



Whitham theory for perturbed Korteweg–de Vries equation



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HIGHLIGHTS

- Averaging Whitham method is applied to perturbed Korteweg–de Vries equation.
- Perturbations of two kinds – gradient and non-gradient – are distinguished.
- Method of elimination of gradient perturbations is suggested.
- The theory is applied to steady-state solutions of a perturbed KdV equation.

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ABSTRACT

Original Whitham's method of derivation of modulation equations is applied to systems whose dynamics is described by a perturbed Korteweg–de Vries equation. Two situations are distinguished: (i) the perturbation leads to appearance of right-hand sides in the modulation equations so that they become non-uniform; (ii) the perturbation leads to modification of the matrix of Whitham velocities. General form of Whitham modulation equations is obtained in both cases. The essential difference between them is illustrated by an example of so-called 'generalized Korteweg–de Vries equation'. Method of finding steady-state solutions of perturbed Whitham equations in the case of dissipative perturbations is considered.

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1. Introduction

In his seminal paper [1] Whitham introduced into nonlinear wave theory several fundamental ideas which formed the basis for development of a vast theory called now *Whitham theory*. First, he generalized the idea of slow evolution of envelopes of linear harmonic wave trains ('wave packets') to description of evolution of nonlinear modulated wave trains whose dynamics is governed by nonlinear wave equations. This idea implies that in the problem under consideration there are two different scales of space and time: the field variables $u(x, t)$ of the nonlinear 'carrier' wave oscillate at the scales of wavelength L and period T , whereas such parameters of the wave as, e.g., wavelength L , amplitude a , phase velocity V , etc., change slowly at the space scale $x \gg L$ and the time scale $t \gg T$. This leads to the second idea of averaging of the conservation laws of the evolution equation over fast local oscillations analogously to the Krylov–Bogoliubov averaging technique developed in the theory of nonlinear vibrations. However, in contrast

with dynamical time-dependent systems, now the field variables depend on time and one (or more) space coordinates and, as a result of averaging of conservation laws, Whitham obtained the system of first order partial differential equations now called *Whitham equations*. Whitham compared this approach with transition from 'microscopic' description of gas dynamics to averaged hydrodynamic description ("Indeed, the present work is in much the same spirit as the derivation of continuum fluid mechanics from kinetic theory", see page 242 in [1]) and this suggested the third idea of application of the averaged equations to description of such physical phenomena as, for example, water undular bores and collisionless shocks in plasma. At last, as the fourth idea, Whitham supposed that his modulation equations can be transformed, by analogy with compressible fluid dynamics, to the diagonal Riemann form and he realized this idea by means of very skillful calculations for the case of modulated nonlinear 'cnoidal' wave whose evolution is governed by the Korteweg–de Vries (KdV) equation.

In the first paper [1] Whitham assumed that the modulation equations he derived are hyperbolic. As was noticed soon after that (see [2]) on the basis of the Whitham approach reformulated in terms of Lagrangian method [3], the Whitham modulation equations can be elliptic and that indicates that the corresponding

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periodic wave is unstable. This observation was done in connection with the experimental discovery of instability of deep water waves [4,5] that was studied theoretically by Whitham [6] with the use of his approach. (About the early history of this discovery and the important role played in it by the Whitham theory one can find in [7,8].)

Thus, Whitham formulated in [1,3,6] the general method for studying modulated nonlinear waves and illustrated fruitfulness of his approach by important nontrivial examples. Richness of ideas introduced in [1,3,6] has been spectacularly confirmed by impressive development of the Whitham theory in the past 50 years.

The first important contribution into the Whitham theory after appearance of the papers [1,3,6] was done by Gurevich and Pitaevskii [9] who showed that a collisionless shock (now commonly called *dispersive shock wave (DSW)*) described by the KdV equation can be represented as an expanding oscillating structure which can be approximated by a modulated cnoidal wave whose evolution is governed by the Whitham equations. At one of its edges the dispersive shock approaches to a soliton train and at the opposite edge it tends to a small amplitude harmonic wave. Gurevich and Pitaevskii studied self-similar solutions of the Whitham equations for a typical examples of evolutions of an initial step-like distribution and of the general wave breaking situation.

Analytical theory developed by Gurevich and Pitaevskii was based on a specific diagonal form of modulation equations obtained by Whitham for modulated KdV cnoidal waves. However, such a form for other nonlinear wave equations was not known and it was not easy to find it by the direct method used by Whitham. Actually, as it became clear later, Whitham had succeeded in finding the Riemann invariants for the KdV equation case because this equation belongs to a very special class of so-called ‘completely integrable equations’ whose solutions can be found by the inverse scattering transform (IST) method discovered independently of the Whitham theory [10–12]. It turned out that this method generalized on quasi-periodic situations [13,14] yields quasi-periodic solutions of the KdV equation which are parameterized directly by Riemann invariants having in this case very simple mathematical meaning: they are the edge points of gaps in the spectrum of the linear (Schrödinger) equation related with the KdV equation in the IST method. As a result, multi-phase averaging method for the quasi-periodic solutions of the KdV equation was developed by Flaschka, Forest and McLaughlin [15] where the Whitham equations were derived in Riemann diagonal form for $2N + 1$ dependent variables (Riemann invariants), N being the number of phases ($N = 1$ for the simplest cnoidal wave case considered by Whitham in [1]). The general method of derivation of the Whitham equations for a wide class of completely integrable equations was suggested by Krichever [16]. The generalized hodograph method of integration of diagonal Whitham equations was developed by Tsarev [17]. This progress in mathematical theory of integrability of nonlinear wave equations and of corresponding Whitham modulation equations has led to a number of applications to physical problems related with formation of DSWs and deeper understanding of qualitative properties of this phenomenon. At the same time, it became quite desirable to extend the theory of DSWs on situations often met in physical applications when wave motion is not described by the completely integrable equations.

From physical point of view, it seems clear that the phenomenon of formation of DSWs is related with effects of dispersion in nonlinear wave systems and is not conditioned by the complete integrability of the corresponding evolution equations. Actually, the Whitham theory was developed in [1] in very general setting under supposition of existence of periodic solutions of evolution equations and only application of this theory to the KdV cnoidal

wave was related implicitly in this paper with the complete integrability of the KdV equation. Therefore for treatment of DSWs in general situation one should resort to analysis of Whitham equations in a non-diagonal form when they do not have Riemann invariants and cannot be integrated by Tsarev’s generalized hodograph method. Such an analysis was done by El [18] in an important particular case of evolution of an initial step-like distributions whose dynamics is governed by non-dissipative nonlinear wave equations. This ingenious method has found a number of interesting applications in which the problem can be reduced to the study of evolution of step-like initial conditions.

Another typical situation appears when the evolution equation differs little from an integrable one. For example, such a difference can appear due to small dissipation effects or weak non-uniformity of the medium through which the wave propagates. As was indicated already by Whitham in [1], these effects lead to modulation of nonlinear periodic waves and can be considered in framework of the averaged modulation equations. In such situations, the Whitham modulation equations can be modified by perturbations in two possible ways: (i) the equations for Riemann invariants $\lambda_i(x, t)$ of unperturbed equations have now ‘right-hand sides’ depending on λ_i , that is these equations become non-uniform; (ii) the additional terms caused by perturbations contribute to fluxes of the conserved quantities leading to appearance in the Whitham equations of terms proportional to the derivatives $\partial \lambda_i / \partial x$, that is the matrix of ‘velocities’ becomes non-diagonal and λ_i are not Riemann invariants anymore, but the non-diagonal terms as well as corrections to the diagonal ones are small and the modulation equations remain uniform. Of course, one can imagine situations when both types of corrections appear in perturbed Whitham equations.

So far, mainly the first type of corrections has been considered. Physically, such corrections appear very naturally when irreversible processes are taken into account. The Whitham averaging method for systems with small dissipation was developed by Jimenez and Whitham [19] in general form without transition to Riemann invariants. Whitham equations for N -phase KdV wave trains in presence of small perturbations were derived in [20], however in a form not convenient enough for applications. More practical and instructive example of one-phase modulated KdV wave trains with account of small Burgers viscosity was considered by Gurevich and Pitaevskii [21] and by Avilov, Krichever and Novikov [22] (earlier the steady-state solution of this problem had been studied by Johnson [23] by a direct perturbation technique). They derived the Whitham equations for the Riemann invariants λ_i of unperturbed problem and showed that small Burgers viscosity results in non-zero right-hand sides of Whitham equations which provide additional contribution into evolution of these modulation parameters λ_i . The analysis presented there showed that although the perturbation is small compared with the main terms in the KdV equation, that does not mean that its contribution into evolution of the Riemann invariants is also small. Indeed, in this case the perturbation should be compared with a small parameter which characterizes ‘slowness’ of modulation rather than with terms which determine fast oscillations of the cnoidal wave. If the cnoidal wave is not modulated at all, then dissipative terms make the only contribution into changes of the Riemann invariants and determine slow evolution of a uniform cnoidal wave (see, e.g., [24,25]). Effects of non-local damping were considered by Gurevich and Pitaevskii in [26] and more general forms of local dissipation were considered by Myint and Grimshaw in [27]. Quite general approach applicable to the Ablowitz–Kaup–Newell–Segur (AKNS) class [28] of completely integrable equations was developed by the author for non-perturbed [29] and perturbed [30] cases. In combination with simplified version [31] of the finite-gap integration method which yields the periodic solutions in a ‘real’ form not-constrained by any

additional ‘reality conditions’, this approach turned out to be quite effective and it has found several non-trivial applications including propagation of KdV wave trains through a non-uniform medium (see, e.g., [32–35]).

However, the approach described above is not applicable to situations when perturbations change the matrix of Whitham velocities although such situations are encountered quite often. For example, if in the perturbed KdV equation

$$u_t + 6uu_x + u_{xxx} = R[u] \quad (1)$$

the perturbation term has the form $R[u] = \epsilon F'(u)u_x$, ($\epsilon \ll 1$), then the general formulae obtained in [27,30] lead to vanishing right-hand sides in the ‘perturbed’ Whitham equations what means that such a perturbation belongs to the type (ii) and a different perturbation scheme should be developed for finding the corrected matrix of Whitham velocities. This difference between two types of perturbations is clearly illustrated by a simple example of perturbation $R[u] = \text{const} \cdot u^2 u_x$ when (1) reduces to the so-called Gardner equation. This equation is also completely integrable, the corresponding Riemann invariants and Whitham equations can be obtained without any approximations (see [36,37]), and they do not reduce to appearance of the right-hand sides in Whitham equations in the KdV limit of the Gardner equation.

Thus, we arrive at the problem of derivation of the approximate Whitham equations for two different situations when either the Whitham equations acquire the right-hand side terms, or the Whitham velocities are modified by perturbations. Here we shall confine ourselves to a simple example of the KdV equation (1) under supposition that the perturbation term is small,

$$|R[u]| \ll \min\{u^2/L, |u|/L^3\}. \quad (2)$$

Hence, we can approximate locally the solution of (1) by the cnoidal wave solution of unperturbed KdV equation and apply the original method of Whitham [1] to this more general situation. In the next section we shall illustrate the method by its application to the already studied earlier situation of perturbations of type (i) and then generalize it to perturbations of type (ii). In Section 3 we shall show that a specific structure of perturbation terms leads to a simple method of finding the steady-state solutions of the Whitham equations. We conclude by the remark that the direct Whitham approach to obtaining the modulation equations is effective enough and it can be successfully used for studying quite complicated nonlinear wave problems.

2. Perturbed Whitham equations

Traveling wave solution of the unperturbed KdV equation

$$u_t + 6uu_x + u_{xxx} = 0 \quad (3)$$

is obtained by means of a simple substitution $u = u(\xi)$, $\xi = x - Vt$, so that after two obvious integrations we get the ordinary differential equation

$$\frac{1}{2}u_\xi^2 = f(u), \quad (4)$$

where $f(u)$ is a third degree polynomial ($\alpha \geq \beta \geq \gamma$),

$$\begin{aligned} f(u) &= -A + Bu + \frac{1}{2}Vu^2 - u^3 \\ &= -(u - \alpha)(u - \beta)(u - \gamma), \end{aligned} \quad (5)$$

where V, A , and B are the integration constants related with the zeros α, β, γ of the polynomial $f(u)$ by the formulae

$$\begin{aligned} V &= 2(\alpha + \beta + \gamma), & A &= -\alpha\beta\gamma, \\ B &= -(\alpha\beta + \beta\gamma + \gamma\alpha). \end{aligned} \quad (6)$$

In a standard way the solution of Eq. (4) can be expressed in terms of Jacobi elliptic sinus function

$$u(x, t) = \alpha - (\alpha - \beta) \text{sn}^2\left(\sqrt{(\alpha - \gamma)/2} \xi, m\right), \quad (7)$$

where the parameter m is equal to

$$m = \frac{\alpha - \beta}{\alpha - \gamma}. \quad (8)$$

This is a periodic solution of the KdV equation (3) and its wavelength is given by the formula

$$\begin{aligned} L &= \frac{1}{k} = \int_0^L d\xi = \oint \frac{du}{u_\xi} \\ &= \frac{1}{\sqrt{2}} \oint \frac{du}{\sqrt{f(u)}} = 2\sqrt{\frac{2}{\alpha - \gamma}} K(m), \end{aligned} \quad (9)$$

where $K(m)$ is the complete elliptic integral of the first kind. Following to Whitham [1], we have introduced in (9) the wavenumber of the nonlinear wave (7) as $k = 1/L$; then at a given point x the wave oscillates with the frequency defined as $\omega = kV$, so that the solution (7) depends on the phase $\theta = kx - \omega t$.

In a modulated wave the parameters V, A, B , or, equivalently, α, β, γ , change little in one wavelength L and one period $T \sim 1/\omega$. According to Whitham, the solution of the (perturbed or unperturbed) KdV equation can be approximated locally by the expression (7) where $\alpha(x, t), \beta(x, t), \gamma(x, t)$ are considered now as slow functions of x and t and the phase $\theta = kx - \omega t$ is replaced now by a general dependence $\theta(x, t)$. Then the wavenumber and the frequency are defined as

$$k = \theta_x, \quad \omega = -\theta_t, \quad (10)$$

and, hence, they must satisfy the compatibility condition $k_t + \omega_x = 0$ which has the meaning of conservation of ‘number of waves’ [1]. In a slowly modulated wave both k and $\omega = kV$ are expressed in terms of the slow parameters α, β, γ (see Eqs. (6) and (9)) so that we arrive at the equation for these parameters,

$$k_t + (kV)_x = 0. \quad (11)$$

All that is applied to any modulated KdV wave train and the modulation can be caused either by a non-uniform initial condition or by a perturbation term in (1). Whitham discussed in [1] the first situation only and we wish here to generalize his approach to the perturbed KdV equation. As was indicated in Section 1, we have to distinguish in this case two different situations which, as we shall see, can be formulated more precisely as follows: (i) neither R nor uR are space derivatives, (ii) R and/or uR can be represented as space derivatives of other functions (say, $R = \mathcal{Q}_{1,x} \equiv \partial \mathcal{Q}_1 / \partial x$ and/or $uR = \mathcal{Q}_{2,x} \equiv \partial \mathcal{Q}_2 / \partial x$; we shall use in what follows this index notation: first index indicates the number of the function and the second index denotes the space derivative). We shall call the first situation as a *non-gradient perturbation* and the second one as a *gradient perturbation*, and we shall begin with discussion of the non-gradient perturbations.

2.1. Whitham equations for the case of non-gradient perturbations

In addition to (11), we need two more equations for three parameters $\alpha(x, t), \beta(x, t), \gamma(x, t)$ and, following Whitham [1], we assume that they can be obtained by means of averaging the conservation laws of the KdV equation (1),

$$\begin{aligned} u_t + (3u^2 + u_{xx})_x &= R, \\ \left(\frac{1}{2}u^2\right)_t + \left(2u^3 + uu_{xx} - \frac{1}{2}u_x^2\right)_x &= uR. \end{aligned} \quad (12)$$

Averaging is defined as taking a mean value of an expression \mathcal{P} along the wavelength,

$$\langle \mathcal{P} \rangle = \frac{1}{L} \int_0^L \mathcal{P} dx = k \oint \mathcal{P} \frac{du}{u_x} = \frac{k}{\sqrt{2}} \oint \frac{\mathcal{P} du}{\sqrt{f(u)}}, \quad (13)$$

where integration is taken over the whole cycle of oscillation of u . Whitham averaged densities and fluxes of the conservation laws (12) (with $R = 0$) and obtained two additional equations for the slow variables. If we average the right-hand sides of Eq. (12) according to the rule (13) then we obtain the perturbed Whitham equations with the right-hand sides. The averaged conservation laws (12) take the form

$$\begin{aligned} \langle u \rangle_t + \langle 3u^2 + u_{xx} \rangle_x &= \langle R \rangle, \\ \langle \frac{1}{2} u^2 \rangle_t + \langle 2u^3 + uu_{xx} - \frac{1}{2} u_x^2 \rangle_x &= \langle uR \rangle, \end{aligned} \quad (14)$$

and the condition that R and uR are not x -derivatives yields, generally speaking, non-zero right-hand sides in these equations. The derivatives u_x^2 , u_{xx} can be excluded with the use of Eqs. (4) and (5): $u_x^2 = 2f(u)$, $u_{xx} = f'(u) = B + Vu - 3u^2$, and, as Whitham indicated, it is convenient to represent the averaged expressions in terms of a single ‘action function’

$$\begin{aligned} W(A, B, V) &= -\sqrt{2} \oint \sqrt{f(u)} du \\ &= -\sqrt{2} \oint \sqrt{-A + Bu + \frac{1}{2}Vu^2 - u^3} du, \end{aligned} \quad (15)$$

and its derivatives with respect to A , B , V . Indeed, the wavelength (9) can be written as

$$L = \frac{1}{k} = \frac{1}{\sqrt{2}} \oint \frac{du}{\sqrt{f(u)}} = \frac{\partial W}{\partial A} \equiv W_A \quad (16)$$

and the necessary averages are given by

$$\begin{aligned} \langle u \rangle &= \frac{k}{\sqrt{2}} \oint \frac{udu}{\sqrt{f(u)}} = -kW_B, \\ \langle \frac{1}{2} u^2 \rangle &= \frac{k}{\sqrt{2}} \oint \frac{(u^2/2)du}{\sqrt{f(u)}} = -kW_V. \end{aligned} \quad (17)$$

As a result we obtain the averaged conservation laws

$$\begin{aligned} (-kW_B)_t + (B - kW_{VB})_x &= \langle R \rangle, \\ (-kW_V)_t + (A - kW_{VV})_x &= \langle uR \rangle, \end{aligned}$$

which can be simplified with the use of Eq. (11). Besides that, we introduce the ‘material derivative’

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + V \frac{\partial}{\partial x} \quad (18)$$

and substitute (16) into (11) to obtain the complete set of the Whitham modulation equations:

$$\begin{aligned} \frac{DW_A}{Dt} &= W_A \frac{\partial V}{\partial x}, \\ \frac{DW_B}{Dt} &= W_A \frac{\partial B}{\partial x} - W_A \langle R \rangle, \\ \frac{DW_V}{Dt} &= W_A \frac{\partial A}{\partial x} - W_A \langle uR \rangle. \end{aligned} \quad (19)$$

Naturally, they differ from the original Whitham equations [1] only by the terms with $\langle R \rangle$ and $\langle uR \rangle$.

A remarkable discovery of Whitham was that the unperturbed modulation equations can be transformed ‘after considerable manipulation’ to the diagonal (Riemann) form and the Riemann invariants are expressed in terms of zeros α , β , γ . In our case the same transformation leads again to the diagonal form of Whitham

equations, however now with small right-hand sides, that is the Whitham equations become non-uniform. The necessary ‘considerable manipulation’ is described in detail in [38] and we indicate here briefly the main steps only.

First, we transform Eq. (19) from the variables A , B , V to the variables α , β , γ with the use of relationships (6),

$$\begin{aligned} W_{A,\alpha} \frac{D\alpha}{Dt} + W_{A,\beta} \frac{D\beta}{Dt} + W_{A,\gamma} \frac{D\gamma}{Dt} &= 2W_A(\alpha_x + \beta_x + \gamma_x), \\ W_{B,\alpha} \frac{D\alpha}{Dt} + W_{B,\beta} \frac{D\beta}{Dt} + W_{B,\gamma} \frac{D\gamma}{Dt} &= -W_A \\ &\quad \times [(\beta + \gamma)\alpha_x + (\alpha + \gamma)\beta_x + (\alpha + \beta)\gamma_x] - W_A \langle R \rangle, \\ W_{V,\alpha} \frac{D\alpha}{Dt} + W_{V,\beta} \frac{D\beta}{Dt} + W_{V,\gamma} \frac{D\gamma}{Dt} &= -W_A \\ &\quad \times [\beta\gamma \cdot \alpha_x + \alpha\gamma \cdot \beta_x + \alpha\beta \cdot \gamma_x] - W_A \langle uR \rangle, \end{aligned} \quad (20)$$

where

$$\begin{aligned} W_{A,\alpha} &= \frac{1}{\sqrt{8}} \oint \frac{du}{(u - \alpha)\sqrt{f(u)}}, \\ W_{B,\alpha} &= -\frac{1}{\sqrt{8}} \oint \frac{udu}{(u - \alpha)\sqrt{f(u)}}, \\ W_{V,\alpha} &= -\frac{1}{\sqrt{8}} \oint \frac{(u^2/2)du}{(u - \alpha)\sqrt{f(u)}}, \end{aligned} \quad (21)$$

and similar expressions can be written for derivatives with respect to β and γ . Next, we multiply the first equation (20) by $p = \alpha\beta + \alpha\gamma - \beta\gamma$, the second equation by $q = 2\alpha$, the third equation by $r = -2$ and add them; then with the use of the identities

$$\begin{aligned} W_{A,\alpha} + W_{A,\beta} + W_{A,\gamma} &= \frac{1}{\sqrt{8}} \oint \frac{f'(u)du}{f^{3/2}(u)} = 0, \\ pW_{A,\alpha} + qW_{B,\alpha} + rW_{V,\alpha} &= -\frac{1}{\sqrt{8}} \oint \frac{d}{du} \left(2\sqrt{\frac{(u - \beta)(u - \gamma)}{-(u - \alpha)}} \right) du = 0 \end{aligned}$$

we obtain

$$\begin{aligned} \frac{D(\beta + \gamma)}{Dt} + \frac{W_A}{W_{A,\alpha}} \frac{\partial(\beta + \gamma)}{\partial x} &= -\frac{1}{(\alpha - \beta)(\alpha - \gamma)} \frac{W_A}{W_{A,\alpha}} (\alpha \langle R \rangle - \langle uR \rangle) \end{aligned} \quad (22)$$

and similar equation can be obtained for the variables $\alpha + \beta$ and $\alpha + \gamma$ by means of cyclic transposition of the parameters α , β , γ . At last, we introduce the Riemann invariants of the unperturbed KdV equation,

$$\begin{aligned} \lambda_1 &= -\frac{1}{2}(\alpha + \beta), & \lambda_2 &= -\frac{1}{2}(\alpha + \gamma), \\ \lambda_3 &= -\frac{1}{2}(\beta + \gamma), & \lambda_1 &\leq \lambda_2 \leq \lambda_3, \end{aligned} \quad (23)$$

and with account of

$$\frac{W_A}{W_{A,\alpha}} = \frac{2W_A}{W_{A,\lambda_3}} = \frac{2L}{\partial L / \partial \lambda_3}$$

and similar formulae for $W_A/W_{A,\beta}$ and $W_A/W_{A,\gamma}$ we arrive at the Whitham equation in the form

$$\frac{\partial \lambda_i}{\partial t} + v_i^{(0)} \frac{\partial \lambda_i}{\partial x} = \frac{L}{\partial L / \partial \lambda_i} \frac{\langle (2\lambda_i - s_1 - u)R \rangle}{4 \prod_{j \neq i} (\lambda_i - \lambda_j)}, \quad i = 1, 2, 3, \quad (24)$$

where velocities $v_i^{(0)}$ are given by

$$v_i^{(0)} = -2s_1 + \frac{2L}{\partial L / \partial \lambda_i}, \quad i = 1, 2, 3, \quad (25)$$

and $s_1 = \lambda_1 + \lambda_2 + \lambda_3$. Here all the variables should be parameterized by the Riemann invariants $\lambda_1, \lambda_2, \lambda_3$. In particular, the periodic solution (7) of the KdV equation takes the form

$$u(x, t) = \lambda_3 - \lambda_2 - \lambda_1 - 2(\lambda_3 - \lambda_2) \operatorname{sn}^2 \left(\sqrt{\lambda_3 - \lambda_1} (x - Vt), m \right), \quad (26)$$

where

$$V = -2s_1 = -2(\lambda_1 + \lambda_2 + \lambda_3), \quad (27)$$

$$m = \frac{\lambda_3 - \lambda_2}{\lambda_3 - \lambda_1}, \quad (28)$$

and the wavelength is given by

$$L = \frac{1}{k} = \frac{2K(m)}{\sqrt{\lambda_3 - \lambda_1}}. \quad (28)$$

Substitution of (28) into (25) gives expressions for the Whitham velocities in their original form [1],

$$\begin{aligned} v_1^{(0)} &= -2s_1 + \frac{4(\lambda_3 - \lambda_1)(1 - m)K(m)}{E(m)}, \\ v_2^{(0)} &= -2s_1 - \frac{4(\lambda_3 - \lambda_2)(1 - m)K(m)}{E(m) - (1 - m)K(m)}, \\ v_3^{(0)} &= -2s_1 + \frac{4(\lambda_3 - \lambda_2)K(m)}{E(m) - K(m)}, \end{aligned} \quad (29)$$

where $E(m)$ is the complete elliptic integral of the second kind. For averaging the perturbation terms, it is convenient to introduce the variable $\mu = (u + s_1)/2$ which changes within the interval $\lambda_2 \leq \mu \leq \lambda_3$ and satisfies the equation

$$\mu_x = 2\sqrt{-P(\mu)}, \quad (30)$$

where

$$\begin{aligned} P(\mu) &= (\mu - \lambda_1)(\mu - \lambda_2)(\mu - \lambda_3) \\ &= \mu^3 - s_1\mu^2 + s_2\mu - s_3, \end{aligned} \quad (31)$$

consequently

$$\langle \mathcal{Q} \rangle = \frac{1}{L} \int_0^L \mathcal{Q} dx = \frac{1}{L} \int_{\lambda_2}^{\lambda_3} \frac{\mathcal{Q} d\mu}{\sqrt{-P(\mu)}}. \quad (32)$$

These formulae easily reproduce the results found in Refs. [21,22,26] with $R = \epsilon u_{xx}$ corresponding to the Burgers viscosity and permitted one to derive Whitham equations for the case of $R = F(t)u - G(t)u^2$ corresponding to shallow water waves over a gradual slope with account of bottom friction [33]. However, as was mentioned in Introduction, this approach fails if averages of R and/or uR vanish what happens when these expressions are the x -derivatives of some other expressions and, hence, they contribute into the fluxes of the conservation laws (12). We shall consider such a situation in the next subsection.

2.2. Whitham equations for the case of gradient perturbations

Here we shall assume that both R and uR are space derivatives

$$R = \mathcal{Q}_{1,x}, \quad uR = \mathcal{Q}_{2,x} \quad (33)$$

and, hence, they make additional contribution into the fluxes in the conservation laws (12). Then all divergence terms in (12) should be treated on the same footing and averaging of the conservation laws yields

$$\begin{aligned} \langle u \rangle_t + \langle 3u^2 + u_{xx} \rangle_x &= \langle \mathcal{Q}_1 \rangle_x, \\ \langle \frac{1}{2}u^2 \rangle_t + \langle 2u^3 + uu_{xx} - \frac{1}{2}u_x^2 \rangle_x &= \langle \mathcal{Q}_2 \rangle_x. \end{aligned} \quad (34)$$

Transformations similar to those which were done above give instead of Eq. (19) the equations

$$\begin{aligned} \frac{DW_A}{Dt} &= W_A \frac{\partial V}{\partial x}, \\ \frac{DW_B}{Dt} &= W_A \frac{\partial B}{\partial x} - W_A \frac{\partial \langle \mathcal{Q}_1 \rangle}{\partial x}, \\ \frac{DW_V}{Dt} &= W_A \frac{\partial A}{\partial x} - W_A \frac{\partial \langle \mathcal{Q}_2 \rangle}{\partial x}, \end{aligned} \quad (35)$$

or, after transition to the variables α, β, γ , the equations

$$\begin{aligned} W_{A,\alpha} \frac{D\alpha}{Dt} + W_{A,\beta} \frac{D\beta}{Dt} + W_{A,\gamma} \frac{D\gamma}{Dt} &= 2W_A(\alpha_x + \beta_x + \gamma_x), \\ W_{B,\alpha} \frac{D\alpha}{Dt} + W_{B,\beta} \frac{D\beta}{Dt} + W_{B,\gamma} \frac{D\gamma}{Dt} &= -W_A \\ &\times [(\beta + \gamma)\alpha_x + (\alpha + \gamma)\beta_x + (\alpha + \beta)\gamma_x] \\ &- W_A(\langle \mathcal{Q}_1 \rangle_\alpha \alpha_x + \langle \mathcal{Q}_1 \rangle_\beta \beta_x + \langle \mathcal{Q}_1 \rangle_\gamma \gamma_x), \\ W_{V,\alpha} \frac{D\alpha}{Dt} + W_{V,\beta} \frac{D\beta}{Dt} + W_{V,\gamma} \frac{D\gamma}{Dt} &= -W_A[\beta\gamma \cdot \alpha_x + \alpha\gamma \cdot \beta_x + \alpha\beta \cdot \gamma_x] \\ &- W_A(\langle \mathcal{Q}_2 \rangle_\alpha \alpha_x + \langle \mathcal{Q}_2 \rangle_\beta \beta_x + \langle \mathcal{Q}_2 \rangle_\gamma \gamma_x). \end{aligned} \quad (36)$$

Their linear combination used above for transition to Eq. (22) now gives

$$\begin{aligned} \frac{D(\beta + \gamma)}{Dt} + \frac{W_A}{W_{A,\alpha}} \frac{\partial(\beta + \gamma)}{\partial x} &= -\frac{1}{(\alpha - \beta)(\alpha - \gamma)} \frac{W_A}{W_{A,\alpha}} \{ (\alpha \langle \mathcal{Q}_1 \rangle_\alpha - \langle \mathcal{Q}_2 \rangle_\alpha) \alpha_x \\ &+ (\alpha \langle \mathcal{Q}_1 \rangle_\beta - \langle \mathcal{Q}_2 \rangle_\beta) \beta_x + (\alpha \langle \mathcal{Q}_1 \rangle_\gamma - \langle \mathcal{Q}_2 \rangle_\gamma) \gamma_x \}. \end{aligned} \quad (37)$$

At last, transformation to the variables $\lambda_1, \lambda_2, \lambda_3$ (see (23)) yields the Whitham equations in the form

$$\begin{aligned} \frac{\partial \lambda_i}{\partial t} + v_i^{(0)} \frac{\partial \lambda_i}{\partial x} &= \frac{1}{4 \prod_{j \neq i} (\lambda_i - \lambda_j)} \frac{L}{\partial L / \partial \lambda_i} \\ &\times \sum_k \left\{ (2\lambda_i - s_1) \frac{\partial \langle \mathcal{Q}_1 \rangle}{\partial \lambda_k} - \frac{\partial \langle \mathcal{Q}_2 \rangle}{\partial \lambda_k} \right\} \frac{\partial \lambda_k}{\partial x}, \quad i = 1, 2, 3. \end{aligned} \quad (38)$$

Here it was supposed that both R and uR are gradients of ‘fluxes’ \mathcal{Q}_1 and \mathcal{Q}_2 , respectively. If only one of the variables R and uR can be represented as a gradient, then only one corresponding term $\langle \mathcal{Q}_k \rangle$ is included in (38) and the other term must be treated as a non-gradient one resulting in the right-hand sides of the Whitham equations, as was shown in the preceding subsection.

Generally speaking, the variables $\lambda_1, \lambda_2, \lambda_3$ in Eq. (38) are not Riemann invariants anymore. Indeed, we have got a non-diagonal matrix of Whitham velocities

$$\frac{\partial \lambda_i}{\partial t} + \sum_j v_{ij} \frac{\partial \lambda_j}{\partial x} = 0, \quad v_{ij} = v_i^{(0)} \delta_{ij} + v_{ij}^{(1)}, \quad (39)$$

where $v_{ij}^{(1)}$ correspond to the perturbation terms. They are much smaller than the contributions $v_i^{(0)} \delta_{ij}$ into the diagonal ones and we can find the characteristic velocities as well as the corresponding eigenvectors in the way similar to the stationary perturbation theory well known in quantum mechanics (see, e.g., Section 38 in [39]) as long as

$$|v_i^{(0)} - v_j^{(0)}| \gg \max |v_{ij}^{(1)}|. \quad (40)$$

In particular, the characteristic velocities are given in our approximation by the formulae

$$v_i \cong v_i^{(0)} + v_i^{(1)}. \quad (41)$$

The condition (40) indicates that Eq. (38) cannot be applied to situations with degeneration of two Whitham velocities, that is, for example, to the edge points of dispersive shock waves. This conclusion is supported by the following observation. As is known (see, e.g., [9]), mean value of the variable u vanishes at the soliton edge of dispersive shock wave according to the law

$$\langle u \rangle \propto \frac{1}{\ln(\lambda_2 - \lambda_1)}. \quad (42)$$

Hence, this mean value has infinite derivatives with respect to Riemann invariants λ_1 and λ_2 , and mean values $\langle u^n \rangle$ have similar singularities at the soliton edge. Since typically the perturbation terms are expressed in terms of such mean values and their derivatives, the characteristic velocities (41) are also singular at the soliton edge what prevents application of Eq. (38) to the theory of DSWs. A possible method of avoiding this difficulty is discussed in the following subsection.

2.3. Elimination of gradient perturbations from Whitham equations

There are situations when the gradient terms can be eliminated from the Whitham equations and the difficulty indicated in the preceding subsection can be avoided by means of the method used first by Marchant and Smyth [40] in application of the extended KdV (or Gardner) equation

$$u_t + 6uu_x + u_{xxx} = 6\epsilon u^2 u_x \quad (43)$$

to formation of undular bores in the resonant flow of a fluid over topography. It is supposed that ϵ is a small parameter, $\epsilon \ll 1$, and Marchant and Smyth showed that with accuracy $O(\epsilon)$ the equation can be reduced to the KdV equation

$$U_t + 6UU_x + U_{xxx} = 0 \quad (44)$$

by means of a simple substitution

$$u = U + \epsilon(U^2 + U_{xx}/2) \quad (45)$$

$$\text{or}$$

$$U = u - \epsilon(u^2 + u_{xx}/2).$$

In fact, Eq. (43) is completely integrable and the Whitham equations can be derived [36,37] beyond the perturbation theory for any value (and sign) of ϵ . Nevertheless, as we shall show, the method of substitutions similar to that of Marchant and Smyth turns out to be quite useful in a situation of the ‘generalized KdV equation’

$$u_t + 6uu_x + u_{xxx} = 6\epsilon F(u)u_x, \quad (46)$$

where $F(u)$ is an analytical function in the region of variations of u .

Let us look at several examples of simple substitutions and the results of the corresponding transformations. Everywhere we neglect the terms smaller than the order $O(\epsilon)$.

A substitution

$$U = u - \epsilon u_{xx} \quad \text{or} \quad u = U + \epsilon U_{xx} \quad (47)$$

leads to an approximate (with accepted here accuracy) identity

$$U_t + 6UU_x + U_{xxx} = u_t + 6uu_x + u_{xxx} + 12\epsilon u_x u_{xx}.$$

Hence, the perturbed KdV equation

$$u_t + 6uu_x + u_{xxx} = -12\epsilon u_x u_{xx} \quad (48)$$

is reduced to Eq. (44) by means of the substitution (47).

A substitution

$$U = u - \epsilon u^2 \quad \text{or} \quad u = U + \epsilon U^2 \quad (49)$$

transforms in a similar way the perturbed KdV equation

$$u_t + 6uu_x + u_{xxx} = 6\epsilon(u^2 u_x + u_x u_{xx}) \quad (50)$$

to Eq. (44). Composition of substitutions (47) and (49) yields the substitution (45) of Marchant and Smyth. In this particular case the generalized KdV equation (46) is reduced to the non-perturbed KdV equation (44), however such a reduction is generally impossible. For example, a substitution

$$U = u - \epsilon u^3 \quad \text{or} \quad u = U + \epsilon U^3 \quad (51)$$

reduces the equation

$$u_t + 6uu_x + u_{xxx} = 6\epsilon u^3 u_x \quad (52)$$

to

$$U_t + 6UU_x + U_{xxx} = -6\epsilon(U_x^3 + 3UU_x U_{xx}), \quad (53)$$

and here the terms in the right-hand side cannot be eliminated by additional substitutions. In the general case a substitution

$$U = u - \epsilon F(u) \quad \text{or} \quad u = U + \epsilon F(U) \quad (54)$$

transforms (46) to

$$U_t + 6UU_x + U_{xxx} = -\epsilon(F'''(U)U_x^3 + 3F''(U)U_x U_{xx}). \quad (55)$$

Thus, these substitutions transform a perturbed equation to another perturbed equation, however there is an important difference between the initial and reduced forms: the perturbation in Eq. (46) is a gradient one whereas in Eq. (55) it is non-gradient and therefore these perturbations should be treated by different types of Whitham equations discussed above. (Whether there is deep physical difference between situations when perturbation $F(U)$ can be eliminated by some substitution and when it cannot be eliminated is an open question.)

Another important feature of the perturbation terms in (55) is that they do not contribute into the right-hand sides of the Whitham equation (24) which coincide, hence, with unperturbed Whitham equations for the KdV equation. Let us demonstrate this for the averaged value $\langle R \rangle$. Simple integration by parts shows that

$$\int F'' U_x U_{xx} dx = -\frac{1}{2} \int F'''(U) U_x^3 dx$$

that is contribution of the second term in the right-hand side of Eq. (55) reduces after averaging (up to the factor $-1/2$) to the contribution of the first term. But its average vanishes, as shows a simple calculation; indeed, this integral

$$\begin{aligned} \int F'''(U) U_x^3 dx &= \oint F'''(U) U_x^2 dU \\ &= 2 \oint F'''(U) f(U) dU = 0 \end{aligned}$$

equals to zero by the Cauchy theorem applied to an analytical function $F'''(U)f(U)$ with $f(U)$ equal to the polynomial (5) and under supposition that $F'''(U)$ does not have any singular points on the real axis, that is inside the integration contour. Thus, we can use the unperturbed Whitham equations for Eq. (55) and transform the results to Eq. (46) by means of the substitution (54).

Let us illustrate this approach by its application to the Gurevich–Pitaevskii problem of evolution of initial step-like distribution

$$u(x, 0) = \begin{cases} u_- & \text{for } x < 0, \\ 0 & \text{for } x > 0, \end{cases} \quad (56)$$

according to Eq. (46). The substitution (54) transforms this problem to the same problem for Eq. (55) with $U_- = u_- - \epsilon F(u_-)$. The well-known solution of the Gurevich–Pitaevskii problem for unperturbed Whitham equations applicable to (55) yields, in particular, the speeds of edges of the dispersive shock wave (see, e.g., [9,38]),

$$\begin{aligned} s_- &= -6U_- = -6(u_- - \epsilon F(u_-)), \\ s_+ &= 4U_- = 4(u_- - \epsilon F(u_-)), \end{aligned} \quad (57)$$

where s_- is the speed of the small-amplitude edge and s_+ is the speed of the soliton edge. For $F(u) = u^2$ these formulae coincide with the results of exact theory developed in [37] for the Gardner equation and it is instructive to compare them with the results of El's method [18] applied to Eq. (46). According to this method, the speeds of the shock edges are equal to (see Section IV in [18])

$$s_- = \frac{\partial \omega_0}{\partial k}(u_-, k_-), \quad s_+ = \frac{\tilde{\omega}_s(u_+, \tilde{k}_s)}{\tilde{k}_s}, \quad (58)$$

where $\omega_0(u, k)$ and $\tilde{\omega}_s(u, \tilde{k})$ are the linear and the 'soliton' dispersion laws, respectively, given in our case by

$$\begin{aligned} \omega_0(u, k) &= V(u)k - k^3, \\ \tilde{\omega}_s(u, \tilde{k}) &= V(u)\tilde{k} + \tilde{k}^3. \end{aligned} \quad (59)$$

and $V(u) = 6(u - \epsilon F(u))$. In the problem (56) we have $u_+ = 0$ and k_- and \tilde{k}_s should be found by solving the differential equations

$$\begin{aligned} \frac{dk}{du} &= \frac{\partial \omega_0 / \partial u}{V(u) - \partial \omega_0 / \partial k}, \quad k(u_+) = 0, \\ \frac{d\tilde{k}}{du} &= \frac{\partial \tilde{\omega}_s / \partial u}{V(u) - \partial \tilde{\omega}_s / \partial \tilde{k}}, \quad \tilde{k}(u_-) = 0. \end{aligned} \quad (60)$$

Easy integration yields

$$k_- = \tilde{k}_s = 2\sqrt{u_- - \epsilon F(u_-)} \quad (61)$$

and substitution of these values into (58) gives Eq. (57).

It is remarkable that although in derivation of (57) the approximate transformations (54) were used, the final result coincides in this case with the exact results. This may be of practical importance in applications of the perturbation theory to real physical problems. Besides that, in framework of the perturbation theory under consideration the dissipative effects can be easily taken into account by adding corresponding terms to Eq. (46) and by calculation of their contribution into the right-hand sides of the Whitham equation (24). One may suppose that this approach can be useful for consideration of the problems of the type considered in a recent preprint [41] where a combined action of nonlinear, dispersive and dissipation effects should be taken into account.

At last, the perturbation theory is not limited to the step-like initial conditions. For example, if one wishes to consider evolution after wave breaking described by the Gardner equation (43), so that the initial condition can be reduced to the form (see, e.g., [38])

$$x - 6(u - \epsilon u^2)t = -u^3, \quad (62)$$

then the substitution (45) transforms the problem to solving Eq. (44) with the initial condition

$$x - 6Ut = -U^3 - 3\epsilon U^4. \quad (63)$$

Here the Whitham equations correspond to the unperturbed KdV case. Their solution with the initial condition (63) for $\epsilon = 0$ was found by Potemin [42], and its generalization to the case $\epsilon \neq 0$ can be done by including higher commuting flows of the KdV hierarchy into the standard method of integration of unperturbed Whitham equations (see, e.g., [38]).

3. Steady state solutions of perturbed Whitham equations

In dissipative–dispersive systems one can distinguish several characteristic stages of evolution of an initial distribution. For example, if we consider evolution of a step-like initial distribution (56), then it is natural to expect that the dispersion term with the third order x -derivative is much more important at the initial stage of evolution than the dissipative perturbation term typically proportional to the second order derivative u_{xx} (or even u itself; see, e.g., [33]). Hence, if we introduce a small parameter $R \sim \epsilon$, then for time $t \lesssim 1/\epsilon$ we can neglect the dissipative perturbation and evolution is described by the classical solution of the Gurevich–Pitaevskii problem [9]. However, for $t \gtrsim 1/\epsilon$ the damping of solitons in the dispersive shock wave train becomes essential and this damping can be compensated by the non-zero boundary condition at the small amplitude limit $x \rightarrow -\infty$ where $u \rightarrow u_-$. Thus, for $t \gg 1/\epsilon$ we arrive at the steady-state solution of the perturbed KdV equation with dissipation balanced by the non-zero boundary condition. If damping is small enough, then the steady-state dispersive shock can be described by a modulated cnoidal wave solution (26) where λ_i are the functions of $\xi = x - Vt$ and satisfy the perturbed Whitham equations (24).

The stationary Whitham equations take simpler form if one notices that $s_1 = \lambda_1 + \lambda_2 + \lambda_3$ is their integral $s_1 = \text{const}$ provided $V = -2s_1$; under this supposition the equations can be cast to the form

$$\frac{d\lambda_i}{d\xi} = -\frac{\langle (2\lambda_i - s_1 - u)R \rangle}{8 \prod_{j \neq i} (\lambda_i - \lambda_j)}, \quad i = 1, 2, 3. \quad (64)$$

It is easy to check that these equations indeed have the integral $s_1 = \text{const}$ so that the above supposition is justified. Easy calculation shows that two other symmetric functions $s_2 = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3$ and $s_3 = \lambda_1\lambda_2\lambda_3$ (see (30)) satisfy the equations

$$\frac{ds_2}{d\xi} = \frac{1}{4}\langle R \rangle, \quad \frac{ds_3}{d\xi} = \frac{1}{8}[s_1\langle R \rangle + \langle uR \rangle]. \quad (65)$$

Thus, we have reduced the problem to the system of two ordinary differential equations for s_2 , s_3 and λ_i are considered as functions of s_2 , s_3 being the roots of the algebraic equation

$$\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0. \quad (66)$$

Especially simple and practically important situation realizes when $\langle R \rangle = 0$, hence we have the second integral $s_2 = \text{const}$ and the problem reduces to a single differential equation

$$\frac{ds_3}{d\xi} = \frac{1}{8}\langle uR \rangle. \quad (67)$$

Instead of introduction of new dependent variable s_3 we can return in this case to Eq. (64) and consider, say, λ_1 and λ_2 as functions of λ_3 where $\lambda_3 = \lambda_3(\xi)$. Then we get

$$\frac{d\lambda_1}{d\lambda_3} = \frac{\lambda_3 - \lambda_2}{\lambda_2 - \lambda_1}, \quad \frac{d\lambda_2}{d\lambda_3} = -\frac{\lambda_3 - \lambda_1}{\lambda_2 - \lambda_1}, \quad (68)$$

and this system has, as we know, two integrals

$$\begin{aligned} \lambda_1 + \lambda_2 + \lambda_3 &= s_1 = \text{const}, \\ \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 &= s_2 = \text{const}. \end{aligned} \quad (69)$$

Consequently, λ_1 and λ_2 as functions of λ_3 are two roots of the quadratic equation

$$\lambda^2 - (s_1 - \lambda_3)\lambda + s_2 - (s_1 - \lambda_3)\lambda_3 = 0. \quad (70)$$

They are ordered according to the inequality $\lambda_1 \leq \lambda_2$ and the integration constants s_1 , s_2 can be found from the boundary conditions. Substitution of expressions for $\lambda_1 = \lambda_1(\lambda_3)$ and $\lambda_2 = \lambda_2(\lambda_3)$

into differential equation for $\lambda_3(\xi)$ (see (64)) with known $\langle uR \rangle$ solves in principle the problem. However, some important consequences can be obtained without integration of this differential equation. Let us illustrate this by application of the derived formulae to the steady-state dispersive shock wave evolved from the initial step-like distribution (56).

The initial distribution (56) suggests that at $x \rightarrow -\infty$ the shock has a form of a small-amplitude wave with $m \rightarrow 0$, $\lambda_2 \rightarrow \lambda_3$, so that here according to (26) we have $\lambda_1 = \lambda_1^- = -u_-$ and the integrals take the form

$$s_1 = -u_- + 2\lambda_3^-, \quad s_2 = -2u_- \lambda_3^- + (\lambda_3^-)^2. \quad (71)$$

At the soliton edge $m \rightarrow 1$, $\lambda_2 \rightarrow \lambda_1$ the wave (26) reduces to

$$u(x, t) = -\lambda_3 + \frac{2(\lambda_3 - \lambda_1)}{\cosh^2[\sqrt{\lambda_3 - \lambda_1}(x - Vt)]}$$

and since from (56) we have the boundary condition $u(x, t) \rightarrow 0$ as $x \rightarrow \infty$, we get here

$$\lambda_1^+ = \lambda_2^+, \quad \lambda_3^+ = 0. \quad (72)$$

Then the leading soliton at the soliton edge is described by the equation

$$u(x, t) = \frac{-2\lambda_1}{\cosh^2[\sqrt{-\lambda_1}(x + 4\lambda_1 t)]}. \quad (73)$$

Now we substitute the boundary conditions (72) into Eq. (70) to obtain the relation

$$s_1^2 - 4s_2 = 0 \quad (74)$$

between the integration constants which must be fulfilled along the whole dispersive shock wave. Its application to Eq. (71) yields at the small amplitude edge $\lambda_1^- = -u_-$, $\lambda_2^- = \lambda_3^- = -u_-/4$ and hence

$$s_1 = -\frac{3}{2}u_-, \quad s_2 = \frac{9}{16}u_-^2. \quad (75)$$

Then Eq. (70) gives at the soliton edge where $\lambda_3^+ = 0$ the value of a double root $\lambda_1^+ = \lambda_2^+ = -3u_-/4$ and as a result we find the expressions for the soliton amplitude $a_s = -2\lambda_1$ and the velocity $V = -2s_1$ of the shock wave in terms of a given value of the initial step amplitude,

$$a_s = \frac{3}{2}u_-, \quad V = 3u_-. \quad (76)$$

These expressions were obtained long ago by Johnson [23] for a particular case of Burgers dissipation $R = \epsilon u_{xx}$ in framework of direct perturbation technique. As we see, they can be reproduced quite easily by the Whitham method under more general assumptions about the form of dissipative perturbation of the KdV equation. It is remarkable that under certain conditions not only the velocity $V = 3u_-$ of the shock does not depend on the details of irreversible processes but also the amplitude of the leading soliton has a 'universal' value $a_s = (3/2)u_-$.

4. Conclusion

During past fifty years the Whitham theory has developed into a vast branch of applied mathematics with various applications to real physical processes related with nonlinear wave propagation. In spite of such a progress, as we have demonstrated in this paper, the original Whitham approach remains quite effective. Here we have shown its fruitfulness for a perturbed KdV equation, however it is clear that it is not confined to this single application. In fact, the main difficulty in the original direct Whitham's approach was

the problem of finding the Riemann invariants and Whitham found these Riemann invariants for the KdV equation case due to clever insight and skillful calculations. Now, due to discovered after publication of Whitham's paper relationship between the Whitham theory and the finite-gap integration method of completely integrable equations, the Riemann invariants have been found for many equations which belong to the AKNS scheme [28] (see, e.g., [38]). Therefore we can use the averaged conservation laws in any parameterization with account of perturbation terms and after that transform these equations to the known Riemann invariants of the unperturbed system arriving at perturbed Whitham equations. Thus, one may hope that the direct Whitham method can find in future many interesting and important applications.

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