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Multi-soliton, multi-breather and higher order rogue wave solutions to the complex short pulse equation

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HIGHLIGHTS

- We construct a generalized Darboux transformation (gDT) to the CSP and CCD equations.
- We derive multi-bright soliton solution to the CSP equation based on the gDT.
- We construct a single and multi-breather solutions to the CSP equation.
- First and higher order rogue wave solutions to the CSP equation are constructed.

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ABSTRACT

In the present paper, we are concerned with the general analytic solutions to the complex short pulse (CSP) equation including soliton, breather and rogue wave solutions. With the aid of a generalized Darboux transformation, we construct the N-bright soliton solution in a compact determinant form, the N-breather solution including the Akhmediev breather and a general higher order rogue wave solution. The first and second order rogue wave solutions are given explicitly and analyzed. The asymptotic analysis is performed rigorously for both the N-soliton and the N-breather solutions. All three forms of the analytical solutions admit either smoothed-, cusped- or looped-type ones for the CSP equation depending on the parameters. It is noted that, due to the reciprocal (hodograph) transformation, the rogue wave solution to the CSP equation can be a smoothed, cusponed or a looped one, which is different from the rogue wave solution found so far.

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1. Introduction

The nonlinear Schrödinger (NLS) equation, as one of the universal models that describe the evolution of slowly varying packets of quasi-monochromatic waves in weakly nonlinear dispersive media, plays a key role in nonlinear optics [1,2]. Recently, several reported experiments were related to the modulational instability (MI) and the breather solution of the NLS equation in nonlinear optics [3,4]. The Akhmediev breather (periodic in space but localized in time) [5], the Peregrine soliton or rogue wave (RW) solution (time and space homoclinic) [6] and the Kuznetsov–Ma soliton (periodic in time but localized in space) [7] have recently been experimentally observed in optical fibers [8–10] in succession. Besides the experimental observation in optical fibers, the RWs have also been observed in water-wave tanks [11] and plasmas [12].

However, in the regime of ultra-short pulses where the width of optical pulse is in the order of femtosecond (10^{-15} s), the quasi-monochromatic assumption to derive the NLS equation is not valid anymore [13]. Description of ultra-short processes requires a modification of standard slow varying envelope models based on the NLS equation. There are usually two ways to satisfy this requirement in the literature. The first one is to add several higher-order dispersive terms to yield higher-order NLS equation [2]. The second one is to construct a suitable fit to the frequency-dependent dielectric constant $\varepsilon(\omega)$ in the desired spectral range. Several models have been proposed by the latter approach such as the short-pulse (SP) equation [14–16].

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Recently, Schäfer and Wayne derived a short pulse (SP) equation [14]

$$u_{xt} = u + \frac{1}{6}(u^3)_{xx}, \tag{1}$$

to describe the propagation of ultra-short optical pulses in nonlinear media. Here, $u = u(x, t)$ is a real-valued function, representing the magnitude of the electric field. The SP equation (1) is shown to be completely integrable [17–21]. The periodic and soliton solutions of the SP equation (1) were found in [22–24]. The connection between the SP equation (1) and the sine-Gordon equation through the reciprocal transformation was firstly discovered in [19], then was further clarified and used to find two-loop and breather solutions in [22] by the same authors. The general N-soliton solution including multi-loop and multi-breather ones was given in [25,26] by using Hirota's bilinear method [27]. The integrable discretization and the geometric interpretation of the SP equation were given in [28,29].

Most recently, one of the authors proposed a complex short pulse (CSP) equation [30,31]

$$q_{xt} + q + \frac{1}{2}(|q|^2 q_x)_x = 0, \tag{2}$$

and a coupled complex short pulse (CCSP) equation [31]

$$q_{1,xt} + q_1 + \frac{1}{2}\left((|q_1|^2 + |q_2|^2)q_{1,x}\right)_x = 0, \tag{3}$$

$$q_{2,xt} + q_2 + \frac{1}{2}\left((|q_1|^2 + |q_2|^2)q_{2,x}\right)_x = 0, \tag{4}$$

that govern the propagation of ultra short pulse packet along optical fibers. There are several advantages in using complex representation description of wave phenomenon, especially of the optical waves [32]. Firstly, amplitude and phase are two fundamental characteristics for a wave packet, the information of these two factors are nicely combined into a single complex-valued function. Secondly, the use of complex representation can make a lot of manipulations including soliton interactions much easier. Such advantages can be observed in many analytical results related to the NLS equation, the complex short pulse equation and their coupled models. As is shown in [30,31,33], in contrast with the fact that one-soliton solution to the SP equation is always a loop soliton without physical meaning (1), the one-soliton solution to the CSP equation (2) is an envelope soliton with a few optical cycles. In some sense, the CSP equation can be viewed as an analogue of the NLS equation in ultra-short pulse regime.

As a matter of fact, a complex-valued short pulse equation, slightly different from the CSP equation (2), has been studied previously (see Eq. (8) in [34]). Whereas the integrability of Eq. (8) in [34] remains unclear, the integrability and the general N-soliton solution of the CSP and the CCSP equations were firstly clarified in [31]. We note that the CSP equation (2) and its soliton solution were also investigated recently in [35]. It should be pointed out that the CSP equation is mathematically related to a two-component short pulse (2-SP) equation proposed by Dimakis and Müller–Hoissen [36] and Matsuno [37] independently. The bi-Hamiltonian structure of this 2-SP equation was formulated by Brunelli and Sakovich [38]. If we take $u = \mathrm{Re}(q)$ and $v = \mathrm{Im}(q)$, then the 2-SP equation in [36,37] becomes the CSP equation. The breather solution found in [37] is actually the bright soliton solution either in terms of pfaffian form [31] or in determinant form [30,33]. As a counterpart of the NLS equation in ultra-short pulse regime, it is natural to ask a question: are there also breather and rogue wave solutions to the CSP equation (2) in addition to the bright soliton solution? It is the aim of the present paper to investigate all kinds of solutions of the CSP equation by generalized Darboux transformation.

Based on the previous study [30,33], it is known that the CSP equation (2) is linked to a complex coupled dispersionless (CCD) equation [39]

$$\begin{aligned} &q_{ys} = \rho q, \\ &\rho_s + \frac{1}{2}(|q|^2)_y = 0, \end{aligned} \tag{5}$$

through the following reciprocal (hodograph) transformation

$$\mathrm{d}x = \rho \mathrm{d}y - \frac{1}{2}|q|^2 \mathrm{d}s, \qquad \mathrm{d}t = -\mathrm{d}s. \tag{6}$$

The CCD equation (5) is the first negative flow of the Landau–Lifschitz hierarchy while the SP and the CSP equations being the first negative flow of Wadati–Konno–Ichikawa (WKI) hierarchy [40,41]. By constructing a generalized Darboux transformation to the CCD equation and integrating the integrals exactly involved in the reciprocal (hodograph) transformation, we are able to construct the general analytical solutions to the CSP equation including the N-bright soliton, N-breather solution and higher order rogue wave solutions.

It should be pointed out that the compact formulas for these solutions are more convenient for us to perform the asymptotic analysis. Recently the modulational instability has been also considered as a wave breaking mechanism [42]. Indeed, if the initial steepness of the monochromatic wave is large, during the process of modulational instability, one wave will start growing and will soon reach the limiting steepness, and break before becoming a rogue wave. The NLS theory does not predict the breaking or overturning of the waves [43]. Different from previous research regarding the rogue wave solution to the NLS equation, we find that there exists the wave breaking phenomenon in the rogue wave theory of the CSP equation (2). These results could deepen our understanding about the MI mechanism [44].

The outline of the present paper is organized as follows. In Section 2, the generalized Darboux transformation [45–47] of the CCD equation is derived through loop group method [48]. Based on the generalized Darboux transformation, we obtain the general soliton formulas for the CCD equation. Further, by integrating the reciprocal transformation exactly, we construct the general soliton formulas for the CSP equation. In Section 3, the N-bright soliton solution and the N-breather solution are constructed, and their asymptotic analyses are performed. In Section 4, we construct the rogue wave solution including the first order and general higher order rogue wave solutions. Section 4 is devoted to conclusions and some discussions. In Appendices, we give the details involving the proofs of asymptotic analysis and the modulational instability analysis.

2. Generalized Darboux transformation for the CSP equation

Prior to giving the Darboux transformation (DT) for the CSP equation (2), we briefly review the link between the CSP equation and the CCD equation. It is known that the CCD equation (5) admits the following Lax pair

$$\begin{aligned} &\Psi_y = U(\rho, q; \lambda)\Psi, \\ &\Psi_s = V(q; \lambda)\Psi, \end{aligned} \tag{7}$$

where

$$\begin{aligned} &U(\rho, q; \lambda) = \begin{bmatrix} -\frac{\mathrm{i}\rho}{\lambda} & -\frac{q_y^*}{\lambda} \\ \frac{q_y}{\lambda} & \frac{\mathrm{i}\rho}{\lambda} \end{bmatrix}, \\ &V(q; \lambda) = \left(\frac{\mathrm{i}}{4}\lambda\sigma_3 + \frac{\mathrm{i}}{2}Q\right), \qquad \sigma_3 = \mathrm{diag}(1, -1), \\ &Q = \begin{bmatrix} 0 & q^* \\ q & 0 \end{bmatrix}, \end{aligned} \tag{8}$$

and * represents the complex conjugate. Through the reciprocal transformation (6), one can obtain the CSP equation (2) and its Lax pair:

$$\begin{aligned}\Psi_x &= \begin{bmatrix} -\frac{\mathrm{i}}{\lambda} & -\frac{q_x^*}{\lambda} \\ \frac{q_x}{\lambda} & \frac{\mathrm{i}}{\lambda} \end{bmatrix} \Psi, \\ \Psi_t &= \begin{bmatrix} -\frac{\mathrm{i}}{4}\lambda + \frac{\mathrm{i}|q|^2}{2\lambda} & -\frac{\mathrm{i}q^*}{2} + \frac{|q|^2 q_x^*}{2\lambda} \\ -\frac{\mathrm{i}q}{2} - \frac{|q|^2 q_x}{2\lambda} & \frac{\mathrm{i}}{4}\lambda - \frac{\mathrm{i}|q|^2}{2\lambda} \end{bmatrix} \Psi. \end{aligned} \tag{9}$$

On the contrary, the CSP equation (2) can be transformed into the CCD equation (5). Note that the CSP equation (2) can be rewritten as the following conservative form

$$\left(\sqrt{1+|q_x|^2}\right)_t + \frac{1}{2}\left(|q|^2\sqrt{1+|q_x|^2}\right)_x = 0, \tag{10}$$

thus, by letting $\rho^{-1} = \sqrt{1+|q_x|^2}$ and defining an inverse reciprocal transformation

$$\mathrm{d}y = \rho^{-1}\mathrm{d}x - \frac{1}{2}\rho^{-1}|q|^2\mathrm{d}t, \qquad \mathrm{d}s = -\mathrm{d}t, \tag{11}$$

we can convert system (9) into system (7). The equivalence between the CSP and the CCD equations is kind of formal under the reciprocal and inverse reciprocal transformations. The rigorous equivalence is valid only if $\rho \neq 0$ for $(y, s) \in \mathbb{R}^2$, or $|q_x| \neq \infty$ for $(x, t) \in \mathbb{R}^2$.

To construct the soliton and rogue wave solutions for the CSP equation (2), we give the following proposition

Proposition 1. *The Darboux matrix*

$$T = I + \frac{\lambda_1^* - \lambda_1}{\lambda - \lambda_1^*}P_1, \quad P_1 = \frac{|y_1\rangle\langle y_1|}{\langle y_1|y_1\rangle}, \quad \langle y_1| = |y_1\rangle^\dagger,$$

$$|y_1\rangle = \begin{bmatrix} \psi_1(y, s; \lambda_1) \\ \phi_1(y, s; \lambda_1) \end{bmatrix} \tag{12}$$

where $|y_1\rangle$ is a special solution for linear system (7) with $\lambda = \lambda_1$, can convert system (7) into a new system

$$\begin{aligned}\Psi[1]_y &= U(\rho[1], q[1]; \lambda)\Psi[1], \\ \Psi[1]_s &= V(\rho[1], q[1]; \lambda)\Psi[1]. \end{aligned} \tag{13}$$

The Bäcklund transformations between $(\rho[1], q[1])$ and (ρ, q) are given through

$$\begin{aligned}\rho[1] &= \rho - 2\ln_{ys}\left(\frac{\langle y_1|y_1\rangle}{\lambda_1^* - \lambda_1}\right), \\ q[1] &= q + \frac{(\lambda_1^* - \lambda_1)\psi_1^*\phi_1}{\langle y_1|y_1\rangle}, \\ |q[1]|^2 &= |q|^2 + 4\ln_{ss}\left(\frac{\langle y_1|y_1\rangle}{\lambda_1^* - \lambda_1}\right). \end{aligned} \tag{14}$$

Proof. The Darboux transformation for the system (7) is a standard one for the AKNS system with $SU(2)$ symmetry. The rest of the proposition is to prove the formulas (14), which carry on some ideas from the classical monograph [49].

Suppose there is a holomorphic solution for Lax pair equation (7) in some punctured neighborhood of infinity on the Riemann surface, smoothing depending on y and s. Thus, we may assume the following asymptotical expansion as $\lambda \to \infty$.

$$\begin{bmatrix} \psi_1 \\ \phi_1 \end{bmatrix} = \left[\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \sum_{i=1}^{\infty} \Psi_i \lambda^{-i}\right] \exp\left(\frac{\mathrm{i}}{4}\lambda s\right), \tag{15}$$

for the wave function Ψ and

$$T = I + \sum_{i=1}^{\infty} T^{[i]}\lambda^{-i}, \tag{16}$$

for the Darboux matrix T. Since T is the Darboux matrix, it satisfies the following relation

$$T_y + TU = U[1]T. \tag{17}$$

By comparing the entries of the matrices, we get

$$\begin{aligned} q_y[1] &= q_y + \left(T_{2,1}^{[1]}\right)_y, \\ \rho[1] &= \rho + \mathrm{i}\left(T_{1,1}^{[1]}\right)_y. \end{aligned} \tag{18}$$

Integrating the first equation with respect to y, we have the second equation in (14). Let

$$H \equiv q^*\frac{\phi_1}{\psi_1} = \sum_{i=1}^{\infty} H_i \lambda^{-i},$$

we then have

$$\begin{aligned}(\ln H)_s &= \frac{\phi_{1,s}}{\phi_1} - \frac{\psi_{1,s}}{\psi_1} + (\ln q^*)_s \\ &= -\frac{\mathrm{i}}{2}\lambda - \frac{\mathrm{i}}{2}H + \frac{\mathrm{i}}{2}|q|^2 H^{-1} + (\ln q^*)_s \end{aligned}$$

from the first equation of (7). Thus

$$H_s = \frac{\mathrm{i}}{2}|q|^2 - \frac{\mathrm{i}}{2}\lambda H - \frac{\mathrm{i}}{2}H^2 + (\ln q^*)_s H.$$

Then the coefficient H_i can be determined as follows:

$$H_1 = |q|^2, \qquad H_2 = 2\mathrm{i}q_s q^*,$$

$$H_{i+1} = 2\mathrm{i}q^*\left(\frac{H_i}{q^*}\right)_s - \sum_{j=1}^{i-1} H_j H_{i-j}, \quad i \geq 2.$$

On the one hand, the first equation of (7) can be rewritten as

$$\psi_{1,s} = \left(\frac{\mathrm{i}}{4}\lambda + \frac{\mathrm{i}}{2}\sum_{i=1}^{\infty} H_i \lambda^{-i}\right)\psi_1.$$

Substituting the asymptotical expansion (15)

$$\psi_1 = \left(1 + \sum_{i=1}^{\infty} \Psi_i^{[1]}\lambda^{-i}\right)\exp\left(\frac{\mathrm{i}}{4}\lambda s\right)$$

into above equation, where superscript $^{[1]}$ represents the first component of the vector, we then have

$$\Psi_{1,s}^{[1]} = \frac{\mathrm{i}}{2}H_1 = \frac{\mathrm{i}}{2}|q|^2. \tag{19}$$

Similarly, by assuming an asymptotical expansion

$$\begin{bmatrix} \psi_1[1] \\ \phi_1[1] \end{bmatrix} = \left[\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \sum_{i=1}^{\infty} \Psi[1]_i \lambda^{-i}\right]\exp\left(\frac{\mathrm{i}}{4}\lambda s\right), \quad \lambda \to \infty, \tag{20}$$

we have

$$\Psi[1]_{1,s}^{[1]} = \frac{\mathrm{i}}{2}|q[1]|^2. \tag{21}$$

Moreover, by Darboux transformation

$$\begin{bmatrix}\psi_1[1]\\ \phi_1[1]\end{bmatrix} = \left(I + \sum_{i=1}^{\infty} T^{[i]}\lambda^{-i}\right)\left[\begin{bmatrix}1\\0\end{bmatrix} + \sum_{i=1}^{\infty}\Psi_i\lambda^{-i}\right]\exp\left(\frac{\mathrm{i}}{4}\lambda s\right),$$

one can obtain

$$\left(T_{1,1}^{[1]}\right)_s + \Psi_{1,s}^{[1]} = \frac{\mathrm{i}}{2}|q[1]|^2, \tag{22}$$

where the element $T_{i,j}^{[1]}$ denotes the (i,j)th entry of matrix $T^{[1]}$. Together with (19), we can obtain

$$|q[1]|^2 = |q|^2 - 2\mathrm{i}\left(T_{1,1}^{[1]}\right)_s. \tag{23}$$

Next, we proceed to the calculation of $\left(T_{1,1}^{[1]}\right)_s$ and $\left(T_{1,1}^{[1]}\right)_y$. Since

$$|y_1\rangle_s = \left(\frac{\mathrm{i}}{4}\lambda_1\sigma_3 + \frac{\mathrm{i}}{2}Q\right)|y_1\rangle,$$

$$-\langle y_1|_s\sigma_3 = \langle y_1|\sigma_3\left(\frac{\mathrm{i}}{4}\lambda_1^*\sigma_3 + \frac{\mathrm{i}}{2}Q\right),$$

which originates from the Lax pair of the CSP equation (2), we then have

$$\left(\frac{\langle y_1|y_1\rangle}{\lambda_1^* - \lambda_1}\right)_s = \frac{\mathrm{i}}{4}(-|\psi_1|^2 + |\phi_1|^2). \tag{24}$$

On the other hand,

$$\langle y_1|y_1\rangle = |\psi_1|^2 + |\phi_1|^2,$$

which implies

$$\left(\frac{|\psi_1|^2}{\langle y_1|\sigma_3|y_1\rangle}\right)_y = -\left(\frac{|\phi_1|^2}{\langle y_1|\sigma_3|y_1\rangle}\right)_y.$$

Thus, we have

$$(T_{1,1}^{[1]})_y = \left(\frac{|\psi_1|^2}{\frac{\langle y_1|y_1\rangle}{\lambda_1^*-\lambda_1}}\right)_y = \left(\frac{|\psi_1|^2 - |\phi_1|^2}{2\frac{\langle y_1|y_1\rangle}{\lambda_1^*-\lambda_1}}\right)_y = 2\mathrm{i}\ln_{ys}\left(\frac{\langle y_1|y_1\rangle}{\lambda_1^* - \lambda_1}\right).$$

Similarly, we could derive

$$(T_{1,1}^{[1]})_s = 2\mathrm{i}\ln_{ss}\left(\frac{\langle y_1|y_1\rangle}{\lambda_1^* - \lambda_1}\right). \tag{25}$$

Finally, combining (23) and (18), we obtain the last two formulas in (14). This completes the proof. □

To construct a general Darboux matrix, the following identities will be used. Suppose M is a $N\times N$ matrix, ϕ, ψ are $1\times N$ column vectors, then we have the following identities

$$\begin{aligned}\phi M^{-1}\psi^\dagger &= \frac{\begin{vmatrix} M & \psi^\dagger\\ -\phi & 0\end{vmatrix}}{|M|},\\ 1 + \phi M^{-1}\psi^\dagger &= \frac{\begin{vmatrix} M & \psi^\dagger\\ -\phi & 1\end{vmatrix}}{|M|} = \frac{\det(M+\psi^\dagger\phi)}{\det(M)},\end{aligned} \tag{26}$$

where † represents the Hermite conjugate. Then we have the following proposition which gives the N-fold Darboux transformation and the generalized N-fold Darboux transformation for the CSP equation

Proposition 2. *The N-fold Darboux transformation for the CCD equation can be represented as*

$$T_N = I + YM^{-1}D^{-1}Y^\dagger, \tag{27}$$

where $Y = [|y_1\rangle, |y_2\rangle, \ldots, |y_N\rangle]$, *and*

$$M = \left(\frac{\langle y_i|y_j\rangle}{\lambda_i^* - \lambda_j}\right)_{1\le i,j\le N},$$

$$D = \mathrm{diag}\left(\lambda - \lambda_1^*, \lambda - \lambda_2^*, \ldots, \lambda - \lambda_N^*\right).$$

Moreover, the general Darboux matrix is

$$T_N = I + YM^{-1}D^{-1}Y^\dagger, \tag{28}$$

where

$$Y = \left[|y_1^{[0]}\rangle, |y_1^{[1]}\rangle, \ldots, |y_1^{[n_1-1]}\rangle, \ldots, |y_r^{[0]}\rangle, |y_r^{[1]}\rangle, \ldots, |y_r^{[n_r-1]}\rangle\right],$$

$$M = \begin{bmatrix} M_{11} & M_{12} & \cdots & M_{1r}\\ M_{21} & M_{22} & \cdots & M_{2r}\\ \vdots & \vdots & \ddots & \vdots\\ M_{21} & M_{22} & \cdots & M_{2r}\end{bmatrix},$$

$$M_{ij} = \begin{bmatrix} M_{ij}^{[1,1]} & M_{ij}^{[1,2]} & \cdots & M_{ij}^{[1,n_j]}\\ M_{ij}^{[2,1]} & M_{ij}^{[2,2]} & \cdots & M_{ij}^{[2,n_j]}\\ \vdots & \vdots & \ddots & \vdots\\ M_{ij}^{[n_i,1]} & M_{ij}^{[n_i,2]} & \cdots & M_{ij}^{[n_i,n_j]}\end{bmatrix},$$

$$D = \mathrm{diag}\left(D_1, D_2, \ldots, D_r\right), \qquad D_i = \begin{bmatrix} D_i^{[0]} & \cdots & D_i^{[n_i-1]}\\ 0 & \ddots & \vdots\\ 0 & 0 & D_i^{[0]}\end{bmatrix}$$

and

$$|y_i(\lambda_i + \alpha_i\epsilon_i)\rangle = \sum_{k=0}^{n_i-1}|y_i^{[k]}\rangle\epsilon_i^k + O(\epsilon_i^{n_i}),$$

$$\frac{1}{\lambda - \lambda_i^* - \alpha_i\epsilon_i^*} = \sum_{k=0}^{n_i-1} D_i^{[k]}\epsilon_i^{*k} + O(\epsilon_i^{*n_i}),$$

$$\frac{\langle y_i(\lambda_i + \alpha_i\epsilon_i)|y_j(\lambda_j + \alpha_j\epsilon_j)\rangle}{\lambda_i^* - \lambda_j + \alpha_i^*\epsilon_i^* - \alpha_j\epsilon_j} = \sum_{k=1}^{n_i}\sum_{l=1}^{n_j} M_{ij}^{[k,l]}\epsilon_i^{*k}\epsilon_j^l + O(\epsilon_i^{*n_i}, \epsilon_j^{n_j}).$$

The general Bäcklund transformations are

$$\begin{aligned}\rho[N] &= \rho - 2\ln_{ys}(\det(M)),\\ q[N] &= q + \frac{\det(G)}{\det(M)},\\ |q[N]|^2 &= |q|^2 + 4\ln_{ss}(\det(M)),\end{aligned} \tag{29}$$

where $G = \begin{bmatrix} M & Y_1^\dagger\\ -Y_2 & 0\end{bmatrix}$, Y_k *represents the kth row of matrix* Y.

Proof. Through the standard iterated step for DT [47], we can obtain the N-fold DT. Next, by using the following equalities

$$\begin{aligned}(T_{1,1}^{[1]})_y &= \left(Y_1M^{-1}Y_1^\dagger\right)_y = \left(-Y_2M^{-1}Y_2^\dagger\right)_y\\ &= \left(\frac{Y_1M^{-1}Y_1^\dagger - Y_2M^{-1}Y_2^\dagger}{2}\right)_y = 2\mathrm{i}\ln_{ys}\left(\det(M)\right),\end{aligned}$$

$$(T_{1,1}^{[1]})_s = 2\mathrm{i}\ln_{ss}\left(\det(M)\right),$$

we can obtain the formula (29) from the above N-fold DT (27). To complete the generalized DT, we set

$$\begin{aligned}
\lambda_{r+1} &= \lambda_1 + \alpha_1\varepsilon_{1,1},\ |y_{r+1}\rangle = |y_1(\lambda_{r+1})\rangle;\ \ldots;\ \lambda_{r+n_1-1}\\
&= \lambda_1 + \alpha_1\varepsilon_{1,n_1-1},\ |y_{r+n_1-1}\rangle = |y_1(\lambda_{r+n_1-1})\rangle;\\
\lambda_{r+n_1} &= \lambda_2 + \alpha_2\varepsilon_{2,1},\ |y_{r+n_1}\rangle = |y_2(\lambda_{r+n_1})\rangle;\ \ldots,\ \lambda_{r+n_1+n_2-2}\\
&= \lambda_2 + \alpha_2\varepsilon_{2,n_2-1},\ |y_{r+n_1+n_2-2}\rangle = |y_2(\lambda_{r+n_1+n_2-2})\rangle;\\
&\vdots\\
\lambda_{N-n_r+1} &= \lambda_r + \alpha_r\varepsilon_{r,1},\ |y_{N-n_r+1}\rangle = |y_r(\lambda_{N-n_r+1})\rangle;\ \ldots;\ \lambda_N\\
&= \lambda_r + \alpha_r\varepsilon_{r,n_r-1},\ |y_N\rangle = |y_r(\lambda_N)\rangle.
\end{aligned}$$

Taking limit $\varepsilon_{i,j} \to 0$, we can obtain the generalized DT (28) and formulas (29). □

Recently the generalized DT for the AB system without the first and third relation in (29) was given in Ref. [50] in a different form. Actually, the first and third relations in (29) are the key procedures to construct the exact solution for the CSP equation. In summary, with the aid of reciprocal transformation (6), we obtain the general expression for N-soliton solution of the CSP equation (2):

$$\begin{aligned}
q[N] &= q + \frac{\det(G)}{\det(M)},\\
x &= \int \rho(y,s)\mathrm{d}y\\
&\quad - \frac{1}{2}\int |q(y,s)|^2\mathrm{d}s - 2\ln_s(\det(M)), \quad t = -s.
\end{aligned} \tag{30}$$

3. Multi-soliton and multi-breather solutions to the CSP equation

In this section, we provide multi-soliton and multi-breather solutions to the CSP equation by using formula (30).

3.1. Single soliton solution and N-soliton solution

We start with a seed solution

$$\rho[0] = -\frac{\gamma}{2}, \quad q[0] = 0,\ \gamma > 0. \tag{31}$$

Solving the Lax pair equation (7) with $(\rho, q; \lambda) = (\rho[0], q[0]; \lambda_i)$, we arrive at

$$\Psi_i = \begin{bmatrix} \mathrm{e}^{\theta_i} \\ \mathrm{e}^{-\theta_i} \end{bmatrix}, \quad \theta_i = \frac{\mathrm{i}\gamma}{2\lambda_i}y + \frac{\mathrm{i}\lambda_i}{4}s + a_i, \tag{32}$$

from which, we can obtain the single soliton solution through the formula (30):

$$\begin{aligned}
q[1] &= \lambda_{1,I}\mathrm{sech}(2\theta_{1,R})\mathrm{e}^{-2\mathrm{i}\theta_{1,I}-\frac{\pi\mathrm{i}}{2}},\\
x &= -\frac{\gamma}{2}y + \lambda_{1,I}\tanh(2\theta_{1,R}), \quad t = -s,
\end{aligned} \tag{33}$$

where $\lambda_1 = \lambda_{1,R} + \mathrm{i}\lambda_{1,I}, \theta_1 = \theta_{1,R} + \mathrm{i}\theta_{1,I}$. We comment here that λ_1 is the reciprocal of the wave number p_1 in [29]. As discussed in [29], if $\lambda_{1,R}^2 > \lambda_{1,I}^2$, one has the smooth soliton solution; if $\lambda_{1,R}^2 = \lambda_{1,I}^2$, one has the cusponed soliton solution; if $\lambda_{1,R}^2 < \lambda_{1,I}^2$, one obtains the loop soliton solution.

Furthermore, by using the N-fold DT, we could drive the N-soliton solution through the formula (30):

$$\begin{aligned}
q[N] &= \frac{\det(G)}{\det(M)},\\
x &= -\frac{\gamma}{2}y - 2\ln_s(\det(M)), \quad t = -s,
\end{aligned} \tag{34}$$

where

$$\begin{aligned}
M &= \left(\frac{\mathrm{e}^{\theta_i^*+\theta_j} + \mathrm{e}^{-\theta_i^*-\theta_j}}{\lambda_i^* - \lambda_j}\right)_{1\le i,j\le N}, \qquad G = \begin{bmatrix} M & Y_1^\dagger \\ -Y_2 & 0 \end{bmatrix},\\
Y_1 &= \left[\mathrm{e}^{\theta_1},\ \mathrm{e}^{\theta_2},\ \cdots,\ \mathrm{e}^{\theta_N}\right],\\
Y_2 &= \left[\mathrm{e}^{-\theta_1},\ \mathrm{e}^{-\theta_2},\ \cdots,\ \mathrm{e}^{-\theta_N}\right],
\end{aligned} \tag{35}$$

the expressions θ_i's are given in (32). The dynamics for two soliton is shown in Ref. [31]. Finally, to understand the dynamics of above N-soliton solution (34), we give the following asymptotic analysis and its proof.

Proposition 3. *Suppose* $0 < v_1 < v_2 < \cdots < v_N$. *When* $s \to \pm\infty$, *we have*

$$q[N] = \sum_{k=1}^{N} \lambda_{k,I}\mathrm{sech}(2\theta_{k,R}^{\pm})\mathrm{e}^{-2\mathrm{i}\theta_{k,I}^{\pm}-\frac{\pi\mathrm{i}}{2}} + O(\mathrm{e}^{-c|s|}), \tag{36}$$

where

$$\begin{aligned}
\theta_{k,R}^{\pm} &= \frac{\gamma\lambda_{k,I}}{2|\lambda_k|^2}(y - v_k s) + a_{k,R} \pm \Delta_{k,R}^{\pm},\\
\Delta_{k,R}^{\pm} &= \frac{1}{2}\left(\sum_{l=1}^{k-1}\left|\frac{\lambda_l^* - \lambda_k}{\lambda_l - \lambda_k}\right| - \sum_{l=k+1}^{N}\left|\frac{\lambda_l^* - \lambda_k}{\lambda_l - \lambda_k}\right|\right),\\
\theta_{k,I}^{\pm} &= \frac{\gamma\lambda_{k,R}}{2|\lambda_k|^2}y + \frac{\lambda_{k,R}}{4}s + a_{k,I} \mp \Delta_{k,I}^{\pm},\\
\Delta_{k,I}^{\pm} &= \frac{1}{2}\left(\sum_{l=1}^{k-1}\arg\left(\frac{\lambda_l - \lambda_k}{\lambda_l^* - \lambda_k}\right) - \sum_{l=k+1}^{N}\arg\left(\frac{\lambda_l - \lambda_k}{\lambda_l^* - \lambda_k}\right)\right),
\end{aligned} \tag{37}$$

and $c = \min\left(\left|\frac{\gamma\lambda_{k,I}}{2|\lambda_k|^2}\right|\right)\min_{i\neq j}(|v_i - v_j|),\ v_i = \frac{|\lambda_i|^2}{2\gamma}$.

The proof is given in Appendix A. Next we analyze the coordinates transformation: as $s \to \pm\infty$, along the line $\theta_{k,R}^{\pm} = 0$, we have

$$x = -\frac{\gamma}{2}y - 2\ln_s(M) \to -\frac{\gamma}{2}y \pm \left[\sum_{i=1}^{k-1}\lambda_{i,I} - \sum_{j=k+1}^{N}\lambda_{j,I}\right].$$

It follows that

Proposition 4. *When* $t \to \mp\infty$, *along the trajectory* $\theta_{k,R}^{\pm} = 0$, *we have*

$$q[N] = \sum_{k=1}^{N} \lambda_{k,I}\mathrm{sech}(2\theta_{k,R}^{\pm})\mathrm{e}^{-2\mathrm{i}\theta_{k,I}^{\pm}-\frac{\pi\mathrm{i}}{2}} + O(\mathrm{e}^{-c|t|}),$$

where

$$\begin{aligned}
\theta_{k,R}^{\pm} &= -\frac{\lambda_{k,I}}{|\lambda_k|^2}x + \frac{\lambda_{k,I}}{4}t + a_{k,R}\\
&\quad \pm \frac{\lambda_{k,I}}{|\lambda_k|^2}\left(\sum_{i=1}^{k-1}\lambda_{i,I} - \sum_{j=k+1}^{N}\lambda_{j,I}\right) \pm \Delta_{k,R}^{\pm},\\
\theta_{k,I}^{\pm} &= -\frac{\lambda_{k,R}}{|\lambda_k|^2}x - \frac{\lambda_{k,R}}{4}t + a_{k,I}\\
&\quad \pm \frac{\lambda_{k,I}}{|\lambda_k|^2}\left(\sum_{i=1}^{k-1}\lambda_{i,I} - \sum_{j=k+1}^{N}\lambda_{j,I}\right) \mp \Delta_{k,I}^{\pm}.
\end{aligned}$$

3.2. Single breather and multi-breather solutions

To find a single breather solution, we depart from a seed solution

$$\rho[0] = -\frac{\gamma}{2}, \quad q[0] = \frac{\beta}{2}e^{i\theta}, \quad \theta = y + \frac{\gamma}{2}s,$$
$$\gamma > 0, \quad \beta \geq 0. \tag{38}$$

Then we have the solution for the Lax pair equation (7) with $(q, \rho; \lambda) = (q[0], \rho[0]; \lambda_i)$,

$$|y_i\rangle = KL_iE_i, \quad K = \mathrm{diag}\left(e^{-\frac{i}{2}\theta}, e^{\frac{i}{2}\theta}\right), \ \lambda_i \neq -\gamma + i\beta, \tag{39}$$

where

$$L_i = \begin{bmatrix} 1 & 1 \\ \frac{\beta}{\gamma + \xi_i} & \frac{\beta}{\gamma + \chi_i} \end{bmatrix}, \qquad E_i = \begin{bmatrix} e^{\theta_i} \\ e^{-\theta_i} \end{bmatrix},$$

and

$$\theta_i = \frac{i}{4}\sqrt{\beta^2 + (\lambda_i + \gamma)^2}\left(s + \frac{2}{\lambda_i}y\right) + a_i,$$
$$\xi_i = \lambda_i + \sqrt{\beta^2 + (\lambda_i + \gamma)^2}, \qquad \chi_i = \lambda_i - \sqrt{\beta^2 + (\lambda_i + \gamma)^2}.$$

To avoid the inconvenience of involving the square root of a complex number, we introduce the following transformation:

$$\lambda_i + \gamma = \beta \sinh(\varphi_i), \quad \varphi_i \equiv \varphi_{i,R} + i\varphi_{i,I}, \ (\varphi_{i,R}, \varphi_{i,I}) \in \Omega,$$

where $\Omega = \{(\varphi_R, \varphi_I) | 0 < \varphi_I < \pi$, and $0 < \varphi_R < \infty$, or $\varphi_R = 0$, and $\frac{\pi}{2} \leq \varphi_I < \pi\}$, then

$$\xi_i + \gamma = \beta e^{\varphi_i}, \qquad \chi_i + \gamma = -\beta e^{-\varphi_i}.$$

By some tedious calculations, the single breather solution can be constructed from the formula (30) by using the technique [51]

$$q[1] = \frac{\beta}{2}\left[\frac{\cosh(2\theta_{1,R} - 2i\varphi_{1,I})\cosh(\varphi_{1,R}) + \sin(2\theta_{1,I} + 2i\varphi_{1,R})\sin(\varphi_{1,I})}{\cosh(2\theta_{1,R})\cosh(\varphi_{1,R}) - \sin(2\theta_{1,I})\sin(\varphi_{1,I})}\right]e^{i\theta},$$
$$x = -\frac{\gamma}{2}y - \frac{\beta^2}{8}s - 2\ln_s\left[\cosh(2\theta_{1,R})\cosh(\varphi_{1,R}) - \sin(2\theta_{1,I})\sin(\varphi_{1,I})\right],$$
$$t = -s, \tag{40}$$

where

$$\theta_{1,R} = \delta_1\left(y - \frac{2}{\gamma}v_1 s\right) - \varphi_{1,R} + a_{1,R},$$
$$\theta_{1,I} = \epsilon_1\left(y - \frac{2}{\gamma}w_1 s\right) - \varphi_{1,I} + a_{1,I},$$

and

$$v_1 = \frac{\alpha_1\gamma\sinh(\varphi_{1,R})}{4\left(\gamma\sinh(\varphi_{1,R}) + \beta\cos(\varphi_{1,I})\right)},$$
$$\delta_1 = \frac{2\beta}{\alpha_1}\sin(\varphi_{1,I})\left(\gamma\sinh(\varphi_{1,R}) + \beta\cos(\varphi_{1,I})\right),$$
$$w_1 = \frac{-\alpha_1\gamma\cos(\varphi_{1,I})}{4\left(\beta\sinh(\varphi_{1,R}) - \gamma\cos(\varphi_{1,I})\right)},$$
$$\epsilon_1 = \frac{2\beta}{\alpha_1}\cosh(\varphi_{1,R})\left(\beta\sinh(\varphi_{1,R}) - \gamma\cos(\varphi_{1,I})\right),$$
$$\alpha_1 = \left(\beta\sinh(\varphi_{1,R})\cos(\varphi_{1,I}) - \gamma\right)^2 + \beta^2\cosh^2(\varphi_{1,R})\sin^2(\varphi_{1,I}).$$

If $(\varphi_{1,R}, \varphi_{1,I}) \in \Omega_1 \equiv \{(\varphi_R, \varphi_I) | 0 \leq \varphi_R < \mathrm{arcsinh}(\frac{\beta}{\gamma})$, $\arccos(-\frac{\gamma}{\beta}\sinh(\varphi_R)) < \varphi_I < \pi\}$, then the single breather $|q[1]|^2$ propagates with velocity $\frac{2}{\gamma}v_1 \leq 0$. If $(\varphi_{1,R}, \varphi_{1,I}) \in \Omega_2 \equiv \{(\varphi_R, \varphi_I) | \frac{\pi}{2} \leq \varphi_I < \pi$, $\mathrm{arcsinh}(-\frac{\gamma}{\beta}\cos(\varphi_I)) < \varphi_R\}$, then the single breather $|q[1]|^2$ propagates with velocity $\frac{2}{\gamma}v_1 > 0$. An example of this case is illustrated in Fig. 1(a). If $\gamma\sinh(\varphi_{1,R}) + \beta\cos(\varphi_{1,I}) = 0$, then we can obtain the so-called Akhmediev breather, which is periodic in space and localized in time. Fig. 1(b) shows an example of Akhmediev breather.

The bright soliton can be derived from the breather solution by taking a limit $\beta \to 0$ and setting $e^{2a_1} = e^{\varphi_1} = \frac{\lambda_1 + \gamma}{\beta} + \sqrt{1 + \left(\frac{\lambda_1 + \gamma}{\beta}\right)^2} \to \infty$, but, as discussed previously, the bright soliton solution can be constructed in an easier way. To analyze the dynamics of the breather solution for the CSP equation (2), we need to solve the relation between (x, t) and (y, s). Although it is not possible in general, we can obtain the relation at special location $\theta_{1,R} = 0$ and $\theta_{1,I} = \kappa\pi + \frac{\pi}{4}$, that is, $s = -t$ and $y = -\frac{2}{\gamma}(x - \frac{\beta^2}{8}t)$. It follows that

$$\theta_{1,R} = -\frac{\gamma}{2}\delta_1\left[x - \left(v_1 + \frac{\beta^2}{8}\right)t\right] - \varphi_{1,R} + a_{1,R},$$
$$\theta_{1,I} = -\frac{\gamma}{2}\epsilon_1\left[x - \left(w_1 + \frac{\beta^2}{8}\right)t\right] - \varphi_{1,I} + a_{1,I}.$$

The breather solution $|q[1]|^2$ propagates with the velocity $v_1 + \frac{\beta^2}{8}$ (Fig. 1(a)). If $\delta_1 = 0$, we can obtain the Akhmediev breather (Fig. 1(b)). The periodic in x-direction is $\frac{2\pi}{\gamma|\epsilon_1|}$, and the periodic in t-direction is $\frac{2\pi}{\gamma|\epsilon_1|\left(w_1 + \frac{\beta^2}{8}\right)}$. The peak value of $|q[1]|^2$ is located at

$$x = \frac{1}{v_1 - w_1}\left[-\frac{2\left(v_1 + \frac{\beta^2}{8}\right)}{\gamma\epsilon_1}\left(\frac{\pi}{4} + \kappa\pi + (\varphi_{1,I} - a_{1,I})\right)\right.$$
$$\left. - \frac{2}{\gamma}\frac{a_{1,R} - \varphi_{1,R}}{\delta_1}\left(w_1 + \frac{\beta^2}{8}\right)\right],$$
$$t = \frac{1}{v_1 - w_1}\left[-\frac{2}{\gamma\epsilon_1}\left(\frac{\pi}{4} + \kappa\pi + (\varphi_{1,I} - a_{1,I})\right) - \frac{2}{\gamma}\frac{a_{1,R} - \varphi_{1,R}}{\delta_1}\right].$$

Similar to three cases of the single soliton solution, we can classify the single breather solution by defining

$$M_1 = \frac{1}{2}\beta^3\sinh(2\varphi_{1,R})\sin(2\varphi_{1,I})$$
$$- \gamma\left[(\cosh(\varphi_{1,R}) - \cos(\varphi_{1,I}))^2\right.$$
$$\left. + 2\cosh(\varphi_{1,R})\sin(\varphi_{1,I})\right]\beta^2$$
$$- 2\beta\gamma^2\sinh(\varphi_{1,R})\cos(\varphi_{1,I}) + \gamma^3. \tag{41}$$

It can be shown that if $M_1 > 0$, the breather solution is a smooth one; if $M_1 = 0$, the breather becomes a cusponed one, in which $|q_x| \to \infty$ at the peak point; if $M_1 < 0$, then we have a looped breather, which is a multi-valued solution.

Generally, through the formula (30) we have the following N-breather solution:

$$q[N] = \frac{\beta}{2}\left[\frac{\det(G)}{\det(M)}\right]e^{i\theta},$$
$$x = -\frac{\gamma}{2}y - \frac{\beta^2}{8}s - 2\ln_s(\det(M)), \quad t = -s, \tag{42}$$

where

$$M=\left(\left[\frac{\mathrm{e}^{2(\theta_i^*+\theta_j)}}{\xi_i^*-\xi_j}+\frac{\mathrm{e}^{2\theta_i^*}}{\xi_i^*-\chi_j}+\frac{\mathrm{e}^{2\theta_j}}{\chi_i^*-\xi_j}+\frac{1}{\chi_i^*-\chi_j}\right]\mathrm{e}^{-(\theta_i^*+\theta_j)}\right)_{1\le i,j\le N},$$

$$G=\left(\left[\frac{\xi_i^*+\gamma}{\xi_j+\gamma}\frac{\mathrm{e}^{2(\theta_i^*+\theta_j)}}{\xi_i^*-\xi_j}+\frac{\xi_i^*+\gamma}{\chi_j+\gamma}\frac{\mathrm{e}^{2\theta_i^*}}{\xi_i^*-\chi_j}+\frac{\chi_i^*+\gamma}{\xi_j+\gamma}\frac{\mathrm{e}^{2\theta_j}}{\chi_i^*-\xi_j}+\frac{\chi_i^*+\gamma}{\chi_j+\gamma}\frac{1}{\chi_i^*-\chi_j}\right]\mathrm{e}^{-(\theta_i^*+\theta_j)}\right)_{1\le i,j\le N}.$$

An example of two breather solution is shown in Fig. 1(c). The dynamics for N-breather solution is a very interesting topic. It is naturally to conjecture that the N-breather solution possesses the same law as the N-bright soliton solution. To understand the N-breather solution for the CSP equation (42), we first give the following asymptotical analysis for the CCD equation (5):

Proposition 5. *Suppose $v_1<v_2<\cdots<v_l\le 0<v_N<v_{N-1}<\cdots<v_{l+1}$. When $s\to-\infty$, we have*

$$\begin{aligned}q[N]=\frac{\beta}{2}\Big[&q_1^-+\left(q_2^--\mathrm{e}^{-2\mathrm{i}\varphi_{1,I}}\Theta_1^-\right)+\cdots\\&+\left(q_k^--\mathrm{e}^{-2\mathrm{i}\varphi_{k-1,I}}\Theta_{k-1}^-\right)+\left(q_N^--\mathrm{e}^{-2\mathrm{i}\varphi_{k,I}}\Theta_k^-\right)\\&+\left(q_{N-1}^--\mathrm{e}^{2\mathrm{i}\varphi_{N,I}}\Theta_N^-\right)+\cdots\\&+\left(q_{k+1}^--\mathrm{e}^{2\mathrm{i}\varphi_{k+1,I}}\Theta_{k+2}^-\right)\Big]\mathrm{e}^{\mathrm{i}\theta}+O(\mathrm{e}^{-c|s|}),\end{aligned}\tag{43}$$

where $c=\frac{2}{\gamma}\min(\delta_i)\min_{i\ne j}(|v_i-v_j|)$. When $s\to+\infty$, we have

$$\begin{aligned}q[N]=\frac{\beta}{2}\Big[&q_1^++\left(q_2^+-\mathrm{e}^{2\mathrm{i}\varphi_{1,I}}\Theta_1^+\right)+\cdots\\&+\left(q_k^+-\mathrm{e}^{2\mathrm{i}\varphi_{k-1,I}}\Theta_{k-1}^+\right)+\left(q_N^+-\mathrm{e}^{2\mathrm{i}\varphi_{k,I}}\Theta_k^+\right)\\&+\left(q_{N-1}^+-\mathrm{e}^{-2\mathrm{i}\varphi_{N,I}}\Theta_N^+\right)+\cdots\\&+\left(q_{k+1}^+-\mathrm{e}^{-2\mathrm{i}\varphi_{k+1,I}}\Theta_{k+2}^+\right)\Big]\mathrm{e}^{\mathrm{i}\theta}+O(\mathrm{e}^{-c|s|}),\end{aligned}\tag{44}$$

where

$$q_k^\pm=\Theta_k^\pm\left[\frac{\cosh(2\theta_{k,R}^\pm-2\mathrm{i}\varphi_{k,I})\cosh(\varphi_{k,R})+\sin(2\theta_{k,I}^\pm+2\mathrm{i}\varphi_{k,I})\sin(\varphi_{k,I})}{\cosh(2\theta_{k,R}^\pm)\cosh(\varphi_{k,R})-\sin(2\theta_{k,I}^\pm)\sin(\varphi_{k,I})}\right],$$

and

$$\begin{aligned}&\theta_{k,R}^+=\theta_{k,R}+\Delta_{k,R}^+,\\&\Delta_{k,R}^+=\frac{1}{2}\left[\bigstar\ln\left|\frac{\chi_n-\xi_k}{\chi_n^*-\xi_k}\right|\left|\frac{\chi_n^*-\chi_k}{\chi_n-\chi_k}\right|+\blacklozenge\ln\left|\frac{\xi_n-\xi_k}{\xi_n^*-\xi_k}\right|\left|\frac{\xi_n^*-\chi_k}{\xi_n-\chi_k}\right|\right],\\&\theta_{k,I}^+=\theta_{k,I}+\Delta_{k,I}^+,\\&\Delta_{k,I}^+=\frac{1}{2}\left[\bigstar\arg\left(\frac{\chi_n-\xi_k}{\chi_n^*-\xi_k}\frac{\chi_k^*-\chi_n^*}{\chi_k^*-\chi_n}\right)+\blacklozenge\arg\left(\frac{\xi_n-\xi_k}{\xi_n^*-\xi_k}\frac{\chi_k^*-\xi_n^*}{\chi_k^*-\xi_n}\right)\right],\\&\theta_{k,R}^-=\theta_{k,R}+\Delta_{k,R}^-,\\&\Delta_{k,R}^-=\frac{1}{2}\left[\bigstar\ln\left|\frac{\xi_n-\xi_k}{\xi_n^*-\xi_k}\right|\left|\frac{\xi_n^*-\chi_k}{\xi_n-\chi_k}\right|+\blacklozenge\ln\left|\frac{\chi_n-\xi_k}{\chi_n^*-\xi_k}\right|\left|\frac{\chi_n^*-\chi_k}{\chi_n-\chi_k}\right|\right],\\&\theta_{k,I}^-=\theta_{k,I}+\Delta_{k,I}^-,\\&\Delta_{k,I}^-=\frac{1}{2}\left[\bigstar\arg\left(\frac{\xi_n-\xi_k}{\xi_n^*-\xi_k}\frac{\chi_k^*-\xi_n^*}{\chi_k^*-\xi_n}\right)+\blacklozenge\arg\left(\frac{\chi_n-\xi_k}{\chi_n^*-\xi_k}\frac{\chi_k^*-\chi_n^*}{\chi_k^*-\chi_n}\right)\right],\\&\Theta_k^+=\exp\left(\bigstar2\mathrm{i}\varphi_{n,I}-\blacklozenge2\mathrm{i}\varphi_{n,I}\right),\qquad\Theta_k^-=\exp\left(-\bigstar2\mathrm{i}\varphi_{n,I}+\blacklozenge2\mathrm{i}\varphi_{n,I}\right),\end{aligned}\tag{45}$$

if $1\le k\le l$, then $\bigstar=\left(\sum_{n=1}^{k-1}+\sum_{n=l+1}^{N}\right)$, $\blacklozenge=\sum_{n=k+1}^{l}$; if $l<k\le N$, then $\bigstar=\sum_{n=1}^{k-1}$, $\blacklozenge=\sum_{n=k+1}^{N}$; and

$$\theta_{i,R}=\delta_i\left(y-\frac{2}{\gamma}v_is\right)-\varphi_{i,R}+a_{i,R},$$

$$\theta_{i,I}=\epsilon_i\left(y-\frac{2}{\gamma}w_is\right)-\varphi_{i,I}+a_{i,I},$$

and

$$v_i=\frac{\alpha_i\gamma\sinh\left(\varphi_{i,R}\right)}{4\left(\gamma\sinh\left(\varphi_{i,R}\right)+\beta\cos\left(\varphi_{i,I}\right)\right)},$$

$$\delta_i=\frac{2\beta}{\alpha_i}\sin\left(\varphi_{i,I}\right)\left(\gamma\sinh\left(\varphi_{i,R}\right)+\beta\cos\left(\varphi_{i,I}\right)\right),$$

$$w_i=\frac{-\alpha_i\gamma\cos(\varphi_{i,I})}{4\left(\beta\sinh\left(\varphi_{i,R}\right)-\gamma\cos(\varphi_{i,I})\right)},$$

$$\epsilon_i=\frac{2\beta}{\alpha_i}\cosh(\varphi_{i,R})\left(\beta\sinh\left(\varphi_{i,R}\right)-\gamma\cos(\varphi_{i,I})\right),$$

$$\alpha_i=\left(\beta\sinh\left(\varphi_{i,R}\right)\cos\left(\varphi_{i,I}\right)-\gamma\right)^2+\beta^2\cosh^2\left(\varphi_{i,R}\right)\sin^2\left(\varphi_{i,I}\right).$$

Based on the above proposition, we can obtain the dynamics of N-breather solution for the CSP equation (2). In general, the dynamics of N-breather solution for the CSP equation (2) cannot be solved analytically. However, in some special location, we can analyze them by the coordinate transformation. When $s\to\pm\infty$, $\theta_{k,R}^\pm=0$ and $\theta_{k,I}^\pm=\frac{\pi}{4}+\kappa\pi$, we have

$$\begin{aligned}x&=-\frac{\gamma}{2}y-\frac{\beta^2}{8}s-2\ln_s(\det(M))\\&\to-\frac{\gamma}{2}y-\frac{\beta^2}{8}s-2\ln_s\left(\exp\left[\pm2\left(-\bigstar\theta_{n,R}+\blacklozenge\theta_{n,R}\right)\right]\right.\\&\quad\left.\times\exp\left[-2\theta_{k,R}\right]\det(M_k)\right),\\&=-\frac{\gamma}{2}y-\frac{\beta^2}{8}s\pm\tau_k,\end{aligned}$$

where

$$\tau_k=\beta\left(\bigstar\sin(\varphi_{n,I})\sinh(\varphi_{n,R})-\blacklozenge\sin(\varphi_{n,I})\sinh(\varphi_{n,R})\right).$$

Proposition 6. *When $t\to\mp\infty$, along the trajectory $\theta_{k,R}^\pm=0$ and $\theta_{k,I}^\pm=\frac{\pi}{4}+\kappa\pi$, we have*

$$q[N]=\frac{\beta}{2}q_k^\pm\mathrm{e}^{\mathrm{i}\theta}+O(\mathrm{e}^{-c|t|}),$$

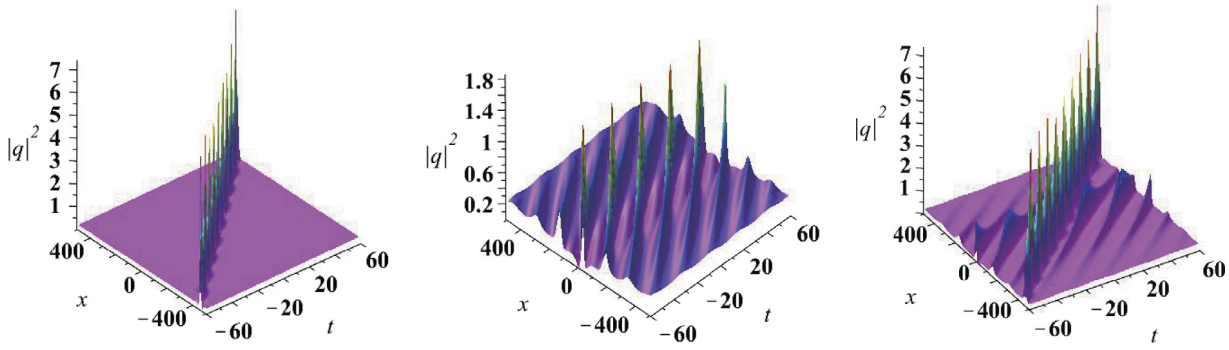
where

$$\theta_{k,R}^\pm=-\frac{\gamma}{2}\delta_k\left[x-\left(v_k+\frac{\beta^2}{8}\right)t\mp\tau_k\right]-\varphi_{k,R}+a_{k,R}+\Delta_{k,R}^\pm,$$

$$\theta_{k,I}^\pm=-\frac{\gamma}{2}\epsilon_k\left[x-\left(w_k+\frac{\beta^2}{8}\right)t\mp\tau_k\right]-\varphi_{k,I}+a_{k,I}+\Delta_{k,I}^\pm.$$

4. General rogue wave solution to the CSP equation

In previous section, we solved the linear system (7) with plane wave seed solution under the restriction $\lambda_i\ne-\gamma+\mathrm{i}\beta$. It is natural to ask what happens if $\lambda_i=-\gamma+\mathrm{i}\beta$. Actually, we can obtain the rogue wave solution and higher order rogue wave solutions



(a) $|q|^2$. (b) $|q|^2$. (c) $|q|^2$.

Fig. 1. (Color online): Parameters $\beta = 1, \gamma = 5$; (a) single breather solution, parameters: $a_1 = 0, \varphi_1 = \ln(2+\sqrt{5}) + \frac{\pi}{2}\mathrm{i}$; (b) Akhmediev breather solution, parameters: $a_1 = 0, \varphi_1 = \frac{\pi}{3}\mathrm{i}$; (c) two breather solution, parameters: $\varphi_1 = \ln(2+\sqrt{5}) + \frac{\pi}{2}\mathrm{i}, \varphi_2 = \frac{\pi}{3}\mathrm{i}, a_1 = a_2 = 0$.

under this special condition. The general procedure to yield these solutions was proposed in [46,47].

Starting from the linear system (7) with $(q, \rho, \lambda) = (q[0], \rho[0], -\gamma + \mathrm{i}\beta)$, where $q[0]$ and $\rho[0]$ are given in Eq. (38), then one can firstly obtain the quasi-rational solution, from which the first order rogue wave solution can be obtained through formula (30). However, the higher order RW solution cannot be constructed in the same way. To find the general higher order rogue wave solution, we need to solve the linear system (7) with $(q, \rho, \lambda) = (q[0], \rho[0], -\gamma + \mathrm{i}\beta - \frac{\mathrm{i}\epsilon^2}{2\beta})$, where ϵ is a small parameter.

To this end, we give the following lemma.

Lemma 1. *Denote*

$$\lambda_1 = -\gamma + \mathrm{i}\beta - \frac{\mathrm{i}\epsilon^2}{2\beta}, \qquad \mu_1 = \epsilon\sqrt{1-\left(\frac{\epsilon}{2\beta}\right)^2}, \tag{46}$$

$$\xi_1 = \lambda_1 + \mu_1,$$

then the following parameters can be expanded in terms of a small parameter ϵ

$$\mu_1 = \sum_{n=0}^{\infty} \mu_1^{[n]} \epsilon^{2n+1},$$

$$\frac{1}{\xi_1^* - \xi_1} = \sum_{i=0,j=0}^{\infty,\infty} F^{[i,j]} \epsilon^{*i} \epsilon^j,$$

$$\frac{1}{\xi_1 + \gamma} \equiv \frac{1}{\mathrm{i}\beta(\sqrt{1-\epsilon^2} - \mathrm{i}\epsilon)^2} = \sum_{i=0}^{\infty} J^{[i]} \epsilon^i,$$

where

$$\mu_1^{[n]} = \binom{\frac{1}{2}}{n}\left(\frac{-1}{4\beta^2}\right)^n, \quad \binom{\frac{1}{2}}{n} = \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\cdots\left(\frac{1}{2}-n+1\right)}{n!},$$

$$F^{[i,j]} = \frac{\mathrm{i}}{i!j!\beta}\frac{\partial^{i+j}}{\partial\epsilon^{*i}\partial\epsilon^j}\left(\left[\exp\left(2\mathrm{i}\arcsin\left(\frac{\epsilon^*}{2\beta}\right)\right) + \exp\left(-2\mathrm{i}\arcsin\left(\frac{\epsilon}{2\beta}\right)\right)\right]^{-1}\right)_{|_{\epsilon^*=0,\epsilon=0}},$$

$$J^{[0]} = \frac{1}{\mathrm{i}\beta}, \qquad J^{[1]} = \frac{1}{\beta^2}, \qquad J^{[2]} = \frac{\mathrm{i}}{2\beta^3},$$

$$J^{[2i+1]} = \frac{(-1)^i}{\beta^2}\binom{\frac{1}{2}}{i}\left(\frac{1}{2\beta}\right)^{2i}, \quad J^{[2i+2]} = 0,\ i \geq 1.$$

With the aid of above lemma, we have the following expansion

$$\begin{aligned} X_1 &\equiv \frac{\mathrm{i}}{4}\mu_1\left(s + \frac{2y}{\lambda} + \sum_{i=1}^{\infty}(a_i + \mathrm{i}b_i)\epsilon^{2i}\right) + \frac{1}{2}\ln\left(\frac{\mu_1+\lambda_1+\gamma}{\mathrm{i}\beta}\right) \\ &= \mathrm{i}\epsilon\left(\sum_{i=1}^{\infty}\frac{\mu_1^{[i]}}{4}\epsilon^{2i}\right)\left(\sum_{i=0}^{\infty}K^{[i]}\epsilon^{2i}\right) - \mathrm{i}\arcsin\left(\frac{\epsilon}{2\beta}\right) \\ &= \mathrm{i}\epsilon\sum_{k=0}^{\infty}X_1^{[2k+1]}\epsilon^{2k}, \end{aligned}$$

where

$$X_1^{[2k+1]} = \left[\sum_{j=0}^{k}\frac{1}{4}K^{[j]}\mu_1^{[k-j]} - \frac{(-1)^k}{2k+1}\binom{-\frac{1}{2}}{k}\right],$$

$$K^{[k]} = \begin{cases} s - \dfrac{2(\gamma+\mathrm{i}\beta)y}{\beta^2+\gamma^2}, & k = 0, \\ \left(\dfrac{-2(\gamma+\mathrm{i}\beta)y}{\gamma^2+\beta^2}\right)\left(\dfrac{(\gamma+\mathrm{i}\beta)}{2\beta(\gamma^2+\beta^2)\mathrm{i}}\right)^k + a_k + \mathrm{i}b_k, & k \geq 1. \end{cases}$$

Furthermore we have

$$\mathrm{e}^{X_1} = \sum_{i=0}^{\infty} S_i(\mathbf{X}_1)\epsilon^i, \quad \mathbf{X}_1 = \left(X_1^{[1]}, X_1^{[2]}, \ldots\right),\ X_1^{[2k]} = 0, \quad k \geq 1,$$

where $S_i(\mathbf{X}_1)$ are elementary Schur polynomials

$$S_0(\mathbf{X}_1) = 1, \qquad S_1(\mathbf{X}_1) = X_1^{[1]}, \qquad S_2(\mathbf{X}_1) = X_1^{[2]} + \frac{(X_1^{[1]})^2}{2},$$

$$S_3(\mathbf{X}_1) = X_1^{[3]} + X_1^{[1]}X_1^{[2]} + \frac{(X_1^{[1]})^3}{6}, \ldots$$

$$S_i(\mathbf{X}_1) = \sum_{l_1+2l_2+\cdots+kl_k=i}\frac{(X_1^{[1]})^{l_1}(X_1^{[2]})^{l_2}\cdots(X_1^{[k]})^{l_k}}{l_1!l_2!\cdots l_k!}.$$

Since $KE_1(\epsilon)$ satisfies the Lax equation (7), then $KE_1(-\epsilon)$ also satisfies the Lax equation (7). To obtain the general higher order rogue wave solution, we choose the general special solution

$$|y_1\rangle = \frac{K}{2\epsilon}\left[E_1(\epsilon) - E_1(-\epsilon)\right] \equiv K\begin{bmatrix}\varphi_1 \\ \beta\psi_1\end{bmatrix}, \quad E_1 = \begin{bmatrix}\mathrm{e}^{X_1} \\ \dfrac{\beta\mathrm{e}^{X_1}}{\xi_1+\gamma}\end{bmatrix}.$$

Finally, we have

$$\begin{aligned} \frac{\langle y_1|y_1\rangle}{2(\lambda_1^*-\lambda_1)} &= \frac{1}{4}\left[\frac{\mathrm{e}^{X_1^*+X_1}}{\xi_1^*-\xi_1} - \frac{\mathrm{e}^{X_1^*-X_1}}{\xi_1^*-\chi_1} - \frac{\mathrm{e}^{-X_1^*+X_1}}{\chi_1^*-\xi_1} + \frac{\mathrm{e}^{-X_1^*-X_1}}{\chi_1^*-\chi_1}\right] \\ &= \sum_{m=1,n=1}^{\infty,\infty} M^{[m,n]}\epsilon^{*2(m-1)}\epsilon^{2(n-1)}, \end{aligned} \tag{47}$$

where $\chi_1 = \xi_1(-\epsilon)$,

$$M^{[m,n]} = \sum_{i=0}^{2m-1}\sum_{j=0}^{2n-1} F^{[i,j]} S_{2n-i-1}(X_1) S_{2m-j-1}(X_1^*).$$

On the other hand, by using Lemma 1, we have the following expansion

$$\begin{aligned} \varphi_1 &= \frac{1}{2}\left(\mathrm{e}^{X_1} - \mathrm{e}^{-X_1}\right) = \sum_{n=1}^{\infty} \varphi_1^{[n]} \epsilon^{2(n-1)}, \\ \psi_1 &= \frac{1}{2}\left(\frac{\mathrm{e}^{X_1}}{\xi_1+\gamma} - \frac{\mathrm{e}^{-X_1}}{\chi_1+\gamma}\right) = \sum_{n=1}^{\infty} \psi_1^{[n]} \epsilon^{2(n-1)}, \end{aligned} \tag{48}$$

where

$$\varphi_1^{[n]} = S_{2n-1}(\mathbf{X}_1), \qquad \psi_1^{[n]} = \sum_{k=0}^{2n-1} S_k(\mathbf{X}_1) J_1^{[2n-1-k]}.$$

Based on the expansion Eqs. (47)–(48), and formulas (29)–(30)–(26), we can obtain the general rogue wave solutions:

Proposition 7. *The general higher order rogue wave solution for the CSP equation (2) can be represented as*

$$\begin{aligned} q[N] &= \frac{\beta}{2}\left[\frac{\det(G)}{\det(M)}\right]\mathrm{e}^{\mathrm{i}\theta}, \\ x &= -\frac{\gamma}{2}y - \frac{\beta^2}{8}s - 2\ln_s(\det(M)), \quad t = -s, \end{aligned} \tag{49}$$

where

$$M = \left(M^{[m,n]}\right)_{1\le m,n\le N}, \qquad G = \left(M^{[m,n]} + \varphi_1^{[m]*}\psi_1^{[n]}\right)_{1\le m,n\le N}.$$

Specifically, the first order rogue wave solution can be written explicitly through formula (49)

$$\begin{aligned} q[1] &= \frac{\beta}{2}\left[1 + \frac{16(\mathrm{i}\beta^2 y - \beta^2 - \gamma^2)}{\beta^2(2y-\gamma s)^2 + \beta^4 s^2 + 4\gamma^2 + 4\beta^2}\right]\mathrm{e}^{\mathrm{i}\theta}, \\ x &= -\frac{\gamma}{2}y - \frac{\beta^2}{8}s \\ &\quad - \frac{4\beta^2\left(\gamma^2 s + \beta^2 s - 2\gamma y\right)}{\beta^2(2y-\gamma s)^2 + \beta^4 s^2 + 4\gamma^2 + 4\beta^2}, \quad t = -s. \end{aligned} \tag{50}$$

It can be shown that if $\beta^2 < \frac{\gamma^2}{3}$, then one has the regular rogue wave solution (Fig. 2); if $\beta^2 = \frac{\gamma^2}{3}$, then one obtains the cusponed rogue wave solution, in which $|q_x| \to \infty$ at the peak point (Fig. 3); if $\beta^2 > \frac{\gamma^2}{3}$, then we has the looped rogue wave solution (Fig. 4). Although both the NLS and the CSP equations possess the modulational instability (see the Appendix), the rogue wave solution of the CSP equation (2) could yield the singularity which is different from the one for the NLS equation. This solution may be related to the wave breaking in the CSP equation. By the formula (30), the second order rogue wave solution can be calculated as

$$\begin{aligned} q[2] &= \frac{\beta}{2}\left[1 + \frac{G_2}{M_2}\right]\mathrm{e}^{\mathrm{i}\theta}, \\ x &= -\frac{\gamma}{2}y - \frac{\beta^2}{8}s - 2\ln_s(M_2), \quad t = -s, \end{aligned} \tag{51}$$

where

$$\begin{aligned} M_2 &= \beta^6 A\hat{y}^6 + \left[3\beta^6 A\hat{s}^2 + 12\beta^6 + 108\beta^4\gamma^2\right]\hat{y}^4 \\ &\quad + \left[-288\gamma\beta^5\hat{s} - 96b_1 A\beta^6\right]\hat{y}^3 \\ &\quad + \left[3\beta^6 A\hat{s}^4 + \left(-72\beta^4\gamma^2 + 216\beta^6\right)\hat{s}^2\right. \\ &\quad \left. - 288a_1\beta^6 A\hat{s} + 432\beta^4 + 1584\gamma^2\beta^2\right]\hat{y}^2 \\ &\quad + \left[96\gamma\beta^5\hat{s}^3 + 288Ab_1\beta^6\hat{s}^2 - 1152\gamma\beta^3\hat{s}\right. \\ &\quad \left. + 4608a_1\beta^5\gamma + 1152b_1\beta^6 - 3456b_1\beta^4\gamma^2\right]\hat{y} \\ &\quad + A\left[\beta^6\hat{s}^6 + 12\beta^4\hat{s}^4 + 96a_1\beta^6\hat{s}^3 + 432\beta^2\hat{s}^2\right. \\ &\quad \left. - 1152a_1\beta^4\hat{s} + 576 + 2304\left(a_1{}^2 + b_1{}^2\right)\beta^6\right], \\ G_2 &= a_1\beta^4 A\left(2304\mathrm{i}\beta\hat{y} + 4068\right)\hat{s} + A\left[1152\mathrm{i}\beta^5\left(\hat{y}^2 - \hat{s}^2\right)\right. \\ &\quad \left. + 4068\beta^4\hat{y} - 4608\mathrm{i}\beta^3\right]b_1 \\ &\quad - 24\mathrm{i}A\beta^5\hat{y}^5 - 240\beta^4 A\hat{y}^4 \\ &\quad + \left[-48\mathrm{i}A\beta^5\hat{s}^2 - 192\mathrm{i}\gamma^2\beta^3 + 960\mathrm{i}\beta^5\right]\hat{y}^3 \\ &\quad + \left[-288\beta^4 A\hat{s}^2 + 2304\mathrm{i}\gamma\beta^4\hat{s}\right. \\ &\quad \left. - 3456\gamma^2\beta^2 + 1152\beta^4\right]\hat{y}^2 \\ &\quad + \left[-24\mathrm{i}A\beta^5\hat{s}^4 + 576\mathrm{i}\left(\gamma^2 - \beta^2\right)\beta^3\hat{s}^2\right. \\ &\quad \left. + 4608\beta^3\gamma\hat{s} + 5760\mathrm{i}\gamma^2\beta + 1152\mathrm{i}\beta^3\right]\hat{y} \\ &\quad - A(48\beta^4\hat{s}^4 + 1152\beta^2\hat{s}^2 - 2304) \end{aligned} \tag{52}$$

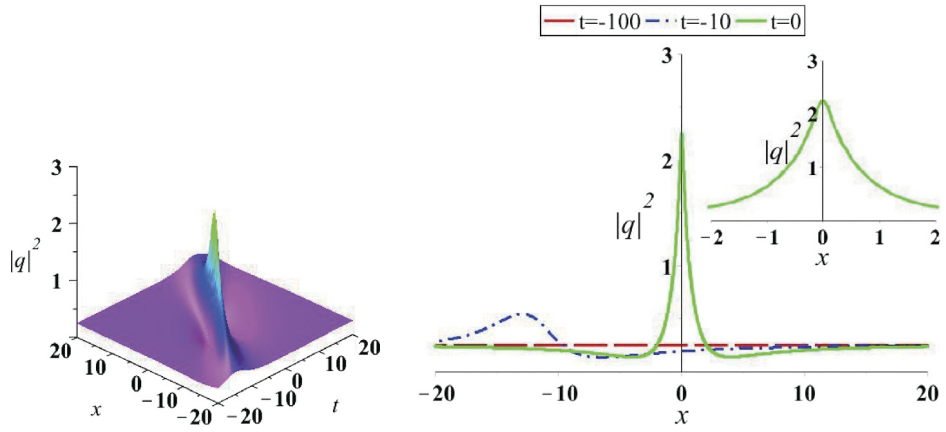
and

$$\hat{s} = s - \frac{2\gamma y}{A}, \qquad \hat{y} = \frac{-2\beta y}{A}, \quad A = \beta^2 + \gamma^2.$$

Here β and γ represent the amplitude and the angular frequency of the background plane wave, respectively, a_1, b_1 are two free parameters which determine the distribution of peaks. The spatio-temporal pattern of the second order RW solution for the CSP equation is similar to the ones for the NLS equation [46] or for the derivative NLS equation [47]. An example is shown in (Fig. 5(b)). For the general case, it is extremely tedious and not feasible to completely describe their dynamics analytically because of too many parameters involved. However for the symmetric case of $a_1 = b_1 = 0$ where the RW has maximum peak, we can show that if $\beta^2 < \left(1 - \frac{2\sqrt{5}}{5}\right)\gamma^2$, one obtains the smoothed rogue wave (Fig. 5(a)); if $\beta^2 = \left(1 - \frac{2\sqrt{5}}{5}\right)\gamma^2$, one can obtain the cusponed rogue wave; if $\beta^2 > \left(1 - \frac{2\sqrt{5}}{5}\right)\gamma^2$, one arrives at the looped rogue wave (Fig. 6). On the other hand, for the case of $a_1^2 + b_1^2 \gg 1$, then it follows the same classification as first order RW solution. In other words, if $\beta^2 < \frac{1}{3}\gamma^2$ one obtains the smoothed rogue wave triplets [52] (see Fig. 5(b)); if $\beta^2 > \frac{1}{3}\gamma^2$, one obtains the looped rogue wave. The detailed analysis is too tedious to be included. The expression for the higher order rogue wave solution $N \ge 3$ becomes very complicated. Here, we only illustrate a third order rogue wave solution (Fig. 7) without providing an analytical expression.

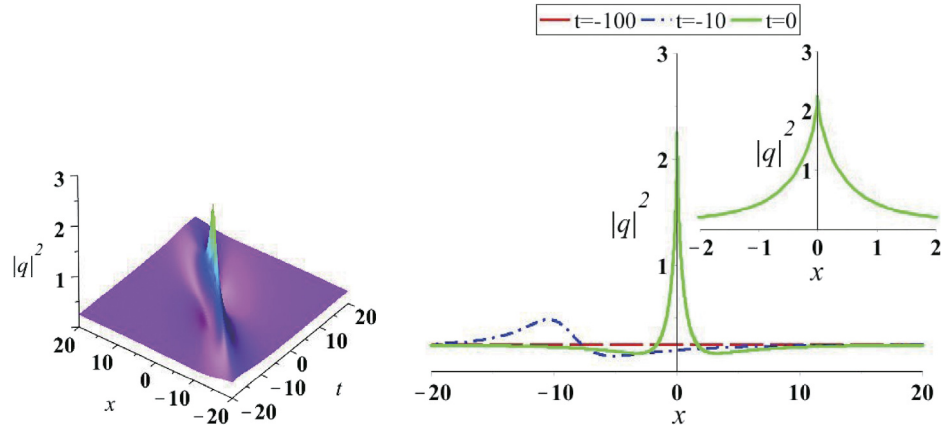
5. Conclusions and discussions

In the present paper, we study the general analytic solutions to the complex short pulse (CSP) equation by the Darboux transformation method. We firstly develop a generalized Darboux



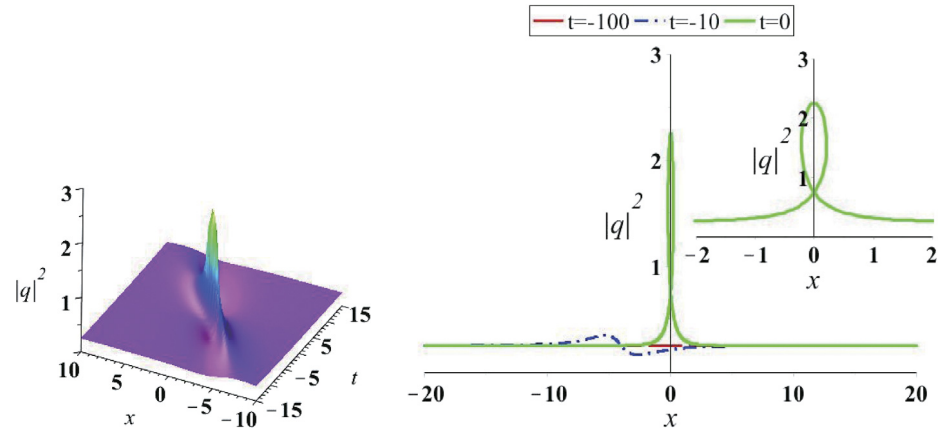
(a) $|q|^2$. (b) $|q|^2$.

Fig. 2. (Color online): Parameters: $\beta = 1, \gamma = 2$; (a) the spatio-temporal pattern for the regular first order RW; (b) the profile of $|q[1]|^2$ at different time.



(a) $|q|^2$. (b) $|q|^2$.

Fig. 3. (Color online): Parameters: $\beta = 1, \gamma = \sqrt{3}$; (a) the spatio-temporal pattern for the first order cusponed RW; (b) the profile of $|q[1]|^2$ at different time.

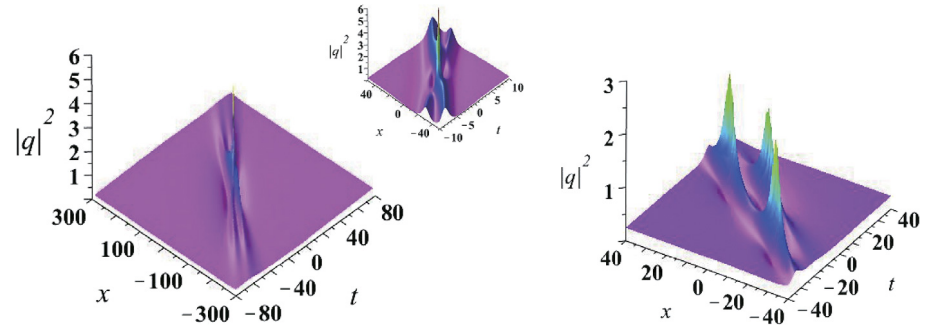


(a) $|q|^2$. (b) $|q|^2$.

Fig. 4. (Color online): Parameters: $\beta = 1, \gamma = 1$; (a) the spatio-temporal pattern for the first order loop RW; (b) the profile of $|q[1]|^2$ at different time.

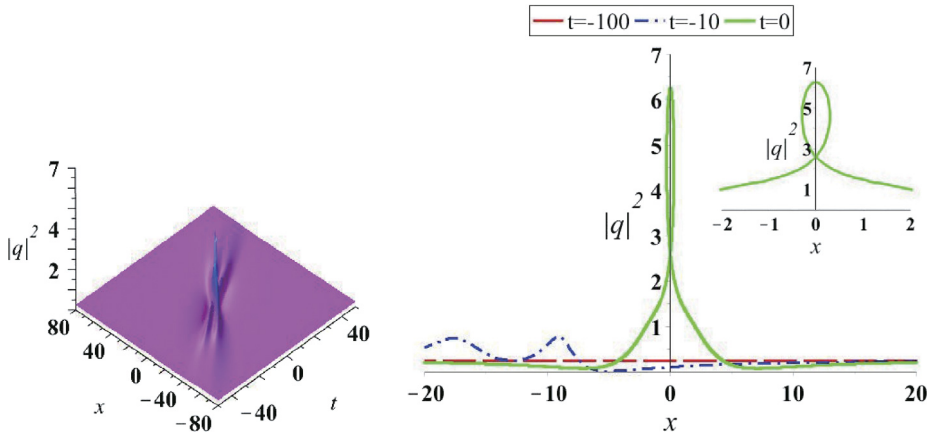
transformation (DT) and associated Bäcklund transformation for the complex coupled dispersionless (CCD) equation, which leads to a general soliton formulas for the CCD equation. Then by integrating the reciprocal transformation exactly, the N-bright soliton solution in a compact determinant form to the CSP equation is constructed. Furthermore, the N-breather solution and higher order rogue wave solution to the CSP equation are constructed by a delicate limiting process.

The N-bright soliton solution should be equivalent to the ones found by one of the authors [31,33], the N-breather solution and higher order rogue wave solution to the CSP equation are found for the first time and deserve further study. Especially, this is the first example for the existence of rogue wave solution in a nonlinear wave equation possessing reciprocal (hodograph) transformation. Due to this reciprocal transformation, the analytical solutions including the bright, breather and rogue wave ones can be



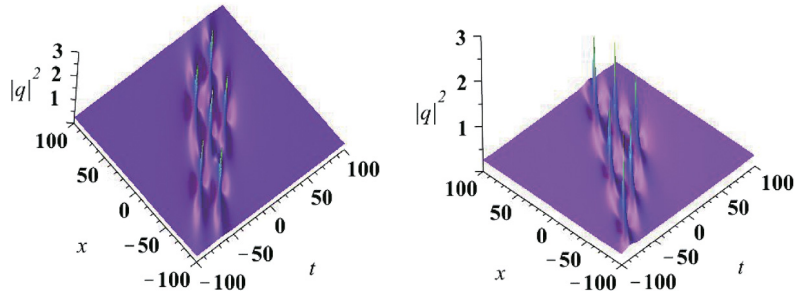
(a) $|q|^2$. (b) $|q|^2$.

Fig. 5. (Color online): (a) Second order regular RW, parameters: $\beta = 1, \gamma = 4, a_1 = b_1 = 0$; (b) second order regular RW with triplet, parameters: $\beta = 1, \gamma = 2, a_1 = 20, b_1 = 0$.



(a) $|q|^2$. (b) $|q|^2$.

Fig. 6. (Color online): Parameters: $\beta = 1, \gamma = 2, a_1 = b_1 = 0$, (a) second order loop RW; (b) profile of $|q[2]|^2$ at different time.



(a) $|q|^2$. (b) $|q|^2$.

Fig. 7. (Color online): (a) Third order RW with pentagon arrangement, parameters: $\beta = 1, \gamma = 2, a_2 = 500, a_1 = 0, b_1 = b_2 = 0$; (b) third order RW with triangle arrangement, $\beta = 1, \gamma = 2, a_2 = 0, a_1 = 100, b_1 = b_2 = 0$.

either smoothed, cusponed or looped ones. Based on the compact determinant form of the solutions, we perform an asymptotic analysis for the N-bright soliton and N-breather solutions. It should be pointed out that the method for the asymptotical analysis can be extended to other integrable equations as well. In compared to the NLS equation, the rogue wave solution for the CSP equation (2) could develop into wave-breaking. This illustrates that the modulational instability for the CSP equation (2) is stronger than the NLS equation.

Finally, the CSP equation could be of defocusing type, which admits the dark soliton solution, same as the NLS equation. It turns out that this is indeed the case. The complex short pulse equation of both focusing and defocusing types can be derived from the context of nonlinear optics. The results are summarized in a separate work [53].

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Appendix A. Proof of Proposition 3

Proof. Fixed $y - v_k s = \text{const}$, and $s \to -\infty$, it follows that $\theta_1, \theta_2, \ldots, \theta_{k-1} \to -\infty$; $\theta_{k+1}, \theta_{k+2}, \ldots, \theta_N \to +\infty$. On the other

hand, $q[N]$ can be rewritten as

$$q[N] = -\frac{\det(\widehat{G})}{\det(\widehat{M})}, \tag{53}$$

where

$$\widehat{M} = \left(\frac{e^{2(\theta_i^*+\theta_j)}+1}{\lambda_i^*-\lambda_j}\right)_{1\le i,j\le N}, \qquad \widehat{G} = \begin{bmatrix} \widehat{M} & \widehat{Y}_1^\dagger \\ \widehat{Y}_2 & 0 \end{bmatrix},$$

$$\widehat{Y}_1 = \begin{bmatrix} e^{2\theta_1} & e^{2\theta_2} & \cdots & e^{2\theta_N} \end{bmatrix}, \qquad \widehat{Y}_2 = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}.$$

It follows that

$$\det(\widehat{M}) = e^{2(\theta_{k+1}+\theta_{k+2}+\cdots+\theta_N)}\left[\det(M_k)+O(e^{-c|s|})\right],$$

$$\det(\widehat{G}) = e^{2(\theta_{k+1}+\theta_{k+2}+\cdots+\theta_N)}\left[\det(G_k)+O(e^{-c|s|})\right],$$

where M_k and G_k are given in Box I. By direct calculation, we have

$$\begin{aligned}\det(G_k) &= (-1)^{k+N+1}\begin{vmatrix} \frac{1}{\lambda_1^*-\lambda_1} & \cdots & \frac{1}{\lambda_1^*-\lambda_{k-1}} & \frac{1}{\lambda_1^*-\lambda_k} \\ \vdots & \ddots & \vdots & \vdots \\ \frac{1}{\lambda_{k-1}^*-\lambda_1} & \cdots & \frac{1}{\lambda_{k-1}^*-\lambda_{k-1}} & \frac{1}{\lambda_{k-1}^*-\lambda_k} \\ 1 & \cdots & 1 & 1 \end{vmatrix} \\ &\quad\times \begin{vmatrix} \frac{e^{2\theta_k^*}}{\lambda_k^*-\lambda_{k+1}} & \cdots & \frac{e^{2\theta_k^*}}{\lambda_k^*-\lambda_N} & e^{2\theta_k^*} \\ \frac{1}{\lambda_{k+1}^*-\lambda_{k+1}} & \cdots & \frac{1}{\lambda_{k+1}^*-\lambda_N} & 1 \\ \vdots & \ddots & \vdots & \vdots \\ \frac{1}{\lambda_N^*-\lambda_{k+1}} & \cdots & \frac{1}{\lambda_N^*-\lambda_N} & 1 \end{vmatrix} \\ &= -\prod_{l=1}^{k-1}\left(\frac{\lambda_l-\lambda_k}{\lambda_l^*-\lambda_k}\right)\prod_{l=k+1}^{N}\left(\frac{\lambda_l^*-\lambda_k^*}{\lambda_l-\lambda_k^*}\right) \\ &\quad\times C(\lambda_1^*,\lambda_2^*,\ldots,\lambda_{k-1}^*)C(\lambda_{k+1}^*,\lambda_{k+2}^*,\ldots,\lambda_N^*)e^{2\theta_k^*};\end{aligned}$$

and $\det(M_k)$ is given in Box II, where $C(a_1^*,a_2^*,\ldots,a_m^*) = \det\left(\frac{1}{a_i^*-a_j}\right)_{1\le i,j\le m}$ represents the determinant of a Cauchy matrix. Thus, along the trajectory $y - v_k s = \text{const}$, we have $q[N] = \lambda_{k,I}\operatorname{sech}(2\theta_{k,R}^-)e^{-2i\theta_{k,I}^- - \frac{\pi i}{2}} + O(e^{-c|s|})$.

For the general case $y - vs = \text{const}$, $v \neq v_k$ $(k = 1, 2, \ldots, N)$, we have $q[N] = O(e^{-c|s|})$. Thus we obtain the asymptotic behavior (36) when $s \to -\infty$.

By the same procedure as above, we can obtain the asymptotical behavior (36) when $s \to +\infty$. Finally, we have obtained the N-soliton's asymptotic behavior (36). □

Appendix B. Proof of Proposition 5

Proof. Fixed $y - \frac{2}{\gamma}v_k s = \text{const}$, $l < k \le N$, and $s \to +\infty$, it follows that $\theta_1, \theta_2, \ldots, \theta_{k-1} \to -\infty$; $\theta_{k+1}, \theta_{k+2}, \ldots, \theta_N \to +\infty$. It follows that

$$\det(M) = \exp\left[\sum_{i=k+1}^{N}(\theta_i+\theta_i^*) - \sum_{i=1}^{k}(\theta_i+\theta_i^*)\right] \times \left(\det(M_k)+O(e^{-c|s|})\right)$$

$$\det(G) = \exp\left[\sum_{i=k+1}^{N}(\theta_i+\theta_i^*) - \sum_{i=1}^{k}(\theta_i+\theta_i^*)\right] \times \left(\det(G_k)+O(e^{-c|s|})\right)$$

where M_k and G_k are given in Box III, and $m_k = \frac{e^{2(\theta_k^*+\theta_k)}}{\xi_k^*-\xi_k} + \frac{e^{2\theta_k^*}}{\xi_k^*-\chi_k} + \frac{e^{2\theta_k}}{\chi_k^*-\xi_k} + \frac{1}{\chi_k^*-\chi_k}$, $g_k = \frac{e^{2(\theta_k^*+\theta_k)}}{\xi_k^*-\xi_k}\frac{\xi_k^*+\gamma}{\xi_k+\gamma} + \frac{e^{2\theta_k^*}}{\xi_k^*-\chi_k}\frac{\xi_k^*+\gamma}{\chi_k+\gamma} + \frac{e^{2\theta_k}}{\chi_k^*-\xi_k}\frac{\chi_k^*+\gamma}{\xi_k+\gamma} + \frac{1}{\chi_k^*-\chi_k}\frac{\chi_k^*+\gamma}{\chi_k+\gamma}$. By direct calculation, we have

$$\begin{aligned}\det(M_k) &= \det(M_k^{[1]})e^{2(\theta_k^*+\theta_k)} + \det(M_k^{[2]})e^{2\theta_k^*} \\ &\quad + \det(M_k^{[3]})e^{2\theta_k} + \det(M_k^{[4]})\end{aligned}$$

where

$$M_k^{[1]} = \begin{bmatrix} \frac{1}{\chi_1^*-\chi_1} & \cdots & \frac{1}{\chi_1^*-\chi_{k-1}} & \frac{1}{\chi_1^*-\xi_k} & \frac{1}{\chi_1^*-\xi_{k+1}} & \cdots & \frac{1}{\chi_1^*-\xi_N} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\chi_{k-1}^*-\chi_1} & \cdots & \frac{1}{\chi_{k-1}^*-\chi_{k-1}} & \frac{1}{\chi_{k-1}^*-\xi_k} & \frac{1}{\chi_{k-1}^*-\xi_{k+1}} & \cdots & \frac{1}{\chi_{k-1}^*-\xi_N} \\ \frac{1}{\xi_k^*-\chi_1} & \cdots & \frac{1}{\xi_k^*-\chi_{k-1}} & \frac{1}{\xi_k^*-\xi_k} & \frac{1}{\xi_k^*-\xi_{k+1}} & \cdots & \frac{1}{\xi_k^*-\xi_N} \\ \frac{1}{\xi_{k+1}^*-\chi_1} & \cdots & \frac{1}{\xi_{k+1}^*-\chi_{k-1}} & \frac{1}{\xi_{k+1}^*-\xi_k} & \frac{1}{\xi_{k+1}^*-\xi_{k+1}} & \cdots & \frac{1}{\xi_{k+1}^*-\xi_N} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\xi_N^*-\chi_1} & \cdots & \frac{1}{\xi_N^*-\chi_{k-1}} & \frac{1}{\xi_N^*-\xi_k} & \frac{1}{\xi_N^*-\xi_{k+1}} & \cdots & \frac{1}{\xi_N^*-\xi_N} \end{bmatrix},$$

$$M_k^{[2]} = \begin{bmatrix} \frac{1}{\chi_1^*-\chi_1} & \cdots & \frac{1}{\chi_1^*-\chi_{k-1}} & \frac{1}{\chi_1^*-\chi_k} & \frac{1}{\chi_1^*-\xi_{k+1}} & \cdots & \frac{1}{\chi_1^*-\xi_N} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\chi_{k-1}^*-\chi_1} & \cdots & \frac{1}{\chi_{k-1}^*-\chi_{k-1}} & \frac{1}{\chi_{k-1}^*-\chi_k} & \frac{1}{\chi_{k-1}^*-\xi_{k+1}} & \cdots & \frac{1}{\chi_{k-1}^*-\xi_N} \\ \frac{1}{\xi_k^*-\chi_1} & \cdots & \frac{1}{\xi_k^*-\chi_{k-1}} & \frac{1}{\xi_k^*-\chi_k} & \frac{1}{\xi_k^*-\xi_{k+1}} & \cdots & \frac{1}{\xi_k^*-\xi_N} \\ \frac{1}{\xi_{k+1}^*-\chi_1} & \cdots & \frac{1}{\xi_{k+1}^*-\chi_{k-1}} & \frac{1}{\xi_{k+1}^*-\chi_k} & \frac{1}{\xi_{k+1}^*-\xi_{k+1}} & \cdots & \frac{1}{\xi_{k+1}^*-\xi_N} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\xi_N^*-\chi_1} & \cdots & \frac{1}{\xi_N^*-\chi_{k-1}} & \frac{1}{\xi_N^*-\chi_k} & \frac{1}{\xi_N^*-\xi_{k+1}} & \cdots & \frac{1}{\xi_N^*-\xi_N} \end{bmatrix},$$

$$M_k^{[3]} = \begin{bmatrix} \frac{1}{\chi_1^*-\chi_1} & \cdots & \frac{1}{\chi_1^*-\chi_{k-1}} & \frac{1}{\chi_1^*-\xi_k} & \frac{1}{\chi_1^*-\xi_{k+1}} & \cdots & \frac{1}{\chi_1^*-\xi_N} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\chi_{k-1}^*-\chi_1} & \cdots & \frac{1}{\chi_{k-1}^*-\chi_{k-1}} & \frac{1}{\chi_{k-1}^*-\xi_k} & \frac{1}{\chi_{k-1}^*-\xi_{k+1}} & \cdots & \frac{1}{\chi_{k-1}^*-\xi_N} \\ \frac{1}{\chi_k^*-\chi_1} & \cdots & \frac{1}{\chi_k^*-\chi_{k-1}} & \frac{1}{\chi_k^*-\xi_k} & \frac{1}{\chi_k^*-\xi_{k+1}} & \cdots & \frac{1}{\chi_k^*-\xi_N} \\ \frac{1}{\xi_{k+1}^*-\chi_1} & \cdots & \frac{1}{\xi_{k+1}^*-\chi_{k-1}} & \frac{1}{\xi_{k+1}^*-\xi_k} & \frac{1}{\xi_{k+1}^*-\xi_{k+1}} & \cdots & \frac{1}{\xi_{k+1}^*-\xi_N} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\xi_N^*-\chi_1} & \cdots & \frac{1}{\xi_N^*-\chi_{k-1}} & \frac{1}{\xi_N^*-\xi_k} & \frac{1}{\xi_N^*-\xi_{k+1}} & \cdots & \frac{1}{\xi_N^*-\xi_N} \end{bmatrix},$$

$$M_k = \begin{bmatrix} \frac{1}{\lambda_1^*-\lambda_1} & \cdots & \frac{1}{\lambda_1^*-\lambda_{k-1}} & \frac{1}{\lambda_1^*-\lambda_k} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\lambda_{k-1}^*-\lambda_1} & \cdots & \frac{1}{\lambda_{k-1}^*-\lambda_{k-1}} & \frac{1}{\lambda_{k-1}^*-\lambda_k} & 0 & \cdots & 0 \\ \frac{1}{\lambda_k^*-\lambda_1} & \cdots & \frac{1}{\lambda_k^*-\lambda_{k-1}} & \frac{e^{2(\theta_k^*+\theta_k)}+1}{\lambda_k^*-\lambda_k} & \frac{e^{2\theta_k^*}}{\lambda_k^*-\lambda_{k+1}} & \cdots & \frac{e^{2\theta_k^*}}{\lambda_k^*-\lambda_N} \\ 0 & \cdots & 0 & \frac{e^{2\theta_k}}{\lambda_{k+1}^*-\lambda_k} & \frac{1}{\lambda_{k+1}^*-\lambda_{k+1}} & \cdots & \frac{1}{\lambda_{k+1}^*-\lambda_N} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \frac{e^{2\theta_k}}{\lambda_N^*-\lambda_k} & \frac{1}{\lambda_N^*-\lambda_{k+1}} & \cdots & \frac{1}{\lambda_N^*-\lambda_N} \end{bmatrix},$$

$$G_k = \begin{bmatrix} \frac{1}{\lambda_1^*-\lambda_1} & \cdots & \frac{1}{\lambda_1^*-\lambda_{k-1}} & \frac{1}{\lambda_1^*-\lambda_k} & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{\lambda_{k-1}^*-\lambda_1} & \cdots & \frac{1}{\lambda_{k-1}^*-\lambda_{k-1}} & \frac{1}{\lambda_{k-1}^*-\lambda_k} & 0 & \cdots & 0 & 0 \\ \frac{1}{\lambda_k^*-\lambda_1} & \cdots & \frac{1}{\lambda_k^*-\lambda_{k-1}} & \frac{e^{2(\theta_k^*+\theta_k)}+1}{\lambda_k^*-\lambda_k} & \frac{e^{2\theta_k^*}}{\lambda_k^*-\lambda_{k+1}} & \cdots & \frac{e^{2\theta_k^*}}{\lambda_k^*-\lambda_N} & e^{2\theta_k^*} \\ 0 & \cdots & 0 & \frac{e^{2\theta_k}}{\lambda_{k+1}^*-\lambda_k} & \frac{1}{\lambda_{k+1}^*-\lambda_{k+1}} & \cdots & \frac{1}{\lambda_{k+1}^*-\lambda_N} & 1 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & \frac{e^{2\theta_k}}{\lambda_N^*-\lambda_k} & \frac{1}{\lambda_N^*-\lambda_{k+1}} & \cdots & \frac{1}{\lambda_N^*-\lambda_N} & 1 \\ 1 & \cdots & 1 & 1 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

Box I.

$$M_k^{[4]} = \begin{bmatrix} \frac{1}{\chi_1^*-\chi_1} & \cdots & \frac{1}{\chi_1^*-\chi_{k-1}} & \frac{1}{\chi_1^*-\chi_k} & \frac{1}{\chi_1^*-\xi_{k+1}} & \cdots & \frac{1}{\chi_1^*-\xi_N} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\chi_{k-1}^*-\chi_1} & \cdots & \frac{1}{\chi_{k-1}^*-\chi_{k-1}} & \frac{1}{\chi_{k-1}^*-\chi_k} & \frac{1}{\chi_{k-1}^*-\xi_{k+1}} & \cdots & \frac{1}{\chi_{k-1}^*-\xi_N} \\ \frac{1}{\chi_k^*-\chi_1} & \cdots & \frac{1}{\chi_k^*-\chi_{k-1}} & \frac{1}{\chi_k^*-\chi_k} & \frac{1}{\chi_k^*-\xi_{k+1}} & \cdots & \frac{1}{\chi_k^*-\xi_N} \\ \frac{1}{\xi_{k+1}^*-\chi_1} & \cdots & \frac{1}{\xi_{k+1}^*-\chi_{k-1}} & \frac{1}{\xi_{k+1}^*-\chi_k} & \frac{1}{\xi_{k+1}^*-\xi_{k+1}} & \cdots & \frac{1}{\xi_{k+1}^*-\xi_N} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\xi_N^*-\chi_1} & \cdots & \frac{1}{\xi_N^*-\chi_{k-1}} & \frac{1}{\xi_N^*-\chi_k} & \frac{1}{\xi_N^*-\xi_{k+1}} & \cdots & \frac{1}{\xi_N^*-\xi_N} \end{bmatrix}.$$

On the other hand, we have $\det(M_k^{[1]})$ which is given in Box IV, moreover $\Delta_k^{[1]}$ is given in Box V, where $C(\cdot,\cdot,\ldots,\cdot)$ represents the determinant of a Cauchy matrix. Thus, we have

$$\begin{aligned}\det(M_k^{[1]}) &= \frac{1}{\xi_k^*-\xi_k}\left(\prod_{l=1}^{k-1}\frac{\chi_l-\xi_k}{\chi_l^*-\xi_k}\frac{\xi_k^*-\chi_l^*}{\xi_k^*-\chi_l}\right)\\ &\quad\times\left(\prod_{l=k+1}^{N}\frac{\xi_l-\xi_k}{\xi_l^*-\xi_k}\frac{\xi_k^*-\xi_l^*}{\xi_k^*-\xi_l}\right)\\ &\quad\times C(\chi_1^*,\chi_2^*,\ldots,\chi_{k-1}^*)C(\xi_{k+1}^*,\xi_{k+2}^*,\ldots,\xi_N^*).\end{aligned}$$

Similarly, we have

$$\begin{aligned}\det(M_k^{[2]}) &= \frac{1}{\xi_k^*-\chi_k}\left(\prod_{l=1}^{k-1}\frac{\chi_l-\chi_k}{\chi_l^*-\chi_k}\frac{\xi_k^*-\chi_l^*}{\xi_k^*-\chi_l}\right)\\ &\quad\times\left(\prod_{l=k+1}^{N}\frac{\xi_l-\chi_k}{\xi_l^*-\chi_k}\frac{\xi_k^*-\xi_l^*}{\xi_k^*-\xi_l}\right)\\ &\quad\times C(\chi_1^*,\chi_2^*,\ldots,\chi_{k-1}^*)C(\xi_{k+1}^*,\xi_{k+2}^*,\ldots,\xi_N^*),\\ \det(M_k^{[3]}) &= \frac{1}{\chi_k^*-\xi_k}\left(\prod_{l=1}^{k-1}\frac{\chi_l-\xi_k}{\chi_l^*-\xi_k}\frac{\chi_k^*-\chi_l^*}{\chi_k^*-\chi_l}\right)\\ &\quad\times\left(\prod_{l=k+1}^{N}\frac{\xi_l-\xi_k}{\xi_l^*-\xi_k}\frac{\chi_k^*-\xi_l^*}{\chi_k^*-\xi_l}\right)\\ &\quad\times C(\chi_1^*,\chi_2^*,\ldots,\chi_{k-1}^*)C(\xi_{k+1}^*,\xi_{k+2}^*,\ldots,\xi_N^*),\\ \det(M_k^{[4]}) &= \frac{1}{\chi_k^*-\chi_k}\left(\prod_{l=1}^{k-1}\frac{\chi_l-\chi_k}{\chi_l^*-\chi_k}\frac{\chi_k^*-\chi_l^*}{\chi_k^*-\chi_l}\right)\\ &\quad\times\left(\prod_{l=k+1}^{N}\frac{\xi_l-\chi_k}{\xi_l^*-\chi_k}\frac{\chi_k^*-\xi_l^*}{\chi_k^*-\xi_l}\right)\\ &\quad\times C(\chi_1^*,\chi_2^*,\ldots,\chi_{k-1}^*)C(\xi_{k+1}^*,\xi_{k+2}^*,\ldots,\xi_N^*).\end{aligned}$$

Moreover, we have

$$\det(M_k) = C(\chi_1^*,\chi_2^*,\ldots,\chi_{k-1}^*)C(\xi_{k+1}^*,\xi_{k+2}^*,\ldots,\xi_N^*)$$

$$
\begin{aligned}
\det(M_k) = &\begin{vmatrix}
\frac{1}{\lambda_1^*-\lambda_1} & \cdots & \frac{1}{\lambda_1^*-\lambda_k} & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\frac{1}{\lambda_{k-1}^*-\lambda_1} & \cdots & \frac{1}{\lambda_{k-1}^*-\lambda_k} & 0 & \cdots & 0 \\
\frac{1}{\lambda_k^*-\lambda_1} & \cdots & \frac{1}{\lambda_k^*-\lambda_k} & \frac{e^{2\theta_k^*}}{\lambda_k^*-\lambda_{k+1}} & \cdots & \frac{e^{2\theta_k^*}}{\lambda_k^*-\lambda_N} \\
0 & \cdots & 0 & \frac{1}{\lambda_{k+1}^*-\lambda_{k+1}} & \cdots & \frac{1}{\lambda_{k+1}^*-\lambda_N} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & \frac{1}{\lambda_N^*-\lambda_{k+1}} & \cdots & \frac{1}{\lambda_N^*-\lambda_N}
\end{vmatrix} \\
&+ \begin{vmatrix}
\frac{1}{\lambda_1^*-\lambda_1} & \cdots & \frac{1}{\lambda_1^*-\lambda_{k-1}} & 0 & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\frac{1}{\lambda_{k-1}^*-\lambda_1} & \cdots & \frac{1}{\lambda_{k-1}^*-\lambda_{k-1}} & 0 & 0 & \cdots & 0 \\
\frac{1}{\lambda_k^*-\lambda_1} & \cdots & \frac{1}{\lambda_k^*-\lambda_{k-1}} & \frac{e^{2(\theta_k^*+\theta_k)}}{\lambda_k^*-\lambda_k} & \frac{e^{2\theta_k^*}}{\lambda_k^*-\lambda_{k+1}} & \cdots & \frac{e^{2\theta_k^*}}{\lambda_k^*-\lambda_N} \\
0 & \cdots & 0 & \frac{e^{2\theta_k}}{\lambda_{k+1}^*-\lambda_k} & \frac{1}{\lambda_{k+1}^*-\lambda_{k+1}} & \cdots & \frac{1}{\lambda_{k+1}^*-\lambda_N} \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & \frac{e^{2\theta_k}}{\lambda_N^*-\lambda_k} & \frac{1}{\lambda_N^*-\lambda_{k+1}} & \cdots & \frac{1}{\lambda_N^*-\lambda_N}
\end{vmatrix} \\
&= C(\lambda_1^*, \lambda_2^*, \ldots, \lambda_k^*)C(\lambda_{k+1}^*, \lambda_{k+2}^*, \ldots, \lambda_N^*) + C(\lambda_1^*, \lambda_2^*, \ldots, \lambda_{k-1}^*)C(\lambda_k^*, \lambda_{k+1}^*, \ldots, \lambda_N^*)e^{2(\theta_k^*+\theta_k)} \\
&= \frac{1}{\lambda_k^*-\lambda_k}C(\lambda_1^*, \lambda_2^*, \ldots, \lambda_{k-1}^*)C(\lambda_{k+1}^*, \lambda_{k+2}^*, \ldots, \lambda_N^*)\left(\prod_{l=1}^{k-1}\left|\frac{\lambda_l-\lambda_k}{\lambda_l^*-\lambda_k}\right|^2 + \prod_{l=k+1}^{N}\left|\frac{\lambda_l-\lambda_k}{\lambda_l^*-\lambda_k}\right|^2 e^{2(\theta_k^*+\theta_k)}\right),
\end{aligned}
$$

Box II.

$$
\begin{aligned}
&\times \left[\frac{1}{\xi_k^*-\xi_k}\left(\prod_{l=1}^{k-1}\left|\frac{\chi_l-\xi_k}{\chi_l^*-\xi_k}\right|^2\right)\right. \\
&\times \left(\prod_{l=k+1}^{N}\left|\frac{\xi_l-\xi_k}{\xi_l^*-\xi_k}\right|^2\right)e^{2(\theta_k^*+\theta_k)} \\
&+ \frac{1}{\xi_k^*-\chi_k}\left(\prod_{l=1}^{k-1}\frac{\chi_l-\chi_k}{\chi_l^*-\chi_k}\frac{\xi_k^*-\chi_l^*}{\xi_k^*-\chi_l}\right) \\
&\times \left(\prod_{l=k+1}^{N}\frac{\xi_l-\chi_k}{\xi_l^*-\chi_k}\frac{\xi_k^*-\xi_l^*}{\xi_k^*-\xi_l}\right)e^{2\theta_k^*} \\
&+ \frac{1}{\chi_k^*-\xi_k}\left(\prod_{l=1}^{k-1}\frac{\chi_l-\xi_k}{\chi_l^*-\xi_k}\frac{\chi_k^*-\chi_l^*}{\chi_k^*-\chi_l}\right) \\
&\times \left(\prod_{l=k+1}^{N}\frac{\xi_l-\xi_k}{\xi_l^*-\xi_k}\frac{\chi_k^*-\xi_l^*}{\chi_k^*-\xi_l}\right)e^{2\theta_k} \\
&+ \frac{1}{\chi_k^*-\chi_k}\left(\prod_{l=1}^{k-1}\left|\frac{\chi_l-\chi_k}{\chi_l^*-\chi_k}\right|^2\right) \\
&\times \left.\left(\prod_{l=k+1}^{N}\left|\frac{\xi_l-\chi_k}{\xi_l^*-\chi_k}\right|^2\right)\right].
\end{aligned}
$$

Similar procedure as above, we have

$$
\begin{aligned}
\det(G_k) = &\left(\prod_{l=1}^{k-1}\frac{\chi_l^*+\gamma}{\chi_l+\gamma}\right)\left(\prod_{l=k+1}^{N}\frac{\xi_l^*+\gamma}{\xi_l+\gamma}\right) \\
&\times \left[\frac{\xi_k^*+\gamma}{\xi_k+\gamma}\det(M_k^{[1]})e^{2(\theta_k^*+\theta_k)} + \frac{\xi_k^*+\gamma}{\chi_k+\gamma}\det(M_k^{[2]})e^{2\theta_k^*}\right. \\
&\left. + \frac{\chi_k^*+\gamma}{\xi_k+\gamma}\det(M_k^{[3]})e^{2\theta_k} + \frac{\chi_k^*+\gamma}{\chi_k+\gamma}\det(M_k^{[4]})\right].
\end{aligned}
$$

Finally, as $s \to +\infty$, along the trajectory $y - v_k s = \text{const}$, we have

$$q[N] = \frac{\beta}{2}q_k^+ e^{i\theta} + O(e^{-c|s|}),$$

where $\theta_{k,R}^+, \theta_{k,I}^+$ are given in Eqs. (45).

For the general case $y - vs = \text{const}$, $v \neq v_k$ $(k = 1, 2, \ldots, N)$, if $v < v_1$ then $q[N] = \frac{\beta}{2}\Theta_1^+ e^{i\theta - 2i\varphi_{1,I}} + O(e^{-c|s|})$; if $v_{m-1} < v < v_m$ $(m = 2, 3, \ldots, l)$, then $q[N] = \frac{\beta}{2}\Theta_m^+ e^{i\theta - 2i\varphi_{m,I}} + O(e^{-c|s|})$; if $v_l < v < v_N$ then $q[N] = \frac{\beta}{2}\Theta_l^+ e^{i\theta + 2i\varphi_{l,I}} + O(e^{-c|s|})$; if $v_{m+1} < v < v_m$ $(m = l+1, l+2, \ldots, N-1)$, then $q[N] = \frac{\beta}{2}\Theta_m^+ e^{i\theta + 2i\varphi_{m,I}} +$

$$M_k = \begin{bmatrix} \frac{1}{\chi_1^*-\chi_1} & \cdots & \frac{1}{\chi_1^*-\chi_{k-1}} & \frac{e^{2\theta_k}}{\chi_1^*-\xi_k}+\frac{1}{\chi_1^*-\chi_k} & \frac{1}{\chi_1^*-\xi_{k+1}} & \cdots & \frac{1}{\chi_1^*-\xi_N} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\chi_{k-1}^*-\chi_1} & \cdots & \frac{1}{\chi_{k-1}^*-\chi_{k-1}} & \frac{e^{2\theta_k}}{\chi_{k-1}^*-\xi_k}+\frac{1}{\chi_{k-1}^*-\chi_k} & \frac{1}{\chi_{k-1}^*-\xi_{k+1}} & \cdots & \frac{1}{\chi_{k-1}^*-\xi_N} \\ \frac{e^{2\theta_k^*}}{\xi_k^*-\chi_1}+\frac{1}{\chi_k^*-\chi_1} & \cdots & \frac{e^{2\theta_k^*}}{\xi_k^*-\chi_{k-1}}+\frac{1}{\chi_k^*-\chi_{k-1}} & m_k & \frac{e^{2\theta_k^*}}{\xi_k^*-\xi_{k+1}}+\frac{1}{\chi_k^*-\xi_{k+1}} & \cdots & \frac{e^{2\theta_k^*}}{\xi_k^*-\xi_N}+\frac{1}{\chi_k^*-\xi_N} \\ \frac{1}{\xi_{k+1}^*-\chi_1} & \cdots & \frac{1}{\xi_{k+1}^*-\chi_{k-1}} & \frac{e^{2\theta_k}}{\xi_{k+1}^*-\xi_k}+\frac{1}{\xi_{k+1}^*-\chi_k} & \frac{1}{\xi_{k+1}^*-\xi_{k+1}} & \cdots & \frac{1}{\xi_{k+1}^*-\xi_N} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\xi_N^*-\chi_1} & \cdots & \frac{1}{\xi_N^*-\chi_{k-1}} & \frac{e^{2\theta_k}}{\xi_N^*-\xi_k}+\frac{1}{\xi_N^*-\chi_k} & \frac{1}{\xi_N^*-\xi_{k+1}} & \cdots & \frac{1}{\xi_N^*-\xi_N} \end{bmatrix},$$

$$G_k = \begin{bmatrix} \frac{1}{\chi_1^*-\chi_1}\frac{\chi_1^*+\gamma}{\chi_1+\gamma} & \cdots & \frac{1}{\chi_1^*-\chi_{k-1}}\frac{\chi_1^*+\gamma}{\chi_{k-1}+\gamma} & \frac{e^{2\theta_k}}{\chi_1^*-\xi_k}\frac{\chi_1^*+\gamma}{\xi_k+\gamma}+\frac{1}{\chi_1^*-\chi_k}\frac{\chi_1^*+\gamma}{\chi_k+\gamma} \\ \vdots & \ddots & \vdots & \vdots \\ \frac{1}{\chi_{k-1}^*-\chi_1}\frac{\chi_{k-1}^*+\gamma}{\chi_1+\gamma} & \cdots & \frac{1}{\chi_{k-1}^*-\chi_{k-1}}\frac{\chi_{k-1}^*+\gamma}{\chi_{k-1}+\gamma} & \frac{e^{2\theta_k}}{\chi_{k-1}^*-\xi_k}\frac{\chi_{k-1}^*+\gamma}{\xi_k+\gamma}+\frac{1}{\chi_{k-1}^*-\chi_k}\frac{\chi_{k-1}^*+\gamma}{\chi_k+\gamma} \\ \frac{e^{2\theta_k^*}}{\xi_k^*-\chi_1}\frac{\xi_k^*+\gamma}{\chi_1+\gamma}+\frac{1}{\chi_k^*-\chi_1}\frac{\chi_k^*+\gamma}{\chi_1+\gamma} & \cdots & \frac{e^{2\theta_k^*}}{\xi_k^*-\chi_{k-1}}\frac{\xi_k^*+\gamma}{\chi_{k-1}+\gamma}+\frac{1}{\chi_k^*-\chi_{k-1}}\frac{\chi_k^*+\gamma}{\chi_{k-1}+\gamma} & g_k \\ \frac{1}{\xi_{k+1}^*-\chi_1}\frac{\xi_{k+1}^*+\gamma}{\chi_1+\gamma} & \cdots & \frac{1}{\xi_{k+1}^*-\chi_{k-1}}\frac{\xi_{k+1}^*+\gamma}{\chi_{k-1}+\gamma} & \frac{e^{2\theta_k}}{\xi_{k+1}^*-\xi_k}\frac{\xi_{k+1}^*+\gamma}{\xi_k+\gamma}+\frac{1}{\xi_{k+1}^*-\chi_k}\frac{\xi_{k+1}^*+\gamma}{\chi_k+\gamma} \\ \vdots & \ddots & \vdots & \vdots \\ \frac{1}{\xi_N^*-\chi_1}\frac{\xi_N^*+\gamma}{\chi_1+\gamma} & \cdots & \frac{1}{\xi_N^*-\chi_{k-1}}\frac{\xi_N^*+\gamma}{\chi_{k-1}+\gamma} & \frac{e^{2\theta_k}}{\xi_N^*-\xi_k}\frac{\xi_N^*+\gamma}{\xi_k+\gamma}+\frac{1}{\xi_N^*-\chi_k}\frac{\xi_N^*+\gamma}{\chi_k+\gamma} \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{\chi_1^*-\xi_{k+1}}\frac{\chi_1^*+\gamma}{\xi_{k+1}+\gamma} & \cdots & \frac{1}{\chi_1^*-\xi_N}\frac{\chi_1^*+\gamma}{\xi_N+\gamma} \\ \vdots & \ddots & \vdots \\ \frac{1}{\chi_{k-1}^*-\xi_{k+1}}\frac{\chi_{k-1}^*+\gamma}{\xi_{k+1}+\gamma} & \cdots & \frac{1}{\chi_{k-1}^*-\xi_N}\frac{\chi_{k-1}^*+\gamma}{\xi_N+\gamma} \\ \frac{e^{2\theta_k^*}}{\xi_k^*-\xi_{k+1}}\frac{\xi_k^*+\gamma}{\xi_{k+1}+\gamma}+\frac{1}{\chi_k^*-\xi_{k+1}}\frac{\chi_k^*+\gamma}{\xi_{k+1}+\gamma} & \cdots & \frac{e^{2\theta_k^*}}{\xi_k^*-\xi_N}\frac{\xi_k^*+\gamma}{\xi_N+\gamma}+\frac{1}{\chi_k^*-\xi_N}\frac{\chi_k^*+\gamma}{\xi_N+\gamma} \\ \frac{1}{\xi_{k+1}^*-\xi_{k+1}}\frac{\xi_{k+1}^*+\gamma}{\xi_{k+1}+\gamma} & \cdots & \frac{1}{\xi_{k+1}^*-\xi_N}\frac{\xi_{k+1}^*+\gamma}{\xi_N+\gamma} \\ \vdots & \ddots & \vdots \\ \frac{1}{\xi_N^*-\xi_{k+1}}\frac{\xi_N^*+\gamma}{\xi_{k+1}+\gamma} & \cdots & \frac{1}{\xi_N^*-\xi_N}\frac{\xi_N^*+\gamma}{\xi_N+\gamma} \end{bmatrix}$$

Box III.

$O(e^{-c|s|})$; if $v_{l+1} < v$ then $q[N] = \frac{\beta}{2}\Theta_{l+1}^{+}e^{i\theta-2i\varphi_{l+1,l}}+O(e^{-c|s|})$. Thus we have the asymptotic behavior (44). By the same procedure as above, we can obtain the asymptotic behavior (43) when $s \to -\infty$. So we complete the proof. □

Appendix C. Modulational instability analysis for plane wave solution

The simplest exact solution to the CSP equation (2) is the plane wave—a constant amplitude, exponential wavetrain,

$$q_0 = \frac{\beta}{2}e^{-i\left(\frac{2}{\gamma}x-\omega t\right)}, \quad \omega = \frac{1}{4}\left(\frac{\beta^2}{\gamma^2}-2\right)\gamma \tag{54}$$

where β, γ are real constants. The linearized stability of the plane wave is easily obtained from Fourier analysis [54]. It proves most convenient to introduce the disturbance quantities $\tilde{q}$ as multiplicative perturbations to the plane wave

$$q = \left(\frac{\beta}{2}+\tilde{q}\right)e^{-i\left(\frac{2}{\gamma}x-\omega t\right)} \tag{55}$$

since this results in a convenient simplification upon linearization. Keeping only terms linear in $\tilde{q}$ after direct substitution of (55) into the CSP (2), the linearized disturbance equations become

$$\tilde{q}_t + \frac{\beta^2}{8}\tilde{q}_x^* + \frac{1}{4}\left(\beta^2+\gamma^2\right)\tilde{q}_x + \frac{i\gamma}{2}\tilde{q}_{xt} + \frac{i\gamma\beta^2}{16}\tilde{q}_{xx} - \frac{i\beta^2}{4\gamma}\left(\tilde{q}^*+\tilde{q}\right) = 0. \tag{56}$$

$$\det(M_k^{[1]}) = \begin{vmatrix} \frac{1}{\chi_1^*-\chi_1}\frac{\chi_1-\xi_k}{\chi_1^*-\xi_k} & \cdots & \frac{1}{\chi_1^*-\chi_{k-1}}\frac{\chi_{k-1}-\xi_k}{\chi_1^*-\xi_k} & \frac{1}{\chi_1^*-\xi_k} & \frac{1}{\chi_1^*-\xi_{k+1}}\frac{\xi_{k+1}-\xi_k}{\chi_1^*-\xi_k} & \cdots & \frac{1}{\chi_1^*-\xi_N}\frac{\xi_N-\xi_k}{\chi_1^*-\xi_k} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\chi_{k-1}^*-\chi_1}\frac{\chi_1-\xi_k}{\chi_{k-1}^*-\xi_k} & \cdots & \frac{1}{\chi_{k-1}^*-\chi_{k-1}}\frac{\chi_{k-1}-\xi_k}{\chi_{k-1}^*-\xi_k} & \frac{1}{\chi_{k-1}^*-\xi_k} & \frac{1}{\chi_{k-1}^*-\xi_{k+1}}\frac{\xi_{k+1}-\xi_k}{\chi_{k-1}^*-\xi_k} & \cdots & \frac{1}{\chi_{k-1}^*-\xi_N}\frac{\xi_N-\xi_k}{\chi_{k-1}^*-\xi_k} \\ \frac{1}{\xi_k^*-\chi_1}\frac{\chi_1-\xi_k}{\xi_k^*-\xi_k} & \cdots & \frac{1}{\xi_k^*-\chi_{k-1}}\frac{\chi_{k-1}-\xi_k}{\xi_k^*-\xi_k} & \frac{1}{\xi_k^*-\xi_k} & \frac{1}{\xi_k^*-\xi_{k+1}}\frac{\xi_{k+1}-\xi_k}{\xi_k^*-\xi_k} & \cdots & \frac{1}{\xi_k^*-\xi_N}\frac{\xi_N-\xi_k}{\xi_k^*-\xi_k} \\ \frac{1}{\xi_{k+1}^*-\chi_1}\frac{\chi_1-\xi_k}{\xi_{k+1}^*-\xi_k} & \cdots & \frac{1}{\xi_{k+1}^*-\chi_{k-1}}\frac{\chi_{k-1}-\xi_k}{\xi_{k+1}^*-\xi_k} & \frac{1}{\xi_{k+1}^*-\xi_k} & \frac{1}{\xi_{k+1}^*-\xi_{k+1}}\frac{\xi_{k+1}-\xi_k}{\xi_{k+1}^*-\xi_k} & \cdots & \frac{1}{\xi_{k+1}^*-\xi_N}\frac{\xi_N-\xi_k}{\xi_{k+1}^*-\xi_k} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\xi_N^*-\chi_1}\frac{\chi_1-\xi_k}{\xi_N^*-\xi_k} & \cdots & \frac{1}{\xi_N^*-\chi_{k-1}}\frac{\chi_{k-1}-\xi_k}{\xi_N^*-\xi_k} & \frac{1}{\xi_N^*-\xi_k} & \frac{1}{\xi_N^*-\xi_{k+1}}\frac{\xi_{k+1}-\xi_k}{\xi_N^*-\xi_k} & \cdots & \frac{1}{\xi_N^*-\xi_N}\frac{\xi_N-\xi_k}{\xi_N^*-\xi_k} \end{vmatrix}$$

$$= \frac{1}{\xi_k^*-\xi_k}\left(\prod_{l=1}^{k-1}\frac{\chi_l-\xi_k}{\chi_l^*-\xi_k}\right)\left(\prod_{l=k+1}^{N}\frac{\xi_l-\xi_k}{\xi_l^*-\xi_k}\right)\Delta_k^{[1]}$$

where

$$\Delta_k^{[1]} = \begin{vmatrix} \frac{1}{\chi_1^*-\chi_1} & \cdots & \frac{1}{\chi_1^*-\chi_{k-1}} & 1 & \frac{1}{\chi_1^*-\xi_{k+1}} & \cdots & \frac{1}{\chi_1^*-\xi_N} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\chi_{k-1}^*-\chi_1} & \cdots & \frac{1}{\chi_{k-1}^*-\chi_{k-1}} & 1 & \frac{1}{\chi_{k-1}^*-\xi_{k+1}} & \cdots & \frac{1}{\chi_{k-1}^*-\xi_N} \\ \frac{1}{\xi_k^*-\chi_1} & \cdots & \frac{1}{\xi_k^*-\chi_{k-1}} & 1 & \frac{1}{\xi_k^*-\xi_{k+1}} & \cdots & \frac{1}{\xi_k^*-\xi_N} \\ \frac{1}{\xi_{k+1}^*-\chi_1} & \cdots & \frac{1}{\xi_{k+1}^*-\chi_{k-1}} & 1 & \frac{1}{\xi_{k+1}^*-\xi_{k+1}} & \cdots & \frac{1}{\xi_{k+1}^*-\xi_N} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\xi_N^*-\chi_1} & \cdots & \frac{1}{\xi_N^*-\chi_{k-1}} & 1 & \frac{1}{\xi_N^*-\xi_{k+1}} & \cdots & \frac{1}{\xi_N^*-\xi_N} \end{vmatrix}$$

Box IV.

$$\Delta_k^{[1]} = \begin{vmatrix} \frac{1}{\chi_1^*-\chi_1}\frac{\xi_k^*-\chi_1^*}{\xi_k^*-\chi_1} & \cdots & \frac{1}{\chi_1^*-\chi_{k-1}}\frac{\xi_k^*-\chi_1^*}{\xi_k^*-\chi_{k-1}} & 0 & \frac{1}{\chi_1^*-\xi_{k+1}}\frac{\xi_k^*-\chi_1^*}{\xi_k^*-\xi_{k+1}} & \cdots & \frac{1}{\chi_1^*-\xi_N}\frac{\xi_k^*-\chi_1^*}{\xi_k^*-\xi_N} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\chi_{k-1}^*-\chi_1}\frac{\xi_k^*-\chi_{k-1}^*}{\xi_k^*-\chi_1} & \cdots & \frac{1}{\chi_{k-1}^*-\chi_{k-1}}\frac{\xi_k^*-\chi_{k-1}^*}{\xi_k^*-\chi_{k-1}} & 0 & \frac{1}{\chi_{k-1}^*-\xi_{k+1}}\frac{\xi_k^*-\chi_{k-1}^*}{\xi_k^*-\xi_{k+1}} & \cdots & \frac{1}{\chi_{k-1}^*-\xi_N}\frac{\xi_k^*-\chi_{k-1}^*}{\xi_k^*-\xi_N} \\ \frac{1}{\xi_k^*-\chi_1} & \cdots & \frac{1}{\xi_k^*-\chi_{k-1}} & 1 & \frac{1}{\xi_k^*-\xi_{k+1}} & \cdots & \frac{1}{\xi_k^*-\xi_N} \\ \frac{1}{\xi_{k+1}^*-\chi_1}\frac{\xi_k^*-\xi_{k+1}^*}{\xi_k^*-\chi_1} & \cdots & \frac{1}{\xi_{k+1}^*-\chi_{k-1}}\frac{\xi_k^*-\xi_{k+1}^*}{\xi_k^*-\chi_{k-1}} & 0 & \frac{1}{\xi_{k+1}^*-\xi_{k+1}}\frac{\xi_k^*-\xi_{k+1}^*}{\xi_k^*-\xi_{k+1}} & \cdots & \frac{1}{\xi_{k+1}^*-\xi_N}\frac{\xi_k^*-\xi_{k+1}^*}{\xi_k^*-\xi_N} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\xi_N^*-\chi_1}\frac{\xi_k^*-\xi_N^*}{\xi_k^*-\chi_1} & \cdots & \frac{1}{\xi_N^*-\chi_{k-1}}\frac{\xi_k^*-\xi_N^*}{\xi_k^*-\chi_{k-1}} & 0 & \frac{1}{\xi_N^*-\xi_{k+1}}\frac{\xi_k^*-\xi_N^*}{\xi_k^*-\xi_{k+1}} & \cdots & \frac{1}{\xi_N^*-\xi_N}\frac{\xi_k^*-\xi_N^*}{\xi_k^*-\xi_N} \end{vmatrix}$$

$$= \left(\prod_{l=1}^{k-1}\frac{\xi_k^*-\chi_l^*}{\xi_k^*-\chi_l}\right)\left(\prod_{l=k+1}^{N}\frac{\xi_k^*-\xi_l^*}{\xi_k^*-\xi_l}\right)C(\chi_1^*,\chi_2^*,\ldots,\chi_{k-1}^*)C(\xi_{k+1}^*,\xi_{k+2}^*,\ldots,\xi_N^*)$$

Box V.

Because of the conjugates in (56), the eigenfunctions are most conveniently expressed as linear combinations of pure Fourier modes,

$$\tilde{q} = f_+ e^{i\kappa(x+\Omega t)} + f_-^* e^{-i\kappa(x+\Omega^* t)}. \tag{57}$$

$$\begin{bmatrix} 8k^2\gamma^2\Omega + \left(4 + k^2\gamma^2\right)\beta^2 - 4\gamma\kappa\left(4\,\Omega + \gamma^2 + \beta^2\right) & 2\beta^2(2 - \gamma\kappa) \\ -2\beta^2(2 + \gamma\kappa) & -8k^2\gamma^2\Omega - \left(4 + k^2\gamma^2\right)\beta^2 - 4\gamma\kappa\left(4\,\Omega + \gamma^2 + \beta^2\right) \end{bmatrix} \begin{bmatrix} f_+ \\ f_- \end{bmatrix} = 0. \tag{58}$$

Box VI.

These eigenmodes are parameterized by the real wavenumber κ of the disturbance and the complex phase velocity Ω, where a positive imaginary part indicates a pure temporal growth mode of instability in positive time. Substitution into the linearized PDEs (55) and collection of resonant terms results in four linear homogeneous equations for the Fourier amplitudes $f_\pm$, Eq. (58) is given in Box VI. Solvability for this system requires that the determinant of the matrix of coefficients vanishes—this determines the dispersion relation for linearized disturbances

$$16\,\gamma^2\kappa^2\left(4\,\Omega + \gamma^2 + \beta^2\right)^2 + 16\,\beta^4 - \left[8\,\kappa^2\gamma^2\Omega + \left(4 + \kappa^2\gamma^2\right)\beta^2\right]^2 - 4\,\gamma^2\beta^4\kappa^2 = 0.$$

Ones can readily obtain that two roots for above square equation:

$$\Omega = \frac{(8 - \beta^2\kappa^2)\gamma^2 + 4\,\beta^2 \pm 4\gamma\sqrt{\gamma^2\kappa^2(\beta^2 + \gamma^2) - 4\beta^2}}{8(\kappa^2\gamma^2 - 4)}.$$

So when $\kappa^2 < \frac{4\beta^2}{\gamma^2(\beta^2+\gamma^2)}$, those roots with nonzero imaginary part correspond to linearly unstable modes, with growth rate $\kappa|\mathrm{Im}(\Omega)| = \gamma\kappa\sqrt{4\beta^2 - \gamma^2\kappa^2(\beta^2 + \gamma^2)}/[2|\kappa^2\gamma^2 - 4|]$. Then the baseband MI yields the rogue wave solution [55].

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