{g}^n$ of the algebra $\mathfrak{g}$ is $\mathfrak{g}^n = [\mathfrak{g}^{n-1}, \mathfrak{g}]$ for $n \geqslant 2$ with $\mathfrak{g}^1 = \mathfrak{g}$. One more, less obvious, way of obtaining new megaideals from known ones is given by the following assertion.

Proposition 4. *If $\mathfrak{i}_0$, $\mathfrak{i}_1$ and $\mathfrak{i}_2$ are megaideals of $\mathfrak{g}$, then the set $\mathfrak{s} = \{z \in \mathfrak{i}_0 : [\langle z \rangle, \mathfrak{i}_1] \subseteq \mathfrak{i}_2\}$ is also a megaideal of $\mathfrak{g}$.*

Within the megaideal-based version of the algebraic method, one uses the condition $\mathcal{T}_* \mathfrak{i} = \mathfrak{i}$ for each megaideal $\mathfrak{i}$ from a certain set of known megaideals of the maximal Lie invariance algebra $\mathfrak{g}$ of the system $\mathcal{L}$. Megaideals that are sums of other megaideals give constraints that are consequences of constraints that are derived from the consideration of the summands. This is why such decomposed megaideals should be neglected in the course of the computation. Upon having derived the constraints for a point symmetry transformation $\mathcal{T}$ implied by the megaideal invariance, one completes the computation of the complete point symmetry group with the direct method.

4. Complete point symmetry group

We compute the complete point symmetry group of the system (4) using the megaideal-based version of the algebraic method.

The nonzero commutation relations among generating elements (8) of the maximal Lie invariance algebra $\mathfrak{g}$ of the system $\mathcal{S}$ are exhausted by

$$[\mathcal{P}^t, \mathcal{D}] = \mathcal{P}^t, \quad [\mathcal{P}^x, \mathcal{D}] = \mathcal{P}^x, \quad [\mathcal{P}^t, \mathcal{G}] = \mathcal{P}^x,$$
$$[\mathcal{W}(\Omega^1), \mathcal{W}(\Omega^2)] = \mathcal{W}(\Omega^1 \Omega_w^2 - \Omega^2 \Omega_w^1).$$

Therefore, the algebra $\mathfrak{g}$ is the direct sum of its finite-dimensional and infinite-dimensional parts, $\mathfrak{g} = \langle \mathcal{D}, \mathcal{G}, \mathcal{P}^t, \mathcal{P}^x, \mathcal{P}^v \rangle \oplus \langle \mathcal{W}(\Omega) \rangle$.

Moreover, the finite-dimensional part can be split into a direct sum as well, so $\mathfrak{g} = \langle \mathcal{D}, \mathcal{G}, \mathcal{P}^t, \mathcal{P}^x \rangle \oplus \langle \mathcal{P}^v \rangle \oplus \langle \mathcal{W}(\Omega) \rangle$.

We now construct a list of megaideals for the algebra $\mathfrak{g}$. First, the derivatives of $\mathfrak{g}$ are megaideals of $\mathfrak{g}$, so $\mathfrak{g}' = \langle \mathcal{P}^t, \mathcal{P}^x, \mathcal{W}(\Omega) \rangle$ and $\mathfrak{g}'' = \langle \mathcal{W}(\Omega) \rangle$ are megaideals, with $\mathfrak{g}^{(i)} = \mathfrak{g}''$ for $i \geqslant 2$. The center $\mathcal{Z}(\mathfrak{g}) = \langle \mathcal{P}^v \rangle$ of the algebra $\mathfrak{g}$ is also its megaideal.

Lemma 5. *The radical $\mathfrak{r}$ of $\mathfrak{g}$ coincides with the finite-dimensional part of $\mathfrak{g}$,*

$$\mathfrak{r} = \langle \mathcal{D}, \mathcal{G}, \mathcal{P}^t, \mathcal{P}^x, \mathcal{P}^v \rangle.$$

Proof. We temporarily denote by $\mathfrak{s}$ the finite-dimensional part of $\mathfrak{g}$, $\mathfrak{s} = \langle \mathcal{D}, \mathcal{G}, \mathcal{P}^t, \mathcal{P}^x, \mathcal{P}^v \rangle$. The subspace $\mathfrak{s}$ is an ideal of $\mathfrak{g}$, which is solvable since $\mathfrak{s}'' = \{0\}$. Therefore, it is contained in the radical $\mathfrak{r}$ of $\mathfrak{g}$, $\mathfrak{s} \subseteq \mathfrak{r}$. If an ideal of $\mathfrak{g}$ contains a vector field $\mathcal{W}(\Omega^0)$ for a particular nonvanishing $\Omega^0 = \Omega^0(w)$ substituted for Ω, then it contains the entire infinite-dimensional part $\langle \mathcal{W}(\Omega) \rangle$ and hence it is not solvable. This means that $\mathfrak{r} \cap \langle \mathcal{W}(\Omega) \rangle = \{0\}$. Therefore, $\mathfrak{r} = \mathfrak{s}$. $\square$

Thus, we find a Levi decomposition of the infinite-dimensional algebra $\mathfrak{g}$, $\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{g}''$, where $\mathfrak{r}$ is the (finite-dimensional) radical and $\mathfrak{g}''$ is an (infinite-dimensional) simple subalgebra, which is also an (mega)ideal of $\mathfrak{g}$ but contains no proper subideals.

Lemma 6. *The nilradical $\mathfrak{n}$ of $\mathfrak{g}$ is spanned by the vector fields $\mathcal{G}$, $\mathcal{P}^t$, $\mathcal{P}^x$ and $\mathcal{P}^v$,*

$$\mathfrak{n} = \langle \mathcal{G}, \mathcal{P}^t, \mathcal{P}^x, \mathcal{P}^v \rangle.$$

Proof. The nilradical of $\mathfrak{g}$ is contained in the radical $\mathfrak{r}$ of $\mathfrak{g}$. We temporarily denote by $\mathfrak{s}$ the span of the vector fields $\mathcal{G}$, $\mathcal{P}^t$, $\mathcal{P}^x$ and $\mathcal{P}^v$, $\mathfrak{s} = \langle \mathcal{G}, \mathcal{P}^t, \mathcal{P}^x, \mathcal{P}^v \rangle$. The subspace $\mathfrak{s}$ is an ideal of $\mathfrak{g}$, and it is nilpotent since $\mathfrak{s}^2 = \{0\}$. Moreover, $\mathfrak{s}$ is the maximal nilpotent ideal of $\mathfrak{g}$ since the only subspace of $\mathfrak{r}$ properly containing $\mathfrak{s}$ is the radical $\mathfrak{r}$ itself, which is not nilpotent. Thus, $\mathfrak{n} = \mathfrak{s}$. $\square$

Corollary 7. *The derivatives $\mathfrak{r}' = \langle \mathcal{P}^t, \mathcal{P}^x \rangle$ and $\mathfrak{n}' = \langle \mathcal{P}^x \rangle$ of the radical $\mathfrak{r}$ and the nilradical $\mathfrak{n}$ of $\mathfrak{g}$, respectively, are megaideals of $\mathfrak{g}$.*

Corollary 8. *The ideal $\mathfrak{m}_1 = \langle \mathcal{G}, \mathcal{P}^x, \mathcal{P}^v \rangle$ of the algebra $\mathfrak{g}$ is its megaideal.*

Proof. This is a simple consequence of Proposition 4 for $\mathfrak{i}_0 = \mathfrak{r}$, $\mathfrak{i}_1 = \mathfrak{r}$ and $\mathfrak{i}_2 = \mathfrak{n}'$. $\square$

The nilradical $\mathfrak{n}$ is not essential for the megaideal-based version of the algebraic method since it is the sum of other megaideals of $\mathfrak{g}$, $\mathfrak{n} = \mathfrak{m}_1 + \mathfrak{r}'$. As a result, for finding the complete point symmetry group of the system (4) with the megaideal-based version of the algebraic method we use the following list of megaideals of the algebra $\mathfrak{g}$:

$$\langle \mathcal{D}, \mathcal{G}, \mathcal{P}^t, \mathcal{P}^x, \mathcal{P}^v \rangle, \quad \langle \mathcal{G}, \mathcal{P}^x, \mathcal{P}^v \rangle, \quad \langle \mathcal{P}^t, \mathcal{P}^x \rangle, \quad \langle \mathcal{P}^x \rangle,$$
$$\langle \mathcal{P}^v \rangle, \quad \langle \mathcal{W}(\Omega) \rangle. \tag{9}$$

Theorem 9. *The complete point symmetry group G of the modified no-slip isothermal drift flux model (4) consists of the transformations*

$$\tilde{t} = T^1 t + T^0, \quad \tilde{x} = T^1 x + T^1 U^0 t + X^0,$$
$$\tilde{u} = u + U^0, \quad \tilde{v} = v + V^0, \quad \tilde{w} = W(w), \tag{10}$$

where T^0, T^1, X^0, U^0 and V^0 are arbitrary constants with $T^1 \neq 0$ and W runs through the set of smooth functions of w with $W_w \neq 0$.

Proof. The general form of a point symmetry transformation for the system (4) is

$$\mathcal{T}\colon\ (\tilde{t}, \tilde{x}, \tilde{u}, \tilde{v}, \tilde{w}) = (T, X, U, V, W),$$

where T, X, U, V and W are functions of t, x, u, v and w with nonvanishing Jacobian. To obtain constraints for a point symmetry transformation $\mathcal{T}$, we push forward each of the Lie symmetry generators (8), Q, by this transformation and use the invariance, with respect to the pushforward $\mathcal{T}_*$, of the minimal megaideal from the list (9) that contains Q. This leads to the following conditions:

$$
\begin{aligned}
\mathcal{T}_*\mathcal{D} &= a_{11}(\tilde{t}\partial_{\tilde{t}} + \tilde{x}\partial_{\tilde{x}}) + a_{21}(\tilde{t}\partial_{\tilde{x}} + \partial_{\tilde{u}}) + a_{31}\partial_{\tilde{t}} + a_{41}\partial_{\tilde{x}} + a_{51}\partial_{\tilde{v}}, \\
\mathcal{T}_*\mathcal{G} &= a_{22}(\tilde{t}\partial_{\tilde{x}} + \partial_{\tilde{u}}) + a_{42}\partial_{\tilde{x}} + a_{52}\partial_{\tilde{v}}, \\
\mathcal{T}_*\mathcal{P}^t &= T_t\partial_{\tilde{t}} + X_t\partial_{\tilde{x}} + U_t\partial_{\tilde{u}} + V_t\partial_{\tilde{v}} + W_t\partial_{\tilde{w}} = a_{33}\partial_{\tilde{t}} + a_{43}\partial_{\tilde{x}}, \\
\mathcal{T}_*\mathcal{P}^x &= T_x\partial_{\tilde{t}} + X_x\partial_{\tilde{x}} + U_x\partial_{\tilde{u}} + V_x\partial_{\tilde{v}} + W_x\partial_{\tilde{w}} = a_{44}\partial_{\tilde{x}}, \\
\mathcal{T}_*\mathcal{P}^v &= T_v\partial_{\tilde{t}} + X_v\partial_{\tilde{x}} + U_v\partial_{\tilde{u}} + V_v\partial_{\tilde{v}} + W_v\partial_{\tilde{w}} = a_{55}\partial_{\tilde{v}}, \\
\mathcal{T}_*\mathcal{W}(\Omega) &= \Omega(T_w\partial_{\tilde{t}} + X_w\partial_{\tilde{x}} + U_w\partial_{\tilde{u}} + V_w\partial_{\tilde{v}} + W_w\partial_{\tilde{w}}) = \tilde{\Omega}^\Omega\partial_{\tilde{w}},
\end{aligned}
\tag{11}
$$

where all a's are constants and $\tilde{\Omega}^\Omega$ is a smooth function of $\tilde{w}$ depending on the parameter function $\Omega = \Omega(w)$.

We collect components of vector fields in the last four conditions of (11) and then take into account the resulting equations when expanding and componentwise splitting the first two conditions. As a result, we arrive at the following system:

$$
\begin{aligned}
&T_t = a_{33}, \quad X_t = a_{43}, \quad U_t = V_t = W_t = 0; \\
&T_x = U_x = V_x = W_x = 0, \quad X_x = a_{44}; \\
&T_v = X_v = U_v = W_v = 0, \quad V_v = a_{55}; \\
&T_w = X_w = U_w = V_w = 0, \quad \Omega W_w = \tilde{\Omega}^\Omega(W); \\
&tT_t = a_{11}T + a_{31}, \quad tX_t + xX_x = a_{11}X + a_{41}, \quad a_{21} = a_{51} = 0; \\
&T_u = W_u = 0, \quad tX_x + X_u = a_{22}T + a_{42}, \quad U_u = a_{22}, \quad V_u = a_{52}.
\end{aligned}
$$

Equations of this system are partitioned into groups according to their source conditions in (11). The general solution of this system is given by

$$
\begin{aligned}
T &= a_{33}t - a_{31}, \quad X = a_{22}a_{33}x + a_{43}t - a_{41}, \\
U &= a_{22}u + U^0, \quad V = a_{55}v + a_{52}u + V^0, \quad W = W(w),
\end{aligned}
\tag{12}
$$

where U^0 and V^0 are arbitrary constants, and additionally $a_{22}a_{33}a_{55} \neq 0$, $a_{11} = 1$, $a_{44} = a_{22}a_{33}$ and $a_{42} = a_{31}a_{22}$.

Now the direct method of computing complete point symmetry groups should be applied. To this end, we apply a transformation of the form (12) to the system $\mathcal{S}$. To do this, the derivative operators with respect to the new independent variables, $\partial_{\tilde{t}}$ and $\partial_{\tilde{x}}$, have to be determined,

$$
\partial_{\tilde{t}} = \frac{1}{a_{33}}\left(\partial_t - \frac{a_{42}}{a_{22}a_{33}}\partial_x\right), \quad \partial_{\tilde{x}} = \frac{1}{a_{22}a_{33}}\partial_x.
$$

Employing these derivative operators to express the derivatives of the new variables, and enforcing the symmetry condition requires that $a_{42} = a_{33}U^0$, $a_{22} = a_{55} = 1$, $a_{52} = 0$.

Re-denoting parameter constants completes the proof of the theorem. $\square$

Corollary 10. *The modified no-slip isothermal drift flux model (4) possesses two independent (up to combining with each other and with continuous symmetries) discrete point symmetries given by the reflections*

$$(t, x, u, v, w) \to (-t, -x, u, v, w),$$
$$(t, x, u, v, w) \to (t, x, u, v, -w).$$

Remark 11. As we have already mentioned in Section 3, the automorphism-based version of the algebraic method for finding the complete point symmetry group requires knowing the automorphism group of the corresponding maximal invariance algebra. For the system $\mathcal{S}$, this algebra is infinite-dimensional, and therefore the computation of its automorphism group is complicated. However, it could be useful to look for the automorphism group $\mathrm{Aut}(\mathfrak{r})$ of the radical $\mathfrak{r}$ which coincides, in view of Lemma 5, with the finite-dimensional part of the algebra $\mathfrak{g}$. Note that the algebra $\mathfrak{r}$ is isomorphic to the algebra $A_{4.8}^0 \oplus A_1$ from the classification list of five-dimensional real Lie algebras presented in [36,37]. In the basis $(\mathcal{D}, \mathcal{G}, \mathcal{P}^t, \mathcal{P}^x, \mathcal{P}^v)$, the automorphism group $\mathrm{Aut}(\mathfrak{r})$ group can be identified with the matrix group constituted by the matrices of the form

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & b_{22} & 0 & 0 & 0 \\
b_{31} & 0 & b_{33} & 0 & 0 \\
b_{41} & b_{31}b_{22} & b_{43} & b_{22}b_{33} & 0 \\
b_{51} & b_{52} & 0 & 0 & b_{55}
\end{pmatrix}
\quad \text{with} \quad b_{22}b_{33}b_{55} \neq 0.
$$

The knowledge of $\mathrm{Aut}(\mathfrak{r})$ allows us to set constraints on the constants a's in the conditions (11) before analyzing these conditions,

$$a_{11} = 1, \quad a_{22}a_{33}a_{55} \neq 0, \quad a_{21} = 0,$$
$$a_{44} = a_{22}a_{33}, \quad a_{42} = a_{31}a_{22}.$$

The structure of the automorphism matrices obviously implies that there is one more megaideal of $\mathfrak{r}$ and, therefore, of $\mathfrak{g}$, $\mathfrak{m}_2 = \langle \mathcal{D}, \mathcal{P}^t, \mathcal{P}^x, \mathcal{P}^v \rangle$. This completes the description of megaideals of the algebra $\mathfrak{g}$. More specifically, the megaideals of $\mathfrak{g}$ are exhausted by the essential megaideals $\mathfrak{n}'$, $\mathcal{Z}(\mathfrak{g})$, $\mathfrak{r}'$, $\mathfrak{m}_1$, $\mathfrak{m}_2$ and $\mathfrak{g}''$ and their sums. The megaideal $\mathfrak{m}_2$ cannot be found using the means presented in Section 3. The presence of this megaideal explains the first of the above constraints on the constants a's whilst the rest of these constraints cannot be obtained using the megaideal-based version of algebraic method, being, at the same time, a direct consequence of its automorphism-based counterpart applied to $\mathfrak{r}$. In fact, this discussion shows a possibility for combining the automorphism- and megaideal-based versions of algebraic method for finding the complete point symmetry groups of systems of differential equations.

Remark 12. There is one constraint for the constants a's, $a_{51} = 0$, among those derived from the conditions (11) that cannot be obtained from the structure of automorphism matrices of the radical $\mathfrak{r}$. This means that there exist automorphisms of the entire algebra $\mathfrak{g}$ that are not induced by point transformations of (t, x, u, v, w).

5. Classification of subalgebras

In order to efficiently perform group-invariant reductions for finding exact solutions of the system (4), it is necessary to construct an optimal list of G-inequivalent one- and two-dimensional subalgebras of the maximal Lie invariance algebra $\mathfrak{g}$ admitted by this system [24]. Since the algebra $\mathfrak{g}$ is infinite-dimensional, for constructing the adjoint representation of G on $\mathfrak{g}$ we use the method based on computing pushforwards of vector fields in $\mathfrak{g}$ by transformations in G [38,39].

Any transformation $\mathcal{T}$ from G can be represented as a composition of elementary symmetry transformations,

$$\mathcal{T} = \mathcal{D}(T^1)\mathcal{P}(T^0)\mathcal{P}(X^0)\mathcal{P}(V^0)\mathcal{G}(U^0)\mathcal{W}(W)$$

with

$$
\begin{aligned}
\mathcal{P}^t(T^0)\colon\ & (\tilde{t}, \tilde{x}, \tilde{u}, \tilde{v}, \tilde{w}) = (t + T^0, x, u, v, w), \\
\mathcal{P}^x(X^0)\colon\ & (\tilde{t}, \tilde{x}, \tilde{u}, \tilde{v}, \tilde{w}) = (t, x + X^0, u, v, w),
\end{aligned}
$$

$\mathcal{P}^v(V^0)$: $(\tilde{t}, \tilde{x}, \tilde{u}, \tilde{v}, \tilde{w}) = (t, x, u, v + V^0, w)$,

$\mathcal{D}(T^1)$: $(\tilde{t}, \tilde{x}, \tilde{u}, \tilde{v}, \tilde{w}) = (T^1 t, T^1 x, u, v, w)$,

$\mathcal{G}(U^0)$: $(\tilde{t}, \tilde{x}, \tilde{u}, \tilde{v}, \tilde{w}) = (t, x + U^0 t, u + U^0, v, w)$,

$\mathcal{W}(W)$: $(\tilde{t}, \tilde{x}, \tilde{u}, \tilde{v}, \tilde{w}) = (t, x, u, v, W(w))$,

where, as in Theorem 9, T^0, T^1, X^0, U^0 and V^0 are arbitrary constants with $T^1 \neq 0$ and $W(w)$ runs through the set of smooth functions of w with $W_w \neq 0$. The nonidentity actions of push-forwards of elementary symmetry transformations on generating elements of $\mathfrak{g}$ are exhausted by the following:

$\mathcal{P}^t_*(T^0)\mathcal{D} = \mathcal{D} - T^0 \mathcal{P}^t$, $\mathcal{P}^t_*(T^0)\mathcal{G} = \mathcal{G} - T^0 \mathcal{P}^x$,

$\mathcal{P}^x_*(X^0)\mathcal{D} = \mathcal{D} - X^0 \mathcal{P}^x$,

$\mathcal{D}_*(T^1)\mathcal{P}^t = T^1 \mathcal{P}^t$, $\mathcal{D}_*(T^1)\mathcal{P}^x = T^1 \mathcal{P}^x$,   (13)

$\mathcal{G}_*(U^0)\mathcal{P}^t = \mathcal{P}^t + U^0 \mathcal{P}^x$,

$\mathcal{W}_*(W)\mathcal{W}(\Omega) = \mathcal{W}(\tilde{\Omega})$,

where the function $\tilde{\Omega} = \tilde{\Omega}(\tilde{w})$ is related to $\Omega = \Omega(w)$ by $\tilde{\Omega}(W(w)) = W_w(w)\Omega(w)$.

Theorem 13. *An optimal list of one-dimensional subalgebras of the maximal Lie invariance algebra $\mathfrak{g}$ of the system* (4) *is exhausted by the subalgebras*

$\langle \mathcal{D} + a\mathcal{G} + b\mathcal{P}^v + \mathcal{W}(\delta_1) \rangle$, $\langle \mathcal{G} + \delta_2 \mathcal{P}^t + b\mathcal{P}^v + \mathcal{W}(\delta_1) \rangle$,

$\langle \mathcal{P}^t + \delta_2 \mathcal{P}^v + \mathcal{W}(\delta_1) \rangle$, $\langle \mathcal{P}^x + \delta_2 \mathcal{P}^v + \mathcal{W}(\delta_1) \rangle$,   (14)

$\langle \mathcal{P}^v + \mathcal{W}(\delta_1) \rangle$, $\langle \mathcal{W}(1) \rangle$,

where $\delta_1, \delta_2 \in \{0, 1\}$, and a and b are arbitrary constants.

Proof. We start with the most general form of a basis vector field of a one-dimensional subalgebra of the algebra $\mathfrak{g}$,

$Q = a_1 \mathcal{D} + a_2 \mathcal{G} + a_3 \mathcal{P}^t + a_4 \mathcal{P}^x + a_5 \mathcal{P}^v + \mathcal{W}(\Omega)$,

where a_i, $i = 1, \dots, 5$, are constants, Ω is a smooth function of w, and at least one of these quantities does not vanish. We simplify Q using the adjoint action of G and scaling of the entire Q. In all cases when $\Omega \neq 0$, the parameter function Ω can be set to 1 by applying the pushforward $\mathcal{W}_*\left(\int 1/\Omega \, dw\right)$. In other words, we can always suppose $\Omega(w) = \delta_1 \in \{0, 1\}$. Moreover, if $a_5 \neq 0$, then the summand with $\mathcal{P}^v$ cannot be deleted using the adjoint action of G.

We order the elements spanning $\mathfrak{g}$ in the following way:

$$\mathcal{D} \succ \mathcal{G} \succ \mathcal{P}^t \succ \mathcal{P}^x \succ \mathcal{P}^v \succ \mathcal{W}(\Omega).$$

This partial order is justified by the hierarchy of megaideals of the algebra $\mathfrak{g}$. Subsequent analysis is done recursively according to this order: regarding the leading coefficient of Q as nonzero, we scale this coefficient, if it is a constant, to 1 by scaling the entire vector field Q, further simplify the form of Q by the adjoint action of G and re-denote the parameters preserved in the final form of Q. As a result, we reduce the classification to the following inequivalent cases.

1. $a_1 = 1$. Using successively the pushforwards $\mathcal{P}^t_*(a_3)$ and $\mathcal{P}^x_*(a_4 - a_2 a_3)$ we set $a_3 = a_4 = 0$. This yields the first subalgebra (more precisely, the first family of subalgebras) of the list (14).

2. $a_1 = 0$, $a_2 = 1$. We act by $\mathcal{P}^t_*(a_4)$ and, if $a_3 \neq 0$, by $\mathcal{D}_*(a_3^{-1})$ to set $a_4 = 0$ and $a_3 = 1$, respectively, which leads to the second subalgebra.

3. $a_1 = a_2 = 0$, $a_3 = 1$. Then we act by $\mathcal{G}_*(-a_4)$ to set $a_4 = 0$. If $a_5 \neq 0$, then we can also set $a_5 = 1$ by acting $\mathcal{D}_*(a_5)$ and rescaling Q. This gives the third subalgebra.

4. $a_1 = a_2 = a_3 = 0$, $a_4 = 1$. Upon acting by $\mathcal{D}_*(a_5)$ and rescaling Q, we derive the fourth subalgebra.

5. $a_i = 0$, $i = 1, \dots, 4$, $a_5 = 1$. This yields the fifth subalgebra.

6. $a_i = 0$, $i = 1, \dots, 5$. Then Ω is nonzero, which corresponds to the last subalgebra.   □

Theorem 14. *An optimal list of two-dimensional subalgebras of the maximal algebra of invariance $\mathfrak{g}$ of the system* (4) *is given by*

$\langle \mathcal{D} + a\mathcal{P}^v + \mathcal{W}(\delta_1), \mathcal{G} + b\mathcal{P}^v + \mathcal{W}(\delta_2) \rangle$,

$\langle \mathcal{D} + a\mathcal{P}^v + \mathcal{W}(\delta_5), \mathcal{P}^t \rangle$, $\langle \mathcal{D} + a\mathcal{P}^v + \mathcal{W}(w), \mathcal{P}^t + \mathcal{W}(1) \rangle$,

$\langle \mathcal{D} + a\mathcal{G} + b\mathcal{P}^v + \mathcal{W}(\delta_5), \mathcal{P}^x \rangle$,

$\langle \mathcal{D} + a\mathcal{G} + b\mathcal{P}^v + \mathcal{W}(w), \mathcal{P}^x + \mathcal{W}(1) \rangle$,

$\langle \mathcal{D} + a\mathcal{G} + \mathcal{W}(\delta_1), \mathcal{P}^v + \mathcal{W}(\delta_2) \rangle$,

$\langle \mathcal{G} + \delta_3 \mathcal{P}^t + a\mathcal{P}^v + \mathcal{W}(\delta_1), \mathcal{P}^x + \delta_4 \mathcal{P}^v + \mathcal{W}(\delta_2) \rangle$,

$\langle \mathcal{G} + \delta_5 \mathcal{P}^t + \mathcal{W}(\delta_1), \mathcal{P}^v + \mathcal{W}(\delta_2) \rangle$,   (15)

$\langle \mathcal{P}^t + \delta_3 \mathcal{P}^v + \mathcal{W}(\delta_1), \mathcal{P}^x + \delta_4 \mathcal{P}^v + \mathcal{W}(\delta_2) \rangle$,

$\langle \mathcal{P}^t + \mathcal{W}(\delta_1), \mathcal{P}^v + \mathcal{W}(\delta_2) \rangle$, $\langle \mathcal{P}^x + \mathcal{W}(\delta_1), \mathcal{P}^v + \mathcal{W}(\delta_2) \rangle$,

$\langle \mathcal{D} + a\mathcal{G} + b\mathcal{P}^v + c\mathcal{W}(w), \mathcal{W}(1) \rangle$,

$\langle \mathcal{G} + \delta_5 \mathcal{P}^t + b\mathcal{P}^v + c\mathcal{W}(w), \mathcal{W}(1) \rangle$,

$\langle \mathcal{P}^t + \delta_1 \mathcal{P}^v + \delta_2 \mathcal{W}(w), \mathcal{W}(1) \rangle$,

$\langle \mathcal{P}^x + \delta_1 \mathcal{P}^v + \delta_2 \mathcal{W}(w), \mathcal{W}(1) \rangle$,

$\langle \mathcal{P}^v + c\mathcal{W}(w), \mathcal{W}(1) \rangle$, $\langle \mathcal{W}(w), \mathcal{W}(1) \rangle$,

where $\delta_1, \dots, \delta_4$, a, b and c are arbitrary constants such that one of nonzero components in each of the pairs (δ_1, δ_2) and (δ_3, δ_4) can be set to be equal 1, and $\delta_5 \in \{0, 1\}$. In the seventh family of subalgebras, either $a = 0$ if $\delta_4 \neq 0$ or $\delta_1 = 0$ if $\delta_2 \neq 0$.

Proof. We start with a two-dimensional subalgebra $\mathfrak{s} := \langle Q_1, Q_2 \rangle$ spanned by two linearly independent vector fields from $\mathfrak{g}$ of the most general form,

$Q_1 = a_1 \mathcal{D} + a_2 \mathcal{G} + a_3 \mathcal{P}^t + a_4 \mathcal{P}^x + a_5 \mathcal{P}^v + \mathcal{W}(\Omega^1)$,

$Q_2 = b_1 \mathcal{D} + b_2 \mathcal{G} + b_3 \mathcal{P}^t + b_4 \mathcal{P}^x + b_5 \mathcal{P}^v + \mathcal{W}(\Omega^2)$.

Within the classification procedure, we should take into account the condition of closedness of $\mathfrak{s}$ with respect to the Lie bracket, $[Q_1, Q_2] \in \mathfrak{s}$, which we will briefly call the $\mathfrak{s}$-condition below. Since $\mathfrak{g} = \langle \mathcal{D}, \mathcal{G}, \mathcal{P}^t, \mathcal{P}^x \rangle \oplus \langle \mathcal{P}^v \rangle \oplus \langle \mathcal{W}(\Omega) \rangle$, summands with $\mathcal{P}^v$ or $\mathcal{W}(\Omega)$, if they are in Q^1 or Q^2, cannot be deleted by the adjoint action of G. Nevertheless, we can always set $\Omega^i = 1$ for nonzero Ω^i with fixed $i \in \{1, 2\}$ by the action of $\mathcal{W}_*\left(\int 1/\Omega^i \, dw\right)$. If the parameter functions Ω^1 and Ω^2 are linearly independent, linearly recombining Q_1 and Q_2, which is consistent with the further classification procedure, reduces the $\mathfrak{s}$-condition to $[Q_1, Q_2] \in \langle Q_2 \rangle$. Setting $\Omega^2 = 1$, we can thus assume $\Omega^1 = c_1 w + c_2$, where c_1 and c_2 are constants. In particular, $\Omega^2 = \text{const}$ if Q^1 and Q^2 commute. We can consider only Ω's of the above form from the very beginning. If $a_1 \neq 0$, we can set $a_3 = a_4 = 0$ successively using $\mathcal{P}^t_*(a_3)$ and $\mathcal{P}^x_*(a_4 - a_2 a_3)$.

For the sake of efficient classification of two-dimensional subalgebras of the algebra $\mathfrak{g}$, it is convenient to consider the coefficient matrix

$$A = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 & b_4 & b_5 \end{pmatrix}.$$

The classification splits into two cases, $\text{rk}\, A = 2$ and $\text{rk}\, A < 2$.

I. rk $A = 2$. On the set of index pairs $J := \{(i, j) \mid 1 \leqslant i < j \leqslant 5\}$ we introduce the lexicographical order with the alphabet ordering $1 \succ 2 \succ 3 \succ 4 \succ 5$, which is consistent with the ordering of vector fields spanning the algebra $\mathfrak{g}$. Then we further split the case $\text{rk}\, A = 2$ into subcases that are labeled by index pairs in J and each of which is singled out by the additional condition $\text{rk}\, A_{i', j'} < 2$, $(i', j') \succ (i, j)$, $\text{rk}\, A_{i,j} = 2$ with the associated index pair (i, j). Here $A_{i,j}$ denotes the 2×2 matrix composed by the ith and jth columns of A. Note that $a_i = b_i = 0$ if both $\text{rk}\, A_{i,j} < 2$ for $i < j$ and $\text{rk}\, A_{j,i} < 2$ for $j < i$. In the (i, j)th case, up to linearly recombining Q_1 and Q_2 we can assume that $A_{i,j} = I$ and

$b_1 = \cdots = b_{j'} = 0$, $j' < j$. Here and in what follows I is the 2×2 unit matrix. As a result, we should consider the following cases.

1. $A_{1,2} = I$, $a_3 = a_4 = 0$. The $\mathfrak{s}$-condition yields $b_3 = b_4 = 0$, which results in the first subalgebra (more precisely, the first family of subalgebras) in the list (15).

2. $A_{1,3} = I$, $b_2 = 0$, $a_4 = 0$. Using $\mathcal{G}_*(b_4)$ we set $b_4 = 0$. Then the $\mathfrak{s}$-condition implies $a_2 = b_5 = 0$ and $\Omega^2 \Omega_w^1 - \Omega^1 \Omega_w^2 = \Omega^2$. Note that here we use the $\mathfrak{s}$-condition before employing the pushforward $\mathbb{W}_*$. This leads to the second and third subalgebras, depending on whether Ω^2 vanishes or not; for the latter we should additionally act on $\mathfrak{s}$ with the pushforward of a shift of w.

3. $A_{1,4} = I$, $b_2 = b_3 = 0$, $a_3 = 0$. We employ the $\mathfrak{s}$-condition in the same way as in the previous case, obtaining the fourth and fifth subalgebras.

In all the other cases with $\mathrm{rk}\,A = 2$, the basis elements Q_1 and Q_2 commute in view of the $\mathfrak{s}$-condition. Hence $\Omega^1 \Omega_w^2 - \Omega^2 \Omega_w^1 = 0$, and we can simultaneously set both the coefficients Ω^1 and Ω^2 to be constants.

4. $A_{1,5} = I$, $b_2 = b_3 = b_4 = 0$, $a_3 = a_4 = 0$. Thus, we derive the sixth subalgebra.

5. $A_{2,3} = I$, $a_1 = b_1 = 0$. $[Q_1, Q_2] = \mathcal{P}^x + \mathcal{W}(\Omega^1 \Omega_w^2 - \Omega^2 \Omega_w^1) \notin \langle Q_1, Q_2 \rangle$ and hence $\langle Q_1, Q_2 \rangle$ is not a subalgebra.

6. $A_{2,4} = I$, $a_1 = b_1 = b_3 = 0$. One of the coefficients a_3 or b_5 can be scaled to 1 (if nonzero) using $\mathcal{D}_*(a_3^{-1})$ for a_3 or $\mathcal{D}_*(b_5)$ and rescaling of Q^2 for b_5. We obtain the seventh algebra. The specific gauge for parameters of this case from the theorem is set by composing the pushforward $\mathcal{P}_*^t(c)$ with the replacement Q^1 by $Q^1 + cQ^2$, where $c = -a/\delta_4$ or $c = -\delta_1/\delta_2$, respectively.

7. $A_{2,5} = I$ and $a_1 = b_1 = b_3 = b_4 = 0$. We act on $\mathfrak{s}$ by $\mathcal{P}_*^t(a_4)$ and, if $a_3 \neq 0$, by $\mathcal{D}_*(a_3^{-1})$ to set $a_4 = 0$ and $a_3 = 1$, respectively. This gives the eighth subalgebra.

8. $A_{3,4} = I$, $a_i = b_i = 0$, $i = 1, 2$. One of a_5 and b_5 (if nonzero) can be scaled to 1 acting on $\mathfrak{s}$ with $\mathcal{D}_*(c)$ and replacing the basis element Q_i by $c^{-1}Q_i$, where $c = a_5$, $i = 1$ and $c = b_5$, $i = 2$, respectively. As a result, we have the ninth subalgebra.

9. $A_{3,5} = I$, $a_i = b_i = 0$, $i = 1, 2$, $b_4 = 0$. Setting $a_4 = 0$ with $\mathcal{G}_*(-a_4)$ gives the tenth subalgebra.

10. $A_{4,5} = I$, $a_i = b_i = 0$, $i = 1, 2, 3$. This case is associated with the eleventh subalgebra.

II. rk A < 2. Linearly combining Q_1 and Q_2 in this case, we can set $b_1 = \cdots = b_5 = 0$. Since then $\Omega^2 \neq 0$, we can set $\Omega^2 = 1$ using the pushforward $\mathbb{W}_* (\int 1/\Omega^2 \, dw)$. Up to linearly recombining Q_1 and Q_2, the $\mathfrak{s}$-condition implies $\Omega^1 = cw$ for a constant c. The further classification of inequivalent forms for Q_1, which gives the rest of listed subalgebras, is similar to the classification of one-dimensional subalgebras, and thus we omit it. $\square$

6. Reductions

Although in Section 8 we describe the entire set of local solutions of the system $\mathcal{S}$ in implicit form in terms of solutions of the telegraph equation, it is instructive to find invariant solutions as well. We employ the optimal lists of one- and two-dimensional subalgebras of the algebra $\mathfrak{g}$ from Theorems 13 and 14, respectively, to obtain Lie reductions associated with them. Lie ansatzes constructed using one-dimensional subalgebras reduce the system $\mathcal{S}$ with the two independent variables (t, x) to systems of ODEs. The last two families of subalgebras from Theorem 13 are not relevant to the framework of Lie reductions since the transversality condition does not hold for them.

Lie ansatzes constructed using two-dimensional subalgebras reduce the system $\mathcal{S}$ to systems of algebraic equations. Note that constraints imposed by invariance with respect to two-parameter transformation groups are too restrictive for solutions of PDEs with two independent variables. This is why such solutions are not of much significance. Nevertheless, we are going to perform the Lie reduction with respect to a two-dimensional subalgebra of the algebra $\mathfrak{g}$ as an example. It is obvious that in the list (15) of families of inequivalent two-dimensional subalgebras of the algebra $\mathfrak{g}$, only subalgebras from the first to fifth, and seventh and ninth families are appropriate for Lie reduction. Subalgebras from other families, which do not satisfy the transversality condition, can be useful for constructing partially invariant solutions [25], which we also illustrate by an example.

Within this section, c_1, c_2 and c_3 are arbitrary constants, and prime denotes the derivative with respect to ω.

1. $\langle \mathcal{D} + a\mathcal{G} + b\mathcal{P}^v + \mathcal{W}(\delta_1) \rangle$. A Lie ansatz constructed with this subalgebra has the form $u = \phi(\omega) + x/t + a$, $v = \chi(\omega) + b \ln |t|$, $w = \psi(\omega) + \delta_1 \ln |t|$, where $\omega = x/t - a \ln |t|$. The corresponding reduced system of ODEs is

$$\phi'\phi + \chi' + \phi + a = 0, \quad \phi\chi' + \phi' + b + 1 = 0, \quad \phi\psi' + \delta_1 = 0.$$

For integrating this system, it is convenient to separately consider the singular and general cases.

A. $\phi^2 + a\phi - b - 1 = 0$, i.e., $\phi = \mu$, where μ is a root of this algebraic equation. Then $\chi = -(a + \mu)\omega + c_2$. For ψ we additionally have two cases. If $\mu = 0$, then ψ is an arbitrary function of ω, as well as $b = -1$ and $\delta_1 = 0$ are the only possible values of these parameters. Otherwise, $\psi = -\delta_1\omega/\mu + c_3$.

B. Otherwise, $\phi^2 + a\phi - b - 1 \neq 0$. We also have $\phi^2 \neq 1$, and the reduced system is equivalent to the system $\phi' = (\phi^2 + a\phi - b - 1)/(1 - \phi^2)$, $\chi' = (b\phi - a)/(1 - \phi^2)$, $\psi' = -\delta_1/\phi$. As a result, we construct the general solution of the reduced system in parametric form,

$$
\omega = \int \frac{(1 - \phi^2)\, d\phi}{\phi^2 + a\phi - b - 1}, \quad \chi = \int \frac{(b\phi - a)\, d\phi}{\phi^2 + a\phi - b - 1},
$$
$$
\hat{\psi} = \int \frac{(1 - \phi^2)\, d\phi}{\phi(\phi^2 + a\phi - b - 1)},
\tag{16}
$$

and $\psi = -\delta_1 \hat{\psi}$. All the three integrals are integrals of rational functions, which can be easily computed.

2. $\langle \mathcal{G} + \delta_2 \mathcal{P}^t + b\mathcal{P}^v + \mathcal{W}(\delta_1) \rangle$. Let us consider separately the cases $\delta_2 = 1$ and $\delta_2 = 0$.

A. $\delta_2 = 1$. A Lie ansatz constructed for this value of δ_2 and the associated reduced system of ODEs are $u = \phi(\omega) + t$, $v = \chi(\omega) + bt$, $w = \psi(\omega) + \delta_1 t$, where $\omega = x - t^2/2$, and

$$\phi\phi' + \chi' + 1 = 0, \quad \phi\chi' + \phi' + b = 0, \quad \phi\psi' + \delta_1 = 0.$$

For $b = \varepsilon := \pm 1$, the reduced system admits the singular solution $\phi = \varepsilon$, $\chi = -\omega + c_2$ and $\psi = -\delta_1\varepsilon\omega + c_3$. In the general case, we exclude χ' from the first two equations of the system and derive the equation $\phi' = (\phi - b)/(1 - \phi^2)$. Integrating this equation and then using the obtained solution for integrating the equations for χ and ψ, we find the general solution of the reduced system in parametric form,

$$\omega = -\frac{\phi^2}{2} - b\phi - (b^2 - 1)\ln|\phi - b| + c_1,$$

$$\chi = b\phi + (b^2 - 1)\ln|\phi - b| + c_2,$$

$$\psi = \delta_1\phi + \frac{\delta_1}{b}\ln|\phi| + \frac{\delta_1}{b}(b^2 - 1)\ln|\phi - b| + c_3.$$

B. $\delta_2 = 0$. A Lie ansatz constructed for this value of δ_2 and the associated reduced system of ODEs are $u = \phi(\omega) + x/t$, $v = \chi(\omega) + bx/t$, $w = \psi(\omega) + \delta_1 x/t$, where $\omega = t$, and

$$\omega\phi' + \phi + b = 0, \quad \omega\chi' + b\phi + 1 = 0, \quad \omega\psi' + \delta_1\phi = 0.$$

This yields $\phi = c_1/\omega - b$, $\chi = (b^2 - 1)\ln|\omega| + c_1 b/\omega + c_2$ and $\psi = \delta_1 b \ln|\omega| + c_1/\omega + c_3$.

3. $\langle \mathcal{P}^t + \delta_2 \mathcal{P}^v + \mathcal{W}(\delta_1) \rangle$. A Lie ansatz $u = \phi(\omega)$, $v = \chi(\omega) + \delta_2 t$, $w = \psi(\omega) + \delta_1 t$, where $\omega = x$, constructed with this subalgebra reduces the system (4) to the system of ODEs

$$\phi\phi' + \chi' = 0, \quad \phi\chi' + \phi' + \delta_2 = 0, \quad \phi\psi' + \delta_1 = 0.$$

Combining the first two equations to exclude χ', we derive the equations $\delta_2 + \phi' = \phi^2 \phi'$, which integrates to $\phi^3/3 - \phi = \delta_2 \omega + c_1$.

If $\delta_2 = 0$, then ϕ is just a constant satisfying the algebraic equation. Hence χ is another constant. If $\phi \neq 0$, then ψ is linear function with respect to ω. Otherwise, δ_1 vanishes and $\psi(\omega)$ is an arbitrary smooth function of its argument.

If $\delta_2 \neq 0$, we construct the general solution of the reduced system in parametric form

$$\omega = \frac{\phi^3}{3\delta_2} - \frac{\phi}{\delta_2} - \frac{c_1}{\delta_2}, \quad \chi = \frac{\phi^2}{2} + c_2, \quad \psi = \frac{\delta_1}{2}\phi^2 - \delta_1 \ln|\phi| + c_3.$$

4. $\langle \mathcal{D} + a\mathcal{G} + b\mathcal{P}^v + \mathcal{W}(\delta_1), \mathcal{P}^x \rangle$. The ansatz $u = a \ln|t| + c_1$, $v = b \ln|t| + c_2$, $w = \delta_1 \ln|t| + c_3$ constructed with this subalgebra gives only trivial constant solutions since the corresponding reduced system is $a = b = \delta_1 = 0$, which is rather the condition for consistency of this ansatz with the system $\mathcal{S}$.

5. $\langle \mathcal{D} + a\mathcal{G} + b\mathcal{P}^v, \mathcal{W}(1) \rangle$. This subalgebra does not satisfy the transversality condition in view of the form of the second basis vector field $\mathcal{W}(1)$. Therefore, this subalgebra cannot be used within the framework of Lie reductions. At the same time, it can be used for finding partially invariant solutions of the system (4). The existence of partially invariant solutions for the system (4) is guaranteed by the fact that this system is only partially coupled (see Section 7 for details). We can reduce the essential subsystem (4a)–(4b) with a Lie ansatz constructed using the one-dimensional subalgebra of $\mathfrak{g}$ spanned by the first basis vector field of the two-dimensional algebra under study. Then we find solutions of the reduced system of ODEs, merge them with the corresponding ansatz to construct solutions of (4a)–(4b) and substitute the later solutions to Eq. (4c). The resulting equation is a first-order linear partial differential equation with respect to w, which is integrated separately. For constructing the required solutions of the essential subsystem (4a)–(4b), it in fact suffices to use the expressions for (u, v) obtained in the course of the first reduction of this section.

A. $u = x/t + a + \mu$, $v = -(a+\mu)x/t + (b + a^2 + a\mu) \ln|t| + c_2$. Then the function w satisfies the equation $w_t + (x/t + a + \mu)w_x = 0$, the general solution of which is $w = \psi(\varpi)$, where ψ runs through the set of smooth functions of $\varpi = x/t - (a + \mu) \ln|t|$.

B. $u = \phi(\omega) + x/t + a$, $v = \chi(\omega) + b \ln|t|$, where $\omega = x/t - a \ln|t|$, and ϕ and χ are defined by (16). After changing the independent variables (t, x) to (τ, ω), where $\tau = \ln|t|$, Eq. (4c) with the above value of u takes the form $w_\tau + \phi(\omega)w_\omega = 0$, and thus its general solution is $w = \psi(\varpi)$, where ψ runs through the set of smooth functions of $\varpi = \tau - \int d\omega/\phi = \ln|t| - \hat{\psi}$, and $\hat{\psi}$ is defined by (16) as well.

7. Group analysis of the essential subsystem

The system of Eqs. (4a)–(4b), which we denote for brevity by $\mathcal{S}_0$, does not involve the unknown function w, i.e., the system $\mathcal{S}$ is only partially coupled. This is why we can solve the system $\mathcal{S}_0$ in the first place and substitute the obtained u into Eq. (4c), which then becomes a linear first-order PDE with respect to w. Finding a partial solution of Eq. (4c) and acting on it by the transformation $\mathcal{W}(W)$, we obtain a family of solutions parameterized by an arbitrary function. Moreover, the system $\mathcal{S}_0$ is simpler for finding exact solutions than $\mathcal{S}$ because it has fewer dependent variables and admits a larger Lie symmetry algebra.

To compute Lie symmetries of the system $\mathcal{S}_0$, we again apply the Lie infinitesimal method, cf. Section 2. We look for the generator $Q = \tau \partial_t + \xi \partial_x + \eta \partial_u + \theta \partial_v$ of a one-parameter point symmetry group, where the components τ, ξ, η and θ depend on t, x, u and v. For the system $\mathcal{S}_0$ and the generator Q, the infinitesimal invariance criterion reads

$$Q^{(1)}(u_t + uu_x + v_x)|_{\mathcal{S}_0} = 0, \qquad Q^{(1)}(v_t + uv_x + u_x)|_{\mathcal{S}_0} = 0.$$

The first prolongation $Q^{(1)}$ of the vector field Q is given by

$$Q^{(1)} = Q + \eta^{(1,0)}\partial_{u_t} + \eta^{(0,1)}\partial_{u_x} + \theta^{(1,0)}\partial_{v_t} + \theta^{(0,1)}\partial_{v_x},$$

where the components $\eta^{(1,0)}$, $\eta^{(0,1)}$, $\theta^{(1,0)}$ and $\theta^{(0,1)}$ of the prolonged vector field $Q^{(1)}$ are obtained from the general prolongation formula [24,25]. Then the infinitesimal invariance criterion implies

$$\eta^{(1,0)} + u\eta^{(0,1)} + \eta u_x + \theta^{(0,1)} = 0,$$
$$\theta^{(1,0)} + u\theta^{(0,1)} + \eta v_x + \eta^{(0,1)} = 0,$$

when substituting $u_t = -uu_x - v_x$ and $v_t = -uv_x - u_x$. Splitting these equations with respect to the parametric derivatives u_x and v_x yields the following system of determining equations

$$
\begin{aligned}
&R^1 := \eta_t + u\eta_x + \theta_x = 0, \quad && R^2 := \theta_t + u\theta_x + \eta_x = 0,\\
&R^3 := \eta_u - \theta_v = 0, && R^4 := \theta_u - \eta_v = 0,\\
&R^5 := \tau_t + 2u\tau_x - \xi_x = 0, && R^6 := \xi_t + (u^2 - 1)\tau_x - \eta = 0,\\
&R^7 := u\tau_u - \tau_v - \xi_u = 0, && R^8 := u\tau_v - \tau_u - \xi_v = 0.
\end{aligned}
\tag{17}
$$

We derive differential consequences of the system (17), which are then (mainly, implicitly) employed for simplifying successive differential consequences of this system,

$$R^1_u - R^2_v - R^3_t - uR^3_x - R^4_x = \eta_x = 0,$$
$$R^2_u - R^1_v - R^4_t - uR^4_x - R^3_x = \theta_x = 0,$$
$$R^9 := R^7_v - R^8_u = \tau_{uu} - \tau_{vv} - \tau_v = 0,$$
$$R^{10} := R^8_v - R^7_u + uR^9 = \xi_{uu} - \xi_{vv} - \tau_u - u\tau_v = 0,$$
$$R^5_{uu} - R^5_{vv} - R^5_v - R^9_t - 2uR^9_x + R^8_x = 2\tau_{xu} = 0,$$
$$R^5_u - R^7_x = \tau_{tu} + 2\tau_x + \tau_{xv} = 0,$$
$$R^5_v - R^8_x = \tau_{tv} + u\tau_{xv} = 0,$$
$$R^3_u + R^4_v + R^6_{uu} - R^6_{vv} - R^{10}_t - u^2 R^9_x = \tau_{tu} + 2\tau_x - \tau_{xv} = 0,$$
$$R^6_u + R^7_t = u(\tau_{tu} + 2\tau_x) - \tau_{tv} - \eta_u = 0,$$
$$R^6_v + R^8_t = -\eta_v - \tau_{tu} = 0,$$

and hence $R^1 = \eta_t = 0$, $R^2 = \theta_t = 0$, whereas the recombination of the last five differential consequences of the above ones gives $\tau_{tv} = 0$, $\tau_{xv} = 0$, $\tau_{tu} + 2\tau_x = 0$, $\eta_u = 0$ and $\eta_v = 2\tau_x$. Differentiating the last equation with respect to t and x, we get $\tau_{xx} = \tau_{tx} = 0$, i.e., $\tau_x = \text{const}$. Finally, we obtain the consequences

$$R^3 = -\theta_v = 0, \quad R^4 = \theta_u - 2\tau_x = 0,$$
$$R^6_t = \xi_{tt} = 0, \quad R^6_x = \xi_{tx} = 0, \quad R^5_t = \tau_{tt} = 0.$$

Hence, the general solution of the system (17) is

$$\tau = A_4(x - 2ut) + A_1 t + \tau^0(u, v),$$
$$\xi = A_4 t(2v - u^2 + 1) + A_2 t + A_1 x + \xi^0(u, v),$$
$$\eta = 2A_4 v + A_2, \quad \theta = 2A_4 u + A_3,$$

where $A_1, \ldots, A_4$ are arbitrary constants and (τ^0, ξ^0) runs through the solution set of the system $u\tau^0_u - \tau^0_v = \xi^0_u$, $u\tau^0_v - \tau^0_u = \xi^0_v$. This proves the following assertion.

Theorem 15. *The maximal Lie invariance algebra $\mathfrak{g}_0$ of the system $\mathcal{S}_0$ is spanned by the vector fields*

$$\check{\mathcal{D}} = t\partial_t + x\partial_x, \quad \check{\mathcal{G}} = t\partial_x + \partial_u,$$

$$\check{\mathcal{P}}(\tau^0, \xi^0) = \tau^0(u,v)\partial_t + \xi^0(u,v)\partial_x, \quad \check{\mathcal{P}}^v = \partial_v, \qquad (18)$$

$$\check{\mathcal{J}} = \left(\tfrac{1}{2}x - tu\right)\partial_t + t\left(v - \tfrac{1}{2}u^2 + \tfrac{1}{2}\right)\partial_x + v\partial_u + u\partial_v,$$

where (τ^0, ξ^0) runs through the solution set of the system

$$u\tau_u^0 - \tau_v^0 = \xi_u^0, \quad u\tau_v^0 - \tau_u^0 = \xi_v^0.$$

    Note that in terms of Riemann invariants the system for (τ^0, ξ^0) reads $\xi_1^0 = V^2\tau_1^0$, $\xi_2^0 = V^1\tau_2^0$.

Remark 16. It is obvious that for any Ω the system $\mathcal{S}_0$ cannot admit a counterpart of the Lie symmetry vector field $\mathcal{W}(\Omega)$ of the system $\mathcal{S}$. The algebra $\mathfrak{g}/\langle\mathcal{W}(\Omega)\rangle$ is isomorphic to the proper subalgebra $\langle\check{\mathcal{D}}, \check{\mathcal{G}}, \check{\mathcal{P}}(1,0), \check{\mathcal{P}}(0,1), \check{\mathcal{P}}^v\rangle$ of $\mathfrak{g}_0$, where the spanning vector fields correspond to $\mathcal{D}, \mathcal{G}, \mathcal{P}^t, \mathcal{P}^x, \mathcal{P}^v$, respectively.

    By virtue of the structure of the algebra $\mathfrak{g}_0$, the system $\mathcal{S}_0$ can be linearized by a hodograph transformation, cf. [40]. In general, every $(1 + 1)$-dimensional system of hydrodynamic type with two dependent variables can be linearized by a two-dimensional hodograph transformation [11]. A hodograph transformation exchanges the roles of dependent and independent variables. For the system $\mathcal{S}_0$, the pairs of independent and dependent variables are to be interchanged, i.e., we set $y = u$, $z = v$, $p = t$, $q = x$, where y and z are the new independent variables and p and q are the new dependent variables. Then differentiating the equality $p(u, v) = t$, $q(u, v) = x$ with respect to t and x using the chain rule, we obtain the system of linear algebraic equations for the first derivatives of (u, v)

$$p_y u_t + p_z v_t = 1, \quad p_y u_x + p_z v_x = 0,$$
$$q_y u_t + q_z v_t = 0, \quad q_y u_x + q_z v_x = 1.$$

When assuming the nondegeneracy condition $\Delta := p_y q_z - p_z q_y \neq 0$, which is equivalent to $u_t v_x - u_x v_t \neq 0$, the solution of this system is

$$u_t = \frac{q_z}{\Delta}, \quad u_x = -\frac{p_z}{\Delta}, \quad v_t = -\frac{q_y}{\Delta}, \quad v_x = \frac{p_y}{\Delta}. \qquad (19)$$

In the new variables, the system $\mathcal{S}_0$ takes the form

$$q_z - yp_z + p_y = 0, \quad -q_y + yp_y - p_z = 0. \qquad (20)$$

As expected, the system (20) coincides with the system for (τ^0, ξ^0) from Theorem 15. The cross-differentiation with respect to y and z leads to a differential consequence of the system (20), which is the telegraph equation for p alone,

$$p_{zz} + p_z = p_{yy}. \qquad (21)$$

In other words, the system (20) is a potential system for Eq. (21). The substitution $p = e^{-z/2}\tilde{p}$ reduces (21) to the Klein–Gordon equation

$$\tilde{p}_{yy} = \tilde{p}_{zz} - \frac{\tilde{p}}{4}. \qquad (22)$$

    The maximal Lie invariance algebra $\mathfrak{g}_{KG}$ of the Klein–Gordon equation (22) is well known [41]. It is spanned by the vector fields

$$\hat{\mathcal{P}}^y = \partial_y, \quad \hat{\mathcal{P}}^z = \partial_z, \quad \hat{\mathcal{J}} = y\partial_z + z\partial_y, \quad \hat{\mathcal{D}} = \tilde{p}\partial_{\tilde{p}},$$

$$\hat{\mathcal{P}}(\psi) = \tilde{\psi}(y,z)\partial_{\tilde{p}},$$

where $\tilde{\psi}$ runs through the solution set of (22). The maximal Lie invariance algebra of the telegraph equation (21) is spanned by the pushforwards of these vector fields under the transformation $p = e^{-z/2}\tilde{p}$, where the variables y and z are not changed,

$$\partial_y, \quad \partial_z - \frac{1}{2}p\partial_p, \quad y\partial_z + z\partial_y - \frac{1}{2}yp\partial_p, \quad p\partial_p, \quad \psi(y,z)\partial_p,$$

where $\psi = e^{-z/2}\tilde{\psi}(y,z)$ runs through the solution set of (21). Although the maximal Lie invariance algebras $\mathfrak{g}_0$ and $\mathfrak{g}_{KG}$ of the system $\mathcal{S}_0$ and the Klein–Gordon equation (22) look similar and there seems to be an obvious relation among the generating elements, $\check{\mathcal{D}} \sim \hat{\mathcal{D}}$, $\check{\mathcal{G}} \sim \hat{\mathcal{P}}^y$, $\check{\mathcal{P}}(\tau^0, \xi^0) \sim \hat{\mathcal{P}}(e^{z/2}\tau^0)$, $\check{\mathcal{P}}^v \sim \hat{\mathcal{P}}^z + \frac{1}{2}\hat{\mathcal{D}}$, $\check{\mathcal{J}} \sim \hat{\mathcal{J}}$, these algebras are in fact not isomorphic. This can be explained using the following arguments. Since the systems $\mathcal{S}_0$ and (20) are related by a hodograph transformation, the maximal Lie invariance algebra $\tilde{\mathfrak{g}}_0$ of the system (20) is the pushforward of $\mathfrak{g}_0$ under this transformation. Hence the algebras $\mathfrak{g}_0$ and $\tilde{\mathfrak{g}}_0$ coincide up to re-denoting variables. The transition from the system (20) to Eq. (21) requires the projection $(y, z, p, q) \mapsto (y, z, p)$. At the same time, the vector field $\check{\mathcal{P}}(0, 1)$ is projected to the zero vector field. Moreover, the vector field $\check{\mathcal{J}}$ is not projectable, and thus the commutators $[\check{\mathcal{J}}, \check{\mathcal{P}}(\tau^0, \xi^0)]$ and $[\hat{\mathcal{J}}, \hat{\mathcal{P}}(e^{z/2}\tau^0)]$ are not related to each other in the above way. This is why there is no direct relation between the set of Lie solutions of the systems $\mathcal{S}_0$ and that of Eq. (22) (resp. of Eq. (21)).

8. Solution through linearization of the essential subsystem

    Since the system $\mathcal{S}$ is partially coupled and the subsystem $\mathcal{S}_0$ can be linearized, we can construct an implicit representation for the general solution of the system $\mathcal{S}$ in terms of the general solution of the telegraph equation (21) or, equivalently, of the Klein–Gordon equation (22). Consider the potential system $p_y = \Upsilon_z + \Upsilon$, $p_z = \Upsilon_y$ of Eq. (21). The "pseudopotential" $\Upsilon = \Upsilon(y, z)$ is in fact related to a usual potential of Eq. (21): the function $e^z\Upsilon$ is the potential associated with the conservation-law characteristic e^z of Eq. (21). Since the equation $p_z = \Upsilon_y$ is in conserved form, we can introduce the second-level potential $\Phi = \Phi(y, z)$ defined by the system $\Phi_y = p$, $\Phi_z = \Upsilon$. The equation $p_y = \Upsilon_z + \Upsilon$ implies that the potential Φ satisfies the same telegraph equation (21) as p, $\Phi_{zz} + \Phi_z = \Phi_{yy}$. Substituting the expression $p = \Phi_y$ into the system (20) and using the equation for Φ, we obtain a compatible system $(q - y\Phi_y + \Phi_z + \Phi)_y = 0$, $(q - y\Phi_y + \Phi_z + \Phi)_z = 0$ for q. The potential Φ is defined up to a constant summand. Hence we can assume that $q = y\Phi_y - \Phi_z - \Phi$. Then $\Delta := p_y q_z - p_z q_y = (\Phi_{yy})^2 - (\Phi_{yz})^2$, and the nondegeneracy condition $\Delta \neq 0$ reduces, on the solution set of the equation for Φ, to the condition $\Phi_{yy} \neq \Phi_{yz}$ or, equivalently, $\Phi_{yy} \neq -\Phi_{yz}$.[1] To find the component of the general solution for w, we write Eq. (4c) in the new variables, $(q_z - yp_z)w_y - (q_y - yp_y)w_z = 0$, or $p_y w_y - p_z w_z = 0$ when taking into account the system (20). We substitute the expression $p = \Phi_y$ into the last equation multiplied by e^z and take into account the equation for Φ. As a result, we derive the equation $(e^z\Phi_z)_z w_y - (e^z\Phi_z)_y w_z = 0$ meaning, in view of $e^z\Phi_z \neq$ const that w is a function of $e^z\Phi_z$ only. Re-writing the expressions obtained for p, q and w in terms of the initial variables, we construct the following implicit representation of all solutions of the system $\mathcal{S}$ with $u_t v_x - u_x v_t \neq 0$: $t = \Phi_u$, $x = u\Phi_u - \Phi_v - \Phi$, $w = W(e^v\Phi_v)$, where the function $\Phi = \Phi(u, v)$ runs through the solution set of the equation $\Phi_{vv} + \Phi_v = \Phi_{uu}$, and W is an arbitrary function of its argument $e^v\Phi_v$. This representation can also be interpreted as parametric.

    The condition $u_t v_x - u_x v_t = 0$ for singular solutions of the system $\mathcal{S}$ is easily seen to be equivalent to the constraint $v_x^2 = u_x^2$, i.e., $v_x = \varepsilon u_x$ with $\varepsilon = \pm 1$. Then the subsystem $\mathcal{S}_0$ also implies $v_t = \varepsilon u_t$. Therefore, $v = \varepsilon u + c$, where c is an arbitrary constant, and the subsystem $\mathcal{S}_0$ reduces to the single equation $u_t + uu_x + \varepsilon u_x = 0$. Suppose that $u \neq$ const, which is equivalent to the condition $u_x \neq 0$. We perform the one-dimensional hodograph

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[1] The overdetermined system $\Phi_{zz} + \Phi_z = \Phi_{yy}$, $\Phi_{yy} = \varepsilon\Phi_{yz}$ with $\varepsilon \in \{-1, 1\}$ implies that $\Phi_{yyy} = \Phi_{yzz} + \Phi_{yz} = \varepsilon\Phi_{yyz} + \varepsilon\Phi_{yy} = \Phi_{yyy} + \varepsilon\Phi_{yy}$ and thus $\Phi_{yy} = \Phi_{yz} = 0$ resulting also to $\Phi_{yy} = -\varepsilon\Phi_{yz}$.

transformation with $s = t$ and $y = u$ being the new independent variables and $q = x$ and w being the new dependent variables. Deriving the expressions for first derivatives of u and w in the hodograph variables,

$$u_x = \frac{1}{q_y}, \quad u_t = -\frac{q_s}{q_y}, \quad w_x = \frac{w_y}{q_y}, \quad w_t = w_s - \frac{q_s}{q_y}w_y,$$

we represent the equations of the reduced system, $u_t + uu_x + \varepsilon u_x = 0$ and $w_t + uw_x = 0$, in these variables, which read $q_s = y + \varepsilon$ and $q_y w_s = \varepsilon w_y$. The equation $q_s = y + \varepsilon$ integrates to $q = (y + \varepsilon)s + \varphi(y)$, where φ is an arbitrary function of y. The general solution of the equation $q_y w_s = \varepsilon w_y$ with respect to w is an arbitrary function of a first integral I of the ODE $ds/dy = -\varepsilon(s + \varphi_y)$. To derive a nice expression for this integral, instead of φ we introduce a new arbitrary function $\Theta = \Theta(y)$ such that $\varphi = e^{-\varepsilon y}\Theta_y$, which gives $I = e^{\varepsilon y}s + \varepsilon\Theta_y - \Theta$. Returning to the initial variables, we obtain $x - (u + \varepsilon)t = e^{-\varepsilon u}\Theta_u$, $v = \varepsilon u + c$ and $w = W(I)$, where c is an arbitrary constant, $\varepsilon = \pm 1$, $\Theta = \Theta(u)$ is an arbitrary function of u and W is an arbitrary function of $I = e^{\varepsilon u}t + \varepsilon\Theta_u - \Theta$.

If $u = $ const, then also $v = $ const, and the system $\mathcal{S}$ reduces to Eq. (4c) whose general solution in this case takes the form $w = W(x - ut)$ with W being an arbitrary function of its argument $x - ut$. We refer to this case as ultra-singular.

We collect all the three cases for solutions of the system $\mathcal{S}$ into a single assertion.

Theorem 17. *Any solution of the system $\mathcal{S}$ (locally) belongs to one of the following families; below W is an arbitrary function of its argument.*

1. The regular family, where $u_t v_x - u_x v_t \neq 0$ (the general solution):

$$t = \Phi_u, \quad x = u\Phi_u - \Phi_v - \Phi, \quad w = W(e^v \Phi_v). \quad (23)$$

Here the function $\Phi = \Phi(u, v)$ runs through the solutions of the telegraph equation $\Phi_{vv} + \Phi_v = \Phi_{uu}$ with $\Phi_{uu} \neq \Phi_{uv}$.

2. The singular family, where $u_t v_x - u_x v_t = 0$ but u and v are not constants:

$$x - (u + \varepsilon)t = e^{-\varepsilon u}\Theta_u, \quad v = \varepsilon u + c, \quad w = W(e^{\varepsilon u}t + \varepsilon\Theta_u - \Theta).$$

Here c is an arbitrary constant, $\varepsilon = \pm 1$, and $\Theta = \Theta(u)$ is an arbitrary function of u.

3. The ultra-singular family, where u and v are arbitrary constants and $w = W(x - ut)$.

In other words, the regular, singular and ultra-singular families of solutions of the system $\mathcal{S}$ are associated with solutions of the subsystem $\mathcal{S}_0$ with rank 2, 1 and 0, respectively; cf. [42].

9. Solution using generalized hodograph transformation

Another approach to finding the general solution of the system (3) is to employ the generalized hodograph method (see [12] and Section 1) for the diagonalized form (5) of (3). In this way, one employs the semi-Hamiltonian property related to the diagonalized form (5) instead of the partial coupling, which emerges in both the simplified forms (4) and (5). For the system (5), the tuple of the parameter functions $W = (W^1, W^2, W^3)$ in the ansatz (2) runs through the solution set of the overdetermined linear systems of first-order partial differential equations

$$W_2^1 = W_1^2 = \frac{1}{2}(W^1 - W^2), \quad W_3^1 = W_3^2 = 0, \quad (24a)$$

$$W_1^3 = W^1 - W^3, \quad W_2^3 = W^3 - W^2. \quad (24b)$$

(Recall that a subscript of a function in $\{1, 2, 3\}$ denotes the derivative with respect to the corresponding Riemann invariant.)

Using the equation $W_2^1 = W_1^2$, which is in conserved form, and the equations $W_3^1 = W_3^2 = 0$, we introduce the potential $\Lambda = \Lambda(\mathfrak{r}^1, \mathfrak{r}^2)$ defined by $\Lambda_1 = W^1$ and $\Lambda_2 = W^2$ and reduce the subsystem (24a) to the single equation

$$2\Lambda_{12} = \Lambda_1 - \Lambda_2. \quad (25)$$

Then we consider the subsystem (24b) as an overdetermined inhomogeneous linear system of first-order partial differential equations with respect to the function W^3. The general solution of this system is represented in the form

$$W^3 = F(\mathfrak{r}^3)e^{\mathfrak{r}^2 - \mathfrak{r}^1} + \Phi(\mathfrak{r}^1, \mathfrak{r}^2),$$

where F is an arbitrary smooth function of $\mathfrak{r}^3$ and $\Phi = \Phi(\mathfrak{r}^1, \mathfrak{r}^2)$ is a particular solution of the subsystem (24b), i.e.,

$$\Phi_1 + \Phi = \Lambda_1, \quad \Phi_2 - \Phi = -\Lambda_2. \quad (26)$$

Therefore, the function Φ satisfies the same equation as the function Λ, $2\Phi_{12} = \Phi_1 - \Phi_2$. It is convenient to assume for Φ to be the principal parameter function and to express Λ in terms of Φ. Then the determinant $\det(V_j^i t + W_j^i)$, where the indices i and j run from 1 to 3, is equal to

$$(\Phi_{11} - \Phi_{22})t + 2(\Phi_{11} + \Phi_{12})(2\Phi_{12} + \Phi_2) - 2(\Phi_{22} + \Phi_{12})(2\Phi_{11} + \Phi_1).$$

It vanishes on solutions of equation for Φ if and only if $\Phi_{11} = \Phi_{22}$ and $\Phi_{11} + \Phi_{12} = 0$, which also gives $\Phi_{22} + \Phi_{12} = 0$. Nevertheless, the equation $\Phi_{11} = \Phi_{22}$ is a differential consequence of the overdetermined system $2\Phi_{12} = \Phi_1 - \Phi_2$, $\Phi_{11} + \Phi_{12} = 0$. The same result holds true if we replace the second equation of this system by $\Phi_{22} + \Phi_{12} = 0$. This is why the nondegeneracy condition $\det(V_j^i t + W_j^i) \neq 0$ is equivalent to the single inequality $\Phi_{11} + \Phi_{12} \neq 0$ (resp. $\Phi_{22} + \Phi_{12} \neq 0$).

As a result, we construct the following implicit representation for the general solution of the system (5):

$$\begin{aligned} x - (\mathfrak{r}^1 + \mathfrak{r}^2 + 1)t &= \Phi + \Phi_1, \\ x - (\mathfrak{r}^1 + \mathfrak{r}^2 - 1)t &= \Phi - \Phi_2, \\ x - (\mathfrak{r}^1 + \mathfrak{r}^2)t &= \Phi + Fe^{\mathfrak{r}^2 - \mathfrak{r}^1}, \end{aligned} \quad (27)$$

where F is an arbitrary smooth function of $\mathfrak{r}^3$ with $F_{\mathfrak{r}^3} \neq 0$ and the function $\Phi = \Phi(\mathfrak{r}^1, \mathfrak{r}^2)$ runs through the set of solutions of the equation $2\Phi_{12} = \Phi_1 - \Phi_2$ with $\Phi_{11} \neq -\Phi_{12}$ or, equivalently, $\Phi_{22} \neq -\Phi_{12}$. The equation for Φ is reduced by the transformation $\tilde{\Phi} = e^{(\mathfrak{r}^1 - \mathfrak{r}^2)/2}\Phi$ to the Klein–Gordon equation $\tilde{\Phi}_{12} = -\tilde{\Phi}/4$.

The representation (27) can be derived from the particular case of the representation (23), where the parameter function W assumed to be nonconstant. It is just necessary to replace dependent variables in (23) by their expressions in terms of the Riemann invariants and re-denote Ψ and the inverse of W by $-\Phi$ and $-F$, respectively.

Families of solutions from Theorem 17 are nicely characterized in terms of conditions for Riemann invariants: none of (resp. exactly one of, resp. both) $\mathfrak{r}^1$ and $\mathfrak{r}^2$ are constants for the regular (resp. singular, resp. ultra-singular) family. To complete the set of solutions of the form (27) to the entire solution set of the system (5), we should add solutions, where $(\mathfrak{r}^1, \mathfrak{r}^2)$ are defined by the first two equations of (27) and $\mathfrak{r}^3$ is a constant, and solutions obtained from the singular and ultra-singular solutions of Theorem 17 by the transition to Riemann invariants. The corresponding representations in terms of Riemann invariants are obvious.

10. First-order generalized symmetries and beyond

To study first-order generalized symmetries[2] of the system (3), it is more convenient to take advantage of its diagonalized form (5) in terms of the Riemann invariants $\mathfrak{r} = (\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^3)$. From now on the repeated indices i and j mean summation over $i, j \in \{1, 2, 3\}$, while the index k is fixed and takes the values 1,2 or 3. Up to equivalence of generalized symmetries, it suffices to consider only evolutionary generalized vector fields [24,43] with characteristics of the form $\eta = (\eta^1(t, x, \mathfrak{r}, \mathfrak{r}_x), \eta^2(t, x, \mathfrak{r}, \mathfrak{r}_x), \eta^3(t, x, \mathfrak{r}, \mathfrak{r}_x))$, where $\mathfrak{r}_x$ denotes the first-order x-derivative of $\mathfrak{r}$.

The first-order evolutionary vector field Q is a generalized symmetry of the system (5) if it satisfies the condition

$$Q^{(1)}(\mathfrak{r}_t^k + V^k \mathfrak{r}_x^k)|_{(5)} = 0,$$

cf. (7) and [24, Definition 5.2], which implies the system $R^k = 0$ for the characteristic η, where

$$R^k := \eta_t^k - \eta_{\mathfrak{r}^j}^k V^j \mathfrak{r}_x^j - \eta_{\mathfrak{r}_x^j}^k \left((\mathfrak{r}_x^1 + \mathfrak{r}_x^2)\mathfrak{r}_x^j + V^j \mathfrak{r}_{xx}^j\right)$$
$$+ (\eta^1 + \eta^2)\mathfrak{r}_x^k + V^k(\eta_x^k + \eta_{\mathfrak{r}^j}^k \mathfrak{r}_x^j + \eta_{\mathfrak{r}_x^j}^k \mathfrak{r}_{xx}^j).$$

Splitting the equation $R^k = 0$ with respect to $\mathfrak{r}_{xx}^j$ readily gives $\eta_{\mathfrak{r}_x^j}^k = 0$ for $j \neq k$, and thus

$$R^k := \eta_t^k + V^k \eta_x^k + (V^k - V^j)\eta_{\mathfrak{r}^j}^k \mathfrak{r}_x^j - \eta_{\mathfrak{r}_x^k}^k (\mathfrak{r}_x^1 + \mathfrak{r}_x^2)\mathfrak{r}_x^k + (\eta^1 + \eta^2)\mathfrak{r}_x^k.$$

Then the equations $R_{\mathfrak{r}_x^1 \mathfrak{r}_x^1}^3 = 0$ and $R_{\mathfrak{r}_x^2 \mathfrak{r}_x^2}^3 = 0$ are equivalent to the equations $\eta_{\mathfrak{r}_x^1 \mathfrak{r}_x^1}^1 = 0$ and $\eta_{\mathfrak{r}_x^2 \mathfrak{r}_x^2}^2 = 0$, respectively, and hence the characteristic components η^1 and η^2 can be represented in the form

$$\eta^k = \theta^k(t, x, \mathfrak{r})\mathfrak{r}_x^k + \zeta^k(t, x, \mathfrak{r}), \quad k = 1, 2.$$

Using this representation, we further split the entire system $R^k = 0$ with respect to $(\mathfrak{r}_x^1, \mathfrak{r}_x^2)$ and, additionally, the equations $R^1 = 0$ and $R^2 = 0$ with respect to $\mathfrak{r}_x^3$. As a result, we obtain the system

$$2\theta_{\mathfrak{r}^2}^1 = 2\theta_{\mathfrak{r}^1}^2 = \theta^1 - \theta^2, \quad \zeta_{\mathfrak{r}^2}^1 = \zeta_{\mathfrak{r}^1}^2 = 0,$$
$$\theta_{\mathfrak{r}^3}^k = \zeta_{\mathfrak{r}^3}^k = 0, \quad H^k := \theta_t^k + V^k \theta_x^k + \zeta^1 + \zeta^2 = 0,$$
$$\zeta_t^k + V^k \zeta_x^k = 0, \quad k = 1, 2, \qquad (28)$$
$$\mathfrak{r}_x^3 \eta_{\mathfrak{r}^3}^3 + \eta_{\mathfrak{r}^1}^3 = \theta^1 \mathfrak{r}_x^3, \quad \mathfrak{r}_x^3 \eta_{\mathfrak{r}^3}^3 - \eta_{\mathfrak{r}^2}^3 = \theta^2 \mathfrak{r}_x^3,$$
$$\eta_t^3 + V^3 \eta_x^3 + (\zeta^1 + \zeta^2)\mathfrak{r}_x^3 = 0.$$

In view of $\zeta_{\mathfrak{r}^2}^1 = \zeta_{\mathfrak{r}^1}^2 = 0$, the equation $\zeta_t^1 + V^k \zeta_x^1 = 0$ (resp. $\zeta_t^2 + V^k \zeta_x^2 = 0$) splits with respect to $\mathfrak{r}^2$ (resp. $\mathfrak{r}^1$) into the equations $\zeta_t^1 = 0$, $\zeta_x^1 = 0$ (resp. $\zeta_t^2 = 0$, $\zeta_x^2 = 0$). We denote $K^1 := \mathfrak{r}_x^3 \partial_{\mathfrak{r}^3} + \partial_{\mathfrak{r}^1}$, $K^2 := \mathfrak{r}_x^3 \partial_{\mathfrak{r}^3} - \partial_{\mathfrak{r}^2}$, $B := \partial_t + V^3 \partial_x$ and derive the following differential consequences of the system (28):

$$\partial_{\mathfrak{r}^2}(\theta_t^1 + V^1 \theta_x^1 + \zeta^1 + \zeta^2) = \theta_x^1 - \theta_x^2 + \zeta_{\mathfrak{r}^2}^2 = 0,$$
$$\partial_{\mathfrak{r}^1}(\theta_t^2 + V^2 \theta_x^2 + \zeta^1 + \zeta^2) = \theta_x^2 - \theta_x^1 + \zeta_{\mathfrak{r}^1}^1 = 0,$$
$$(K^1 B - B K^1)\eta^3 = \eta_x^3 = (\theta_x^1 - \zeta_{\mathfrak{r}^1}^1)\mathfrak{r}_x^3,$$
$$(B K^2 - K^2 B)\eta^3 = \eta_x^3 = (\theta_x^2 - \zeta_{\mathfrak{r}^2}^2)\mathfrak{r}_x^3.$$

Therefore, $\theta_x^1 - \theta_x^2 = \zeta_{\mathfrak{r}^1}^1 = -\zeta_{\mathfrak{r}^2}^2 = \zeta_{\mathfrak{r}^1}^1 - \zeta_{\mathfrak{r}^2}^2$, i.e., $\zeta_{\mathfrak{r}^1}^1 = \zeta_{\mathfrak{r}^2}^2 = 0$, $\theta_x^1 = \theta_x^2$, and thus $\eta_x^3 = \theta_x^1 \mathfrak{r}_x^3$. We can conclude that ζ^1 and ζ^2 are constants. Differentiating the equation $2\theta_{\mathfrak{r}^2}^1 = 2\theta_{\mathfrak{r}^1}^2 = \theta^1 - \theta^2$ with respect to x gives $2\theta_{x\mathfrak{r}^2}^1 = 2\theta_{x\mathfrak{r}^1}^2 = \theta_x^1 - \theta_x^2 = 0$, and hence $\theta_{x\mathfrak{r}^1}^1 = \theta_{x\mathfrak{r}^1}^2 = 0$ and $\theta_{x\mathfrak{r}^2}^1 = \theta_{x\mathfrak{r}^2}^2 = 0$. The differential consequence

$\partial_x(H^1 - H^2) = 0$ of (28) is equivalent to $\theta_{xx}^1 = \theta_{xx}^2 = 0$. Then the equations $\partial_x H^1 = 0$ and $\partial_x H^2 = 0$ reduce to $\theta_{tx}^1 = \theta_{tx}^2 = 0$. As a result, we have

$$\theta_x^k = \gamma = \text{const}, \quad \theta_t^k = -\gamma V^k - (\zeta^1 + \zeta^2),$$
$$2\theta_{\mathfrak{r}^2}^1 = 2\theta_{\mathfrak{r}^1}^2 = \theta^1 - \theta^2,$$
$$\eta_x^3 = \gamma \mathfrak{r}_x^3, \quad \eta_t^3 = -\gamma V^3 \mathfrak{r}_x^3 - (\zeta^1 + \zeta^2)\mathfrak{r}_x^3,$$
$$\mathfrak{r}_x^3 \eta_{\mathfrak{r}^3}^3 + \eta_{\mathfrak{r}^1}^3 = \theta^1 \mathfrak{r}_x^3, \quad \mathfrak{r}_x^3 \eta_{\mathfrak{r}^3}^3 - \eta_{\mathfrak{r}^2}^3 = \theta^2 \mathfrak{r}_x^3.$$

Integrating this system, we derive explicit expressions for θ^1, θ^2 and η^3 and, therefore, for the entire characteristic η of Q,

$$\eta^1 = \left(\gamma x - \gamma t V^1 - (\zeta^1 + \zeta^2)t + \Phi + \Phi_{\mathfrak{r}^1}\right)\mathfrak{r}_x^1 + \zeta^1,$$
$$\eta^2 = \left(\gamma x - \gamma t V^2 - (\zeta^1 + \zeta^2)t + \Phi - \Phi_{\mathfrak{r}^2}\right)\mathfrak{r}_x^2 + \zeta^2,$$
$$\eta^3 = \left(\gamma x - \gamma t V^3 - (\zeta^1 + \zeta^2)t + \Phi\right)\mathfrak{r}_x^3 + \Omega,$$

where γ, ζ^1 and ζ^2 are arbitrary constants, the function $\Omega = \Omega\left(\mathfrak{r}^3, e^{\mathfrak{r}^2 - \mathfrak{r}^1}\mathfrak{r}_x^3\right)$ runs through the set of smooth functions of $\left(\mathfrak{r}^3, e^{\mathfrak{r}^2 - \mathfrak{r}^1}\mathfrak{r}_x^3\right)$, the function $\Phi = \Phi(\mathfrak{r}^1, \mathfrak{r}^2)$ runs through the solution set of the equation $2\Phi_{\mathfrak{r}^1 \mathfrak{r}^2} = \Phi_{\mathfrak{r}^1} - \Phi_{\mathfrak{r}^2}$. This proves the following theorem.

Theorem 18. *The algebra Σ^1 of first-order reduced generalized symmetries of the system (5) is spanned by evolutionary vector fields of the form*

$$\check{\mathcal{D}} = \left(x - (\mathfrak{r}^1 + \mathfrak{r}^2 + 1)t\right)\mathfrak{r}_x^1 \partial_{\mathfrak{r}^1} + \left(x - (\mathfrak{r}^1 + \mathfrak{r}^2 - 1)t\right)\mathfrak{r}_x^2 \partial_{\mathfrak{r}^2}$$
$$+ \left(x - (\mathfrak{r}^1 + \mathfrak{r}^2)t\right)\mathfrak{r}_x^3 \partial_{\mathfrak{r}^3},$$
$$\check{\mathcal{G}}_1 = (t\mathfrak{r}_x^1 - 1)\partial_{\mathfrak{r}^1} + t\mathfrak{r}_x^2 \partial_{\mathfrak{r}^2} + t\mathfrak{r}_x^3 \partial_{\mathfrak{r}^3}, \quad \check{\mathcal{G}}_2 = \partial_{\mathfrak{r}^1} - \partial_{\mathfrak{r}^2},$$
$$\check{\mathcal{P}}(\Phi) = (\Phi + \Phi_{\mathfrak{r}^1})\mathfrak{r}_x^1 \partial_{\mathfrak{r}^1} + (\Phi - \Phi_{\mathfrak{r}^2})\mathfrak{r}_x^2 \partial_{\mathfrak{r}^2} + \Phi \mathfrak{r}_x^3 \partial_{\mathfrak{r}^3},$$
$$\check{\mathcal{W}}(\Omega) = \Omega \partial_{\mathfrak{r}^3},$$

where the parameter function $\Omega = \Omega\left(\omega^0, \omega^1\right)$ runs through the set of smooth functions of $\omega^0 := \mathfrak{r}^3$ and $\omega^1 := e^{\mathfrak{r}^2 - \mathfrak{r}^1}\mathfrak{r}_x^3$, and the parameter function $\Phi = \Phi(\mathfrak{r}^1, \mathfrak{r}^2)$ runs through the set of solutions of the equation $2\Phi_{\mathfrak{r}^1 \mathfrak{r}^2} = \Phi_{\mathfrak{r}^1} - \Phi_{\mathfrak{r}^2}$, which is reduced by the substitution $\tilde{\Phi} = e^{(\mathfrak{r}^1 - \mathfrak{r}^2)/2}\Phi$ to the Klein–Gordon equation $\tilde{\Phi}_{\mathfrak{r}^1 \mathfrak{r}^2} = -\tilde{\Phi}/4$.

Remark 19. The space of first-order reduced generalized symmetries of the system (5) is closed with respect to the Lie bracket of generalized vector fields, and hence we can call it an algebra. This property is shared by all strictly hyperbolic diagonalizable hydrodynamic-type systems. Up to skew-symmetry of Lie bracket, nonzero commutation relations among the above vector fields are exhausted by the following ones:

$$[\check{\mathcal{D}}, \check{\mathcal{P}}(\Phi)] = \check{\mathcal{P}}(\Phi), \quad [\check{\mathcal{G}}_1, \check{\mathcal{P}}(\Phi)] = \check{\mathcal{P}}(-\Phi_{\mathfrak{r}^1}),$$
$$[\check{\mathcal{G}}_2, \check{\mathcal{P}}(\Phi)] = \check{\mathcal{P}}(\Phi_{\mathfrak{r}^1} - \Phi_{\mathfrak{r}^2}),$$
$$[\check{\mathcal{D}}, \check{\mathcal{W}}(\Omega)] = \check{\mathcal{W}}(\omega^1 \Omega_{\omega^1}), \quad [\check{\mathcal{G}}_1, \check{\mathcal{W}}(\Omega)] = \check{\mathcal{W}}(\omega^1 \Omega_{\omega^1}),$$
$$[\check{\mathcal{G}}_2, \check{\mathcal{W}}(\Omega)] = -2\check{\mathcal{W}}(\omega^1 \Omega_{\omega^1}),$$
$$[\check{\mathcal{W}}(\Omega^1), \check{\mathcal{W}}(\Omega^2)]$$
$$= \check{\mathcal{W}}\left(\Omega_{\omega^0}^1(\omega^1 \Omega_{\omega^1}^2 - \Omega^2) - \Omega_{\omega^0}^2(\omega^1 \Omega_{\omega^1}^1 - \Omega^1)\right).$$

Therefore, the subspaces $\mathcal{I}^1$ and $\mathcal{I}^2$ that consist of all generalized vector fields of the forms $\check{\mathcal{P}}(\Phi)$ and $\check{\mathcal{W}}(\Omega)$ from the algebra Σ^1, respectively, are (infinite-dimensional) ideals of Σ^1. Moreover, the ideal $\mathcal{I}^1$ is commutative. Since

$$\check{\mathcal{P}}(e^{\mathfrak{r}^2 - \mathfrak{r}^1}) = \check{\mathcal{W}}(\omega^1) = e^{\mathfrak{r}^2 - \mathfrak{r}^1}\mathfrak{r}_x^3 \partial_{\mathfrak{r}^3},$$

the above ideals are not disjoint, $\mathcal{I}^1 \cap \mathcal{I}^2 = \langle e^{\mathfrak{r}^2 - \mathfrak{r}^1}\mathfrak{r}_x^3 \partial_{\mathfrak{r}^3}\rangle$.

Remark 20. Amongst the generalized vector fields presented in Theorem 18, there are evolutionary forms of Lie symmetries of the system (5) (cf. Remark 3) as well as genuinely generalized symmetries. The generalized vector fields $\check{\mathcal{D}}$, $\check{\mathcal{G}}_1$, $\check{\mathcal{G}}_2$, $\check{\mathcal{P}}(\mathfrak{r}^1 + \mathfrak{r}^2)$, $\check{\mathcal{P}}(1)$ and $\check{\mathcal{W}}(\Omega)$ with Ω depending on $\mathfrak{r}^3$ alone are evolutionary forms of the vector fields $-\hat{\mathcal{D}}$, $-\frac{1}{2}\hat{\mathcal{G}} - \frac{1}{2}\hat{\mathcal{P}}^v$, $\hat{\mathcal{P}}^v$, $\hat{\mathcal{P}}^t$, $-\hat{\mathcal{P}}^x$ and $\hat{\mathcal{W}}(\Omega)$, respectively. The vector fields $\check{\mathcal{W}}(\Omega)$ with Ω not being a function of $\mathfrak{r}^3$ only are genuine generalized symmetries. The existence of generalized symmetries $\check{\mathcal{P}}(\Phi)$ with $\Phi \notin \mathrm{span}(1, \mathfrak{r}^1 + \mathfrak{r}^2)$ can be explained by observing that the pushforward of the vector field $\check{\mathcal{P}}(\Phi)$ by the projection to $(t, x, \mathfrak{r}^1, \mathfrak{r}^2)$ is the evolutionary form of the Lie symmetry vector field $\check{\mathcal{P}}(\tau^0, \xi^0)$ of the essential subsystem $\mathcal{S}_0$ (re-written in terms of $(\mathfrak{r}^1, \mathfrak{r}^2)$) with $\tau^0 = \frac{1}{2}(\Phi_{\mathfrak{r}^1} + \Phi_{\mathfrak{r}^2})$ and $\xi^0 = \frac{1}{2}(\Phi_{\mathfrak{r}^1} + \Phi_{\mathfrak{r}^2})(\mathfrak{r}^1 + \mathfrak{r}^2 - 1) - \Phi + \Phi_{\mathfrak{r}^2}$. Therefore, Lie symmetries of the essential subsystem that are lost upon transition to the entire system are recovered as generalized first-order symmetries of the entire system. In contrast to the family $\{\check{\mathcal{P}}(\tau^0, \xi^0)\}$, the other distinguished Lie symmetry of the essential subsystem $\mathcal{S}_0$, $\check{\mathcal{J}}$, is related to $\zeta_{\mathfrak{r}^1}^1$, which vanishes in the proof of Theorem 18, and therefore this symmetry has no counterpart amongst first-order symmetries of the system $\mathcal{S}$.

First-order symmetries from the subspace $\{\check{\mathcal{W}}(\Omega)\}$ can be directly generalized to an arbitrary order.

Proposition 21. *The system (5) admits generalized symmetries of arbitrarily high order of the form* $\check{\mathcal{W}}(\Omega) = \Omega \partial_{\mathfrak{r}^3}$, *where* Ω *runs through the set of smooth functions of a finite, but unspecified number of* $\omega^\iota = (e^{\mathfrak{r}^2 - \mathfrak{r}^1} D_x)^\iota \mathfrak{r}^3$, $\iota \in \mathbb{N}_0$, *i.e.,* $\Omega = \Omega(\omega^0, \dots, \omega^\kappa)$ *with* $\kappa \in \mathbb{N}_0$.

Proof. The differential operator $A = e^{\mathfrak{r}^2 - \mathfrak{r}^1} D_x$ commutes with the operator $B = D_t + (\mathfrak{r}^1 + \mathfrak{r}^2) D_x$ on solutions of the system (5) since

$$[B, A] = (B\mathfrak{r}^2 - \mathfrak{r}_x^2 - B\mathfrak{r}^1 - \mathfrak{r}_x^1)A.$$

Therefore, the operator A maps solutions of Eq. (5c) to solutions of the same equation. In particular, the equations $B\omega^\kappa = 0$ are differential consequences of the system (5). This observation as well as the fact that Eq. (5c) is both linear and of first order with respect to $\mathfrak{r}^3$ hints that $\check{\mathcal{W}}(\Omega)$ is a generalized symmetry of the system (5). Indeed, it satisfies the invariance criterion. □

In fact, the system (5) admits another family of generalized symmetries of arbitrarily high order, which essentially differs from the family $\{\check{\mathcal{W}}(\Omega)\}$ and is obtained by the prolongation of generalized symmetries of the essential subsystem to $\mathfrak{r}^3$. In the next paper on the system (5), we shall exhaustively describe all generalized symmetries of this system.

11. Hydrodynamic conservation laws and their generalizations

Here we find the complete space of zeroth-order conservation laws of the system (5), which is in fact exhausted by linear combinations of a single non-translation-invariant conservation law with hydrodynamic conservation laws. Recall that a conservation law is called *hydrodynamic* if its density ρ is a function of dependent variables only. Following [44], we also present the complete set of first-order conservation laws whose densities do not involve independent variables and a family of conservation laws of arbitrarily high order.

Theorem 22. *The space of zeroth-order conservation laws of the system (5) is spanned by the single non-translation-invariant conservation law with the conserved current*

$$e^{\mathfrak{r}^1 - \mathfrak{r}^2}(x - V^3 t,\ V^3(x - V^3 t) - t)$$

and the space of hydrodynamic conservation laws of this system, which is naturally isomorphic to the space of conserved currents of the general form

$$\left(e^{\mathfrak{r}^1 - \mathfrak{r}^2}\Omega + \Psi_{\mathfrak{r}^1} - \Psi_{\mathfrak{r}^2},\ (\mathfrak{r}^1 + \mathfrak{r}^2)e^{\mathfrak{r}^1 - \mathfrak{r}^2}\Omega + V^1\Psi_{\mathfrak{r}^1} - V^2\Psi_{\mathfrak{r}^2}\right).$$

Here Ω *is an arbitrary smooth function of* $\mathfrak{r}^3$ *and the parameter function* $\Psi = \Psi(\mathfrak{r}^1, \mathfrak{r}^2)$ *runs through the solution set of the equation* $2\Psi_{\mathfrak{r}^1 \mathfrak{r}^2} = \Psi_{\mathfrak{r}^2} - \Psi_{\mathfrak{r}^1}$, *which is reduced to the Klein–Gordon equation* $\tilde{\Psi}_{\mathfrak{r}^1 \mathfrak{r}^2} = -\tilde{\Psi}/4$ *by the transformation* $\tilde{\Psi} = e^{(\mathfrak{r}^2 - \mathfrak{r}^1)/2}\Psi$.

Proof. If $\rho = \rho(t, x, \mathfrak{r})$ is a density of a conservation law of the system (5), then $\bar{\mathrm{E}}\bar{\mathrm{D}}_t \rho = 0$, where $\bar{\mathrm{D}}_t = \partial_t - \sum_{i=1}^3 V^i \mathfrak{r}_x^i \partial_{\mathfrak{r}^i}$ is the restricted reduced operator of total derivative with respect to t in view of the system (5), $\bar{\mathrm{E}} = \left(\partial_{\mathfrak{r}^i} - D_x \partial_{\mathfrak{r}_x^i}, j = 1, 2, 3\right)^\mathsf{T}$ is the restricted Euler operator. The above condition is equivalent to the system

$$\sum_{i=1}^3 \left((V^j - V^i)\rho_{\mathfrak{r}^i \mathfrak{r}^j} + V_{\mathfrak{r}^i}^j \rho_{\mathfrak{r}^j} - V_{\mathfrak{r}^j}^i \rho_{\mathfrak{r}^i}\right)\mathfrak{r}_x^i + \rho_{\mathfrak{r}^j t} + V^j \rho_{\mathfrak{r}^j x} = 0, \quad j = 1, 2, 3.$$

Splitting this system with respect to $\mathfrak{r}_x^i$ gives three equations corresponding to summands without derivatives of $\mathfrak{r}$ and a system associated with coefficients of $\mathfrak{r}_x^i$, $i = 1, 2, 3$, whose left-hand side is antisymmetric with respect to the permutation of i and j. Hence the latter system contains only three independent differential equations on ρ. As a result, we derive a system of six determining equations for ρ,

$$\begin{gathered} \rho_{\mathfrak{r}^1 \mathfrak{r}^3} = \rho_{\mathfrak{r}^3}, \quad \rho_{\mathfrak{r}^2 \mathfrak{r}^3} = -\rho_{\mathfrak{r}^3}, \quad 2\rho_{\mathfrak{r}^1 \mathfrak{r}^2} = \rho_{\mathfrak{r}^2} - \rho_{\mathfrak{r}^1}, \\ \rho_{t \mathfrak{r}^j} + V^j \rho_{x \mathfrak{r}^j} = 0, \quad j = 1, 2, 3. \end{gathered} \tag{29}$$

Denoting $R^0 := \rho_{\mathfrak{r}^1 \mathfrak{r}^3} - \rho_{\mathfrak{r}^3}$ and $R^j := \rho_{t \mathfrak{r}^j} + V^j \rho_{x \mathfrak{r}^j}, j = 1, 2, 3$, we derive the differential consequence

$$\partial_{\mathfrak{r}^3}R^1 - (\partial_t + V^1 \partial_x)R^0 - R^3 = \rho_{\mathfrak{r}^3 x} = 0,$$

and thus also $\rho_{\mathfrak{r}^3 t} = 0$. The first two equations of the system (29) can be rewritten as $(e^{\mathfrak{r}^2 - \mathfrak{r}^1}\rho_{\mathfrak{r}^3})_{\mathfrak{r}^1} = (e^{\mathfrak{r}^2 - \mathfrak{r}^1}\rho_{\mathfrak{r}^3})_{\mathfrak{r}^2} = 0$. Therefore, the general solution of (29) admits the representation

$$\rho = e^{\mathfrak{r}^1 - \mathfrak{r}^2}\Omega + \tilde{\rho}(t, x, \mathfrak{r}^1, \mathfrak{r}^2),$$

where $\Omega = \Omega(\mathfrak{r}^3)$ is an arbitrary smooth function of $\mathfrak{r}^3$, and $\tilde{\rho} = \tilde{\rho}(t, x, \mathfrak{r}^1, \mathfrak{r}^2)$ is the general solution of the reduced system

$$2\tilde{\rho}_{\mathfrak{r}^1 \mathfrak{r}^2} = \tilde{\rho}_{\mathfrak{r}^2} - \tilde{\rho}_{\mathfrak{r}^1}, \quad \tilde{\rho}_{t \mathfrak{r}^j} + V^j \tilde{\rho}_{x \mathfrak{r}^j} = 0, \quad j = 1, 2. \tag{30}$$

The last two equations of (30) integrate to $\tilde{\rho}_{\mathfrak{r}^j} = f^j(y_j, \mathfrak{r}^1, \mathfrak{r}^2)$, $j = 1, 2$, where f^1 and f^2 are smooth functions of their arguments, and $y_j = x - V^j t$. The consistency of these representations for the derivatives $\tilde{\rho}_{\mathfrak{r}^1}$ and $\tilde{\rho}_{\mathfrak{r}^2}$ with each other and with the first equations of (30) leads to equations for f^1 and f^2,

$$(y_1 - y_2)f_{y_1}^1 + 2f_{\mathfrak{r}^2}^1 = (y_1 - y_2)f_{y_2}^2 + 2f_{\mathfrak{r}^1}^2 = f^2 - f^1. \tag{31}$$

Successively differentiating these equations with respect to y_1 and y_2, we get $f_{y_1 y_1}^1 = f_{y_2 y_2}^2 = 0$, i.e., $f^j = g^j(\mathfrak{r}^1, \mathfrak{r}^2)y_j + h^j(\mathfrak{r}^1, \mathfrak{r}^2)$ for some smooth functions g^j and h^j of $(\mathfrak{r}^1, \mathfrak{r}^2), j = 1, 2$. Then the splitting of Eqs. (31) with respect to y_1 and y_2 gives a system

$$g_{\mathfrak{r}^2}^1 = -g^1, \quad g_{\mathfrak{r}^1}^2 = g^2, \quad g^1 = -g^2, \quad 2h_{\mathfrak{r}^2}^1 = 2h_{\mathfrak{r}^1}^2 = h^2 - h^1$$

for the coefficients g^j and h^j. The general solution of this system can be represented in the form

$$g^1 = -g^2 = Ce^{\mathfrak{r}^1 - \mathfrak{r}^2}, \quad h^1 = \Psi_{\mathfrak{r}^1 \mathfrak{r}^2} - \Psi_{\mathfrak{r}^1 \mathfrak{r}^1}, \quad h^2 = \Psi_{\mathfrak{r}^2 \mathfrak{r}^2} - \Psi_{\mathfrak{r}^1 \mathfrak{r}^2},$$

where C is an arbitrary constant and $\Psi = \Psi(\mathfrak{r}^1, \mathfrak{r}^2)$ is an arbitrary solution of the equation $2\Psi_{\mathfrak{r}^1 \mathfrak{r}^2} = \Psi_{\mathfrak{r}^2} - \Psi_{\mathfrak{r}^1}$. Simultaneously integrating the equations $\tilde{\rho}_{\mathfrak{r}^j} = f^j$ in view of the derived expressions for $f_j, j = 1, 2$, we finally obtain

$$\tilde{\rho} = Ce^{\mathfrak{r}^1 - \mathfrak{r}^2}(x - V^3 t) + \Psi_{\mathfrak{r}^1} - \Psi_{\mathfrak{r}^2} + \rho^0$$

with ρ^0 being an arbitrary smooth function of (t, x), which is the density of a null divergence and should thus be neglected. To find the flux σ associated with $\rho = e^{\mathfrak{r}^1 - \mathfrak{r}^2}\Omega + \tilde\rho$, we notice that $D_x\sigma = -\bar D_t\rho = D_x\big((\mathfrak{r}^1 + \mathfrak{r}^2)e^{\mathfrak{r}^1 - \mathfrak{r}^2}\Omega + C(V^3(x - V^3 t) - t) + V^1\Psi_{\mathfrak{r}^1} - V^2\Psi_{\mathfrak{r}^2}\big)$. $\square$

The conservation-law characteristic associated with the above conserved current is

$$(e^{\mathfrak{r}^1 - \mathfrak{r}^2}\Omega + \Psi_{\mathfrak{r}^1\mathfrak{r}^1} - \Psi_{\mathfrak{r}^1\mathfrak{r}^2}, \; -e^{\mathfrak{r}^1 - \mathfrak{r}^2}\Omega + \Psi_{\mathfrak{r}^1\mathfrak{r}^2} - \Psi_{\mathfrak{r}^2\mathfrak{r}^2}, \; e^{\mathfrak{r}^1 - \mathfrak{r}^2}\Omega_{\mathfrak{r}^3}).$$

In addition to the standard equivalence of conservation laws one can consider their equivalence up to the action of a point symmetry group of the system of differential equations under study, see [45,46] for details. A set of conservation laws that generate a space of conservation laws via linear combining and acting with point symmetries is called a *generating set* for this space [45].

There are only two inequivalent values of the parameter function $\Omega(\mathfrak{r}^3)$ up to transformations in the point symmetry group G of the system (5) (see Theorem 9), $\Omega = 1$ and $\Omega = \mathfrak{r}^3$. Moreover, it is easily seen that the conserved current with $(\Omega, \Psi) = (1, 0)$ coincides with the conserved current with $(\Omega, \Psi) = (0, e^{\mathfrak{r}^1 - \mathfrak{r}^2})$.

Corollary 23. *A generating set for the space of hydrodynamic conservation laws of the system* (5) *up to the point symmetry group G of this system consists of the conservation laws with conserved currents*

$$\text{DHC} = \left(e^{\mathfrak{r}^1 - \mathfrak{r}^2}\mathfrak{r}^3, \; (\mathfrak{r}^1 + \mathfrak{r}^2)e^{\mathfrak{r}^1 - \mathfrak{r}^2}\mathfrak{r}^3\right),$$

$$\text{EHC}(\Psi) = \left(\Psi_{\mathfrak{r}^1} - \Psi_{\mathfrak{r}^2}, \; (\mathfrak{r}^1 + \mathfrak{r}^2 + 1)\Psi_{\mathfrak{r}^1} - (\mathfrak{r}^1 + \mathfrak{r}^2 - 1)\Psi_{\mathfrak{r}^2}\right),$$

where $\Psi = \Psi(\mathfrak{r}^1, \mathfrak{r}^2)$ runs through the solution set of the equation $2\Psi_{\mathfrak{r}^1\mathfrak{r}^2} = \Psi_{\mathfrak{r}^2} - \Psi_{\mathfrak{r}^1}$, which is reduced by the transformation $\tilde\Psi = e^{(\mathfrak{r}^2 - \mathfrak{r}^1)/2}\Psi$ to the Klein–Gordon equation $\tilde\Psi_{\mathfrak{r}^1\mathfrak{r}^2} = -\tilde\Psi/4$.

Remark 24. Functions Φ and Ψ parameterizing first-order generalized symmetries and hydrodynamic conservation laws of the system (5) satisfy adjoint differential equations. This can be explained by the fact that characteristics of conservations laws of the system (5) are cosymmetries of this system, cf. [47].

For the system (4), the space of hydrodynamic conservation laws is naturally isomorphic to the space of conserved currents of the general form $(e^v\Omega + \Psi_v, \; ue^v\Omega + u\Psi_v + \Psi_u)$, where $\Omega = \Omega(w)$ is an arbitrary smooth function of w and the parameter function $\Psi = \Psi(u, v)$ runs through the solution set of the telegraph equation $-\Psi_{uu} = \Psi_{vv} - \Psi_v$. The conservation-law characteristic associated with the above conserved current is $(\Psi_{uv}, e^v\Omega + \Psi_{vv}, e^v\Omega_w)$. A generating set of the space of hydrodynamic conservation laws of the system $\mathcal{S}$ consists of the conservation laws with conserved currents $(e^v w, ue^v w)$, $(\Psi_v, u\Psi_v + \Psi_u)$, where the parameter function $\Psi = \Psi(u, v)$ runs through the solution set of the telegraph equation $\Psi_{uu} = \Psi_{vv} - \Psi_v$.

Many semi-Hamiltonian systems, including the system (5), also possess first-order conservation laws [44,48] (see also [49, 50]). To construct such conservation laws for the system (5) following [44, Theorem 5.1], we should first find some smooth nonvanishing functions $G^i(\mathfrak{r})$, $i = 1, 2, 3$, that satisfy the equations

$$\frac{G^i_{v^j}}{G^i} = -\frac{V^i_{v^j}}{V^i - V^j}, \quad i \ne j.$$

We can take $G^1 = e^{-\mathfrak{r}^2/2}$, $G^2 = e^{\mathfrak{r}^1/2}$, $G^3 = e^{\mathfrak{r}^1 - \mathfrak{r}^2}$. Every first-order conserved current of the system (5) with density not involving (t, x) is equivalent to the sum of a hydrodynamic conserved current and a conserved current of the form

$$\left(\sum_{i=1}^{2}\frac{(G^i)^2 f^i}{\mathfrak{r}^i_x} + G^3\Omega, \; \sum_{i=1}^{2}V^i\frac{(G^i)^2 f^i}{\mathfrak{r}^i_x} + V^3 G^3\Omega - 2x\sum_{i=1}^{2}V^i_{\mathfrak{r}^i}\frac{(G^i)^2 f^i}{\mathfrak{r}^i_x}\right),$$

where $\Omega = \Omega(\mathfrak{r}^3, \mathfrak{r}^3_x/G^3)$ is an arbitrary smooth function of its arguments, and $f^1 = f^1(\mathfrak{r}^1)$ and $f^2 = f^2(\mathfrak{r}^2)$ are arbitrary smooth functions of $\mathfrak{r}^1$ and $\mathfrak{r}^2$, respectively, that satisfy the condition

$$\sum_{i=1}^{2}(G^i)^2 f^i = \text{const}.$$

(The above summations are in the range $\{1, 2\}$ since $V^3_{\mathfrak{r}^3} = 0$.) Therefore, $f^1(\mathfrak{r}^1) = ce^{\mathfrak{r}^1}$ and $f^2(\mathfrak{r}^2) = -ce^{-\mathfrak{r}^2}$, where c is an arbitrary constant.

Proposition 25. *The space of conserved currents of conservation laws of the system* (5) *of order not greater than one with (t, x)-independent densities is spanned by a family of conserved currents $C_1(\Omega)$, C_0 and the conserved currents $\text{EHC}(\Psi)$ presented in Corollary 23. Here*

$$C_0 = \left(\left(\frac{1}{\mathfrak{r}^1_x} - \frac{1}{\mathfrak{r}^2_x}\right)e^{\mathfrak{r}^1 - \mathfrak{r}^2}, \; \left(\frac{V^1}{\mathfrak{r}^1_x} - \frac{V^2}{\mathfrak{r}^2_x}\right)e^{\mathfrak{r}^1 - \mathfrak{r}^2}\right),$$

$$C_1(\Omega) = \left(e^{\mathfrak{r}^1 - \mathfrak{r}^2}\Omega, \; (\mathfrak{r}^1 + \mathfrak{r}^2)e^{\mathfrak{r}^1 - \mathfrak{r}^2}\Omega\right),$$

$$(32)$$

and Ω runs through the set of smooth functions of $\omega^0 = \mathfrak{r}^3$ and $\omega^1 = \mathfrak{r}^3_x e^{\mathfrak{r}^2 - \mathfrak{r}^1}$.

Since the system $\mathcal{S}$ is not genuinely nonlinear, Theorem 5.2 of [44] implies the following assertion.

Proposition 26. *The system $\mathcal{S}$ possesses a family of nontrivial conservation laws of arbitrarily high order with conserved currents $C_1(\Omega)$ parameterized by an arbitrary function Ω of a finite, but unspecified number of $\omega^\iota = (e^{\mathfrak{r}^2 - \mathfrak{r}^1}D_x)^\iota\mathfrak{r}^3$, $\iota \in \mathbb{N}_0$.*

Remark 27. In a fashion different from [44], existence of higher-order conservation laws for the system (5) can be explained by actions of generalized symmetries $\check{\mathcal{W}}(\Omega) = \Omega\partial_{\mathfrak{r}^3}$ on a hydrodynamic conserved current DHC, see Proposition 21 and Corollary 23. This claim agrees with Proposition 25.

In fact, the system (5) admits another family of conservation laws of arbitrarily high order, which essentially differs from the family of conserved currents $C_1(\Omega)$ and is obtained by pulling back conservation laws of the essential subsystem with the projection to $(t, x, \mathfrak{r}^1, \mathfrak{r}^2)$. In the next paper on the system (5), we shall exhaustively describe all local conservation laws of this system, including (t, x)-dependent ones.

12. Conclusions

In the present paper we have performed an extended symmetry analysis of the isothermal no-slip drift flux model given by the system (3). It turned out that the form (3) was not convenient for the study within the symmetry framework. In particular, the maximal Lie invariance algebra of the system (3) is difficult to compute even using specialized computer algebra packages, e.g. DESOLVII [51]. Therefore, transforming dependent variables we have represented the model in the form (4). This representation has allowed us to compute the maximal Lie invariance algebra $\mathfrak{g}$ of the initial system. This algebra is infinite-dimensional. Note that the infinite-dimensional part of $\mathfrak{g}$ spanned by $\{\check{\mathcal{W}}(\tilde\Omega)\}$ was missed in [23].

Moreover, we have computed the complete point symmetry group of the system $\mathcal{S}$ using the combined algebraic method. Since the algebra $\mathfrak{g}$ is infinite-dimensional, the straightforward application of the automorphism-based method is not appropriate. For this reason we have chosen to employ its megaideal-based counterpart. However, the fact that the finite-dimensional

part of the algebra $\mathfrak{g}$ coincides with its radical $\mathfrak{r}$, which is a megaideal, allowed us to partially use the automorphism-based version of the algebraic method as well. To this end we have computed the entire automorphism group of the radical $\mathfrak{r}$ of $\mathfrak{g}$. Finally, we have employed the constraints derived from the algebraic method to obtain the complete point symmetry group (10) of the isothermal no-slip drift flux model $\mathcal{S}$ applying the direct method.

Following the standard Lie reduction procedure [25], we have also obtained optimal lists of one- and two-dimensional subalgebras of the algebra $\mathfrak{g}$, which were employed for finding appropriate solution ansatzes, using which we have found families of invariant and partially invariant solutions of the system $\mathcal{S}$.

Sections 7–8 and 9 of the present paper provide two alternative approaches to finding solutions of the system under study. The first of them is based on the fact that the subsystem $\mathcal{S}_0$ is partially decoupled from the rest of the system, so we could construct its solutions first and then solve the remaining equation (4c) in parametric form. This choice is justified by the fact that the system $\mathcal{S}_0$ admits a larger Lie symmetry group than the Lie symmetry group of $\mathcal{S}$, and thus one can find more invariant or partially invariant solutions. Moreover, one should keep in mind the well-known fact that every hydrodynamic-type system in two independent variables, say, x, t, and two dependent variables, say, u, v, can be linearized via the two-dimensional hodograph transformation with $(p, q) = (t, x)$, $(y, z) = (u, v)$ being the new dependent and independent variables, respectively. It is worth noticing that upon having applied the above hodograph transformation to $\mathcal{S}_0$ the equation (4c) in the new variables remains a first-order linear PDE in w which can be easily solved. The above change of variables has led to the potential system (20) of the telegraph equation (21). Using the subsequent change of variables $\tilde{p} = p \exp(-z/2)$ we obtained the famous Klein–Gordon equation (22). As the latter equation is well studied, we could construct large classes of exact solutions thereof, find p and solve equation (4c), which allows us to find solutions of the initial system $\mathcal{S}$ in parametric form.

The second approach to finding solutions for (4) is based on the semi-Hamiltonian property of (5), which has allowed us to employ the generalized hodograph transformation. Just as for the previous method, the problem of finding solutions of the system (4) was reduced to solving the Klein–Gordon equation. Nevertheless, the solutions obtained via generalized hodograph transformations are included in the list of solutions obtained via the decoupling of (4).

We have computed the first-order generalized symmetries for the system (5), amongst which we found both evolutionary forms of point symmetries and genuinely generalized symmetries. Interestingly, some of genuinely generalized symmetries are nothing but evolutionary forms of point symmetries of the essential subsystem. Moreover, the system $\mathcal{S}$ possesses genuinely generalized symmetries beyond the latter, see Remark 20.

We have also constructed the entire space of zero-order conservation laws for the system (4), which is spanned by a single non-translation-invariant conservation law and the (infinite-dimensional) space of hydrodynamic conservation laws of this system. Note that while the existence of hydrodynamic-type systems with an infinite number of independent hydrodynamic conservation laws was known for a long time [52,53], hydrodynamic-type systems with non-hydrodynamic conservation laws are significantly less common, cf. e.g. [54] and references therein. The space of hydrodynamic conservation laws of a diagonalizable semi-Hamiltonian system is parameterized by n arbitrary functions of one variable [13], where n is the number of dependent variables.

For the system (4) two of the arbitrary functions in question are determined by solutions of the telegraph equation $\Phi_{uu} = \Phi_{vv} - \Phi_v$, while the third function is an arbitrary function of w, which can be reduced to 1 by actions of point symmetries unless it is zero. The first-order conservation laws with (t, x)-independent densities have also been classified. Existence of higher-order conservation laws has been justified using the fact that system (5) is not genuinely nonlinear. Such conservation laws have been found using the action of generalized symmetries of the system (5) of arbitrarily high order.

We found only generalized symmetries and conservation laws of the system $\mathcal{S}$ of order not greater than one as well as higher-order generalized symmetries and conservation laws related to the linearly degenerate part of the system $\mathcal{S}$.

In general, it is quite a common situation when a hydrodynamic-type system does not possess generalized symmetries of order greater than one. The key observation facilitating the study of higher-order symmetries for the system $\mathcal{S}$ is the fact that this system is only partially coupled, and the essential subsystem $\mathcal{S}_0$ of the system $\mathcal{S}$ is linearized by the hodograph transformation to the Klein–Gordon equation for which all generalized symmetries are known [55–57]. The main conjecture here is that the entire space of generalized symmetries of the system $\mathcal{S}$ is spanned by generalized symmetries from the family presented in Proposition 21 and prolongations of generalized symmetries of the essential subsystem $\mathcal{S}_0$ to $\mathfrak{r}^3$. The further work is to figure out which of the generalized symmetries of $\mathcal{S}_0$ are locally prolonged to generalized symmetries of the entire system $\mathcal{S}$ and how this prolongation is performed.

Linearizability of the subsystem $\mathcal{S}_0$ can also be applied to classification of all (local) conservation laws of the system $\mathcal{S}$. Our conjecture here is that the entire space of conservation laws of the system $\mathcal{S}$ is spanned by conservation laws from the family presented in Remark 27 and by those being pullbacks of conservation laws the essential subsystem $\mathcal{S}_0$ by the projection to $(t, x, \mathfrak{r}^1, \mathfrak{r}^2)$. Moreover, one does not need to prolong conservation laws of the essential subsystem $\mathcal{S}_0$ to conservation laws of the entire system $\mathcal{S}$, which makes the problem easier than that of finding generalized symmetries.

It is readily verified that system (5) admits a one-parameter family of Hamiltonian operators of Dubrovin–Novikov type (see e.g. the surveys [14,58] for more details on such operators) with the parameter λ,

$$\mathcal{P}_\lambda = e^{\mathfrak{r}^2 - \mathfrak{r}^1} \operatorname{diag}\left(1, -1, \lambda e^{\mathfrak{r}^2 - \mathfrak{r}^1}\right) D_x$$

$$+ \frac{e^{\mathfrak{r}^2 - \mathfrak{r}^1}}{2} \begin{pmatrix} \mathfrak{r}_x^2 - \mathfrak{r}_x^1 & \mathfrak{r}_x^1 - \mathfrak{r}_x^2 & -2\mathfrak{r}_x^3 \\ \mathfrak{r}_x^2 - \mathfrak{r}_x^1 & \mathfrak{r}_x^1 - \mathfrak{r}_x^2 & -2\mathfrak{r}_x^3 \\ 2\mathfrak{r}_x^3 & 2\mathfrak{r}_x^3 & -2\lambda e^{\mathfrak{r}^2 - \mathfrak{r}^1}\left(\mathfrak{r}_x^1 - \mathfrak{r}_x^2\right) \end{pmatrix}, \quad (33)$$

and the associated Hamiltonian representation

$$\mathfrak{r}_t = \mathcal{P}_\lambda(\delta H / \delta \mathfrak{r}),$$

where $\mathfrak{r} = (\mathfrak{r}_1, \mathfrak{r}_2, \mathfrak{r}_3)^{\mathrm{T}}$, the Hamiltonian functional is given by

$$H = -\frac{1}{4} \int e^{\mathfrak{r}_1 - \mathfrak{r}_2} \left((\mathfrak{r}_1 + \mathfrak{r}_2)^2 + 2(\mathfrak{r}_1 - \mathfrak{r}_2)\right) \, dx,$$

$\delta / \delta \mathfrak{r}$ is the variational derivative, and the integral is understood in the formal sense of calculus of variations; cf. e.g. [24, Chapter 4] for the details on the latter.

The Hamiltonian operators (33) can be formally inverted if $\lambda \neq 0$, and in view of this and of the above the ratio $\mathcal{R}_{\mu\nu} = \mathcal{P}_\mu \circ \mathcal{P}_\nu^{-1}$ defines, for $\nu \neq 0$, a hereditary recursion operator for (5); see e.g. [9,24] and references therein for more details on recursion operators in general. Alas, the leading term of $\mathcal{R}_{\mu\nu} - \mathbb{I}$, where $\mathbb{I}$ is the identity operator, is a degenerate matrix, which makes

$\mathcal{R}_{\mu\nu}$ rather uninteresting from the point of view of generating symmetries for (5).

In closing note that (5) also admits infinitely many Hamiltonian operators that are not of Dubrovin–Novikov type, and a recursion operator which is free of the degeneracy of the above kind. Also, it is well known that the Hamiltonian operators map cosymmetries (in particular, characteristics of conservation laws) into symmetries, which could lead to an interesting interplay among the two for the system (5), and hence for $\mathcal{S}$. All of this will be discussed in more detail in the sequel to the present paper.

Acknowledgments

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