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Critical behavior for scalar nonlinear waves

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ABSTRACT

In the long wave regime, nonlinear waves may undergo a phase transition from a smooth behavior to a fast oscillatory behavior. In this study, we consider this phenomenon, which is commonly known as dispersive shock, in the light of Dubrovin's universality conjecture (Dubrovin, 2006; Dubrovin and Elaeva, 2012) and we argue that the transition can be described by a special solution of a model universal partial differential equation. This universal solution is constructed using the string equation. We provide a classification of universality classes and an explicit description of the transition with special functions, thereby extending Dubrovin's universality conjecture to a wider class of equations. In particular, we show that the Benjamin–Ono equation belongs to a novel universality class with respect to those known previously, and we compute its string equation exactly. We describe our results using the language of statistical mechanics, where we show that dispersive shocks share many of the features of the tricritical point in statistical systems, and we also build a dictionary of the relations between nonlinear waves and statistical mechanics.

1. Introduction

Dispersive shock waves in $1+1$ dimensions comprise a class of shock waves, which have been observed recently in a variety of physical situations where the media are dispersive or not strictly diffusive, e.g., plasma physics [1], Bose–Einstein condensates [2], nonlinear optics [3,4] and hydrodynamics [5–8]. Following the shock, the waves experience an abrupt phase transition from a regular behavior to a rapid oscillatory behavior, and this transition has been conjectured to be universal, where it depends only on some general properties of the underlying model as a partial differential equation (PDE). The universality classes observed previously correspond to the class of scalar dispersive waves (Korteweg–de Vries universality class, [9]) and the class of two components focused on dispersive waves (the class of a focusing nonlinear Schrödinger equation, [10]). A similar universality property has also been observed in the case of classical dissipative shock (Burgers universality class, [11,12]).

In the present study, we consider a fairly general model equation for 1-dimensional scalar unidirectional waves in a fluid of the form

$$u_t + a(u)u_x + N[u] = 0, \tag{1}$$

where $a(u)$ is a non-constant function (in most relevant cases $a(u) = u$) and N is a (pseudo) differential operator, which is generally nonlinear. Note that N can be a local operator as well as a non-local operator, such as the Hilbert transform. Although our primary interest is the study of dispersive shocks, this class of equations includes diffusive and mixed dispersive-diffusive models, which can be selected based on different choices of the operator N. More precisely, N models the phenomena under examination by considering relevant physical effects, such as dispersion, dissipation, pressure, or the interfacial interaction between two different fluids. Notable examples of equations that belong to this class include generalized Korteweg–de Vries (KdV) and Burgers equations, the intermediate-long wave and Benjamin–Ono (B–O) equations, and the Benjamin–Bona–Mahony and Camassa–Holm equations. The operators N that correspond to these equations are listed in Table 1.

The critical behavior arises when we consider solutions that vary on a large scale (compared with the natural scale of the system) at time $t = 0$, such as $1/\varepsilon$ with ε small, and we study whether fluctuations on a smaller scale arise at a later time. We assume that the nonlinear operator admits a long wave expansion, i.e., we assume that a real number $\beta > 0$ and an operator $\overline{N}$ exist

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Table 1
Notable examples of equations of the form $u_t + a(u)u_x + N[u] = 0$. For every equation in the list, we give $a(u)$, $N[u]$, $\bar{N}[u]$, and the coefficients β, κ, θ for the corresponding universal Eq. (11).

Equation	$a(u)$	$N[u]$	$\bar{N}[u]$	β	κ	θ
Generalized Burgers	$u^n,\ n > 0$	$-u_{xx}$	$-u_{xx}$	1	1	0
Generalized Korteweg–de Vries	$u^n,\ n > 0$	u_{xxx}	u_{xxx}	2	0	1
Benjamin–Ono	u	$-\mathcal{H}[u_{xx}],\ \mathcal{H}[u](x) = \frac{1}{\pi}\fint \frac{u(y)}{x-y}dy$	$-\mathcal{H}[u_{xx}]$	1	0	1
Intermediate Long Wave	u	$-\frac{1}{2\delta}\partial_x^2 \fint_{\mathbb{R}} \coth\left(\frac{\pi(x-y)}{2\delta}\right) u(y)dy + \frac{1}{\delta}u_x$	$\frac{\delta}{3}u_{xxx}$	2	0	$\frac{\delta}{3}$
Camassa–Holm	u	$(1-\partial_x^2)^{-1}\left(-uu_x + \frac{2}{3}u_x u_{xx} + \frac{1}{3}uu_{xxx}\right) + uu_x$	$-\frac{7}{3}u_x u_{xx} - \frac{2}{3}uu_{xxx}$	2	0	$-\frac{2}{3}u_c$
Benjamin–Bona–Mahony	u	$(1-\partial_x^2)^{-1}(-uu_x) + uu_x$	$-3u_x u_{xx} - uu_{xxx}$	2	0	$-u_c$

such that for any sufficiently smooth function u,

$$NS_\varepsilon[u] = \varepsilon^{\beta+1} S_\varepsilon \overline{N}[u] + o(\varepsilon^{\beta+1}), \quad \beta > 0, \tag{2}$$

where S_ε, $\varepsilon > 0$ is the dilation operator

$$S_\varepsilon[u](x) = u(\varepsilon x). \tag{3}$$

Table 1 shows that assumption (2) holds for any of the specific equations listed above. From (2), it follows that in the long wave regime, by applying the change of variables $(x, t) \to (x\varepsilon, t\varepsilon)$, the wave satisfies the rescaled equation

$$u_t + a(u)u_x + \varepsilon^\beta \overline{N}[u(x)] + o(\varepsilon^\beta) = 0, \quad \varepsilon \to 0,$$

where the initial data are independent of ε. If the latter equation is well-posed, then in this regime, (1) is a small perturbation of the scalar conservation law (or Hopf equation) $u_t + a(u)u_x = 0$, provided that the wave remains smooth. However, when the solution of the Hopf equation develops a shock, i.e., a point with a vertical derivative, the term $N[u]$ is no longer negligible and it makes the wave fluctuate on a smaller scale, provided that the perturbation is not strictly dissipative.

In this study, we use some heuristic techniques, which are reminiscent of [9], to analyze the behavior of waves that are similar to shock for equations of type (1), and we study their universal behavior. We show that there is an emerging meso-scale where the shock is indeed universal, which is described by a particular solution of the universal PDE

$$U_T + UU_X + \int_{-\infty}^{+\infty} e^{ipX} (\kappa + i\theta \operatorname{sign}(p)) |p|^{\beta+1} \hat{U}(p, T)\, dp = 0,$$

where $\hat{U}(p, T)$ is the Fourier transform of $U(X, T)$ and κ, θ, β are specific scalar parameters that can be computed explicitly from the operator N. The universality classes are parameterized by pairs of the form $(\frac{\kappa}{\theta}, \beta)$, where different parameter choices yield different universality classes. Based on this classification, we can answer the question posed by [13]: the B–O equation, although it is a Hamiltonian (conservative) PDE, does not belong to the KdV universality class ($\kappa = 0, \beta = 2$), but instead it represents a new universality class ($\kappa = 0, \beta = 1$).

The second part of this study characterizes the particular solution of the universal PDE that describes the wave at the shock. Our construction is based on the concept of the string equation [9,14], which is a perturbative deformation of the algebraic equation that describes the shock of the Hopf equation. Each universality class is characterized by a particular string equation and the universal solution is a specific solution of a boundary value problem for the corresponding string equation. Using this approach, we can compute a quantity with perturbative techniques, which is beyond all order in the standard perturbative expansion of solutions to (1) in powers of ε. In a few particular cases, the string equation can be computed exactly. For instance, we recover the string equations obtained for the KdV and Burgers classes (an ordinary differential equation (ODE) of Painlevé type and the Pearcey equation, respectively), and we compute the string equation for B–O, which was previously unknown. Therefore, we conjecture that the critical behavior of solutions to the Benjamin–Ono equation can be described by a particular solution of the singular integro-differential equation

Table 2
Dictionary: nonlinear waves and statistical mechanics.

Wave equation in 1 + 1 dimensions	Statistical model
Long wave limit	Thermodynamic limit
Critical point of gradient catastrophe	Tricritical point
Wave amplitude in the Whitham zone	Order parameter
Unfolding of cubic singularity	Mean field of φ^6 model
Meso-scale at the critical point	Renormalization group flow
Scaling linear perturbations of Hopf	Fixed point of renormalization group

$$X - UT + U^3 - 3\,U\mathcal{H}[U_X] - 3\,\mathcal{H}[UU_X] - 4\,U_{XX} = 0,$$

where $\mathcal{H}$ is the Hilbert transform on the line. We study the above equation numerically and we compare our results with the universal behavior of KdV and Burgers classes. The numerical solution of the B–O string equation requires an effective scheme for a non-local boundary value problem with an irregular boundary behavior, but no suitable method has been described in previous studies. Thus, our algorithm uses a novel numerical scheme, which was developed ad-hoc by [15], to effectively evaluate the Hilbert transform for functions that are slowly decaying at infinity.

Before obtaining a precise description of the behavior of waves at the critical points, the next section presents some known results related to the theory of dispersive shocks. We describe these phenomena and the new results obtained in the present study using the language of statistical mechanics, thereby illustrating the strong analogy between the critical behavior of nonlinear waves and the theory of phase-transitions in statistical mechanics, as well as making the dispersive-shock phenomenon easy to understand by any scientist familiar with the latter theory. Furthermore, we build a dictionary based on the relationship between the critical behavior of nonlinear waves and the phase transitions of statistical mechanics, which is summarized in Table 2.[1]

2. Dispersive shock as a tricritical phase transition

To understand the dispersive-shock phase transition, we consider the well-known case of the dispersionless (or semiclassical) limit of KdV [17–19]

$$u_t + uu_x - \varepsilon^2 u_{xxx} = 0, \quad \varepsilon \to 0,\ u(x, t = 0, \varepsilon) = \varphi(x).$$

We assume that the initial data are smooth, positive, rapidly decaying, and they have a single hump. The formal $\varepsilon = 0$ limit, known as the Hopf equation, describes a wave where every particle on the profile travels with constant velocity u, i.e., the solution is constant along the characteristic lines

$$x(t; x_0) = x_0 + \varphi(x_0)t, \qquad u(x(t; x_0), t) = \varphi(x_0). \tag{4}$$

[1] It is interesting to compare our table with Table 1 provided by [16], who considered dissipative shock.

Up to the critical time t_c, all lines are distinct and the solution is uniquely determined, whereas the lines start to intersect afterward and the wave develops a shock, i.e., a point with a vertical derivative, where the contribution $\varepsilon^2 u_{xxx}$ is not negligible, irrespective of how small ε might be. Lax and Levermore [18] showed that in the semiclassical regime, the (x, t)-plane is divided into two zones (see Fig. 3). In the first, which is known as the *semiclassical zone* and that contains the strip $\mathbb{R} \times [0, t_c)$, the limit $\lim_{\varepsilon\to 0} u(x, t, \varepsilon) = u(x, t)$ exists and it corresponds to the solution of the Hopf equation, or one of its branches if it is multi-valued. In the second zone, which is known as the *Whitham zone*, $u(x, t, \varepsilon)$ develops oscillations with a vanishing wavelength $O(\varepsilon)$ (see Fig. 1) and the limit exists only in a weak sense, i.e., a function $\bar{u}(x, t)$ exists that averages the oscillations, which are uniquely defined by the weak limit

$$\lim_{\varepsilon\to 0} \int \psi(x)u(x, t, \varepsilon)dx = \int \psi(x)\bar{u}(x, t)dx,$$

for any test function $\psi(x)$.[2]

The boundary of the Whitham zone depends only on the initial data, i.e., an approximate expression of the boundary up to a certain time beyond the critical time and for certain classes of initial data was described by Grava and Klein [20]. To better understand the transition of the solution from a regular behavior to an oscillatory behavior, we introduce an **order parameter** $W(x, t)$, which measures the amplitude of the oscillations in the Whitham zone:

$$W(x, t) = \lim_{\varepsilon\to 0} \sup |u(x, t, \varepsilon) - \bar{u}(x, t)|. \tag{5}$$

In the KdV case, the function W, as shown in Fig. 2, can be computed exactly from the formulae for u and $\bar{u}$, which were obtained in [20]. Let us fix t at a value higher than t_c. Then, the Whitham zone is an interval $(x_-(t), x_+(t))$ of the real line. The order parameter $W(x, t)$ is zero outside this interval, where it behaves similar to $W(x, t) \sim 1/\log(x_+(t)-x)$ close to the right boundary, whereas it is discontinuous at the left boundary: $\lim_{x\downarrow x_-} W(x, t) > 0$. Therefore, the solution undergoes a second order phase transition at $x = x_+$, where the order parameter is continuous but not differentiable, and a first order phase transition at $x = x_-$, where the order parameter is discontinuous. The boundary of the Whitham zone comprises a curve with second order phase transitions and a curve with first order phase transitions, which meet at a point (x_c, t_c); therefore, this is a **tricritical point** according to the standard theory of phase transitions in statistical mechanics [21] (see Fig. 3).[3]

In the following, we investigate the local behavior of solutions close to the tricritical point for a general PDE (1), where we argue that it is universal and we characterize the universality classes. To avoid cumbersome notations and because the final result is independent of a, we employ $a(u) = u$.[4] For the Hopf equation, the critical point x_c, t_c is the point where the wave breaks and the solution becomes multivalued, which is a singular behavior known as a **gradient catastrophe**. It is well known that the generic singularity is a cubic one. Indeed, it follows from (4), that the solutions can be expressed locally by the implicit formula

$$u(x - \varphi(x)t) = \varphi(x) \quad \text{or} \quad x - ut = f(u),\ f = \varphi^{-1}.$$

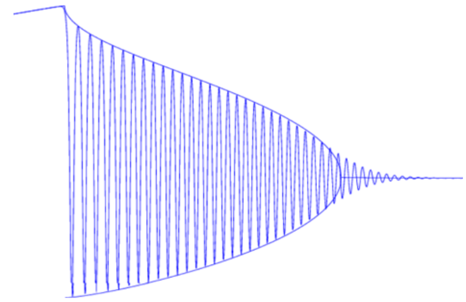

Fig. 1. Profile of the typical solution of KdV in the semiclassical regime (where $\varepsilon = 10^{-2}$) after the critical time.
Source: Figure from the arXiv version of [20].

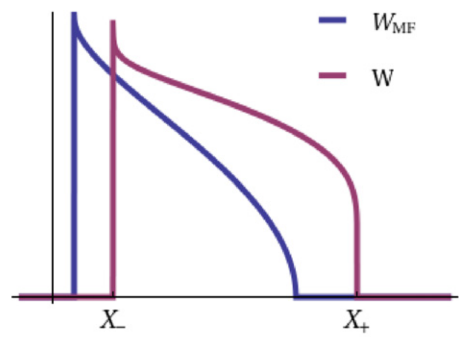

Fig. 2. Profile of the amplitude of oscillations W for a typical solution of KdV and its mean field approximation W_{MF}. W is discontinuous at the left boundary of the shock region and continuous but not differentiable at the right boundary.

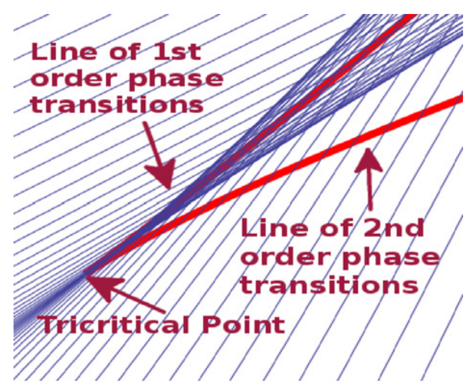

Fig. 3. Phase portrait of the typical solution in the (x, t) plane. The straight lines are characteristics. The thick line is the boundary of the shock region $W > 0$ (or Whitham zone). In general, this does not coincide with the region where the lines of the characteristics intersect.

If we let $u_c = u(x_c, t_c)$ and suppose that $f'''(u_c) \neq 0$, then we can introduce the scale variables

$$X = \frac{x - x_c - u_c(t - t_c)}{\lambda}, \qquad U = \left(\frac{\gamma}{6}\right)^{1/3} \frac{u - u_c}{\lambda^{\frac{1}{3}}},$$

$$T = \left(\frac{6}{\gamma}\right)^{1/3} \frac{(t - t_c)}{\lambda^{2/3}}, \tag{6}$$

with $\gamma = -f'''(u_c) > 0$ and λ, a small parameter, and thus for small $\lambda \to 0^+$, we obtain

$$X - UT + U^3 = 0, \tag{7}$$

which is the miniversal unfolding of the cubic singularity. A good illustration of the transition can be derived from (7). If $\beta_1 \geq \beta_2 \geq \beta_3$ are the three roots of U for $T > 0$, then $W_{MF}(x, t) = 2(\beta_1(x, t) - \beta_2(x, t))$ measures the envelope of the solution. This

2 For times that are sufficiently long after the time of the shock t_c, the solution inside the Whitham zone may undergo further shocks and thus the Whitham zone is divided into subregions with different "genera" [18]. These additional shocks are outside the scope of the present study, where we focus on the behavior of solutions for $t \sim t_c$.

3 In the case of B–O, the Whitham zone coincides with the region of the (x, t) plane where the solution of Hopf is multi-valued [13]. In the general case, e.g., for KdV, this is no longer true.

4 The few modifications required in formulae (6) and (8) can be found in [9].

has the same phase diagram and qualitative behavior as the exact order parameter KdV, but a different exponent $W_{MF} \sim (\tilde{x}_+(t) - x)^{1/2}$. This *mean field* description of the phase transition obtains the correct behavior but it omits the true nature of the oscillations. It has the same structure as the φ^6 mean field theory [21].

3. Universal model PDE at the tricritical point

By generalizing the procedure described in [9], for a quite general perturbation N, we can give a much finer description of the tricritical phase transition, which considers the precise nature of the oscillations and it is remarkably **universal**. In fact, we argue that the local behavior of u around the tricritical point is uniquely characterized by the linearization of $\bar{N}$ at the constant function $u \equiv u_c$:

$$\overline{N}[u_c + \delta u] = \bar{L}_{u_c}[\delta u] + O((\delta u)^2),$$

where $\overline{N}[u_c] = 0$ because of (2). More precisely, at an appropriate scale, the idea is that the wave u satisfies a distinguished universal solution for a model PDE, which is uniquely determined by $\bar{L}$.

In this section, we derive and classify the universal PDEs. In Section 4, we consider the characterization of the particular solution. To set an appropriate scale around the tricritical point, let us change the variables as in (6), but with the scaling parameter $\lambda = \varepsilon^{\frac{1}{\alpha}}$, which depends on ε. A simple computation shows that the wave equation (1) reduces to

$$U_T + UU_X + \varepsilon^{\frac{\beta(\alpha-1)-1/3}{\alpha}} \bar{L}_{u_c}[U] + \text{higher order terms} = 0.$$

The balance of the intrinsic and extrinsic scales ε, λ is achieved when $\alpha = 1 + \frac{1}{3\beta}$, or equivalently if u admits the expansion

$$u(x,t,\varepsilon) \simeq u_c + \varepsilon^{\frac{\beta}{3\beta+1}} U\left(\frac{x - x_c - u_c(t-t_c)}{\varepsilon^{\frac{3\beta}{3\beta+1}}}, \frac{t-t_c}{\varepsilon^{\frac{2\beta}{3\beta+1}}}\right) + O(\varepsilon^{\frac{2\beta}{3\beta+1}}), \quad (8)$$

and the leading term U of (8) is a solution of the linearized perturbation

$$U_T + UU_X + \bar{L}_{u_c}[U] = 0. \quad (9)$$

The universal behavior emerges close to the tricritical point, and thus on a **meso-scale** $\varepsilon^{1/\alpha}$ between the microscopic $O(\varepsilon)$ scale and the macroscopic one $O(\varepsilon^0)$. The reader should compare this situation with the case of a renormalization group in statistical mechanics, where universality arises by *magnifying* the theory at the meso-scale when block spin or phase-space renormalization is performed [22].

Before describing the **distinguished solution** $U(X,T)$ of (9), which gives the universal correction at the tricritical point, we consider the classification of universality classes. Since any positive constant in front of $\bar{L}_{u_c}$ can be factored out trivially, we say that *two nonlinear PDEs N, N' belong to the same universality class if $\bar{L}_{u_c} = \bar{L'}_{u_c}$ up to a (positive) scalar multiple.*

The **classification of universality classes** agrees with the classification of linear operators $\bar{L}_c[U]$, which comprises the linearization of an operator N that admits the long wave expansion (2). By assumption, on N, $\bar{L}_c$ is a linear pseudo-differential operator. If we also assume that L_c is translationally invariant,[5] then it admits the representation $L[U](x) := \int_{-\infty}^{+\infty} e^{ipx} m(p)\hat{U}(p)\,dp$ for some sufficiently regular Fourier multiplier $m(p)$ [23], where $\hat{U}(p)$ denotes the Fourier transform of U. To allow the operator to define a meaningful evolution, it must map real functions onto real functions and it must be either conservative or dissipative, i.e., $\int U(X)L[U(X)]dX \geq 0$, where the two conditions read $m(-p) = m^*(p)$, $Re(m(p)) \geq 0$. A complete characterization of the admissible operators $\bar{L}$ is achieved based on the following fact, which is proved in the Appendix: the scaling assumption (2) on N implies that

$$\bar{N}S_\varepsilon[u] = \varepsilon^{\beta+1}S_\varepsilon\bar{N}[u], \quad (10)$$

for any sufficiently smooth function u, where S_ε is the dilation operator defined in (3). Due to (10), $\bar{L}$ satisfies the same scaling law

$$\bar{L}_{u_c} \circ S_\varepsilon = \varepsilon^{\beta+1}S_\varepsilon \circ \bar{L}_{u_c},$$

and this further constrains the Fourier multiplier to the form $m(p) = \kappa|p|^{\beta+1} + i\theta\, p\,|p|^\beta$, for some $(\kappa,\theta) \in \mathbb{R}^2 \setminus \{0\}$, $\kappa \geq 0$, and $\beta > 0$. Explicitly, we have:

$$U_T + UU_X + \int_{-\infty}^{+\infty} e^{ipX}\,(\kappa + i\theta\,\mathrm{sign}(p))\,|p|^{\beta+1}\hat{U}(p)\,dp = 0. \quad (11)$$

Thus, critical universality classes are characterized by a pair of parameters $(\frac{\kappa}{\theta}, \beta)$. Since the transformation $U(X,T) \to -U(-X,T)$ sends θ to $-\theta$, we can assume that $\theta \geq 0$. Note that if $\theta = 0$, then the perturbation is purely dissipative, whereas if $\kappa = 0$, it is dispersive and possesses the Hamiltonian $H[U] = \int_{-\infty}^{+\infty} U^3 - \theta\, U\, K[U]dX$, where $K[U] = \int e^{ipX}|p|^\beta \hat{U}(p)\,dp$.

Example 1. The conservation laws $u_t + \partial_x f(u, u_x, \ldots) = 0$, with some smooth function f, admit a long wave regime with $\bar{N}[u(x)] = \partial_x(n(u)u_x)$, where $n(u) = \frac{\partial f}{\partial u_x(x)}_{|u_x=u_{xx}=\cdots=0}$. Provided that $n(u_c) \neq 0$, then $\beta = 1$ and $\bar{L}_{u_c} = n(u_c)\,u_{xx}$. Thus, the universal model for these equations is the Burgers equation:

$$U_t + U\,U_x + n(u_c)\,U_{xx} = 0.$$

The critical behavior of these conservation laws is typical for dissipative shocks and it was considered previously by [12,24].

Example 2. Local Hamiltonian PDEs are equations in the form $u_t = \frac{\delta}{\delta u}\int h(u, u_x, \ldots)dx$, for a smooth function h s.t. $h(0,0,\ldots) = 0$. They admit long wave expansions with $\bar{N}[u] = \partial_x(b'(u)\,u_x^2 + 2b(u)\,u_{xx})$, $b(u) = \frac{\partial h}{\partial u_x(x)}_{|u_x=u_{xx}=\cdots=0}$. Provided that $b(u_c) \neq 0$, then $\beta = 2$ and $\bar{L}_{u_c} = b(u_c)\,u_{xxx}$. Therefore, the universal model for these equations is KdV:

$$U_t + U\,U_x + b(u_c)U_{xxx} = 0.$$

The critical behavior of this class was considered by [9].

Example 3. The B–O equation [25,26]

$$u_t + u\,u_x - \mathcal{H}[u_{xx}] = 0,$$

where $\mathcal{H}$ is the Hilbert transform: $\mathcal{H}[u](x) = \frac{1}{\pi}\fint \frac{u(y)}{x-y}dy$, is an integrable Hamiltonian equation such as KdV, but it is non-local. The operator $\mathcal{H}[u_{xx}]$ is a translationally-invariant pseudo-differential operator with the Fourier multiplier $m(p) = i\,\mathrm{sign}(p)p^2$ [23]. Thus, in this case, $N = \bar{L} = \mathcal{H}[u_{xx}]$. Therefore the B–O equation is already in the long wave form (11), with $\beta = 1$, $\kappa = 0$, and $\theta = 1$. It has the same exponent as Burgers but because it is Hamiltonian similar to KdV, its solutions undergo a dispersive shock [13]. Therefore, it corresponds to a novel universality class, which we designate as the *B–O universality class*. All equations of the B–O hierarchy (see [27] for a definition) belong to the B–O universality class.

5 Theoretically, the case of a non-translationally invariant $\bar{N}$ can be dealt with using our methods, but it appears to be less relevant for studying the long wave limit of unidirectional waves.

Example 4. The intermediate long wave equation [28]

$$u_t + u\,u_x + \frac{1}{\delta}\,u_x + \mathcal{T}_\delta[u_{xx}] = 0,$$

where

$$\mathcal{T}_\delta[u(x)] = -\frac{1}{2\delta}\fint_{\mathbb{R}} \coth\left(\frac{\pi(x-\xi)}{2\delta}\right) u(\xi)d\xi,$$

and $\delta \in \mathbb{R}$, is an integrable equation that models nonlinear waves in a fluid of finite depth. Moreover, in the limit $\delta \to 0$, we formally obtain the KdV equation, whereas the limit $\delta \to \infty$ gives the B–O equation. Therefore, it is interesting to check whether this equation belongs to one of the aforementioned universality classes, or possibly to a new one. From the representation $\mathcal{T}_\delta[u(x)] = i\int_{-\infty}^{+\infty} e^{ipx}\coth(\delta p)\hat{u}(p)\,dp$ (see [29]), together with the expansion $\coth(\delta p) = \frac{1}{\delta p} + \frac{\delta}{3}p + O(p^3)$, it follows that, provided that δ remains finite, the intermediate long wave equation belongs to the KdV universality class.

Example 5. The Camassa–Holm equation [30]:

$$u_t - u_{xxt} + uu_x = \frac{2}{3}u_x u_{xx} + \frac{1}{3}uu_{xxx},$$

and the Benjamin–Bona–Mahony equation [31]

$$u_t - u_{xxt} + uu_x = 0$$

can be written in the standard form (1) by inverting $1-\partial_x^2$. Provided that $u_c \neq 0$, they also belong to the KdV universality class [9].

We note that the universal PDE (11) is again of the form of Eq. (1). In many important special cases, such as KdV, Burgers, and B–O, the procedure for rescaling at the tricritical point simply reproduces the original equation. Thus, the special case where $a(u) = u$, $N = \bar{L}_{u_c}$ is **invariant** under rescaling. This explains the naming convention for the universality classes in the examples given above. A comparison between the different universality classes for the examples considered above is given in Table 1

4. Universal correction as a solution of the string equation

In this section, we show how to compute the universal correction $U(X,T)$, which is defined by the multiscale expansion (8) of u at the tricritical point, as a particular solution of the universal model Eq. (9). We argue that U is the solution of a (possibly infinite) deformation of the cubic equation (7), which is known as the **string equation**. Our approach generalizes and simplifies that proposed originally by [9] for the Hamiltonian case and developed mathematically by [32,14] (see [24] for the Burgers universality class). We derive the string equation using a simple principle, which was proved rigorously by [14] with some generality: *in the long wave regime, any solution of Eq. (1), and particularly of Eq. (9), can be uniquely characterized as the fixed point of a symmetry. This symmetry arises as the deformation of a symmetry of the Hopf equation.* This principle can be applied to characterize the universal correction $U(X,T)$. First, we note that the rescaled function $U^\mu(X,T) = \mu^{3\beta+1}U(X/\mu^{9\beta+3}, T/\mu^{\frac{6\beta+2}{3}})$ satisfies the long wave limit of (9), i.e.,

$$U^\mu_T = U^\mu U^\mu_X + \mu\,\bar{L}_{u_c}[U^\mu]. \tag{12}$$

The limit $\mu \to 0$ is well defined because $U^0(X,T)$ coincides with the solution of the cubic equation (7). In addition, the latter is a solution of the Hopf equation that can be characterized as the unique stationary solution (vanishing at $X = T = 0$) of the flow (see Fig. 4)

$$U^0_S = \partial_X(X - U^0 T + (U^0)^3), \tag{13}$$

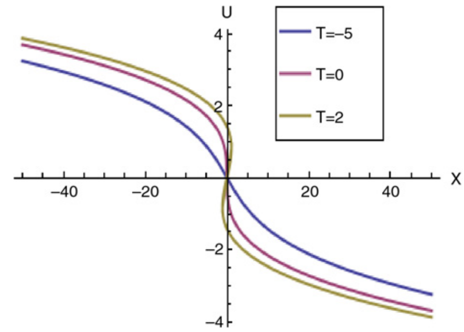

Fig. 4. Solution of the cubic equation, showing solutions of the Hopf equation close to the shock. After the shock $T > 0$, the solution is multi-valued because the Hopf equation is no longer a good model.

which commutes with Hopf. In other words, U^0 is the fixed point of the symmetry generated by the flow (13). Next, we follow the general principle stated above to characterize $U^\mu(X,T)$ as the fixed point of the flow

$$U^\mu_S = \partial_X(X - U^\mu T + (U^\mu)^3 + \mu\,\alpha_1[U^\mu] + \mu^2\alpha_2[U^\mu] + \cdots), \tag{14}$$

obtained as the (unique) power series in μ commuting order by order with (12).[6] By definition, the **string equation** is the equation for the vanishing of the right-hand side of (14). In general, we expect the symmetry to be an infinite (possibly not converging) power series in μ. In this case, the string equation will be valid only asymptotically for small values of μ, or equivalently for $X \gg 0$. However, if the string equation truncates, we can safely use $\mu = 1$ to obtain the exact form of $U \equiv U^1$. Then, we conjecture that the function $U(X,T)$ is uniquely characterized as the solution of the string equation that satisfies the boundary behavior

$$U(X,T) \sim -\operatorname{sign}(X)|X|^{\frac{1}{3}} \quad \text{as } |X| \to \infty,\ \forall T, \tag{15}$$

which ensures the correct long wave ($\mu \to 0$) limit. The string equation is finite for at least three universality classes: Burgers, B–O, and KdV. We note that the string equation method is a valid alternative to the classical approach to shock based on a step-function with the initial data [17] because it contains all of the universal information.

Example 6. Burgers universality class. The equation

$$U_S = \partial_X(X - UT + U^3 - 6UU_X + 4U_{XX}),$$

is a symmetry of Burgers (see [24]) and thus the string equation is

$$X - UT + U^3 - 6UU_X + 4U_{XX} = 0. \tag{16}$$

The unique solution that satisfies (15) can be written explicitly in terms of the Pearcey integral [12,24] and it is plotted in Fig. 5. It has been proved [11] that sufficiently regular solutions of the Burgers equation admit an expansion (8), where $U(X,T)$ is the exact solution of (16).

Example 7. The KdV universality class. The equation

$$U_S = \partial_X\left(X - U\,T + U^3 - 3\,U_X^2 - 6\,U\,U_{XX} + \frac{18}{5}\,U_{XXXX}\right)$$

is a symmetry of the KdV equation [9], and the string equation satisfied by $U(X,T)$ is

$$X - U\,T + U^3 - 3\,U_X^2 - 6\,U\,U_{XX} + \frac{18}{5}\,U_{XXXX} = 0. \tag{17}$$

[6] The existence of such a deformation can be established when L is a differential operator using the method of [33].

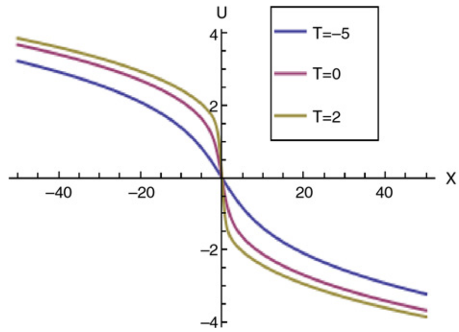

Fig. 5. Solution of Eq. (16), which represents the universal transition from a regular wave ($T < 0$) to a classical shock wave ($T > 0$). As expected, the wave becomes steeper but no oscillations emerge.

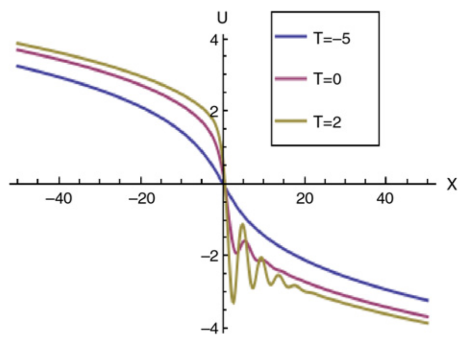

Fig. 6. Solution of the KdV string equation, which represents solutions of equations in the KdV class close to the shock. For $T > 0$, the wave oscillates and exhibits the typical pattern of a dispersive-shock.

The unique solution that satisfies the boundary condition (15) [34] is plotted in Fig. 6. This ODE is known to be the second equation of the Painlevé I hierarchy and it is used in the context of random matrix theory [35,36]. It is known that Painlevé equations can be linearized using an isomonodromy system [37]. It was proved in [38,39] that sufficiently regular solutions of any equation of the KdV hierarchy admit an expansion (8), where $U(X, T)$ is the exact solution of (17). The extension of this result to other local Hamiltonian PDEs, which have yet to be proved, is known as Dubrovin's universality conjecture [9].

Example 8. The B–O string equation for U is finite and it is given by the formula

$$X - UT + U^3 - 3\,U\mathcal{H}[U_X] - 3\,\mathcal{H}[UU_X] - 4\,U_{XX} = 0, \qquad (18)$$

because both $\partial_X(X - UT)$ and $\partial_X(U^3 - 3\,U\mathcal{H}[U_X] - 3\,\mathcal{H}[UU_X] - 4\,U_{XX})$ are symmetries of B–O [27].

A new Painlevé equation?

Eq. (18) is particularly important because this is the first time that a nonlocal ODE resembling a Painlevé equation has been reported. We investigated Eq. (18) numerically using a spectral method where the Hilbert transform was computed following [15]. According to our numerical results (Fig. 7), for any real T, Eq. (18) admits a unique solution that satisfies (15) and thus U solves B–O. We note that Eq. (18) is a candidate for a new class of Painlevé equations. In fact, both (16) and (17) can be linearized, and they satisfy the Painlevé property [37], for any solution that extends to a meromorphic function in the complex plane. Thus, various important questions are raised, as follows. Does the unique solution of (18) that satisfies (15) expand to a meromorphic function? Does (18) admit a linearization using an isomonodromic system? In many cases, we note that the isomondromic system for a string equation that arises from an integrable hierarchy can be constructed from the zero-curvature representation of the latter [40,41]. However, to the best of our knowledge, no zero-curvature representation of B–O has been discovered.

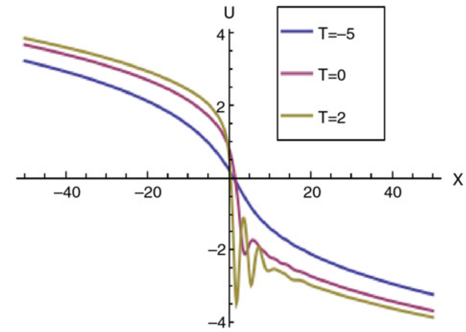

Fig. 7. Solution of the Benjamin–Ono string equation. The shock is again dispersive but the oscillations appear to have shorter a wavelength and a larger amplitude compared with the KdV case.

We conclude this letter by summarizing our results. By introducing an order parameter, i.e., the wave amplitude in the Whitham zone, we analyzed the dispersive-shock-transition based on statistical physics and we showed that it corresponds to a tricritical point. By generalizing an argument of Dubrovin [9], we then refined the coarse description of the transition using the amplitude of the oscillations in the Whitham zone and we determined the precise local behavior of the wave close to the shock with the string equation, which encodes all universal features of the transition. In particular, we obtained an explicit description of the critical behavior of solutions to the B–O equation and we demonstrated that it is a model equation for a new universality class of dispersive shock waves. Although our main focus is the dispersive-shock, our classification of universality classes also helps to understand dissipative and dispersive-dissipative equations, by modeling media where dispersion and diffusion are balanced. Interestingly, their universal classes are PDEs with non-local interactions, similar to those used in experimental analyses of dispersive shocks [4]. At present, we are developing the necessary mathematical tools to explicitly compute the string equation for these more general cases. It would be very interesting to rigorously prove the results claimed in the present study, particularly with respect to the critical behavior of the B–O equation.

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Appendix

We prove formula (10). We suppose that N is a continuous operator on the space of Schwartz functions, which can possibly be extended to a larger space by continuity. We claim that if a $\beta > 0$ and a second operator $\bar{N}$ exist such that N satisfies

$$S_{\varepsilon^{-1}} N S_{\varepsilon} = \varepsilon^{\beta+1}\bar{N} + o(\varepsilon^{\beta}),$$

for any Schwartz function u, then

$$S_{\varepsilon^{-1}} \bar{N} S_{\varepsilon} = \varepsilon^{\beta+1}\bar{N},$$

which is precisely (10). The dilation operator S_ε given above is defined as in (3). To prove our claim, we note that the dilation operator is a continuous operator on the space of Schwartz functions. Therefore, it acts continuously by conjugation on the space of the continuous operators on that space (embedded with a weak topology). Therefore, the thesis emerges as follows.

Lemma 1. *Let $G(\varepsilon)$, $\varepsilon > 0$ be a group of continuous operators on a vector space V. If $G(\varepsilon)v = \varepsilon^\alpha \bar{v} + o(\varepsilon^\alpha)$, then $G(\varepsilon)\bar{v} = \varepsilon^\alpha \bar{v}$.*

Proof. $\bar{v} = \lim_{\varepsilon\to 0} \frac{G(\varepsilon)v}{\varepsilon^\alpha}$. Since G acts continuously, then for every $\delta \in \mathbb{R}^+$, $G(\delta)\bar{v} = \lim_{\varepsilon\to 0} \frac{G(\delta\varepsilon)v}{\varepsilon^\alpha} = \delta^\alpha \bar{v}$. $\square$

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