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Bogoyavlensky–Volterra and Birkhoff integrable systems

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Abstract

In this paper we examine an interesting connection between the generalized Volterra lattices of Bogoyavlensky and a special case of an integrable system defined by Sklyanin. The Sklyanin system happens to be one of the cases in the classification of Kozlov and Treshchev of Birkhoff integrable Hamiltonian systems. Using this connection we demonstrate the integrability of the system and define a new Lax pair representation. In addition, we comment on the bi-Hamiltonian structure of the system.

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1. Introduction—Birkhoff integrable systems

In this paper we examine an interesting connection between the generalized Volterra lattices of Bogoyavlensky [6] and a special case of an integrable system defined by Sklyanin [27]. The Sklyanin system happens to be one of the systems in the classification of Kozlov and Treshchev of Birkhoff integrable Hamiltonian systems. Using this connection we are able to prove the integrability of the system and define a new Lax pair representation different from the one in [26]. In addition, we comment on the multi-Hamiltonian structure of the system. This connection was discovered in an effort to connect the Volterra D_n system with the corresponding Toda D_n system as in the case of the other classical Lie groups. In contrast to the other cases, the Volterra D_n system does not correspond to the Toda D_n system under the procedure of Moser but it actually corresponds to a special case of the Sklyanin system mentioned above. A Miura type transformation between the Volterra and Toda D_n systems is therefore still an open problem.

We begin with the following more general definition which involves systems with exponential interaction: consider a Hamiltonian of the form:

$$H = \frac{1}{2}(\mathbf{p}, \mathbf{p}) + \sum_{i=1}^{N} \mathrm{e}^{(\mathbf{v}_i, \mathbf{q})}, \tag{1}$$

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where $\mathbf{q} = (q_1, \dots, q_n)$, $\mathbf{p} = (p_1, \dots, p_n)$, $\mathbf{v}_1, \dots, \mathbf{v}_N$ are vectors in $\mathbf{R}^n$ and $(\, , \,)$ is the standard inner product in $\mathbf{R}^n$. The set of vectors $\Delta = \{\mathbf{v}_1, \dots, \mathbf{v}_N\}$ is called the spectrum of the system.

Let M be the $N \times N$ matrix whose elements are

$$M_{ij} = (\mathbf{v}_i, \mathbf{v}_j).$$

Hamilton's equations of motion can be transformed by a generalized Flaschka transformation to a polynomial system of $2N$ differential equations. The transformation is defined as follows:

$$a_i = -\mathrm{e}^{(\mathbf{v}_i, \mathbf{q})}, \qquad b_i = (\mathbf{v}_i, \mathbf{p}). \tag{2}$$

We end-up with a system of polynomial differential equations:

$$\dot{a}_k = a_k b_k, \qquad \dot{b}_k = \sum_{i=1}^{N} M_{ki} a_i. \tag{3}$$

Eq. (3) admits the following two integrals:

$$F_1 = \sum_{i=1}^{N} \lambda_i b_i, \qquad F_2 = \prod_{i=1}^{N} a_i^{\lambda_i} \tag{4}$$

provided that there exist constants λ_i such that $\sum_{i=1}^{N} \lambda_i \mathbf{v}_i = 0$. Such integrals always exist for $N > n$. One can define a canonical bracket on the space of variables (a_i, b_i) by the formula:

$$\{b_i, a_j\} = (\mathbf{v}_i, \mathbf{v}_j) a_j$$

and all other brackets equal to zero. The integrals F_1 and F_2 are Casimirs of system (3).

An interesting case of (1) occurs when the spectrum is a system of simple roots for a simple Lie algebra $\mathcal{G}$. In this case $N = l = \operatorname{rank} \mathcal{G}$. It is worth mentioning that the case where N, n are arbitrary is an open and unexplored area of research. The main exception is the work of Kozlov and Treshchev [19] where a classification of system (1) is performed under the assumption that the system possesses n polynomial (in the momenta) integrals. We also note the papers by Ranada [26], Annamalai and Tamizhmani [3] and Emelyanov [10]. Such systems are called Birkhoff integrable. For each Hamiltonian in (1) we associate a Dynkin type diagram as follows: it is a graph whose vertices correspond to the elements of Δ. Each pair of vertices $\mathbf{v}_i$, $\mathbf{v}_j$ are connected by

$$\frac{4(\mathbf{v}_i, \mathbf{v}_j)^2}{(\mathbf{v}_i, \mathbf{v}_i)(\mathbf{v}_j, \mathbf{v}_j)}$$

edges.

Example. The classical Toda lattice corresponds to a Lie algebra of type A_{n-1}. In other words $N = l = n - 1$ and we choose Δ to be the set:

$$\mathbf{v}_1 = (1, -1, 0, \dots, 0), \dots, \mathbf{v}_{n-1} = (0, 0, \dots, 0, 1, -1).$$

The graph is the usual Dynkin diagram of a Lie algebra of type A_{n-1}. The Hamiltonian becomes

$$H(q_1, \dots, q_n, p_1, \dots, p_n) = \sum_{i=1}^{n} \frac{1}{2} p_i^2 + \sum_{i=1}^{n-1} \mathrm{e}^{q_i - q_{i+1}}, \tag{5}$$

which is the well-known classical, non-periodic Toda lattice. This system was investigated in [12,13,15,21,23,28] and numerous other papers that are impossible to list here. This type of Hamiltonian was discovered by Toda [28].

The original Toda lattice can be viewed as a discrete version of the Korteweg–de Vries equation. It is called a lattice as in atomic lattice since interatomic interaction was studied. This system appears also in Cosmology, in the work of Seiberg and Witten on supersymmetric Yang–Mills theories and it has applications in analog computing and numerical computation of eigenvalues. But the Toda lattice is mainly a theoretical mathematical model which is important due to the rich mathematical structure encoded in it. The Toda lattice is integrable in the sense of Liouville. There exist n independent integrals of motion in involution. These integrals are polynomial in the momenta.

As we mentioned earlier, the Toda lattice was generalized to the case where the spectrum corresponds to a root space of an arbitrary simple Lie group. These systems generalize the usual finite, non-periodic Toda lattice (which corresponds to a root system of type A_n). This generalization is due to Bogoyavlensky [4]. These systems were studied extensively in [17] where the solution of the systems was connected intimately with the representation theory of simple Lie groups. There are also studies by Olshanetsky and Perelomov [24] and Adler and van Moerbeke [1].

It is more convenient to work, instead with the space of the natural (q, p) variables, with the Flaschka variables (a, b) which are defined by

$$a_i = \tfrac{1}{2}\mathrm{e}^{1/2(\mathbf{v}_i,\mathbf{q})}, \quad i = 1, 2, \dots, N, \qquad b_i = -\tfrac{1}{2}p_i, \quad i = 1, 2, \dots, n. \tag{6}$$

We end-up with a new set of polynomial equations in the variables (a, b). One can write the equations in Lax pair form, see, for example, [25]. The Lax pair $(L(t), B(t))$ in $\mathcal{G}$ can be described in terms of the root system as follows:

$$L(t) = \sum_{i=1}^{l} b_i(t)h_{\alpha_i} + \sum_{i=1}^{l} a_i(t)(e_{\alpha_i} + e_{-\alpha_i}), \qquad B(t) = \sum_{i=1}^{l} a_i(t)(e_{\alpha_i} - e_{-\alpha_i}).$$

As usual h_{α_i} is an element of a fixed Cartan subalgebra and e_{α_i} is a root vector corresponding to the simple root α_i. The Chevalley invariants of $\mathcal{G}$ provide for the constants of motion.

The first important result in the search for integrable cases of system (1) is due to Adler and van Moerbeke [2]. They considered the special case where the number of elements in the spectrum Δ is $n + 1$ (i.e., $N = n + 1$). Furthermore, they made the assumption that any n vectors in the spectrum are independent. Under these conditions a criterion for algebraic integrability is that

$$\frac{2(\mathbf{v}_i, \mathbf{v}_j)}{(\mathbf{v}_i, \mathbf{v}_i)} \tag{7}$$

should be in the set $\{0, -1, -2, \dots\}$. The classification obtained corresponds to the simple roots of graded Kac–Moody algebras. The associated systems are the periodic Toda lattices of Bogoyavlensky [4]. The complete integrability of these systems using Lax pairs with a spectral parameter was already established in [1]. The method of proof in [2] is based on the classical method of Kovalevskaya.

Sklyanin [27] pointed out another integrable generalization of the Toda lattice:

$$H(q_1, \dots, q_n, p_1, \dots, p_n) = \sum_{i=1}^{n} \frac{1}{2}p_i^2 + \sum_{i=1}^{n-1} \mathrm{e}^{q_i - q_{i+1}} + \alpha_1\,\mathrm{e}^{q_1} + \beta_1\,\mathrm{e}^{2q_1} + \alpha_n\,\mathrm{e}^{-q_n} + \beta_n\,\mathrm{e}^{-2q_n}. \tag{8}$$

He obtained this system by means of the quantum inverse scattering R-matrix method. This potential will be the main focus of this paper.

The next development in the study of system (1) is the work of Kozlov and Treshchev on Birkhoff integrable systems. A system of the form (1) is called Birkhoff integrable if it has n integrals, polynomial in the momenta with coefficients of the form:

$$\sum f_j\,\mathrm{e}^{\langle \mathbf{c}_j, \mathbf{q}\rangle}, \quad f_j \in \mathbf{R}, \quad \mathbf{c}_j \in \mathbf{R}^n,$$

whose leading homogeneous forms are almost everywhere independent. We remark that in the definition given in the book of Kozlov [20] there is no assumption on involutivity of the integrals. In [19] it is proved that the polynomial integrals are in involution. The terminology has its origin in the work of Birkhoff who studied the conditions for the existence of linear and quadratic integrals of general Hamiltonians in two degrees of freedom. A vector in Δ is called maximal if it has the greatest possible length among all the vectors in the spectrum having the same direction. Kozlov and Treshchev proved the following theorem:

Theorem 1. *Assume that the Hamiltonian (1) is Birkhoff integrable. Let $\mathbf{v}_i$ be a maximal vector in Δ and assume that the vector $\mathbf{v}_j \in \Delta$ is linearly independent of $\mathbf{v}_i$. Then*:

$$\frac{2(\mathbf{v}_i, \mathbf{v}_j)}{(\mathbf{v}_i, \mathbf{v}_i)}$$

lies in the set $\{0, -1, -2, \dots\}$.

Note that the condition of the theorem is exactly the same as condition (7) of Adler and van Moerbeke. Of course theorem 1 is more general since there is no restriction on the integer N (the number of summands in the potential of (1)). It turns out, however, that N cannot be much bigger than n. In fact, it follows from the classification that $N \leq n + 3$. A system of the form (1) is called complete if there exist no vector $\mathbf{v}$ such that the set $\Delta \cup \{\mathbf{v}\}$ satisfies the assumptions of Theorem 1. In [19] there is a complete classification of all possible Birkhoff integrable systems based on Theorem 1. The Dynkin type diagram of a complete, irreducible, Birkhoff integrable Hamiltonian system is isomorphic to one of the diagrams shown in Fig. 1.

Remark 1. In the list of diagrams we have omitted some cases that occur as sub-graphs of diagrams (a)–(k) (by truncating one or more vertices). In other words, the spectrum of a Birkhoff integrable Hamiltonian system is obtained from the spectrum of a complete system by dropping part of the elements.

Remark 2. The Dynkin type diagram determines only the angles between pairs of vectors in Δ. In order to reconstruct the ratios of lengths of vectors in Δ we assign to the ith vertex a coefficient proportional to the square of the length of $\mathbf{v}_i$. This explains the numbers appearing on the vertices of the diagrams.

We have to stress that this classification gives only necessary conditions for a system of type (1) to be Birkhoff integrable. The integrability for each system in the list should be established case by case. We give a brief history of the progress in this direction. As we already mentioned, the integrability of systems (a)–(g) was established in [1,4]. The solution of these generalized periodic Toda lattices (associated with affine Lie algebras) was obtained by Goodman and Wallach in [14]. The graph (i) corresponds to a Hamiltonian system in two degrees of freedom with potential:

$$e^{q_1} + e^{q_2} + e^{-q_1-q_2} + e^{-((q_1+q_2)/2)}.$$

The additional integral can be found in [19]. The integrability or non-integrability of systems (j) and (k) is still open. No Lax pair is known for either system. It is believed that system (j) is completely integrable, in fact integrability is established in [10] for the case $n = 4$. It is generally believed that system (k) in non-integrable.

In this paper we deal with the integrability of system (h) which corresponds to the Hamiltonian (8). Sklyanin in [27] indicated this system as another integrable generalization of the Toda lattice. The case $n = 2$ corresponds to the potential:

$$V = e^{q_1-q_2} + c_1 e^{2q_2} + c_2 e^{q_2} + c_3 e^{-q_1} + c_4 e^{-2q_1}.$$

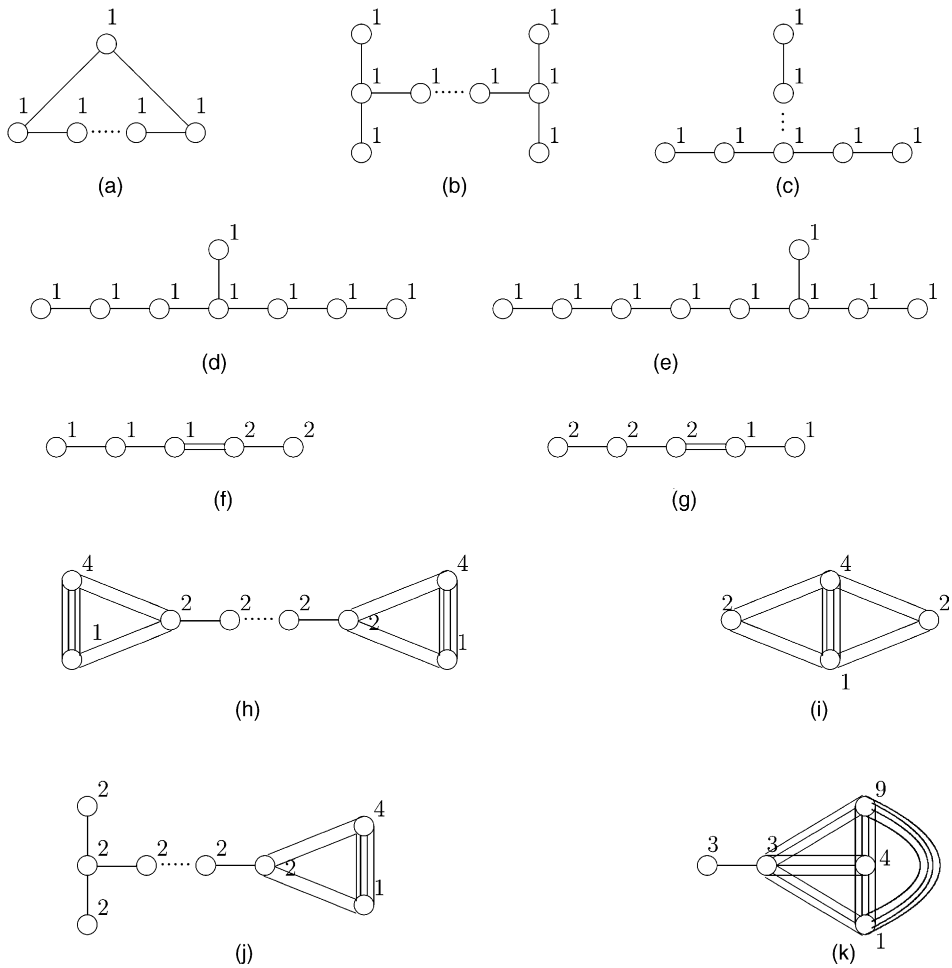


Fig. 1.

Annamalai and Tamizhmani [3] demonstrated the integrability of this particular case by using Noether's theorem. The second integral is of fourth degree in the momenta.

The case $n = 3$ (as well as the general case) is treated in Ranada [26]. Ranada proved integrability by using a Lax pair approach. The additional integrals are of degree 4 and 6.

In this paper we examine the integrability of the system:

$$H(q_1, \dots, q_n, p_1, \dots, p_n) = \sum_{i=1}^{n} \frac{1}{2} p_i^2 + \sum_{i=1}^{n-1} \mathrm{e}^{q_i - q_{i+1}} + \alpha\, \mathrm{e}^{-2q_1} + \beta\, \mathrm{e}^{2q_n}. \tag{9}$$

Without loss of generality we may assume that $\alpha = \beta = 1$. We note that if either α or β are zero then the system reduces to well-known cases of generalized Toda lattices. The technique we use is entirely new and hopefully it will lead to further results for similar systems. The strategy is the following: we consider the D_n Volterra system whose Lax pair formulation and multi-Hamiltonian structure was established recently in Kouzaris [18]. Using a procedure of Moser we transform the system into a new system in some intermediate variables (a_i, b_i):

$$\dot{a}_1 = 2a_1b_1, \qquad \dot{a}_i = a_i(b_i - b_{i-1}), \quad i = 2, 3, \dots, n, \qquad \dot{a}_{n+1} = -2a_{n+1}b_n,$$
$$\dot{b}_i = 2(a_{i+1}^2 - a_i^2), \quad i = 1, 2, \dots, n. \tag{10}$$

We obtain a Lax pair for this system and the integrals of motion. The final step is the construction of a Flaschka transformation from the Hamiltonian system (9) to the system (10). The inverse of Flaschka's transformation provides for the necessary constants of motion in the variables (q, p) for the system (9). Thus, integrability is established.

In Section 2 we describe the construction of Bogoyavlensky–Volterra systems following [5,6]. Bogoyavlensky constructed the systems using the root system of a simple Lie algebra and then through a change of variables (from c_i to u_i) he ended-up with homogeneous polynomial systems in the new variables u_i. The construction of Lax pairs in the variables u_i is in [18].

In Section 3 we present part of the results of [18], namely the D_n case, since it is the only system needed for the purposes of the present paper. We have to point out that in [18] there is a complete treatment of the multi-Hamiltonian structure, Lax pairs and integrability of Bogoyavlensky–Volterra lattices.

The results of Section 4 are entirely new. They can be summarized as two transformations, one Moser-type from the Volterra D_n lattice to system (10) and one Flaschka-type from the Sklyanin system (9) to system (10). Since Flaschka-type transformations are well-known, we describe briefly the Moser approach.

Consider the system:

$$\frac{\mathrm{d}u_i}{\mathrm{d}t} = u_i(u_{i+1} - u_{i-1}), \quad i = 1, \dots, n, \tag{11}$$

where $u_0 = u_{n+1} = 0$. This is the Volterra system, also known as the KM system and is related to the root system of a simple Lie algebra of type A_n. The infinite KM-system was solved by Kac and van Moerbeke [16] using a discrete version of inverse scattering. The Lax pair for system (11) can be found in [22]. The Lax matrix has the form:

$$L = \begin{pmatrix} 0 & a_1 & 0 & 0 & \cdots & \\ a_1 & 0 & a_2 & 0 & \cdots & \\ 0 & a_2 & 0 & a_3 & \cdots & \\ \vdots & \vdots & \vdots & & & \\ & & & & & \\ & & & & & a_{n-1} \\ & & & & a_{n-1} & 0 \end{pmatrix}, \tag{12}$$

where $u_i = 2a_i^2$. Moser in [22] describes a relation between the KM system (11) and the non-periodic Toda lattice. The procedure is the following: form L^2 which is not anymore a tridiagonal matrix but is similar to one. Let $\{e_1, e_2, \dots, e_n\}$ be the standard basis of $\mathbf{R}^n$, and $E_o = \{\mathrm{span}\, e_{2i-1}, i = 1, 2, \dots\}$, $E_e = \{\mathrm{span}\, e_{2i}, i = 1, 2, \dots\}$. Then L^2 leaves E_o, E_e invariant and reduces to each of these spaces to a tri-diagonal symmetric Jacobi matrix. For example, if we omit all even columns and all even rows we obtain a tridiagonal Jacobi matrix and the entries of this new matrix define the transformation from the KM-system to the Toda lattice. We illustrate with a simple example where $n = 5$.

In this case:

$$L = \begin{pmatrix} 0 & a_1 & 0 & 0 & 0 \\ a_1 & 0 & a_2 & 0 & 0 \\ 0 & a_2 & 0 & a_3 & 0 \\ 0 & 0 & a_3 & 0 & a_4 \\ 0 & 0 & 0 & a_4 & 0 \end{pmatrix} \tag{13}$$

and L^2 is the matrix:

$$\begin{pmatrix} a_1^2 & 0 & a_1a_2 & 0 & 0 \\ 0 & a_1^2+a_2^2 & 0 & a_2a_3 & 0 \\ a_1a_2 & 0 & a_2^2+a_3^2 & 0 & a_3a_4 \\ 0 & a_2a_3 & 0 & a_3^2+a_4^2 & 0 \\ 0 & 0 & a_3a_4 & 0 & a_4^2 \end{pmatrix}. \tag{14}$$

Omitting even columns and even rows of L^2 we obtain the matrix:

$$\begin{pmatrix} a_1^2 & a_1a_2 & 0 \\ a_1a_2 & a_2^2+a_3^2 & a_3a_4 \\ 0 & a_3a_4 & a_4^2 \end{pmatrix}. \tag{15}$$

This is a tridiagonal Jacobi matrix. It is natural to define new variables $A_1 = a_1a_2$, $A_2 = a_3a_4$, $B_1 = a_1^2$, $B_2 = a_2^2 + a_3^2$, $B_3 = a_4^2$. The new variables A_1, A_2, B_1, B_2, B_3 satisfy the Toda lattice equations.

This procedure shows that the KM-system and the Toda lattice are closely related: the explicit transformation which is due to Hénon maps one system to the other. The mapping in the general case is given by

$$A_i = -\tfrac{1}{2}\sqrt{u_{2i}u_{2i-1}}, \qquad B_i = \tfrac{1}{2}(u_{2i-1} + u_{2i-2}). \tag{16}$$

The equations satisfied by the new variables A_i, B_i are given by

$$\dot{A}_i = A_i(B_{i+1} - B_i), \qquad \dot{B}_i = 2(A_i^2 - A_{i-1}^2).$$

These are precisely the Toda equations in [12]. We remark that the transformation (16) was first discovered by Hénon. Hénon never published the result but he communicated the formula in a letter to Flaschka in 1973. We refer to paper [7] for more details. In [7] one can find the multiple-Hamiltonian structure, higher Poisson structures and master symmetries for system (11).

We would like to generalize the Hénon correspondence (16) (using the recipe of Moser) from generalized Volterra to generalized Toda systems. The relation between the Volterra systems of type B_n and C_n and the corresponding Toda systems is in [8]. It is natural to attempt to find a similar correspondence between the Volterra lattice of type D_n and the generalized Toda lattice of type D_n. It is a surprising result, and this is the content of the present paper that the Volterra D_n system corresponds not to the Toda D_n system but to a special case of the Sklyanin lattice.

2. Bogoyavlensky–Volterra systems

Bogoyavlensky constructed integrable Hamiltonian systems connected with simple Lie algebras, generalizing the KM-system (11). For more details see Refs. [5,6]. In this section we summarize the construction of Bogoyavlensky.

Let $\mathcal{G}$ be a simple Lie algebra of rank n and $\Pi = \{\omega_1, \omega_2, \dots, \omega_n\}$ the Cartan–Weyl basis of the simple roots in $\mathcal{G}$. There exist unique, positive integers k_i such that

$$k_0\omega_0 + k_1\omega_1 + \cdots + k_n\omega_n = 0,$$

where $k_0 = 1$ and ω_0 is the minimal negative root.

We consider the following Lax pairs:

$$\dot{L} = [B, L], \qquad L(t) = \sum_{i=1}^{n} c_i(t) e_{\omega_i} + e_{\omega_0} + \sum_{1\le i<j\le n} [e_{\omega_i}, e_{\omega_j}], \qquad B(t) = \sum_{i=1}^{n} \frac{k_i}{c_i(t)} e_{-\omega_i} + e_{-\omega_0}. \tag{17}$$

Let $\mathcal{H}$ be a Cartan subalgebra of $\mathcal{G}$. For every root $\omega_a \in \mathcal{H}^*$ there is a unique $H_{\omega_a} \in \mathcal{H}$ such that $\omega(h) = k(H_{\omega_a}, h)\forall h \in \mathcal{H}$, where k is the Killing form. We also have an inner product on $\mathcal{H}^*$ such that $\langle \omega_a, \omega_b \rangle = k(H_{\omega_a}, H_{\omega_b})$. We set

$$c_{ij} = \begin{cases} 1 & \text{if } \langle \omega_i, \omega_j \rangle \neq 0 \text{ and } i < j, \\ 0 & \text{if } \langle \omega_i, \omega_j 0 \rangle \text{ or } i = j, \\ -1 & \text{if } \langle \omega_i, \omega_j \rangle \neq 0 \text{ and } i > j. \end{cases}$$

The matrix equation (17) is equivalent to the dynamical system:

$$\dot{c}_i = -\sum_{j=1}^{n} \frac{k_j c_{ij}}{c_j}. \tag{18}$$

We determine the skew-symmetric variables:

$$x_{ij} = c_{ij} c_i^{-1} c_j^{-1}, \qquad x_{ji} = -x_{ij}, \qquad x_{jj} = 0,$$

which correspond to the edges of the Dynkin diagram for the Lie algebra $\mathcal{G}$, connecting the vertices ω_i and ω_j.

The dynamical system (18) in the variables x_{ij} takes the form:

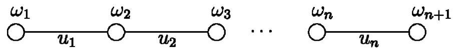
$$\dot{x}_{ij} = x_{ij} \sum_{s=1}^{n} k_s (x_{is} + x_{js}). \tag{19}$$

We recall that the vertices ω_i, ω_j of the Dynkin diagram are joined by edges only if $\langle \omega_i, \omega_j \rangle \neq 0$. Hence $x_{ij} = 0$ if there are no edges connecting the vertices ω_i and ω_j of the diagram. We call Eq. (19) the Bogoyavlensky–Volterra system associated with $\mathcal{G}$ (BV system for short).

We shall now describe the BV system for each simple Lie algebra $\mathcal{G}$. The number of independent variables $x_{ij}(t)$ is equal to $n-1$ and is one less than the number of variables $c_j(t)$. We use the standard numeration of vertices of the Dynkin diagram and define the variables $u_k(t) = x_{ij}(t)$ corresponding to the edges of the Dynkin diagram with increasing order of the vertices $(i < j)$.

The phase space consists of variables u_i, with $u_i > 0$. In the following list we give explicit expressions for the Volterra systems in the variables u_i for each classical simple Lie algebra:

A_{n+1}



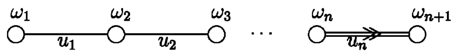
$\omega_0 = -(\omega_1 + \omega_2 + \cdots \omega_{n+1})$
$k_i = 1\ ,\ \ i = 1, \ldots, n+1$

$$c_{ij} = \begin{cases} 0 & |i-j| \neq 1 \\ 1 & j = i+1 \\ -1 & j = i-1 \end{cases}$$

$$\begin{aligned} \dot{u}_1 &= u_1 u_2 \\ \dot{u}_n &= -u_{n-1} u_n \\ \dot{u}_i &= u_i (u_{i+1} - u_{i-1}) \\ &2 \leq i \leq n-1 \end{aligned}$$

$u_i = x_{i,i+1} = \frac{1}{c_i c_{i+1}} \qquad i = 1, \ldots, n$

B_{n+1}



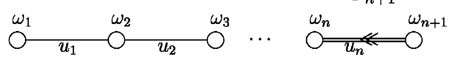
$\omega_0 = -(\omega_1 + 2\omega_2 + \cdots + 2\omega_{n+1})$
$k_1 = 1,\ \ k_i = 2\ ,\ \ i = 2, \ldots, n+1$

$$c_{ij} = \begin{cases} 0 & |i-j| \neq 1 \\ 1 & j = i+1 \\ -1 & j = i-1 \end{cases}$$

$$\begin{aligned} \dot{u}_1 &= u_1 (u_1 + 2u_2) \\ \dot{u}_2 &= u_2 (2u_3 - u_1) \\ \dot{u}_n &= -2u_{n-1} u_n \\ \dot{u}_i &= 2u_i (u_{i+1} - u_{i-1}) \\ &3 \leq i \leq n-1 \end{aligned}$$

$u_i = x_{i,i+1} = \frac{1}{c_i c_{i+1}} \qquad i = 1, \ldots, n$

C_{n+1}



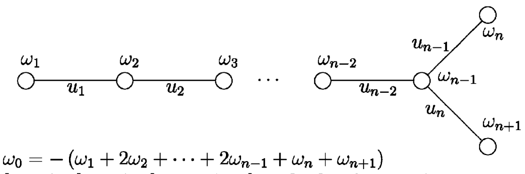
$\omega_0 = -(2\omega_1 + \cdots + 2\omega_n + \omega_{n+1})$
$k_i = 2\ ,\ i = 1, \ldots, n\ ,\ k_{n+1} = 1$

$$c_{ij} = \begin{cases} 0 & |i-j| \neq 1 \\ 1 & j = i+1 \\ -1 & j = i-1 \end{cases}$$

$$\begin{aligned} \dot{u}_1 &= 2u_1 u_2 \\ \dot{u}_{n-1} &= u_{n-1}(u_n - 2u_{n-2}) \\ \dot{u}_n &= -u_n (u_n + 2u_{n-1}) \\ \dot{u}_i &= 2u_i (u_{i+1} - u_{i-1}) \\ &2 \leq i \leq n-2 \end{aligned}$$

$u_i = x_{i,i+1} = \frac{1}{c_i c_{i+1}} \qquad i = 1, \ldots, n$

D_{n+1}



$\omega_0 = -(\omega_1 + 2\omega_2 + \cdots + 2\omega_{n-1} + \omega_n + \omega_{n+1})$
$k_1 = 1\ ,\ k_n = 1\ ,\ k_{n+1} = 1\ \ ,\ k_i = 2\ ,\ 2 \leq i \leq n-1$

$$c_{ij} = -c_{ji} = \begin{cases} 1 & 2 \leq j = i+1 \leq n \\ 0 & (i,j) = (n, n+1) \\ 0 & 3 \leq i+2 \leq j \leq n \\ 1 & (i,j) = (n-1, n+1) \end{cases}$$

$u_i = x_{i,i+1} = \frac{1}{c_i c_{i+1}}\ ,\ i = 1, \ldots, n-1\ ,\ \ u_n = x_{n-1,n+1} = \frac{1}{c_{n-1} c_{n+1}}$

$$\begin{aligned} \dot{u}_1 &= u_1 (2u_2 + u_1)\ ,\quad \dot{u}_2 = u_2 (2u_3 - u_1) \\ \dot{u}_i &= 2u_i (u_{i+1} - u_{i-1})\ \ 3 \leq i \leq n-3 \\ \dot{u}_{n-2} &= u_{n-2} (u_n + u_{n-1} - 2u_{n-3}) \\ \dot{u}_{n-1} &= u_{n-1} (u_n - u_{n-1} - 2u_{n-2}) \\ \dot{u}_n &= -u_n (u_n - u_{n-1} + 2u_{n-2})\ . \end{aligned} \tag{20}$$

3. The Volterra D_n system

Consider the BV D_{n+1} system (20) in the variables u_k. We make a linear change of variables:

$$v_1 = u_1, \qquad v_k = 2u_k, \quad k = 2, \dots, n-2, \qquad v_{n-1} = u_{n-1}, \qquad v_n = u_n,$$

to obtain the equivalent system:

$$\begin{aligned}
&\dot v_1 = v_1(v_1 + v_2), \qquad \dot v_k = v_k(v_{k+1} - v_{k-1}), \quad k = 2, \dots, n-3, \qquad \dot v_{n-2} = v_{n-2}(v_n + v_{n-1} - v_{n-3}),\\
&\dot v_{n-1} = v_{n-1}(v_n - v_{n-1} - v_{n-2}), \qquad \dot v_n = -v_n(v_n - v_{n-1} + v_{n-2}).
\end{aligned} \tag{21}$$

Before giving the Lax pair for the system (21) we introduce some matrix notations:

$$X_k = \begin{pmatrix} \sqrt{v_k} & 0 \\ 0 & \mathrm{i}\sqrt{v_k} \end{pmatrix}, \quad O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad Y_k = \frac{1}{2}\begin{pmatrix} \sqrt{v_k v_{k+1}} & 0 \\ 0 & \sqrt{v_k v_{k+1}} \end{pmatrix}, \quad Y_0 = \frac{\mathrm{i}}{2}\begin{pmatrix} 0 & v_1 \\ -v_1 & 0 \end{pmatrix}.$$

We also set

$$X = \begin{pmatrix} \sqrt{v_n} & \mathrm{i}\sqrt{v_n} \\ -\sqrt{v_{n-1}} & \mathrm{i}\sqrt{v_{n-1}} \end{pmatrix}, \quad Y = \frac{1}{2}\begin{pmatrix} \sqrt{v_{n-2}v_n} & \sqrt{v_{n-2}v_n} \\ -\sqrt{v_{n-2}v_{n-1}} & \sqrt{v_{n-2}v_{n-1}} \end{pmatrix}, \quad W = \frac{\mathrm{i}}{2}\begin{pmatrix} 0 & v_{n-1} - v_n \\ v_n - v_{n-1} & 0 \end{pmatrix}.$$

Eq. (21) can be written in a Lax pair form $\dot L = [L, B]$, where

$$L = \begin{bmatrix}
0 & 0 & \cdots & 0 & \sqrt{v_1} & \mathrm{i}\sqrt{v_1} \\
0 & O & X & O & \cdots & O \\
\vdots & X^t & O & X_{n-2} & \ddots & \vdots \\
0 & O & X_{n-2} & \ddots & \ddots & O \\
\sqrt{v_1} & \vdots & \ddots & \ddots & O & X_2 \\
\mathrm{i}\sqrt{v_1} & O & \cdots & O & X_2 & O
\end{bmatrix},$$

$$B = \begin{bmatrix}
0 & \cdots & \cdots & 0 & -\frac{1}{2}\sqrt{v_1v_2} & -\frac{1}{2}\sqrt{v_1v_2} & 0 & 0 \\
\vdots & O & O & Y & O & \cdots & \cdots & O \\
\vdots & O & W & O & Y_{n-3} & \ddots & & \vdots \\
0 & -Y^t & O & O & \ddots & \ddots & O & \vdots \\
\frac{1}{2}\sqrt{v_1v_2} & O & -Y_{n-3} & \ddots & \ddots & O & Y_3 & O \\
\frac{1}{2}\sqrt{v_1v_2} & \vdots & \ddots & \ddots & O & O & O & Y_2 \\
0 & \vdots & & O & -Y_3 & O & O & O \\
0 & O & \cdots & \cdots & O & -Y_2 & O & Y_0
\end{bmatrix}. \tag{22}$$

We note that the entries of the first row and the first column of L and B are scalars while all the other entries are 2×2 matrices.

The invariant polynomials of this system are given by the functions:

$$\begin{aligned} &H_2, H_4, \dots, H_{n-1}, && \text{when } n \text{ is odd},\\ &H_2, H_4, \dots, H_{n-2}, H_{n-1}, && \text{when } n \text{ is even,} \end{aligned}$$

where $H_k = (1/k)\,\mathrm{Tr}(L^k)$.

We use the variables c_j, $1 \le j \le n+1$ of Eq. (18) in order to find a cubic bracket π_3 of the BV D_{n+1} system. The dynamical system (18) in the case of the Lie algebra of type D_{n+1} can be written in Hamiltonian form $\dot{c}_j = \{c_j, H\}$, with Hamiltonian:

$$H = \log c_1 + 2\sum_{j=2}^{n-1} \log c_j + \log c_n + \log c_{n+1}$$

and Poisson bracket:

$$\{c_j, c_{j+1}\} = -\{c_{j+1}, c_j\} = 1, \quad j = 1, 2, \dots, n-1, \qquad \{c_{n-1}, c_{n+1}\} = -\{c_{n+1}, c_{n-1}\} = 1. \tag{23}$$

All other brackets are zero. In the new variables $v_j (v_1 = c_1^{-1} c_2^{-1}, v_k = 2c_k^{-1} c_{k+1}^{-1}, k = 2, \dots, n-2, v_{n-1} = c_{n-1}^{-1} c_n^{-1}, v_n = c_{n-1}^{-1} c_{n+1}^{-1})$ the above skew-symmetric bracket, which we denote by π_3, is given by

$$\begin{aligned} &\{v_1, v_2\} = v_1 v_2 (2v_1 + v_2), \qquad \{v_i, v_{i+1}\} = v_i v_{i+1}(v_i + v_{i+1}), \quad i = 2, \dots, n-3,\\ &\{v_{n-2}, v_{n-1}\} = v_{n-2} v_{n-1}(2v_{n-1} + v_{n-2}), \qquad \{v_{n-1}, v_n\} = 2v_{n-1} v_n (v_n - v_{n-1}),\\ &\{v_i, v_{i+2}\} = v_i v_{i+1} v_{i+2}, \quad i = 1, \dots, n-3,\\ &\{v_{n-2}, v_n\} = v_{n-2} v_n (v_{n-2} + 2v_n), \qquad \{v_{n-3}, v_n\} = v_{n-3} v_{n-2} v_n. \end{aligned} \tag{24}$$

All other brackets are zero. As in the case of KM system we suppose that n is odd ($n = 2m+1$) and we look for a bracket π_1 which satisfies $\pi_3 \nabla H_2 = \pi_1 \nabla H_4$.

We define

$$\tau_{ij} = -\tau_{ji} = v_{2i-1} \prod_{k=i}^{j-1} \frac{v_{2k+1}}{v_{2k}} \quad \text{for } i < j, \qquad \tau_{ii} = v_{2i-1}$$

and we define the bracket π_1 as follows:

$$\begin{aligned} &\{v_i, v_j\} = (-1)^{i+j-1} \tau_{[i/2]+1,[j+1/2]} \quad \text{for } 1 \le i < j \le n-2,\\ &\{v_i, v_{n-1}\} = \{v_i, v_n\} = \frac{(-1)^{i+n}}{2} \tau_{[i/2]+1,[n/2]} \quad \text{for } i = 1, \dots, n-2,\\ &\{v_{n-1}, v_n\} = -\{v_n, v_{n-1}\} = \frac{1}{2}(v_n - v_{n-1}). \end{aligned} \tag{25}$$

To illustrate, we give the Poisson matrix of the bracket π_1 in the case $n=7$:

$$\pi_1=\begin{bmatrix} 0 & \tau_{11} & -\tau_{12} & \tau_{12} & -\tau_{13} & \frac{1}{2}\tau_{13} & \frac{1}{2}\tau_{13} \\ -\tau_{11} & 0 & \tau_{22} & -\tau_{22} & \tau_{23} & -\frac{1}{2}\tau_{23} & -\frac{1}{2}\tau_{23} \\ \tau_{12} & -\tau_{22} & 0 & \tau_{22} & -\tau_{23} & \frac{1}{2}\tau_{23} & \frac{1}{2}\tau_{23} \\ -\tau_{12} & \tau_{22} & -\tau_{22} & 0 & \tau_{33} & -\frac{1}{2}\tau_{33} & -\frac{1}{2}\tau_{33} \\ \tau_{13} & -\tau_{23} & \tau_{23} & -\tau_{33} & 0 & \frac{1}{2}\tau_{33} & \frac{1}{2}\tau_{33} \\ -\frac{1}{2}\tau_{13} & \frac{1}{2}\tau_{23} & -\frac{1}{2}\tau_{23} & \frac{1}{2}\tau_{33} & -\frac{1}{2}\tau_{33} & 0 & \frac{1}{2}(v_7-v_6) \\ -\frac{1}{2}\tau_{13} & \frac{1}{2}\tau_{23} & -\frac{1}{2}\tau_{23} & \frac{1}{2}\tau_{33} & -\frac{1}{2}\tau_{33} & -\frac{1}{2}(v_7-v_6) & 0 \end{bmatrix}$$

$$=\begin{bmatrix} 0 & v_1 & -\frac{v_1v_3}{v_2} & \frac{v_1v_3}{v_2} & -\frac{v_1v_3v_5}{v_2v_4} & \frac{v_1v_3v_5}{2v_2v_4} & \frac{v_1v_3v_5}{2v_2v_4} \\ -v_1 & 0 & v_3 & -v_3 & \frac{v_3v_5}{v_4} & -\frac{v_3v_5}{2v_4} & -\frac{v_3v_5}{2v_4} \\ \frac{v_1v_3}{v_2} & -v_3 & 0 & v_3 & -\frac{v_3v_5}{v_4} & \frac{v_3v_5}{2v_4} & \frac{v_3v_5}{2v_4} \\ -\frac{v_1v_3}{v_2} & v_3 & -v_3 & 0 & v_5 & -\frac{v_5}{2} & -\frac{v_5}{2} \\ \frac{v_1v_3v_5}{v_2v_4} & -\frac{v_3v_5}{v_4} & \frac{v_3v_5}{v_4} & -v_5 & 0 & \frac{v_5}{2} & \frac{v_5}{2} \\ -\frac{v_1v_3v_5}{2v_2v_4} & \frac{v_3v_5}{2v_4} & -\frac{v_3v_5}{2v_4} & \frac{v_5}{2} & -\frac{v_5}{2} & 0 & \frac{v_7-v_6}{2} \\ -\frac{v_1v_3v_5}{2v_2v_4} & \frac{v_3v_5}{2v_4} & -\frac{v_3v_5}{2v_4} & \frac{v_5}{2} & -\frac{v_5}{2} & -\frac{v_7-v_6}{2} & 0 \end{bmatrix}.$$

The following theorem is from [18].

Theorem 2.

(i) π_1, π_3 *are Poisson*.

(ii) *The function*:

$$\frac{1}{4}H_2=\frac{1}{8}\mathrm{Tr}(L^4)=v_{n-2}v_n+2v_{n-1}v_n+\sum_{i=1}^{n-2}v_iv_{i+1}+\frac{1}{2}\sum_{i=2}^{n-2}v_i^2$$

is the Hamiltonian of the BV D_{n+1} system with respect to the bracket π_1.

(iii) *The function*:

$$F=(v_n-v_{n-1})\prod_{i=1}^{n-2}v_i$$

is the Casimir of the BV D_{n+1} system in the bracket π_1.

(iv) π_1, π_3 *are compatible*.

(v) $\pi_3\nabla H_2=\pi_1\nabla H_4$.

4. From Volterra to Birkhoff

We consider the Volterra D_{n+1} system (21). We assume that n is odd, equal to $2m+1$ and rename again the variables (i.e., use u_k in place of v_k). We recall the equations for the system:

$$\dot{u}_1 = u_1(u_1+u_2), \qquad \dot{u}_k = u_k(u_{k+1}-u_{k-1}) \qquad 2\le k\le n-3, \qquad \dot{u}_{n-2} = u_{n-2}(u_n+u_{n-1}-u_{n-3}),$$
$$\dot{u}_{n-1} = u_{n-1}(u_n-u_{n-1}-u_{n-2}), \qquad \dot{u}_n = -u_n(u_n-u_{n-1}+u_{n-2}). \tag{26}$$

We make the transformation (the analogue of Hénon transformation (16) for the KM-system):

$$a_1 = \frac{\mathrm{i}}{2}(u_n-u_{n-1}), \qquad a_j = \frac{1}{2}\sqrt{u_{n-2j+2}u_{n-2j+1}}, \quad j=2,3,\ldots,m, \qquad a_{m+1} = \frac{\mathrm{i}}{2}u_1,$$
$$b_1 = -\frac{1}{2}(u_n+u_{n-1}+u_{n-2}), \qquad b_j = -\frac{1}{2}(u_{n-2j+1}+u_{n-2j}), \quad j=2,3,\ldots,m. \tag{27}$$

This transformation is derived by mimicking the construction of Moser which takes the A_n Volterra lattice to the Toda lattice, i.e., the construction that was described in the introduction. The formulas (27) are obtained by considering L^2, where L is given by (22), and assigning suitable entries to the variables a_i, b_i.

We calculate

$$\begin{aligned}\dot{a}_1 &= \frac{\mathrm{i}}{2}(\dot{u}_n-\dot{u}_{n-1}) = \frac{\mathrm{i}}{2}[-u_n(u_n-u_{n-1}+u_{n-2})-u_{n-1}(u_n-u_{n-1}-u_{n-2})]\\ &= \frac{\mathrm{i}}{2}(-u_n^2-u_{n-2}u_n+u_{n-1}^2+u_{n-2}u_{n-1}) = \frac{\mathrm{i}}{2}(u_{n-1}-u_n)(u_n+u_{n-1}+u_{n-2}) = 2a_1b_1.\end{aligned}$$

In a similar fashion we obtain the equations of motion in the new variables (a_i, b_i):

$$\dot{a}_1 = 2a_1b_1, \qquad \dot{a}_i = a_i(b_i-b_{i-1}), \quad i=2,3,\ldots,m, \qquad \dot{a}_{m+1} = -2a_{m+1}b_m,$$
$$\dot{b}_i = 2(a_{i+1}^2-a_i^2), \quad i=1,2,\ldots,m. \tag{28}$$

This system can be written in a Lax pair form as follows:

$$L = \begin{pmatrix} b_1 & a_1 & a_2 & 0 & \cdots & \cdots & 0\\ a_1 & -b_1 & 0 & -a_2 & \ddots & & \vdots\\ a_2 & 0 & b_2 & 0 & \ddots & \ddots & \vdots\\ 0 & -a_2 & 0 & -b_2 & \ddots & a_m & 0\\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 & -a_m\\ \vdots & & \ddots & a_m & 0 & b_m & a_{m+1}\\ 0 & \cdots & \cdots & 0 & -a_m & a_{m+1} & -b_m \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & -a_1 & a_2 & 0 & \cdots & \cdots & 0 \\ a_1 & 0 & 0 & a_2 & \ddots & & \vdots \\ -a_2 & 0 & 0 & \ddots & \ddots & \ddots & \vdots \\ 0 & -a_2 & \ddots & \ddots & \ddots & a_m & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 & 0 & a_m \\ \vdots & & \ddots & -a_m & 0 & 0 & a_{m+1} \\ 0 & \cdots & \cdots & 0 & -a_m & -a_{m+1} & 0 \end{pmatrix}. \tag{29}$$

We have that $\dot{L} = [B, L]$ is equivalent to Eq. (28).

The functions:

$$H_{2k} = \frac{1}{2k} \operatorname{tr} L^{2k}, \quad k = 1, 2, \ldots, m$$

are independent constants of motion. There is also a Casimir, which makes the system completely integrable, but these functions are enough for our purpose.

We take H_2 as the Hamiltonian and define a Poisson bracket π_1 as follows: we consider the mapping from $\mathbf{R}^{2m+1}$ to $\mathbf{R}^{2m+1}$ given by (27). In other words it is the mapping which transforms the u_i to the new variables (a_i, b_i). The image of the bracket (25) (up to a constant multiple) is given by

$$\begin{aligned} &\{a_1, b_1\} = a_1, \qquad \{a_i, b_i\} = \tfrac{1}{2}a_i, \quad i = 2, 3, \ldots, m, \qquad \{a_{i+1}, b_i\} = -\tfrac{1}{2}a_i, \quad i = 1, 2, \ldots m-1, \\ &\{a_{m+1}, b_m\} = -a_{m+1}. \end{aligned} \tag{30}$$

All other brackets are zero. The function:

$$C = a_1 a_2^2 a_3^2 \cdots a_m^2 a_{m+1}$$

is a Casimir. This is the Casimir F of theorem 2 in (a, b) coordinates.

We also have involution of invariants, $\{H_i, H_j\} = 0$. We denote this bracket by π_1. We have

$$\pi_1 \nabla H_2$$

is equivalent to Eq. (28).

Define a Hamiltonian system in $\mathbf{R}^{2m}$ with coordinates $(q_1, \ldots, q_m, p_1, \ldots, p_m)$ by

$$H(q, p) = \frac{1}{2}\sum_{j=1}^{m} p_j^2 + \sum_{j=1}^{m-1} \mathrm{e}^{q_j - q_{j+1}} + \mathrm{e}^{-2q_1} + \mathrm{e}^{2q_m}. \tag{31}$$

As in the case of the classical Toda lattice we make a Flaschka-type transformation:

$$\begin{aligned} &a_1 = \frac{1}{\sqrt{2}}\mathrm{e}^{-q_1}, \qquad a_{m+1} = \frac{1}{\sqrt{2}}\mathrm{e}^{q_m}, \qquad a_i = \frac{1}{2}\mathrm{e}^{1/2(q_{i-1} - q_i)}, \quad i = 2, 3, \ldots, m, \\ &b_i = -\frac{1}{2}p_i, \quad i = 1, 2, \ldots, m. \end{aligned} \tag{32}$$

This is a mapping from $\mathbf{R}^{2m} \to \mathbf{R}^{2m+1}$. Note that we are not using transformation (2) but a more traditional variation.

We easily verify that Hamilton's equations for the variables (a_j, b_j) are precisely Eq. (28). For example:

$$\dot{a}_1 = -\frac{1}{\sqrt{2}}\mathrm{e}^{-q_1}\dot{q}_1 = -a_1\frac{\partial H}{\partial p_1} = -a_1 p_1 = -a_1(-2b_1) = 2a_1 b_1.$$

We recall that the system (28) has $H_2, H_4, \ldots, H_{2m}$ as a set of integrals in involution. Reverting back to the original variables (q_i, p_i) in $\mathbf{R}^{2m}$ we obtain m independent integrals in involution, and this proves the integrability of (31). For example:

$$H_2 = \sum_{i=1}^{m} b_i^2 + a_1^2 + 2\sum_{i=2}^{m} a_i^2 + a_{m+1}^2$$

corresponds to the Hamiltonian (31). The integrals are of degrees $2, 4, 6, \ldots, 2m$ in the momenta. The Casimir C reduces to a constant equal to $1/2^{2m-1}$.

We note that for the system (28) we may define a cubic bracket π_3 which satisfies the Lenard relation:

$$\pi_3 \nabla H_2 = \pi_1 \nabla H_4. \tag{33}$$

This bracket is the image of the bracket (24) under the mapping (27). The bracket π_3 is given by

$$\begin{aligned}
&\{a_i, a_{i+1}\} = a_i a_{i+1} b_i, \quad i = 2, 3, \ldots, m-1, \qquad \{a_i, a_{i+1}\} = 2a_i a_{i+1} b_i, \quad i = 1 \text{ and } m,\\
&\{b_i, b_{i+1}\} = 2a_{i+1}^2(b_i + b_{i+1}), \quad i = 1, 2, \ldots, m-1, \qquad \{a_1, b_1\} = 2a_1(a_1^2 + b_1^2),\\
&\{a_i, b_i\} = a_i(a_i^2 + b_i^2), \quad i = 2, 3, \ldots, m-1, \qquad \{a_m, b_m\} = a_m(a_m^2 + b_m^2 - a_{m+1}^2), \quad \{a_1, b_2\} = 2a_2^2 a_1,\\
&\{a_i, b_{i+1}\} = a_{i+1}^2 a_i, \quad i = 2, 3, \ldots, m-1, \qquad \{a_2, b_1\} = -a_2(a_2^2 + b_1^2 - a_1^2),\\
&\{a_{i+1}, b_i\} = -a_{i+1}(a_{i+1}^2 + b_i^2), \quad i = 2, 3, \ldots, m-1, \qquad \{a_{m+1}, b_m\} = -2a_{m+1}(a_{m+1}^2 + b_m^2),\\
&\{a_{i+2}, b_i\} = -a_{i+1}^2 a_{i+2}, \quad i = 1, 2, \ldots, m-2, \qquad \{a_{m+1}, b_{m-1}\} = -2a_m^2 a_{m+1}.
\end{aligned}$$

All other brackets are zero. It is clear, by the way it was constructed, that this bracket is Poisson, compatible with π_1 and that the integrals H_i are all in involution. We close with a few remarks:

1. One could try to obtain a bi-Hamiltonian formulation of the system (28) following the recipe of [9]. The basic steps in the construction are the following: write the second bracket π_3 in (q, p) coordinates (call it J_3) and define a recursion operator in (q, p) space by inverting the standard symplectic bracket (J_1). Define the negative recursion operator in (q, p) space by inverting the second bracket J_3. Define a new rational bracket J_{-1} by $J_{-1} = J_1 J_3^{-1} J_1$. Finally, project the bracket J_{-1} into the (a, b) space to obtain a rational bracket π_{-1}. The result is a bi-Hamiltonian formulation of the system:

   $$\pi_1 \nabla H_2 = \pi_{-1} \nabla H_4.$$

   Note that a recursion operator in the (a, b) space does not exist since π_1 and π_3 are both singular. So far, we were unable to compute the bracket J_3 and we are not certain that it can be computed.
2. It is a straightforward recursive process to solve the u_i as functions of the a_i, b_i using (27). After substitution of the values of a_i, b_i from (32) we obtain an expression of the u_i as functions of q_i, p_i. The formulas are too complicated but in principle the invariants, symmetries and higher Poisson brackets in the (q, p) space of (31) transfer to the corresponding ones for the Volterra system (27) via this mapping.
3. One may predict the degrees of the integrals by computing the Kovalevskaya exponents as in [11]. In our case the degrees of the invariants $2, 4, \ldots$ agree with the predicted values.

4. The Casimir:

$$C = a_1 a_2^2 a_3^2 \dots a_m^2 a_{m+1}$$

may be obtained in a different way using the following observation: the Casimir is the product of all the a_i raised to certain exponents. We note that the exponents (not to be confused with the Kovalevskaya exponents) $(1, 2, 2, \dots, 2, 1)$ can be determined from the condition:

$$\mathbf{v}_1 + 2\mathbf{v}_2 + \cdots + 2\mathbf{v}_{N-1} + \mathbf{v}_N = 0.$$

This is always the case for systems of the form (1) if $N > n$.

5. The procedure of this paper works equally well for the system:

$$H(q_1, \dots, q_n, p_1, \dots, p_n) = \sum_{i=1}^{n} \frac{1}{2} p_i^2 + \sum_{i=1}^{n-1} \mathrm{e}^{q_i - q_{i+1}} + \alpha\, \mathrm{e}^{-2q_1} + \beta\, \mathrm{e}^{q_n}. \tag{34}$$

The only difference is that we use the Volterra D_{n+1} system with $n = 2m$.

6. The Hamiltonian (31) is positive, and the Casimir C keeps the variables a_k from becoming zero. Thus, the energy surface is compact, and solutions lie on tori. This suggests that it should be possible to introduce a spectral parameter into the Lax pair and so get to Riemann surfaces and theta functions.

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