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Physica D 237 (2008) 1006–1012
PHYSICA D
www.elsevier.com/locate/physd

Network synchronizability analysis: The theory of subgraphs and complementary graphs☆

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Received 29 October 2007; received in revised form 6 December 2007; accepted 11 December 2007
Available online 17 December 2007

Communicated by A. Pikovsky

Abstract

In this paper, subgraphs and complementary graphs are used to analyze network synchronizability. Some sharp and attainable bounds are derived for the eigenratio of the network structural matrix, which characterizes the network synchronizability, especially when the network's corresponding graph has cycles, chains, bipartite graphs or product graphs as its subgraphs.
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Keywords: Complex network; Subgraph; Complementary graph; Synchronizability

1. Introduction

Synchronization of complex networks has attracted increasing attention from many scientists, for its important applications in various areas of physical and biological sciences [3,4,6,9,10,26,28]. Synchronization is a ubiquitous phenomenon especially in social and biological networks where, quite often, it is desirable, e.g., in the consensus of multi-agent activities, while in some other cases it is undesirable, e.g., in traffic congestion [6,18,20,21,24,25,27]. The connection structure of a network plays an important role in its synchronization. Of particular interest in this concern is how the synchronizability depends on various structural parameters of the network, such as average distance, clustering coefficient, coupling strength, degree distribution and weight distribution, among others. Some interesting results have been established for such important problems based on the notions of master stability function and synchronized region [3,5,12–14,18,23,31]. Some relationships between synchronizability and structural parameters of complex networks have also been reported, e.g., smaller average network distance does not necessarily mean better synchronizability [21], where it was pointed out that good synchronizability requires a homogeneous degree distribution in [21]. Furthermore, the betweenness centrality was proposed as an indicator for synchronizability [21,10], and two networks with the same degree sequence can have different synchronizabilities [29]. Also, some optimizing rules were established for the network synchronizability [6]. Moreover, how the network statistical properties affect the spectrum of the structural matrix was analyzed and reported in [2]. And the effects of graph operations on the network synchronizability were studied in [1]. Recently, it was shown that the expectation of the largest eigenvalue is well approximated by the lower bound $d_{\max} + 1$ in random scale-free networks [11], and different bounds for the eigenvalues of the Laplacian matrix were established [17]. Last but not least, complementary graphs were used to analyze the network synchronizability in [7]. Motivated by all these research works, this paper attempts to further establish some useful sharp upper bounds for the network synchronizability based on the theory of subgraphs and complementary graphs.

☆ This work was supported by the National Natural Science Foundation of China under grant 60674093, the Key Projects of Educational Ministry under grant 107110, and the City University of Hong Kong under the Research Enhancement Scheme and SRG grant 7002134.
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doi:10.1016/j.physd.2007.12.003

Consider a dynamical network consisting of N coupled identical nodes, with each node being an n-dimensional dynamical system, described by

$$\dot{x}_i = f(x_i) - c\sum_{j=1}^{N} a_{ij}H(x_j), \quad i = 1, 2, \ldots, N, \tag{1}$$

where $x_i = (x_{i1}, x_{i2}, \ldots, x_{in}) \in \mathbb{R}^n$ is the state vector of node i, $f(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth vector-valued function, constant $c > 0$ represents the coupling strength, $H(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is called the inner linking function, and $A = (a_{ij})_{N\times N}$ is called the outer coupling matrix or structural matrix, which represents the coupling configuration of the entire network. This paper only considers the case where the network is diffusive, i.e., the entries of A satisfy $a_{ii} = -\sum_{j=1, j\neq i}^{N} a_{ij}$, $i = 1, 2, \ldots, N$. Further, suppose that, if there is an edge between node i and node j, then $a_{ij} = a_{ji} = -1$, i.e., A is a Laplacian matrix corresponding to network (1). Let all the eigenvalues of A be denoted by

$$0 = \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_N = \lambda_{\max}. \tag{2}$$

It is well known in graph theory that the graph corresponding to A is connected if and only if $\lambda_2 > 0$. For a connected dynamical network (1), it is said to achieve (asymptotical) synchronization if $x_1(t) \to x_2(t) \to \cdots \to x_N(t) \to s(t)$, as $t \to \infty$, where, because of the diffusive coupling configuration, the synchronous state $s(t) \in \mathbb{R}^n$ is a solution of an individual node, i.e., $\dot{s}(t) = f(s(t))$.

It is also well known [3] that for connected networks the eigenratio $r(A) = \frac{\lambda_2}{\lambda_N}$ of the network structural matrix A characterizes the network synchronizability: the larger $r(A)$, the better the synchronizability. In this paper, $r(A)$ will be used as a synchronizability index for the networks with structural matrix A. The relationships between $r(A)$ and network structural characteristics such as average distance, node betweenness, degree distribution, clustering coefficient, etc., have been studied [7,2,29,30]. A well-known upper bound for the eigenratio $r(A)$ in graph theory is $r(A) \leq \frac{d_{\min}}{d_{\max}}$, where $d_{\min}$ and $d_{\max}$ denote the smallest and largest degrees of the graph corresponding to the network [29]. To the best of our knowledge, other than this coarse bound, there are no sharp upper bounds available in the literature today. From both theoretical and practical points of view, some sharp upper bounds for $r(A)$ are derived in this paper by using graph-theoretical methods.

The rest of this paper is organized as follows. In Section 2, some necessary preliminaries in graph theory are provided. In Section 3, for a given graph G, some upper bounds are derived for the eigenratio $r(G)$ when G has cycles as its subgraphs. In Section 4, some lower bounds for the largest eigenvalue of graph G are presented when G has other subgraphs, such as chains, bipartite graphs and product graphs. The conclusions are presented in Section 5.

Throughout this paper, for any given undirected graph G, eigenvalues of G mean eigenvalues of its corresponding Laplacian matrix. Notations for graphs and their corresponding Laplacian matrices are not differentiated, and networks and their corresponding graphs are not distinguished, unless otherwise indicated.

2. Preliminaries

For a given graph G, let $\mathcal{V}$ and $\mathcal{E}$ denote the sets of nodes and edges of G, respectively. A graph G_1 is called an induced subgraph of G, if the node set $\mathcal{V}_1$ of G_1 is a subset of $\mathcal{V}$ and the edges of G_1 are all edges among nodes $\mathcal{V}_1$ in $\mathcal{E}$. In general, if a node g belongs to graphs G and G_1 simultaneously, the degree of g is its degree in G if it is considered as a node of G; otherwise, the degree of g is its degree in G_1 if it is regarded as a node of G_1. The complementary graph of G, denoted by G^c, is the graph containing all the nodes of G and all the edges that are not in G. In this paper, subgraphs and complementary graphs are used to discuss the network synchronizability. For this purpose, the following lemmas are needed.

Lemma 1 (*[15,16]*). *For any given graph G of size N, its nonzero eigenvalues indexed as in (2) grow monotonically with the number of added edges; that is, for any added edge e, $\lambda_i(G + e) \geq \lambda_i(G)$, $i = 1, \ldots, N$.*

Lemma 2 (*[15]*). *For any given connected graph G of size N, its largest eigenvalue λ_N satisfies $\lambda_N \geq d_{\max} + 1$, with equality if and only if $d_{\max} = N - 1$, where $d_{\max}$ is the maximum degree of G.*

Lemma 3 (*[8,29]*). *For any cycle C_N at N (≥ 4) nodes, its eigenvalues are given by $\mu_1, \ldots, \mu_N$ (not necessarily ordered) with $\mu_1 = 0$ and*

$$\mu_{k+1} = 3 - \frac{\sin\left(\frac{3k\pi}{N}\right)}{\sin\left(\frac{k\pi}{N}\right)}, \quad k = 1, \ldots, N-1.$$

From Lemma 3, it follows that, for any cycle C_N at N (≥ 4) nodes, if N is even, its largest eigenvalue is 4; if N is odd, its largest eigenvalue is

$$\lambda_N(C_N) = 3 - \frac{\sin\left(\frac{3(N-1)\pi}{2N}\right)}{\sin\left(\frac{(N-1)\pi}{2N}\right)}.$$

Obviously, for cycles with odd numbers of nodes, the largest eigenvalue converges to 4 as the number of the edges tends to $+\infty$. For eigenvalues of graphs and complementary graphs, the following lemma is useful (see [15,16] and references therein).

Lemma 4. *For any given graph G of size N, the following statements hold:*

(i) *$\lambda_N(G)$, the largest eigenvalue of G, satisfies $\lambda_N(G) \leq N$.*
(ii) *$\lambda_N(G) = N$ if and only if G^c is disconnected.*
(iii) *If G^c is disconnected and has (exactly) q connected components, then the multiplicity of $\lambda_N(G) = N$ is $q - 1$.*
(iv) *$\lambda_i(G^c) + \lambda_{N-i+2}(G) = N$, $2 \leq i \leq N$.*

For a given graph, generally its largest eigenvalue is easier to compute than the smallest nonzero eigenvalue. Hence, by Lemma 4, one can obtain the smallest nonzero eigenvalue by computing the largest eigenvalue of its complementary graph.

Corollary 1. *For any given graph G of size N, if its second smallest eigenvalue equals its smallest node degree, i.e., $\lambda_2(G) = d_{\min}(G)$, then either G or G^c is disconnected; if $\lambda_2(G) > d_{\min}(G)$, then G is a complete graph; if both G and G^c are connected, then $\lambda_2(G) < d_{\min}(G)$.*

Proof. If G^c is connected and $\lambda_2(G) = d_{\min}(G)$, then $\lambda_{\max}(G^c) = N - \lambda_2(G) = N - d_{\min}(G) = d_{\max}(G^c) + 1$. By Lemma 2, $d_{\max}(G^c) = N - 1$, so G is disconnected. Further, if $\lambda_2(G) > d_{\min}(G)$, by Lemma 4, $\lambda_N(G^c) < d_{\max} + 1$. Combining these with Lemma 2, G^c can only have isolated nodes, i.e., G is a complete graph. The third statement holds obviously. □

Combining Lemmas 1 and 2 with Corollary 1, one can easily get the following result.

Corollary 2. *For any given connected graph G and any of its induced subgraphs G_1, one has $\lambda_{\max}(G) \geq \lambda_{\max}(G_1)$, so the synchronizability index of G satisfies $r(G) \leq \frac{d_{\min}(G)}{\lambda_{\max}(G_1)}$; if both G and G^c are connected, then $r(G) < \frac{d_{\min}}{d_{\max}+1}$.*

Since subgraphs have fewer nodes, this corollary is useful when a graph G contains some canonical graphs as subgraphs whose largest eigenvalues can be easily obtained (see the section below for further discussion).

Corollary 3. *For a given graph G, if the largest eigenvalue of G^c is $\lambda_{\max} = d_{\max}(G^c) + \alpha$, then $\lambda_2(G) = d_{\min}(G) + 1 - \alpha$. Consequently, the synchronizability index of G satisfies*

$$r(G) = \frac{d_{\min}(G) + 1 - \alpha}{\lambda_{\max}(G)} \leq \frac{d_{\min}(G) + 1 - \alpha}{d_{\max}(G) + 1}.$$

By Lemma 2, generally $\alpha \geq 1$, so the bound in Corollary 3 is better than the one in Corollary 2.

3. Graphs having cycles as subgraphs

In this section, an even cycle means a cycle with an even number of nodes and an odd cycle means a cycle with an odd number of nodes. On the basis of the discussion in the above section, one can get the following results.

Lemma 5. *For any even cycle, -2 is an eigenvalue of its adjacency matrix. For any odd cycle with n_1 nodes, $-1 + \frac{\sin(\frac{3(n_1-1)\pi}{2n_1})}{\sin(\frac{(n_1-1)\pi}{2n_1})}$ is an eigenvalue of its adjacency matrix.*

Proof. For any even cycle C_{2n}, let $A(C_{2n})$ be its adjacency matrix. Obviously, the Laplacian matrix L of C_{2n} can be written as

$$L = 2I_{2n} - A(C_{2n}),$$

since every node degree of C_{2n} is 2, where I_{2n} denotes the identity matrix of order $2n$. Further, because the largest eigenvalue of C_{2n} is 4 (Lemma 3), one has

$$\det(4I_{2n} - L) = \det(2I_{2n} + A(C_{2n})) = 0.$$

Hence, -2 is an eigenvalue of $A(C_{2n})$. The statement for odd cycles can be similarly verified by Lemma 3. □

Theorem 1. *For any given graph G, suppose G_1 is its induced subgraph composed of all nodes of G with maximum degree $d_{\max}(G)$, and $\mathcal{G}_1$ is the induced subgraph of G^c composed of all nodes of G^c with maximum degree $d_{\max}(G^c)$. Then, the following statements hold:*

(i) *If both G_1 and $\mathcal{G}_1$ have even cycles as induced subgraphs, then $\lambda_{\max}(G) \geq d_{\max}(G)+2$ and $\lambda_{\max}(G^c) \geq d_{\max}(G^c)+2$. Consequently, the synchronizability index of G satisfies $r(G) \leq \frac{d_{\min}(G)-1}{d_{\max}(G)+2}$.*

(ii) *If both G_1 and $\mathcal{G}_1$ have odd cycles as induced subgraphs, and if the longest odd cycle of G_1 has n_1 nodes and the longest odd cycle of $\mathcal{G}_1$ has n_2 nodes, then*

$$\lambda_{\max}(G) \geq d_{\max}(G) + 1 - \frac{\sin\left(\frac{3(n_1-1)\pi}{2n_1}\right)}{\sin\left(\frac{(n_1-1)\pi}{2n_1}\right)}$$

and

$$\lambda_{\max}(G^c) \geq d_{\max}(G^c) + 1 - \frac{\sin\left(\frac{3(n_2-1)\pi}{2n_2}\right)}{\sin\left(\frac{(n_2-1)\pi}{2n_2}\right)}.$$

Consequently, the synchronizability index of G satisfies

$$r(G) \leq \frac{d_{\min}(G) + \frac{\sin\left(\frac{3(n_2-1)\pi}{2n_2}\right)}{\sin\left(\frac{(n_2-1)\pi}{2n_2}\right)}}{d_{\max}(G) + 1 - \frac{\sin\left(\frac{3(n_1-1)\pi}{2n_1}\right)}{\sin\left(\frac{(n_1-1)\pi}{2n_1}\right)}}.$$

(iii) *If G_1 has an even cycle as an induced subgraph, and $\mathcal{G}_1$ has odd cycles as induced subgraphs with the longest odd cycle having n_2 nodes, then $\lambda_{\max}(G) \geq d_{\max}(G) + 2$ and*

$$\lambda_{\max}(G^c) \geq d_{\max}(G^c) + 1 - \frac{\sin\left(\frac{3(n_2-1)\pi}{2n_2}\right)}{\sin\left(\frac{(n_2-1)\pi}{2n_2}\right)}.$$

Consequently, the synchronizability index of G satisfies

$$r(G) \leq \frac{d_{\min}(G) + \frac{\sin\left(\frac{3(n_2-1)\pi}{2n_2}\right)}{\sin\left(\frac{(n_2-1)\pi}{2n_2}\right)}}{d_{\max}(G) + 2}.$$

(iv) *If G_1 has odd cycles as induced subgraphs with the longest odd cycle having n_1 nodes and $\mathcal{G}_1$ has an even cycle as an induced subgraph, then $\lambda_{\max}(G) \geq d_{\max}(G) + 1 - \frac{\sin(\frac{3(n_1-1)\pi}{2n_1})}{\sin(\frac{(n_1-1)\pi}{2n_1})}$ and $\lambda_{\max}(G^c) \geq d_{\max}(G^c) + 2$. Consequently, the synchronizability index of G satisfies*

$$r(G) \leq \frac{d_{\min}(G) - 1}{d_{\max}(G) + 1 - \frac{\sin\left(\frac{3(n_1-1)\pi}{2n_1}\right)}{\sin\left(\frac{(n_1-1)\pi}{2n_1}\right)}}.$$

Proof. (i) By Lemmas 3 and 5, for any even cycle, its largest eigenvalue is 4 and -2 is an eigenvalue of its adjacency matrix. Let L_1 be the sub-matrix of the Laplacian matrix of G related to all the nodes in G_1, and $A(G_1)$ be the adjacency matrix of G_1.

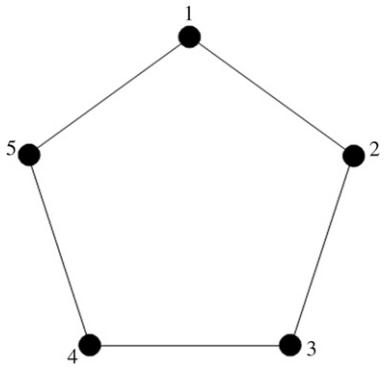


Fig. 1. Graph C_5.

Then, $A(G_1)$ contains a sub-matrix which is the adjacency matrix of the corresponding even cycle. Therefore, one has

$$(d_{\max}(G)+2)I - L_1 = 2I + A(G_1) \not\succ 0.$$

In fact, the sub-matrix of $2I + A(G_1)$ corresponding to the even cycle has a zero eigenvalue. This implies that the largest eigenvalue of G_1 is larger than or equal to $d_{\max}+2$. Then, by Lemma 1, $\lambda_{\max}(G) \geq d_{\max}+2$. Similarly, $\lambda_{\max}(G^c) \geq d_{\max}+2$.

Further, suppose the number of nodes of G is N. Then, the smallest nonzero eigenvalue of G is $\lambda_2 = N - \lambda_{\max}(G^c)$ and the minimum node degree of G is $d_{\min}(G) = N-1-d_{\max}(G^c)$. If $\lambda_{\max}(G^c) \geq d_{\max}(G^c)+2$, then $\lambda_2 \leq N - d_{\max}(G^c) - 2 = d_{\min}(G) - 1$. Therefore, Theorem 1 (i) obviously holds.

(ii) follows easily from (i) and Lemma 5.

(iii) and (iv) can be similarly proved. □

Remark 1. For a given graph G, clearly, the bounds given in Theorem 1 for the synchronizability index $r(G)$ are better than the general bound $r(G) \leq \frac{d_{\min}}{d_{\max}}$. And the existence of cycles can be easily tested by drawing graphs. Since the networks with good synchronizabilities always have homogeneous degree distributions [6,10,21], Theorem 1 is especially useful for estimating the synchronizabilities of homogeneous networks.

Remark 2. The smallest even cycle is cycle C_4, and its complementary graph C_4^c has two separated edges. C_4 and C_4^c are very important in graph theory [16]. A graph has C_4 as an induced subgraph if and only if G^c has C_4^c as an induced subgraph.

Remark 3. The cycle C_3 is the smallest cycle. If a graph G contains only C_3 cycles, its largest eigenvalue satisfies $\lambda_{\max}(G) \geq d_{\max}+1$.

Example 1. Consider cycle C_5 shown in Fig. 1. Its complementary graph is also C_5 (see Fig. 2). Testing the eigenvalues of C_5 and its synchronizability, one finds that they attain the exact bounds in Theorem 1(ii).

Example 2. Consider graph Γ_1 in Fig. 3. Its complementary graph is C_6 (see Fig. 4). Testing the eigenvalues of Γ_1 and its synchronizability, one finds that they attain the exact bounds given in Theorem 1(i).

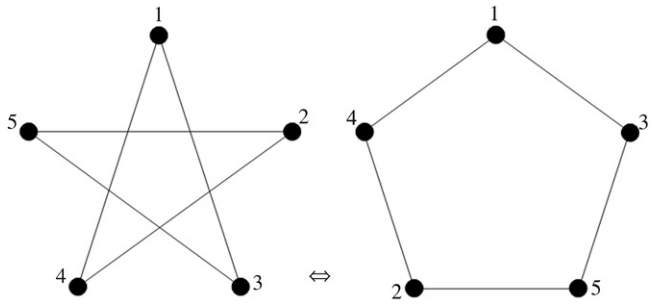


Fig. 2. Graph C_5^c.

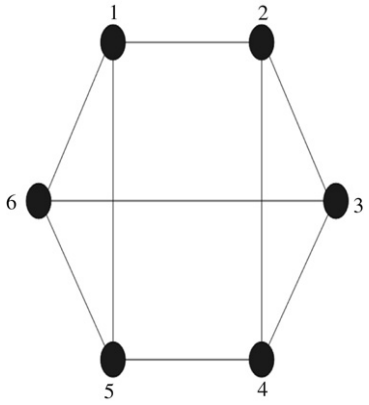


Fig. 3. Graph Γ_1.

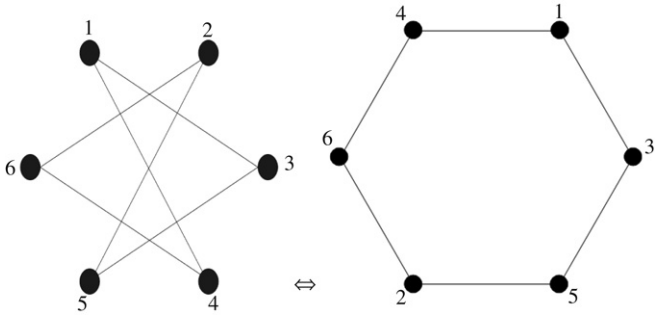


Fig. 4. Graph $\Gamma_1^c = C_6$.

4. Applications of other subgraphs

According to the discussion in the above section, cycles are very important for estimating the network synchronizability. In this section, consider the estimation of the largest eigenvalue of a given graph by its subgraphs. First, the following lemma for adjacency matrices is needed.

Lemma 6 (*[19]*). *Given a graph G, the smallest eigenvalue of its adjacency matrix $A(G)$ satisfies $\lambda_{\min}(A(G)) \leq d_{\max}(G) - \lambda_{\max}(G)$.*

By this lemma, one can get the following result.

Theorem 2. *For a given graph G, let H be a subgraph of G containing all nodes of G with the same node degree d. Let H_1 be a subgraph of H. Then, the largest eigenvalue of G satisfies $\lambda_{\max}(G) \geq d + \lambda_{\max}(H_1) - d_{\max}(H_1)$.*

Proof. It follows from Lemma 6 and the proof of Theorem 1. □

Remark 4. In the above section, it is known that cycles as subgraphs have the same node degrees. So the smallest eigenvalues of the adjacency matrices of cycles can be exactly computed. In Theorem 2, for a general subgraph H_1, its node degrees are not necessarily the same. In this case, Lemma 6 is very useful for estimating the largest eigenvalues.

In what follows, consider some canonical subgraphs which can be used to estimate the largest eigenvalues of the original graphs. All the results can be combined with the results in the above section to obtain upper bounds for the network synchronizability.

4.1. Graphs having chains as subgraphs

A chain is a nearest-neighbor coupled graph without an edge between the first and the last nodes. For a given chain P_N with N nodes, its largest and smallest nonzero eigenvalues are $2(1+\cos(\frac{\pi}{N}))$ and $2(1-\cos(\frac{\pi}{N}))$, respectively [8,15,29]. Obviously, the largest eigenvalue of a chain converges to 4 as the number of edges tends to $+\infty$. According to the above discussion, one can establish the following result.

Theorem 3. *For a given graph G, let G_1 be a subgraph of G containing all nodes of G with the maximum degree. Suppose G_1 contains a chain P_k with k nodes as its subgraph. Then, the largest eigenvalue of G satisfies $\lambda_{\max}(G) \geq d_{\max} + 2\cos(\frac{\pi}{k})$.*

Proof. Theorems 1 and 2 directly lead to the conclusion. □

Note that chain P_4 is a very important graph in graph theory [16]. The complementary graph of P_4 is still a P_4 graph. Therefore, a graph G has P_4 as a subgraph if and only if G^c has P_4 as a subgraph.

4.2. Graphs having bipartite graphs as subgraphs

Let $G_1(\mathcal{V}_1, \mathcal{E}_1)$ and $G_2(\mathcal{V}_2, \mathcal{E}_2)$ be two graphs on disjoint sets of n_1 and n_2 nodes, respectively, where $\mathcal{V}_i$ and $\mathcal{E}_i$ are the corresponding sets of nodes and edges. Their disjoint union $G_1 + G_2$ is the graph $G_1 + G_2 = (\mathcal{V}_1 \bigcup \mathcal{V}_2, \mathcal{E}_1 \bigcup \mathcal{E}_2)$. Then, a bipartite graph $G_1 * G_2$ generated by G_1 and G_2 is a new graph composed of $G_1 + G_2$ and the new edges connecting each node of G_1 to every node of G_2, as detailed in [1,16,22]. It is well known that the largest eigenvalue of the bipartite graph $G_1 * G_2$ is $\lambda_{\max}(G_1 * G_2) = n_1 + n_2$, i.e., the sum of the numbers of nodes of G_1 and G_2. In fact, the complementary graphs of bipartite graphs are always disconnected.

By Lemma 6, for a bipartite graph $G_1 * G_2$, as discussed above, one has $\lambda_{\min}(A(G_1 * G_2)) \leq d_{\max}(G_1 * G_2) - n_1 - n_2$.

Theorem 4. *For a given graph G, let H be a subgraph of G containing all nodes of G with the same degree d. Suppose H contains a bipartite subgraph $H_1 * H_2$, and the numbers of nodes of H_1 and H_2 are n_1 and n_2, respectively. Then, the largest eigenvalue of G satisfies $\lambda_{\max}(G) \geq d + n_1 + n_2 - d_{\max}(H_1 * H_2)$.*

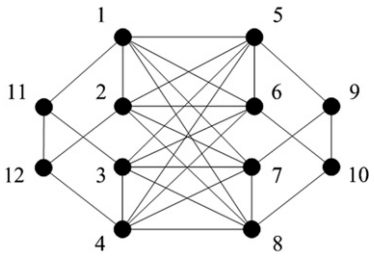


Fig. 5. Graph Γ_2.

Proof. This is a direct result of Theorem 2 for bipartite subgraphs. □

Example 3. Consider graph Γ_2 shown in Fig. 5. Obviously, Γ_2 has a bipartite graph H as its subgraph which is composed of all nodes with degree 6 from Γ_2. The largest eigenvalue of this bipartite graph is 8. So, by Theorem 4, $\lambda_{\max}(\Gamma_2) \geq 9$. On the other hand, the maximum degree of the complementary graph Γ_2^c of Γ_2 is 8. And the nodes with degree 8 in Γ_2^c form a cycle C_4. By Theorem 1, the smallest nonzero eigenvalue of Γ_2 satisfies $\lambda_2(\Gamma_2) \leq d_{\min}(\Gamma_2) - 1 = 2$. So, $r(\Gamma_2) \leq \frac{2}{9}$. Simply computing the Laplacian eigenvalues of Γ_2, one has $\lambda_2 = 1.7251$ and $\lambda_{\max} = 9.2749$. Consequently, $r(\Gamma_2) = \frac{1.7251}{9.2749} \approx 0.176$. Therefore, the theorems in this paper successfully give the upper integer of the largest eigenvalue and the lower integer of the smallest nonzero eigenvalue of Γ_2.

4.3. Graphs having product graphs as subgraphs

Let $G_1(\mathcal{V}_1, \mathcal{E}_1)$ and $G_2(\mathcal{V}_2, \mathcal{E}_2)$ be two nonempty graphs. Their product graph $G_1 \times G_2$ is a graph with node set $\mathcal{V}_1 \times \mathcal{V}_2$, and $(x_1, x_2)(y_1, y_1)$ is an edge in $\mathcal{E}(G_1 \times G_2)$ if and only if either $x_2 = y_2$ with $x_1 y_1 \in \mathcal{E}(G_1)$ or $x_1 = y_1$ with $x_2 y_2 \in \mathcal{E}(G_2)$. One may view $G_1 \times G_2$ as the graph obtained from G_1 by replacing each of its nodes with a copy of G_2 and each of its edges with the number of nodes of G_2 edges joining the corresponding nodes of G_2 in two copies, as detailed in [1, 16].

It is well known that the largest eigenvalue of the product graph $G_1 \times G_2$ is $\lambda_{\max}(G_1 \times G_2) = \lambda_{\max}(G_1) + \lambda_{\max}(G_2)$.

Theorem 5. *For a given graph G, let H be a subgraph of G containing all nodes of G with the same node degree d. Suppose H contains a product graph $H_1 \times H_2$ as its subgraph. Then, the largest eigenvalue of G satisfies $\lambda_{\max}(G) \geq d + \lambda_{\max}(H_1) + \lambda_{\max}(H_2) - d_{\max}(H_1 \times H_2)$.*

Example 4. Consider graph Γ_3 shown in Fig. 6. Obviously, all nodes of Γ_3 have degree 4. And Γ_3 has a product graph $C_4 \times P_3$ (nodes 1 to 12) as its subgraph, where P_3 denotes a chain with three nodes. The largest eigenvalue of this product subgraph is 7. So, by Theorem 5 or Lemma 1, $\lambda_{\max}(\Gamma_3) \geq 7$. On the other hand, the complementary graph Γ_3^c of Γ_3 has a bipartite graph as its subgraph, which is composed of nodes 1 to 4 and nodes 9 to 12. By Theorem 4, like for Example 3, the largest eigenvalue of Γ_3^c satisfies $\lambda(\Gamma_3^c) \geq d_{\max}(\Gamma_3^c) + 3$. Thus, by Corollary 3, the smallest nonzero eigenvalue of Γ_3 satisfies

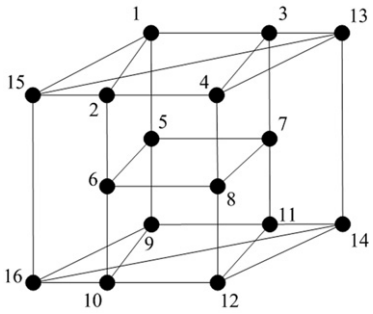


Fig. 6. Graph Γ_3.

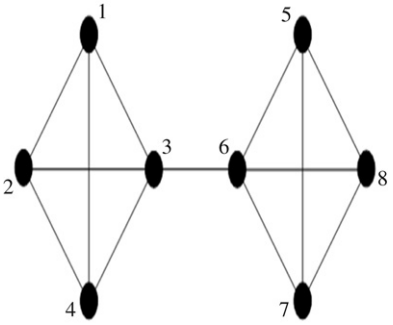


Fig. 7. Graph Γ_4.

$\lambda_2(\Gamma_3) \le d_{\min}(\Gamma_3) - 2 = 2$. So, $r(\Gamma_3) \le \frac{2}{7}$. Simply computing the Laplacian eigenvalues of Γ_3, one has $\lambda_2 = 1.2679$ and $\lambda_{\max} = 7.4142$. Consequently, $r(\Gamma_3) = \frac{1.2679}{7.4142} \approx 0.171$. Like for Example 3, the corresponding theorems proved in this paper successfully give the upper integer of the largest eigenvalue and the lower integer of the smallest nonzero eigenvalue of Γ_3.

4.4. The maximum disconnected subgraph

Given a graph G of size N, suppose G_1 is an induced subgraph of G of size n_1, and G_1 is disconnected. G_1 is called a maximum disconnected subgraph if the node number of any other disconnected subgraph of G is less than or equal to n_1. Note that if G is disconnected, its maximum disconnected subgraph is G itself.

Theorem 6. *For a given connected graph G of size N, if the node number of its maximum disconnected subgraph is n_1, then the smallest nonzero eigenvalue of G satisfies $\lambda_2 \le N - n_1$. Consequently, $r(G) \le \frac{N-n_1}{d_{\max}(G)+1}$.*

Proof. Suppose G_1 is a maximum disconnected subgraph of size n_1. Then, by Lemma 4, $\lambda_{\max}(G_1^c) \ge n_1$. Further, $\lambda_{\max}(G^c) \ge n_1$, so $\lambda_2(G) \le N - n_1$. □

Example 5. Consider graph Γ_4 in Fig. 7. By deleting node 3 or 6, one can verify that the node number of its maximum disconnected subgraph is 7. So, $\lambda_2(\Gamma_4) \le 1$. Combining Lemma 2 and Theorem 6, one has $r(\Gamma_4) < \frac{1}{5}$. Generally, graphs like those shown in Fig. 7 have large node and edge betweenness centralities, and therefore have low synchronizabilities. Theorem 6, based on the theory of subgraphs and complementary graphs, gives an explanation for why such graphs do indeed have low synchronizabilities.

Remark 5. From Examples 3–5, it can be seen that actually one does not need to draw out the complementary graphs; the properties of complementary graphs can be analyzed by using the original graphs directly.

5. Conclusion

Various networks can be represented and analyzed by using graphs. Without considering the dynamical characteristics of network nodes, a network can be viewed as a graph which shows all the structural properties of the underlying network. The basic algebraic graph theory is very useful for understanding network synchronization. For a given network, its corresponding graph as well as its complementary graph and subgraphs can be easily obtained. In this paper, therefore, some relatively tight upper bounds have been obtained for network synchronizability based on the theory of subgraphs and complementary graphs. It has also been shown that better understanding and helpful manipulation of subgraphs and complementary graphs are very useful for enhancing network synchronizability. The graph-theoretic approach provided in this paper is deemed important in the study of network synchronization problems.

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