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NANOPTERON SOLUTIONS OF DIATOMIC FERMI-PASTA-ULAM-TSINGOU LATTICES WITH SMALL MASS-RATIO

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Abstract. Consider an infinite chain of masses, each connected to its nearest neighbors by a (nonlinear) spring. This is a Fermi-Pasta-Ulam-Tsingou lattice. We prove the existence of traveling waves in the setting where the masses alternate in size. In particular we address the limit where the mass ratio tends to zero. The problem is inherently singular and we find that the traveling waves are not true solitary waves but rather "nanopterons", which is to say, waves which asymptotic at spatial infinity to very small amplitude periodic waves. Moreover, we can only find solutions when the mass ratio lies in a certain open set. The difficulties in the problem all revolve around understanding Jost solutions of a nonlocal Schrödinger operator in its semi-classical limit.

Arrange infinitely many particles on a horizontal line, each attached to its nearest neighbors by a spring with a nonlinear restoring force. Constrain the motion of the particles to be within the line. This system is called a Fermi-Pasta-Ulam (FPU) or (more recently [8]) a Fermi-Pasta-Ulam-Tsingou (FPUT) lattice and it is one of the paradigmatic models for nonlinear and dispersive waves. In this article, we consider the existence of traveling waves in a diatomic (or "dimer") FPUT lattice when the ratio of the masses is nearly zero. By this we mean that the masses of the particles alternate between m_1 and m_2 along the chain and

$$\mu := \frac{m_2}{m_1} \ll 1.$$

The springs are all identical materially. The force they exert, when stretched by an amount r from their equilibrium length, is $F_s(r) := -k_s r - b_s r^2$ where $k_s > 0$ and $b_s \neq 0$. See [6] for an overview of this problem's history and [21] for a discussion of technological applications of such a system.

Newton's second law gives the equations of motion. After nondimensionalization, these read

$$\mathfrak{m}_j \ddot{y}_j = -r_{j-1} - r_{j-1}^2 + r_j + r_j^2. \tag{0.1}$$

Here $j \in \mathbf{Z}$ and $r_j := y_{j+1} - y_j$. When j is odd $\mathfrak{m}_j = 1$ and when j is even, $\mathfrak{m}_j = \mu$. In the above, y_j is the nondimensional displacement from equilibrium of the jth particle. See Figure 1 for a schematic.

Key words and phrases. FPU, FPUT, nonlinear hamiltonian lattices, periodic traveling waves, solitary traveling waves, solitons, singular perturbations, homogenization, heterogenous granular media, dimers, polymers, nanopeterons.

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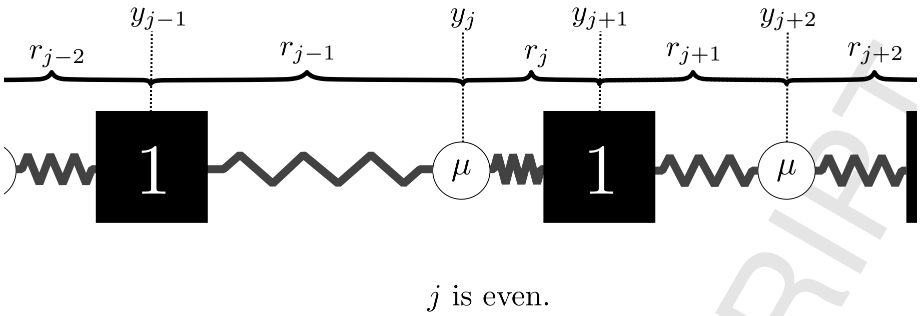

FIGURE 1. *A snippet of the diatomic Fermi-Pasta-Ulam-Tsingou lattice.*

In the monatomic case (where $\mu = 1$), this system famously possesses localized traveling wave solutions. Formal arguments suggesting their existence date back to [18]. The first rigorous proofs can be found in [26] (for a very special alternate nonlinearity) and [14] (for more general convex nonlinearities). The articles [10] [11] [12] [13] demonstrate that these traveling waves are asymptotically stable.

Putting $\mu = 0$ amounts to removing the smaller masses but leaving the springs attached. Which is to say that we have a monatomic lattice with a modified spring force. The results in [14] and [10] apply in this setting as well and so there is a localized traveling wave solution for (0.1) when $\mu = 0$.

The central question of this article is this: *does the* $\mu = 0$ *traveling wave solution persist when* $0 < \mu \ll 1$*?* In (0.1) the small parameter μ multiplies a second derivative and as such the system is singularly perturbed. An attempt to answer the question by way of regular perturbation theory (*i.e.* an implicit function theorem argument) is doomed to failure. Nonetheless, we have an answer: *the localized traveling wave at* $\mu = 0$ *perturbs into a nanopteron*[1] *for* μ *in an open set of positive numbers whose closure contains zero.*

Several recent articles, specifically [27] and [20], have carried out detailed formal asymptotics and performed careful numerics for this problem. They strongly indicate traveling waves solutions for (0.1) are nanopterons, at least for most values of the mass ratio μ; this article represents a rigorous mathematical validation of those predictions. We will in particular comment on the results of [27] below in Remark 3.2 in Section 3.

Nanopteron solutions are one of the many outcomes one may find for singularly perturbed systems of differential equations [5]. The "usual" way to prove their existence for a set of ordinary differential equations is through either geometric singular perturbation theory or matched asymptotics [16]. However, our problem is infinite dimensional which complicates using those sorts of tools. The method by which we prove our main result is a modification of one developed by Beale in [3] to study the existence of traveling waves in the capillary-gravity problem.[2] His method, which is functional analytic in nature, was subsequently deployed to

[1]A nanopteron is the superposition of a localized function and an extremely small amplitude spatially periodic piece.

[2]That is, one-dimensional free surface water waves which are acted by the restoring forces of gravity and surface tension. The existence of nanopteron solutions for capillary-gravity waves was established more or

show the existence of nanopteron solutions in several other singularly perturbed problems (*e.g.* [1]), including one closely related to the one here in [9].[3]

The common feature of problems with nanopteron solutions is that the singular perturbation manifests itself as a high frequency solution of the linearization. This in turn implies that a certain solvability condition must be met by solutions of the nonlinear problem. The key difference between what transpires here versus in [3] [1] [9] is that in our problem the high frequency linear solution is not a pure sinusoid but rather a Jost solution for a nonlocal Schrödinger operator. This is no minor thing: the ability to meet the solvability condition is more subtle and, as a consequence, the method only gives solutions for μ in the aforementioned open set (which we call M_c) as opposed to for all μ sufficiently close to zero. As we shall demonstrate, M_c is an infinite union of finite open intervals which aggregate at $\mu = 0$.

This article is organized as follows:

- Section 1 contains a reformulation of the equations of motion which has a simple form when $\mu = 0$. We state a nontechnical version of our main result in terms of this formulation in Theorem 1.2. Several symmetries of the governing equations are also discussed.
- Section 2 sets up the function analytic framework we work in and contains a number of simple estimates we will use repeatedly.
- Section 3 is a "birds-eye view" of the strategy of our proof, which is based on Beale's work in [3]. This section also discusses the places where and why substantial adjustments to his method have to be made.
- Section 4 is the first technical part of the proof and contains a "refinement" of the $\mu = 0$ monatomic approximation. This is the first building block of the nanopteron solutions.
- Section 5 concerns spatially periodic traveling wave solutions of (0.1) and it is here that we state Theorem 5.1, a novel existence result. These solutions ultimately give rise to the periodic part of the nanopteron, but are of independent interest (see [22] [4]).
- Section 6 we put together the refined leading order limit and periodic solutions and derive the first form of the governing equations for the nanopterons.
- Section 7 deals with the singular part of the linearization. It is in this section where the ultimate form of the nanopteron equations appear.
- Section 8 contains a statement of all the most important estimates (Lemma 8.1), the technical statement of our main result (Theorem 8.2) and the proof of that theorem given the estimates.
- Finally, we have Appendices A-E, which contain all the technical details of the proofs and estimates from the main part of the paper.

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less simultaneously in [3] and by Sun in [25]. Sun's method differs from Beale's in that it relies heavily on reformulating the problem as an integro-differential equation. Subsequently, Lombardi used a combination of harmonic analysis and center manifold theory to prove a very detailed existence/non-existence theorem for such waves in [19].

[3]In that article, the existence of traveling waves of nanopteron type for (0.1) is established but in a rather different limit: $\mu > 0$ is fixed but the the wavespeed is taken just above the lattice's speed of sound.

1. The main result.

1.1. The equations of motion. Before we state our main theorem, we reformulate (0.1). Let

$$y_j(t) = \begin{cases} Y_1(j,t) & \text{when } j \text{ is odd} \\ Y_2(j,t) & \text{when } j \text{ is even.} \end{cases} \tag{1.1}$$

To be clear, $Y_1(j,t)$ is defined only for j odd and $Y_2(j,t)$ for j even. Computing r_j in terms of Y_1 and Y_2 gives

$$r_j = \begin{cases} S^1 Y_2(j) - Y_1(j) & \text{when } j \text{ is odd} \\ S^1 Y_1(j) - Y_2(j) & \text{when } j \text{ is even} \end{cases}$$

where S^d is the "shift by d" map, specifically $S^d f(\cdot) := f(\cdot + d)$.

With this, (0.1) reads

$$\begin{aligned} \ddot{Y}_1 &= -2Y_1 + 2AY_2 + 4(\delta Y_2)(AY_2 - Y_1) \\ \mu \ddot{Y}_2 &= 2AY_1 - 2Y_2 + 4(\delta Y_1)(AY_1 - Y_2) \end{aligned} \tag{1.2}$$

where

$$A := \frac{1}{2}\left(S^1 + S^{-1}\right) \quad \text{and} \quad \delta := \frac{1}{2}\left(S^1 - S^{-1}\right). \tag{1.3}$$

If $\mu = 0$ then the second equation in (1.2) is satisfied when $Y_2 = AY_1$. We want to change variables in a way that exploits this, so we set

$$\rho_2 := Y_2 - AY_1.$$

Notice $\rho_2(j,t)$ is defined for even integers j. Substituting this into (1.2) we get

$$\begin{aligned} \ddot{Y}_1 &= 2\delta^2 Y_1 + 2A\rho_2 + 4(A\delta Y_1 + \delta\rho_2)(\delta^2 Y_1 + A\rho_2) \\ \mu \ddot{Y}_2 &= -2\rho_2 - 4(\delta Y_1)\rho_2 - 2\mu A \left[\delta^2 Y_1 + A\rho_2 + 2(A\delta Y_1 + \delta\rho_2)(\delta^2 Y_1 + A\rho_2)\right]. \end{aligned} \tag{1.4}$$

In computing the above we have made judicious use of the identity $1 + \delta^2 = A^2$.

On the right hand side of (1.4), Y_1 always appears with at least one δ applied. Thus we define

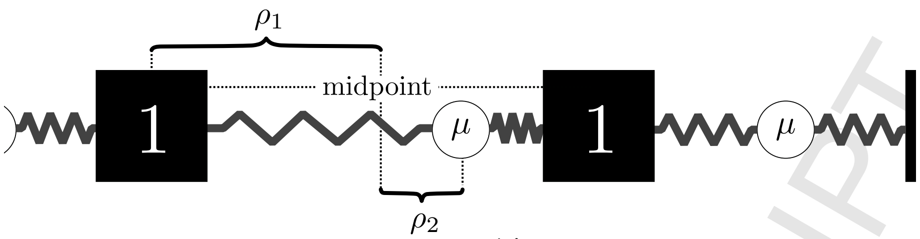
$$\rho_1 := \delta Y_1.$$

Like $\rho_2(j,t)$, $\rho_1(j,t)$ is defined for even integers j. With this, if we apply δ to the first equation we get

$$\begin{aligned} \ddot{\rho}_1 &= 2\delta^2\rho_1 + 2A\delta\rho_2 + 4\delta[(A\rho_1 + \delta\rho_2)(\delta\rho_1 + A\rho_2)] \\ \mu\ddot{\rho}_2 &= -2\rho_2 - 4\rho_1\rho_2 - 2\mu A[\delta\rho_1 + A\rho_2 + 2(A\rho_1 + \delta\rho_2)(\delta\rho_1 + A\rho_2)]. \end{aligned}$$

Next we use the identity $(Ag_1 + \delta g_2)(\delta g_1 + Ag_2) = \dfrac{1}{2}\delta(g_1^2 + g_2^2) + A(g_1 g_2)$ on the right hand side and we arrive at

$$\begin{aligned} \ddot{\rho}_1 &= 2\delta^2\rho_1 + 2A\delta\rho_2 + 2\delta^2(\rho_1^2 + \rho_2^2) + 4A\delta(\rho_1\rho_2) \\ \mu\ddot{\rho}_2 &= -2\mu A\delta\rho_1 - 2(1 + \mu A^2)\rho_2 - 2\mu A\delta(\rho_1^2 + \rho_2^2) - 4(1 + \mu A^2)(\rho_1\rho_2). \end{aligned} \tag{1.5}$$



j is even.

FIGURE 2. *Sketch of ρ_1 and ρ_2.*

This system, which is posed for j in the even lattice $2\mathbf{Z}$, is equivalent to the equations of motion (0.1).

The formulas for the variables ρ_1 and ρ_2 may seem to be somewhat nonintuitive, but they in fact have simple physical meanings. A chase through their definitions shows that ρ_1 and ρ_2 are found from the "stretch" variables $r_j = y_{j+1} - y_j$ by

$$\rho_1(j) = \frac{1}{2}\left(r_j + r_{j-1}\right) \quad \text{and} \quad \rho_2(j) = -\frac{1}{2}\left(r_j - r_{j-1}\right).$$

Since j is an even number in the above, we see that $\rho_1(j)$ is simply half the distance from a heavy mass to the next heavy mass. Which is to say it is the distance to the midway point between heavy masses. And $\rho_2(j)$ measures how far the light mass in between those heavy ones is from this midpoint. See Figure 2. This point of view is why we will sometimes refer to ρ_1 as the "heavy variable"; it is determined fully by the locations of the heavy particles alone. We will also call first equation in (1.5) the "heavy equation." On the other hand, ρ_2 specifies the location of the lighter particles and so we call it the "light variable" and the second equation in (1.5) the "light equation."

1.2. **Our result.** The system (1.5) has a simple structure at $\mu = 0$: the second equation reduces to

$$\rho_2 + 2\rho_1\rho_2 = 0 \tag{1.6}$$

which can be solved by taking $\rho_2(x) \equiv 0$. Physically this means the light (in this case, massless) particles are located exactly halfway between their bigger brethren. The heavy equation is then simply

$$\ddot{\rho}_1 = 2\delta^2\left(\rho_1 + \rho_1^2\right) \tag{1.7}$$

which coincides with the equations of motion for a monatomic lattice with restoring force given by $-2\rho - 2\rho^2$ (see, for instance, [10]).

Since the $\mu = 0$ problem is equivalent to a monatomic FPUT lattice, we can summon the results of [14] and [10] to get an exact traveling wave solution to it.

Theorem 1.1 (Friesecke & Pego)**.** *There exists $c_1 > c_0$, where*

$$c_0 := \text{“the speed of sound”} := \sqrt{2}, \tag{1.8}$$

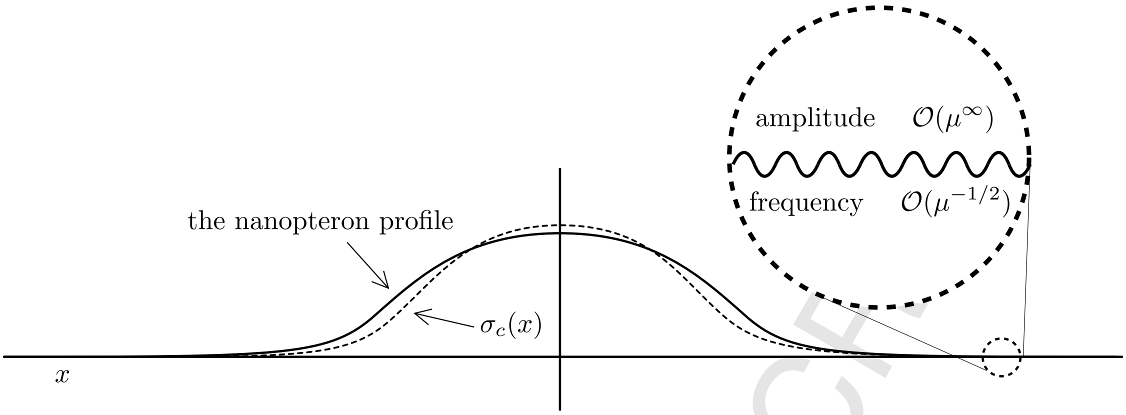


FIGURE 3. *Sketch of the nanopteron profile for the first component, ρ_1. The periodic part is so small as to be invisible relative to the core, hence the inset. By $\mathcal{O}(\mu^{\infty})$ we mean "small beyond all orders of μ."*

for which $|c| \in (c_0, c_1]$ implies the existence of a positive, even, smooth, bounded and unimodal function, $\sigma_c(x)$, such that $\rho_1(j,t) = \sigma_c(j - ct)$ satisfies (1.7). Moreover $\sigma_c(x)$ decays exponentially quickly with respect to x in the following sense: there exists $b_c > 0$ such that, for any $s \geq 0$ and $p \in [1, \infty]$, we have

$$\left\| \cosh^{b_c}(\cdot)\sigma_c^{(s)}(\cdot) \right\|_{L^p(\mathbf{R})} < \infty. \tag{1.9}$$

Thus we see that putting

$$\rho_1(j,t) = \sigma_c(j - ct) \quad \text{and} \quad \rho_2(j,t) = 0 \tag{1.10}$$

solves (1.5) when $\mu = 0$.

Now we can state our main thereom. It says that this solution perturbs into a nanopteron for $\mu > 0$; see Figure 3.

Theorem 1.2. *For $|c| \in (c_0, c_1]$ there exists an open set $M_c \subset \mathbf{R}_+$ whose closure contains zero and for which $\mu \in M_c$ implies the following. There exist smooth functions $\Upsilon_{c,\mu,1}(x)$, $\Upsilon_{c,\mu,2}(x)$, $\Phi_{c,\mu,1}(x)$ and $\Phi_{c,\mu,2}(x)$ such that putting*

$$\rho_1(j,t) = \sigma_c(j - ct) + \Upsilon_{c,\mu,1}(j - ct) + \Phi_{c,\mu,1}(j - ct)$$

and

$$\rho_2(j,t) = \Upsilon_{c,\mu,2}(j - ct) + \Phi_{c,\mu,2}(j - ct)$$

solves (1.5). Moreover, $\Upsilon_{c,\mu,1}(x)$ and $\Upsilon_{c,\mu,2}(x)$ decay exponentially quickly with respect to x and have amplitudes of $\mathcal{O}(\mu)$. Finally, $\Phi_{c,\mu,1}(x)$ and $\Phi_{c,\mu,2}(x)$ are high frequency (specifically $\mathcal{O}(\mu^{-1/2})$) periodic functions of x whose amplitudes are small beyond all orders of μ.

We do not prove this theorem in the coordinates ρ_1 and ρ_2, but rather we make an additional near identity change of variables.

1.3. **"Almost diagonalization".** Denoting[4] $\boldsymbol{\rho} = \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix}$, if we put

(1.11)
$$I_\mu := \mathrm{diag}(1,\mu), \quad \mathcal{D}_\mu := \begin{bmatrix} -2\delta^2 & -2A\delta \\ 2\mu A\delta & 2(1+\mu A^2) \end{bmatrix} \quad \text{and} \quad Q_0(\mathbf{g}, \grave{\mathbf{g}}) = \begin{pmatrix} g_1\grave{g}_1 + g_2\grave{g}_2 \\ g_1\grave{g}_2 + g_2\grave{g}_1 \end{pmatrix}$$

then (1.5) is equivalent to

$$I_\mu \ddot{\boldsymbol{\rho}} + \mathcal{D}_\mu \boldsymbol{\rho} + \mathcal{D}_\mu Q_0(\boldsymbol{\rho}, \boldsymbol{\rho}) = 0. \tag{1.12}$$

Observe that $\mathcal{D}_\mu$ is upper triangular when $\mu = 0$. We can make a simple change of variables that "almost diagonalizes" the linear part of (1.12); this will be advantageous down the line. Let

$$\boldsymbol{\rho} := T_\mu \boldsymbol{\theta} \quad \text{where} \quad T_\mu := \begin{bmatrix} 1 & -\mu A\delta \\ 0 & 1 \end{bmatrix}. \tag{1.13}$$

Since T_μ is a small perturbation of the identity, we will continue to ascribe to θ_1 and θ_2 the physical meaning of ρ_1 and ρ_2; that is θ_1 is "heavy" and θ_2 (which is in fact exactly ρ_2) is "light."

With this, (1.12) becomes

$$I_\mu T_\mu \ddot{\boldsymbol{\theta}} + \mathcal{D}_\mu T_\mu \boldsymbol{\theta} + \mathcal{D}_\mu Q_0(T_\mu \boldsymbol{\theta}, T_\mu \boldsymbol{\theta}) = 0.$$

Since T_μ is invertible for all $\mu \geq 0$ and I_μ is invertible for $\mu > 0$, the above is equivalent to

$$I_\mu \ddot{\boldsymbol{\theta}} + L_\mu \boldsymbol{\theta} + L_\mu Q_\mu(\boldsymbol{\theta}, \boldsymbol{\theta}) = 0 \tag{1.14}$$

where[5]

$$L_\mu := I_\mu T_\mu^{-1} I_\mu^{-1} \mathcal{D}_\mu T_\mu := \begin{bmatrix} -2\delta^2(1-\mu A^2) & -2\mu A\delta\,(1 - 2A^2 + \mu A^2\delta^2) \\ 2\mu A\delta & 2(1 + \mu A^2 - \mu^2 A^2 \delta^2) \end{bmatrix} \tag{1.15}$$

and

$$Q_\mu(\boldsymbol{\theta}, \grave{\boldsymbol{\theta}}) := T_\mu^{-1} Q_0(T_\mu \boldsymbol{\theta}, T_\mu \grave{\boldsymbol{\theta}}).$$

As we advertised above,

$$L_0 = \begin{bmatrix} -2\delta^2 & 0 \\ 0 & 2 \end{bmatrix}$$

is a diagonal operator and thus L_μ is nearly diagonal.

At last we make the traveling wave ansatz:

$$\boldsymbol{\theta}(j,t) = \mathbf{h}(j - ct)$$

[4]Generally, we represent maps from $\mathbf{R} \to \mathbf{R}^2$ by bold letters, for instance $\boldsymbol{\rho}(x)$ or $\mathbf{h}(x)$. Likewise, the first and second components of such functions, as shown here, will be represented in the regular font with a subscript "1" and "2", respectively.

[5]This product defining L_μ is rather painful to multiply out, so we omit showing the details; it can be computed formally by replacing A with $\cos(k)$ and δ with $i\sin(k)$ and asking a computer algebra system to carry out the product. Then you just replace all the cosines with A and sines with $-i\delta$. The reason that this works is that $Ae^{ikx} = \cos(k)e^{ikx}$ and $\delta e^{ikx} = i\sin(k)e^{ikx}$. Which is to say the product is easier on the Fourier side.

where $\mathbf{h}:\mathbf{R}\to\mathbf{R}^2$ and $c\in\mathbf{R}$ is the wave speed. With this we get the following equation for $\mathbf{h}$:[6]

$$\underbrace{c^2 I_\mu \mathbf{h}'' + L_\mu \mathbf{h} + L_\mu Q_\mu(\mathbf{h},\mathbf{h})}_{\mathcal{G}(\mathbf{h},\mu)} = 0. \tag{1.16}$$

The primes denote differentiation with respect to x, the independent variable of $\mathbf{h}$. Of course the operators A and δ (out of which are constructed L_μ and Q_μ) act on functions of $x\in\mathbf{R}$ just as they do on functions of $j\in\mathbf{Z}$. Note that Q_μ is bilinear and symmetric in its arguments.

At $\mu=0$ we have $\boldsymbol{\rho}=\boldsymbol{\theta}$. Thus the line of reasoning that lead to (1.10) tells us that putting[7]

$$\boldsymbol{\sigma}_c := \sigma_c \mathbf{e}_1 \quad \Longrightarrow \quad \mathcal{G}(\boldsymbol{\sigma}_c, 0) = 0. \tag{1.17}$$

1.4. **Some symmetries of $\mathcal{G}$.** We now point out two symmetries possessed by the mapping $\mathcal{G}(\mathbf{h},\mu)$. The first is that if $h_1(x)$ is an even function of x and $h_2(x)$ is odd, then the components of $\mathcal{G}(\mathbf{h},\mu)$ are, respectively, even and odd. This is a consequence of the following simple facts:

- Both ∂_x and δ map even functions to odd ones and vice-versa.
- The map A takes odd functions to odd functions and even to even.
- If g_1 is even and g_2 is odd then g_1^2 and g_2^2 are even while g_1g_2 is odd.

With these in hand, showing that $\mathcal{G}$ maintains "even $\times$ odd" symmetry amounts to just scanning through its definition. Henceforth we assume that the function $\mathbf{h}$ and its descendents will have this even $\times$ odd symmetry.

Moreover, δ annihilates constants, just as ∂_x does. And, also just like ∂_x, we have

$$\int_{\mathbf{R}} \delta f(x)dx = 0 \quad \text{and} \quad \int_{-P}^{P} \delta f(x)dx = 0. \tag{1.18}$$

In the first integral we assume that $f(x)$ is going to zero quickly enough as $|x|\to\infty$ and in the second that f is periodic with period $2P$. Both integrals read as a sort of "mean-zero" condition for δf and so we say that δf is a "mean-zero function."

Now observe that each term in the first row of L_μ has at least one factor of δ exposed. Specifically:

$$\begin{aligned}(L_\mu \mathbf{f})\cdot\mathbf{e}_1 &= -2\delta^2(1-\mu A^2)f_1 - 2\mu A\delta(1-2A^2+\mu A^2\delta^2)f_2\\ &= \delta\left(-2\delta(1-\mu A^2)f_1 - 2\mu A(1-2A^2+\mu A^2\delta^2)f_2\right).\end{aligned}$$

Thus from (1.18) we have

$$\int_{\mathbf{R}} (L_\mu \mathbf{f})(x)\cdot\mathbf{e}_1 dx = 0 \quad \text{and} \quad \int_{-P}^{P} (L_\mu \mathbf{f})(x)\cdot\mathbf{e}_1 dx = 0.$$

[6]As before, we think of $h_1(x)$ as being the heavy variable and $h_2(x)$ as being the light one.

[7]We use $\mathbf{e}_1 := \begin{pmatrix}1\\0\end{pmatrix}$ and $\mathbf{e}_2 := \begin{pmatrix}0\\1\end{pmatrix}$ to denote the usual unit vectors in $\mathbf{R}^2$.

In turn this implies

$$\int_{\mathbf{R}} \mathcal{G}(\mathbf{h},\mu)(x)\cdot \mathbf{e}_1 dx = 0 \quad \text{and} \quad \int_{-P}^{P} \mathcal{G}(\mathbf{h},\mu)(x)\cdot \mathbf{e}_1 dx = 0.$$

Which is to say that the first component of $\mathcal{G}$ has no constant term, or equivalently, that it is mean-zero.

We summarize these symmetries in the informal statement

$$\mathcal{G}(\cdot,\mu) : \{\text{evens}\} \times \{\text{odds}\} \to \{\text{mean-zero evens}\} \times \{\text{odds}\}\,. \tag{1.19}$$

It is worth reiterating that $I_\mu \partial_x^2$, L_μ and $L_\mu Q_\mu$, each on their own, have this same property. Now that we have our traveling wave equation spelled out, and we understand it well at $\mu = 0$, we take the next section to lay out our function spaces, key definitions as well as prove some rudimentary estimates.

2. Functions spaces, notation and basic estimates

2.1. **Periodic functions.** We let $W^{s,p}_{\text{per}} := W^{s,p}(\mathbf{T})$ be the usual "s,p" Sobolev space of 2π-periodic functions. We denote $L^p_{\text{per}} := W^{0,p}_{\text{per}}$ and $H^s_{\text{per}} := W^{s,2}_{\text{per}}$. Put

$$E^s_{\text{per}} := H^s_{\text{per}} \cap \{\text{even functions}\} \quad \text{and} \quad O^s_{\text{per}} := H^s_{\text{per}} \cap \{\text{odd functions}\}\,. \tag{2.1}$$

We will also make use of

$$E^s_{\text{per},0} := E^s_{\text{per}} \cap \left\{ u(X) : \int_{-\pi}^{\pi} u(X)dX = 0 \right\}. \tag{2.2}$$

That is to say, mean-zero even periodic functions. By C^s_{per} we mean the space of $s-$times differentiable 2π-periodic functions and C^∞_{per} is the space of smooth 2π-periodic functions.

2.2. **Functions on R.** We let $W^{s,p} := W^{s,p}(\mathbf{R})$ be the usual "s,p" Sobolev space of functions defined on $\mathbf{R}$. For $b \in \mathbf{R}$ put

$$W^{s,p}_b := \left\{ u \in L^2(\mathbf{R}) : \cosh^b(x)u(x) \in W^{s,p}(\mathbf{R}) \right\}. \tag{2.3}$$

These are Banach spaces with the naturally defined norm. If we say a function is "exponentially localized" we mean that it is in one of these spaces with $b > 0$.

Put $L^p_b := W^{0,p}_b$, $H^s := W^{s,2}$ and $H^s_b := W^{s,2}_b$ and denote $\|\cdot\|_{s,b} := \|\cdot\|_{H^s_b}$. We let

$$E^s_b := H^s_b \cap \{\text{even functions}\} \quad \text{and} \quad O^s_b := H^s_b \cap \{\text{odd functions}\}\,. \tag{2.4}$$

For instance the monatomic wave profile $\sigma_c(x)$, by virtue of (1.9), is in $E^s_{b_c}$ for all s. We will also make use of

$$E^s_{b,0} := E^s_b \cap \left\{ u(x) : \int_{\mathbf{R}} u(x)dx = 0 \right\}. \tag{2.5}$$

This is another space of mean-zero even functions.

2.3. **Big $\mathcal{O}$ and big C notation.** Many of our quantities will depend on the mass ratio μ, the wavespeed c and a decay rate b. Of these, μ is the chief but b will play an important role as well; we generally view c as being fixed and as such we do not usually track dependence on it. In order to simplify the statements of many of our estimates we employ the following conventions. Any exceptions/restrictions will be clearly noted.

Definition 2.1. *Suppose that $q = q(\mu, b, c) \in \mathbf{R}_+$ and $f = f(\mu) \in \mathbf{R}_+$.*

- *We write*

$$q \leq Cf \quad \text{or} \quad q \leq \mathcal{O}(f) \tag{2.6}$$

if, for all $|c| \in (c_0, c_1]$, there exists $\mu_* \in (0,1)$ and $C > 0$ such that $q(\mu, b, c) \leq Cf(\mu)$ for all $\mu \in (0, \mu_*)$ and $b \in [0, b_c]$. That is to say, the constant C in (2.6) may depend on c but not on μ or b.

- *We write*

$$q \leq C_b f \quad \text{or} \quad q \leq \mathcal{O}_b(f) \tag{2.7}$$

if, for all $|c| \in (c_0, c_1]$ and $b \in (0, b_c]$, there exists $\mu_*(b) \in (0,1)$ and $C_b > 0$ such that $q(\mu, b', c) \leq C_b f(\mu)$ for all $\mu \in (0, \mu_*(b)]$ and $b' \in [b, b_c]$. That is to say, the constant C_b in (2.7) may depend on c and b but not on μ.

The point is this: if an estimate depends in a bad way on the decay rate b then we adorn it with the subscript b. Moreover, our definition indicates that C_b increases as $b \to 0^+$.

We will also occasionally write either $q = \mathcal{O}(1)$ or $q = \mathcal{O}(\mu^p)$, as as opposed to $q \leq \mathcal{O}(1)$ or $q \leq \mathcal{O}(\mu^p)$. We use this to indicate that we can control q from above and below by $C\mu^p$. Specifically

Definition 2.2. *Suppose that $q = q(\mu, c) \in \mathbf{R}_+$ and $p \in \mathbf{R}$. We write*

$$q = \mathcal{O}(\mu^p) \tag{2.8}$$

if $|c| \in (c_0, c_1]$ implies there is $\mu_* \in (0,1)$ and $0 < C_1 < C_2$ such that

$$C_1 \mu^p \leq q(\mu, c) \leq C_2 \mu^p$$

for all $\mu \in (0, \mu_*)$.

Lastly, we will encounter terms which are small "beyond all orders of μ." By this we mean the following.

Definition 2.3. *Suppose $q = q(\mu, c) \in \mathbf{R}_+$. We write*

$$q \leq \mathcal{O}(\mu^\infty)$$

if $q \leq \mathcal{O}(\mu^p)$ for all $p \geq 1$.

2.4. **General estimates for $W_b^{s,p}$.** We take for granted the containment estimate:

$$\|f\|_{s,b} \leq \|f\|_{s',b'} \quad \text{when } s \leq s',\ b \leq b'.$$

Likewise, there is the generalization of the Sobolev embedding estimate:

$$\|f\|_{W_b^{s-1,\infty}} \leq C\|f\|_{s,b} \quad \text{when } s \geq 1.$$

We have the following simple estimate:

$$\|fg\|_{W_b^{0,p}} = \|\cosh^b(\cdot)fg\|_{L^p} = \|(\cosh^{b'}(\cdot)f)(\cosh^{b-b'}(\cdot)g)\|_{L^p}$$
$$\leq \|\cosh^{b'}(\cdot)f\|_{L^\infty}\|\cosh^{b-b'}(\cdot)g\|_{L^p} = \|f\|_{W_{b'}^{0,\infty}}\|g\|_{W_{b-b'}^{0,p}}.$$

This in turn implies

$$\|fg\|_{W_b^{s,p}} \leq C\|f\|_{W_{b'}^{s,\infty}}\|g\|_{W_{b-b'}^{s,p}} \tag{2.9}$$

and, using the Sobolev estimate

$$\|fg\|_{s,b} \leq C\|f\|_{s,b'}\|g\|_{s,b-b'}. \tag{2.10}$$

Note that (2.9) holds for $s \geq 0$ but (2.10) only when $s \geq 1$. The important feature of these estimates is that the weight b on the left hand side can be shared between the two functions f and g more or less however we wish. It will even be necessary at several stages for us to have $b - b' < 0$. Note also that (2.10) implies that, if $b \geq 0$ and $s \geq 1$, that H_b^s is an algebra.

We also have the containment $L_b^q \subset L^p$ when $p < q$ and $b > 0$. These follow from the following estimate involving Hölders inequality

$$\|f\|_{L^p} = \|\operatorname{sech}^b(\cdot)\cosh^b(\cdot)f\|_{L^p} \leq \|\operatorname{sech}^b(\cdot)\|_{L^{q'}}\|f\|_{L_b^q} \leq C_b\|f\|_{L_b^q} \tag{2.11}$$

where $\dfrac{1}{q'} = \dfrac{1}{p} - \dfrac{1}{q}$.

2.5. **Simple operator estimates.** The next estimates follow from $u-$substitution and the definition of the shift operator S^d:

$$\|S^d f\|_{W_{\text{per}}^{s,p}} = \|f\|_{W_{\text{per}}^{s,p}} \quad \text{and} \quad \|S^d f\|_{W_b^{s,p}} \leq e^{|db|}\|f\|_{W_b^{s,p}}. \tag{2.12}$$

That is to say S^d is a bounded operators on all of the function spaces we use here; we will treat them as such without comment. These estimates imply

$$\|A\|_{\mathfrak{X}\to\mathfrak{X}} \leq C \quad \text{and} \quad \|\delta\|_{\mathfrak{X}\to\mathfrak{X}} \leq C \tag{2.13}$$

where $\mathfrak{X}$ is either $W_b^{s,p}$ or $W_{\text{per}}^{s,p}$. In turn these, after a quick glance at the definitions of L_μ and T_μ, give

$$\|L_\mu - L_0\|_{\mathfrak{X}\to\mathfrak{X}} \leq C\mu \quad \text{and} \quad \|T_\mu - \mathbf{id}\|_{\mathfrak{X}\to\mathfrak{X}} \leq C\mu \tag{2.14}$$

where $\mathfrak{X}$ is either $W_b^{s,p} \times W_b^{s,p}$ or $W_{\text{per}}^{s,p} \times W_{\text{per}}^{s,p}$. Note that the estimates in (2.14) hold for all $\mu \in (0,1)$.

2.6. **The bilinear map $L_\mu Q_\mu$.** We have not written out $L_\mu Q_\mu(\mathbf{h}, \grave{\mathbf{h}})$ in full detail; there are very many terms and ultimately not much would be learned. But we can give a useful and relatively simple collection of estimates for it. For functions $\mathbf{h} : \mathbf{R} \to \mathbf{R}^n$, define the "windowed absolute value" as

$$|\mathbf{h}(x)|_W := \max_{d\in\mathbf{Z},\ |d|\leq 10} \left\{ |S^d\mathbf{h}(x)|_{\mathbf{R}^n} \right\}.$$

Here $|\cdot|_{\mathbf{R}^n}$ is the just the Euclidean norm on $\mathbf{R}^n$. From the esimates for S^d in (2.12), we can conclude that

$$\||\mathbf{h}(\cdot)|_W\|_{\mathfrak{X}} \leq C\|\mathbf{h}\|_{\mathfrak{X}} \tag{2.15}$$

where $\mathfrak{X} = L^2_b$ or L^2_{per}.

Examining (1.11) and (1.13) gives

$$L_0 Q_0(\mathbf{h}, \grave{\mathbf{h}}) = \begin{bmatrix} -2\delta^2(h_1\grave{h}_1 + h_2\grave{h}_2) \\ 2(h_1\grave{h}_2 + h_2\grave{h}_1) \end{bmatrix}. \tag{2.16}$$

Since δ is made out of the shifts $S^{\pm 1}$, this formula tells us that for all $x \in \mathbf{R}$

$$\left| L_0 Q_0(\mathbf{h}, \grave{\mathbf{h}})(x) \cdot \mathbf{e}_1 \right| \le C \left(|h_1(x)|_W |\grave{h}_1(x)|_W + |h_2(x)|_W |\grave{h}_2(x)|_W \right) \tag{2.17}$$

and

$$\left| L_0 Q_0(\mathbf{h}, \grave{\mathbf{h}})(x) \cdot \mathbf{e}_2 \right| \le C \left(|h_1(x)||\grave{h}_2(x)| + |h_2(x)||\grave{h}_1(x)| \right). \tag{2.18}$$

Similarly, for all $x \in \mathbf{R}$, we have

$$\left| L_\mu Q_\mu(\mathbf{h}, \grave{\mathbf{h}})(x) - L_0 Q_0(\mathbf{h}, \grave{\mathbf{h}})(x) \right|_{\mathbf{R}^2} \le C\mu |\mathbf{h}(x)|_W |\grave{\mathbf{h}}(x)|_W. \tag{2.19}$$

Here is why. The estimates (2.14) and the definition of Q_μ in (1.11) demonstrate that extracting $L_0 Q_0$ from $L_\mu Q_\mu$ leaves only terms with at least one exposed power of μ. The operator $Q_0(\mathbf{h}, \grave{\mathbf{h}})$ is bilinear and this is why we have the product of $|\mathbf{h}(x)|_W$ and $|\grave{\mathbf{h}}(x)|_W$ on the right. And we have the windowed absolute value because T_μ and L_μ are constructed out of A and δ, which themselves are made from $S^{\pm 1}$. Tracking through the definitions shows that the most shifts that could land on one function is ten: two in T_μ, two in T_μ^{-1} and six in L_μ. This is why the window has a radius of ten.

Putting (2.15), (2.17), (2.18) and (2.19) together with (2.9) and various Sobolev estimates yields the following collection of estimates:

$$\begin{aligned} \|L_\mu Q_\mu(\mathbf{h}, \grave{\mathbf{h}}) \cdot \mathbf{e}_1\|_{s,b} \le & C \left(\|h_1\|_{W^{s,\infty}_{b'}} \|\grave{h}_1\|_{s,b-b'} + \|h_2\|_{W^{s,\infty}_{b''}} \|\grave{h}_2\|_{s,b-b''} \right) \\ & + C\mu \left(\|h_1\|_{W^{s,\infty}_{b'''}} \|\grave{h}_2\|_{s,b-b'''} + \|h_2\|_{W^{s,\infty}_{b''''}} \|\grave{h}_1\|_{s,b-b''''} \right) \end{aligned} \tag{2.20}$$

$$\begin{aligned} \|L_\mu Q_\mu(\mathbf{h}, \grave{\mathbf{h}}) \cdot \mathbf{e}_2\|_{s,b} \le & C \left(\|h_1\|_{W^{s,\infty}_{b'}} \|\grave{h}_2\|_{s,b-b'} + \|h_2\|_{W^{s,\infty}_{b''}} \|\grave{h}_1\|_{s,b-b''} \right) \\ & + C\mu \left(\|h_1\|_{W^{s,\infty}_{b'''}} \|\grave{h}_1\|_{s,b-b'''} + \|h_2\|_{W^{s,\infty}_{b''''}} \|\grave{h}_2\|_{s,b-b''''} \right) \end{aligned} \tag{2.21}$$

$$\begin{aligned} \|L_\mu Q_\mu(\mathbf{h}, \grave{\mathbf{h}}) \cdot \mathbf{e}_1\|_{s,b} \le & C \left(\|h_1\|_{s,b'} \|\grave{h}_1\|_{s,b-b'} + \|h_2\|_{s,b''} \|\grave{h}_2\|_{s,b-b''} \right) \\ & + C\mu \left(\|h_1\|_{s,b'''} \|\grave{h}_2\|_{s,b-b'''} + \|h_2\|_{s,b''''} \|\grave{h}_1\|_{s,b-b''''} \right) \end{aligned} \tag{2.22}$$

$$\begin{aligned} \|L_\mu Q_\mu(\mathbf{h}, \grave{\mathbf{h}}) \cdot \mathbf{e}_2\|_{s,b} \le & C \left(\|h_1\|_{s,b'} \|\grave{h}_2\|_{s,b-b'} + \|h_2\|_{s,b''} \|\grave{h}_1\|_{s,b-b''} \right) \\ & + C\mu \left(\|h_1\|_{s,b'''} \|\grave{h}_1\|_{s,b-b'''} + \|h_2\|_{s,b''''} \|\grave{h}_2\|_{s,b-b''''} \right) \end{aligned} \tag{2.23}$$

$$\begin{aligned} \|L_\mu Q_\mu(\mathbf{h}, \grave{\mathbf{h}}) \cdot \mathbf{e}_1\|_{H^s_{\text{per}}} \le & C \left(\|h_1\|_{H^s_{\text{per}}} \|\grave{h}_1\|_{H^s_{\text{per}}} + \|h_2\|_{H^s_{\text{per}}} \|\grave{h}_2\|_{H^s_{\text{per}}} \right) \\ & + C\mu \left(\|h_1\|_{H^s_{\text{per}}} \|\grave{h}_2\|_{H^s_{\text{per}}} + \|h_2\|_{H^s_{\text{per}}} \|\grave{h}_1\|_{H^s_{\text{per}}} \right) \end{aligned} \tag{2.24}$$

and

$$\begin{aligned}\|L_\mu Q_\mu(\mathbf{h},\grave{\mathbf{h}})\cdot \mathbf{e}_2\|_{H^s_{\text{per}}} \leq & C\left(\|h_1\|_{H^s_{\text{per}}}\|\grave{h}_2\|_{H^s_{\text{per}}}+\|h_2\|_{H^s_{\text{per}}}\|\grave{h}_1\|_{H^s_{\text{per}}}\right)\\ & +C\mu\left(\|h_1\|_{H^s_{\text{per}}}\|\grave{h}_1\|_{H^s_{\text{per}}}+\|h_2\|_{H^s_{\text{per}}}\|\grave{h}_2\|_{H^s_{\text{per}}}\right).\end{aligned} \tag{2.25}$$

In the above, b', b'', b''' and b'''' are free to be anything. In (2.22)-(2.25) we need $s \geq 1$, but $s \geq 0$ suffices for (2.20) and (2.21). All of the estimates (2.20)-(2.25) hold for all $\mu \in (0,1)$.

3. The strategy

Here we outline our approach to the proof of Theorem 1.2. The first thing we do is we put $\mathbf{h} = \boldsymbol{\sigma}_c + \boldsymbol{\xi}$ into (1.16). For the time being, we are thinking of $\boldsymbol{\xi}$ as being small and *localized* function, for instance in $E^s_{b_c} \times O^s_{b_c}$. (That is to say $\boldsymbol{\xi}$, like $\mathbf{h}$, has the even $\times$ odd symmetry.) Using (1.17) and some algebra shows that

$$c^2 I_\mu \boldsymbol{\xi}'' + L_0\boldsymbol{\xi} + 2L_0Q_0(\boldsymbol{\sigma}_c,\boldsymbol{\xi}) = R_\mu. \tag{3.1}$$

The left hand side consists of all the leading order terms, plus the singularly perturbed term $c^2\mu\xi_2''$. The right hand side R_μ is everything else. Its exact form is not germaine at this point but it is made up of:

- terms which are linear in $\boldsymbol{\xi}$ but have at least one prefactor of μ (for instance $\mu A^2\delta^2\xi_1$),
- $\mathcal{O}(\mu)$ residuals (for instance $\mu A\delta\sigma_c(x)$) and
- terms which are quadratic in $\boldsymbol{\xi}$.

That is to say, the right hand side is "small."

If we can invert the linear operator of $\boldsymbol{\xi}$ on the left hand side we would have our system written as a $\boldsymbol{\xi} = N(\boldsymbol{\xi})$ where N is an (ostensibly) small operator and then we could use a contraction mapping argument to solve this fixed point equation. A look at the page count of this article makes it clear that we could not get such a strategy to work.

To see why, we write out (3.1) component-wise. The first (or "heavy") component reads

$$\underbrace{c^2\xi_1'' - 2\delta^2[(1+2\sigma_c)\xi_1]}_{\mathcal{H}\xi_1} = R_{\mu,1}. \tag{3.2}$$

The operator $\mathcal{H}$ is closely related to an operator that appears in the analysis of the stability of monatomic FPUT solitary waves [10]. The following result, which states that $\mathcal{H}$ is (more or less) invertible in the class of even functions, can be inferred from results there; we carry out the details in Appendix A.

Proposition 3.1. *The following holds for $|c| \in (c_0, c_1]$*[8]*. The map $\mathcal{H}$ is a homeomorphism of E^{s+2}_b and $E^s_{b,0}$ for any $s \geq 0$ and $b \in (0, b_c]$. Specifically we have*

$$\|\mathcal{H}^{-1}\|_{E^s_{b,0}\to E^{s+2}_b} \leq C_b. \tag{3.3}$$

[8]The constants c_1 and b_c in this proposition may be smaller than their counterparts with the same names in Theorem 1.1, but since that theorem remains true with the smaller values here, we act here (and throughout) as if the constants were the same to begin with.

Note that the result states that the range of $\mathcal{H}$ is the mean-zero even functions as opposed to all even functions. The symmetry properties described in (1.19) imply that $R_{\mu,1}$ is in fact an even mean-zero function. Thus we can write (3.2) as $\xi_1 = \mathcal{H}^{-1}R_{\mu,1}$ and the right hand side of this can be shown to meet the hypotheses of the contraction mapping theorem in the localized spaces. That is, the heavy component of (3.1) poses no problem.

Writing out the second ("light") component of (3.1) gives

$$\underbrace{c^2\mu\xi_2'' + 2(1+2\sigma_c)\xi_2}_{\mathcal{U}_\mu\xi_2} = R_{\mu,2}. \tag{3.4}$$

This operator is a garden-variety second order Schrödinger operator[9] with potential function $2(1+2\sigma_c(x))$. Since we are working with $x \in \mathbf{R}$, standard undergraduate differential equations theory tells us that there are two linearly independent solutions of $\mathcal{U}_\mu\mathfrak{z} = 0$. Call them $\mathfrak{z}_{\mu,0}(x)$ and $\mathfrak{z}_{\mu,1}(x)$. Since $\sigma_c(x)$ is an even function, we can arrange it so that $\mathfrak{z}_{\mu,0}(x)$ is even and $\mathfrak{z}_{\mu,1}(x)$ is odd. And since $\sigma_c(x)$ is positive and converges to zero at infinity we can infer that these functions converge, as $|x| \to \infty$, to solutions of $c^2\mu\mathfrak{z}'' + 2\mathfrak{z} = 0$. Which is to say they are asymptotic as $|x| \to \infty$ to a linear combination of $\sin(\widetilde{\omega}_\mu x)$ and $\cos(\widetilde{\omega}_\mu x)$, with $\widetilde{\omega}_\mu := \sqrt{2/c^2\mu}$. Thus, since μ is small, these have very high frequencies.

The functions $\mathfrak{z}_{\mu,0}$ and $\mathfrak{z}_{\mu,1}$ are the Jost solutions[10] for the operator $\mathcal{U}_\mu$. Note that since these functions do not converge to zero at infinity they are not in the localized spaces H^s_b. And we also know from ODE theory that all solutions of $\mathcal{U}_\mu z = 0$ will be linear combinations of $\mathfrak{z}_{\mu,0}$ and $\mathfrak{z}_{\mu,1}$. All of this implies that the only function $z(x) \in H^s_b$ which solves $\mathcal{U}_\mu z = 0$ is $z(x) \equiv 0$. In this way we can conclude $\mathcal{U}_\mu$, viewed as a map from H^{s+2}_b to H^s_b, is injective, since it has a trivial kernel.

We remark now that an alternate way to prove the injectivity of $\mathcal{U}_\mu$ on the localized spaces is by way of the following coercive estimate: if $f \in H^2_b$ then

$$\|f\|_{0,b} \le C_b\mu^{-1/2}\|\mathcal{U}_\mu f\|_{0,b}. \tag{3.5}$$

We prove a similar estimate for a closely related operator below in Appendix C. The point is that this estimate also implies that the kernel of $\mathcal{U}_\mu$ in H^s_b spaces is trivial and thus $\mathcal{U}_\mu$ is injective.

However $\mathcal{U}_\mu$ is not surjective on those spaces. Now suppose that we have a solution ξ_2 of (3.4) in O^s_b. The embedding (2.11) implies that ξ_2 is in L^1. So if we multiply (3.4) by $\mathfrak{z}_{\mu,1}(x)$ then integrate over $\mathbf{R}$, the integrals will converge. We get

$$\int_{\mathbf{R}} [\mathcal{U}_\mu\xi_2(x)]\mathfrak{z}_{\mu,1}(x)dx = \int_{\mathbf{R}} R_{\mu,2}\mathfrak{z}_{\mu,1}(x)dx.$$

The operator $\mathcal{U}_\mu$ is symmetric with respect to the L^2 inner-product, thanks to integration by parts. Thus, since $\mathcal{U}_\mu\mathfrak{z}_{\mu,1} = 0$, we have

$$\int_{\mathbf{R}} R_{\mu,2}\mathfrak{z}_{\mu,1}(x)dx = 0. \tag{3.6}$$

[9]Note that since $\mu \to 0^+$, we are considering this operator in the "semi-classical limit."

[10]Or, at least, they are linear combinations of them, see [23].

That is to say, we have a third equation[11] we need to solve in addition to (3.2) and (3.4). But we only have two unknowns, ξ_1 and ξ_2.

Remark 3.2. *In* [27]*, the authors study this same problem but with the the restoring force taken to be that of the Toda lattice ($F_s = e^{-r} - 1$) as opposed to the simple force ($F_s = -r - r^2$) we use here. The approach taken there follows the formalism of asymptotics for "fast-slow" dynamical systems. Through the right lens what they find there parallels what we have here. In particular they arrive at an equation equivalent to our* (3.4)*. The Toda lattice is integrable and in that case the function $\sigma_c(x)$ is known explicitly in terms of elementary functions, as is the leading order part of $R_{\mu,2}$. Moreover, they find explicit formulas for their Jost solutions in terms of hypergeometric functions.*

The point is that the (ostensibly) leading order part of their analog of the solvability condition (3.6) *is totally explicit and depends, ultimately, only on μ. It can even be evaluated using residue calculus, though the resulting formula is not so simple to understand. Numerical computation of this formula demonstrates that the leading order part of the solvability condition is met at a countably infinite sequence of values of μ converging to zero.*

Which is to say that perhaps there do exist genuinely localized traveling waves for the problem at (or near) those special values of μ. Our feeling is that if this is true, the method we deploy here is not sufficient to prove it and so we take the cautious route and say only that it is possible.

In [3], Beale found a similar phenomenon in the traveling wave equations for the capillary-gravity problem. In that case, the necessary condition is somewhat simpler and says that (roughly translating his work into the language of our problem) $\int_{\mathbf{R}} R_{\mu,2} \sin(\widetilde{\omega}_\mu x) dx = 0$. That is to say, the Fourier transform of his right hand side has to be equal to zero at a particular frequency. Nevertheless there is enough commonality with his work (and its successors [1] [9]) that we are able to use his method, though there are some substantive complications.

The key idea of his method is to replace $\boldsymbol{\xi}$ with $\boldsymbol{\eta} + a\mathbf{p}_\mu^a$ where $\boldsymbol{\eta}$ is localized and $a\mathbf{p}_\mu^a$ is an exact spatially periodic solution[12] of (1.16) with amplitude a. The frequency of this solution is very close to the asymptotic frequency $\widetilde{\omega}_\mu$ of $\mathfrak{z}_{\mu,1}(x)$. The amplitude a becomes our third variable.

Substituting "Beale's ansatz," $\mathbf{h} = \boldsymbol{\sigma}_c + \boldsymbol{\eta} + a\mathbf{p}_\mu^a$ into (1.16) and (3.6) gets us to

$$\mathcal{H}\eta_1 = \widetilde{R}_{\mu,1}, \quad \mathcal{U}_\mu \eta_2 = \widetilde{R}_{\mu,2} \quad \text{and} \quad \underbrace{\int_{\mathbf{R}} \widetilde{R}_{\mu,2} \mathfrak{z}_{\mu,1}(x) dx}_{\varsigma_\mu(a;\boldsymbol{\eta})} = 0. \tag{3.7}$$

[11]Of course, we could repeat this argument with $\mathfrak{z}_{\mu,1}$ replaced by $\mathfrak{z}_{\mu,0}$. But (1.19) tells us that $R_{\mu,2}$ is an odd function of x and so the resulting integral condition, $\int_{\mathbf{R}} R_{\mu,2}\mathfrak{z}_{\mu,0}(x)dx = 0$, is met automatically since the integrand is odd.

[12]Naturally a big part of our task is to show that such solutions exist; see Section 5.

Here $\widetilde{R}_\mu = (\widetilde{R}_{\mu,1}, \widetilde{R}_{\mu,2})$ is some complicated collection of terms much like those in R_μ from before, but now includes additional terms involving the periodic part $a\mathbf{p}^a_\mu$. The equation $\varsigma_\mu(a;\boldsymbol{\eta}) = 0$ is to be viewed as "the equation for a."

A variation on the Riemann-Lebesgue lemma can be used to show that $\varsigma_\mu(0;0) \le \mathcal{O}(\mu^\infty)$. So long as $\partial_a\varsigma_\mu(0;0) \neq 0$, the inverse function theorem shows that we can solve $\varsigma_\mu(a;\boldsymbol{\eta}) = 0$ for a given $\boldsymbol{\eta}$. Which in turn gives us reason to believe that we can solve $\mathcal{U}_\mu\eta_2 = \widetilde{R}_{\mu,2}$ for η_2. From there we would have everything.

This all worked in [3] [1] and [9].[13] But here something is different: $\partial_a\varsigma_\mu(0;0)$ is identically zero at a countable collection of values of μ converging to zero. Moreover, at values of μ "away" from the zeros[14], we find that $\partial_a\varsigma_\mu(0;0) = \mathcal{O}(\mu^{1/2})$. While this is small, it is still big enough to push through the inverse function theorem argument. But the size of a solution a we get from the inverse function theorem is roughly the same size as $1/\partial_a\varsigma_\mu(0;0) = \mathcal{O}(\mu^{-1/2})$.

Substituting the selected value of a into $R_{\mu,2}$ winds up generating terms which are linear in $\boldsymbol{\eta}$ but come with a prefactor of $\mu^{1/2}$ instead of the $\mathcal{O}(\mu)$ prefactor we had originally in $R_{\mu,2}$. Which is to say the linear parts in the right hand side of the light equation of (3.7) are still small, just not as small as we thought at the outset. Worse, the coercive estimate (3.5) tells us that the size of the inverse of $\mathcal{U}_\mu$ on its range is $\mathcal{O}(\mu^{-1/2})$. Which means that the light equation, after inversion of $\mathcal{U}_\mu$, looks like

$$\eta_2 = \mathcal{U}_\mu^{-1}\widetilde{R}_{\mu,2} = B_\mu\eta_2 + \text{residual and nonlinear terms}$$

where B_μ is an $\mathcal{O}(1)$ linear operator.

Unless we were so lucky as to have the operator norm of B_μ be strictly less than one (and we cannot get that by making μ small), it is not obvious how to solve this equation for η_2. And we are all but certain that B_μ, while $\mathcal{O}(1)$, is rather large in absolute terms. And there are other problematic terms lurking in $\widetilde{R}_\mu$ as well. The general idea is this: *any $\mathcal{O}(\mu)$ term on the right hand side of the second equation in* (3.7) *winds up being an $\mathcal{O}(1)$ term after solving for the amplitude a and inverting $\mathcal{U}_\mu$.*

So we need to do something to get rid of all those deadly $\mathcal{O}(\mu)$ terms. We started the above line of reasoning at (3.1) and in that equation we kept all the $\mathcal{O}(1)$ linear terms on the left (except the singular term) and put all $\mathcal{O}(\mu)$ terms on the right, along with nonlinear terms. Our remedy is a rather brute force one: we will keep all $\mathcal{O}(1)$ and most of the $\mathcal{O}(\mu)$ linear terms on the left and thus the right hand side will no longer have the problematic $\mathcal{O}(\mu)$ linear parts; see Section 6. Once this is done, our general strategy remains as above, but to execute it we must understand the $\mathcal{O}(\mu)$ linear perturbations of $\mathcal{H}$ and $\mathcal{U}_\mu$. The former is

[13] A similar strategy worked in the article [15] to show the existence of heteroclinic traveling waves in monatomic FPUT lattices with non-convex potentials (that is to say, "phase transitions"). Their problem, like ours, exhibits a linear part which is injective but not surjective. In their problem, this is due to the fact that the waves they study are sub-sonic (that is, with speed $|c| < c_0$) whereas here it is tied to the heterogeneity. As in our problem the end result is that the solutions they construct are asymptotic to sinusoids at spatial infinity. One interesting difference between their result and ours is the amplitude of the periodic part at spatial infinity is not small beyond all orders of their small parameter (which measures, roughly speaking, how far their potential is from a piecewise quadratic one analyzed in [24]). This is ultimately due to the fact our problem is singularly perturbed while theirs is not.

[14] This is the origin of the set M_c in Theorem 1.2.

no problem, but the latter is quite complicated since it is where the singular term lives and, moreover, the perturbation is nonlocal; see Appendices A and C. We also need to make a "refined leading order limit" to get rid of $\mathcal{O}(\mu)$ residuals from the right hand side. That is to say, we need to modify $\boldsymbol{\sigma}_c$ slightly to eliminate some problematic terms. This is what we do next.

4. Refined leading order limit.

The estimates (2.14) show that

$$\|\mathcal{G}(\boldsymbol{\sigma}_c,\mu)\|_{H^s_b\times H^s_b} \le C\mu.$$

Recall that the traveling wave equation (1.16) is $\mathcal{G}(\mathbf{h},\mu)=0$. The quantity $\mathcal{G}(\boldsymbol{\sigma}_c,\mu)$ is the residual, or rather, the amount by which the leading order term fails to solve the system at $\mu>0$. We can shrink the size of the residual by slightly modifying $\boldsymbol{\sigma}_c$.

To this end, we let

$$\mathcal{G}_{mod}(\mathbf{h},\mu) := c^2 I_0\mathbf{h}'' + L_\mu\mathbf{h} + L_\mu Q_\mu(\mathbf{h},\mathbf{h}).$$

All we have done here is remove from the singularly perturbed part from $\mathcal{G}(\mathbf{h},\mu)$. We have $\mathcal{G}_{mod}(\boldsymbol{\sigma}_c,0)=\mathcal{G}(\boldsymbol{\sigma}_c,0)=0$. Computing the linearization of $\mathcal{G}_{mod}$ at $(\boldsymbol{\sigma}_c,0)$ we find

$$D_{\mathbf{h}}\mathcal{G}_{mod}(\boldsymbol{\sigma}_c,0)\boldsymbol{\xi} = c^2 I_0\boldsymbol{\xi}'' + L_0\boldsymbol{\xi} + 2L_0Q_0(\boldsymbol{\sigma}_c,\boldsymbol{\xi}) = \begin{bmatrix} \mathcal{H} & 0 \\ 0 & 2+4\sigma_c \end{bmatrix}\boldsymbol{\xi}$$

where $\mathcal{H}$ is taken as above in (3.2).

We saw in Proposition 3.1 that $\mathcal{H}$ was invertible and in Theorem 1.1 that $\sigma_c(x)>0$. Therefore we can conclude $D_{\mathbf{h}}\mathcal{G}_{mod}(\boldsymbol{\sigma}_c,0)$ is an invertible map from $E^{s+2}_{b_c}\times O^s_{b_c}$ to $E^s_{b_c,0}\times O^s_{b_c}$. Though we do not show the details, $G_{mod}(\mathbf{h},\mu)$ is a smooth mapping from $E^{s+2}_b\times O^s_b$ to $E^s_{b,0}\times O^s_b$. And so we can use the implicit function theorem to conclude that there exists a smooth map $\mu\mapsto\boldsymbol{\sigma}_{c,\mu}\in E^{s+2}_{b_c}\times O^s_{b_c}$ for which $\mathcal{G}_{mod}(\boldsymbol{\sigma}_{c,\mu},\mu)=0$ so long as μ is close enough to zero. The implicit function theorem also shows that $\|\boldsymbol{\sigma}_{c,\mu}-\boldsymbol{\sigma}_c\|_{s,b_c}\le\mathcal{O}(\mu)$.

In summary we have

Lemma 4.1. *For all $|c|\in(c_0,c_1]$ there exists $\mu_\xi\in(0,1)$ for which $\mu\in(0,\mu_\xi)$ implies the existence of $\boldsymbol{\xi}_\mu\in\bigcap_{s\ge0}\left(E^s_{b_c}\times O^s_{b_c}\right)$ such that $\boldsymbol{\sigma}_{c,\mu}:=\boldsymbol{\sigma}_c+\mu\boldsymbol{\xi}_\mu$ satisfies*

$$\mathcal{G}_{mod}(\boldsymbol{\sigma}_{c,\mu},\mu)=0$$

and

$$\|\xi_{\mu,1}\|_{s,b}+\|\xi_{\mu,2}\|_{s,b}\le C. \tag{4.1}$$

The important consequence of the above is that

$$\mathcal{G}(\boldsymbol{\sigma}_{c,\mu},\mu)=\mathcal{G}(\boldsymbol{\sigma}_{c,\mu},\mu)-\mathcal{G}_{mod}(\boldsymbol{\sigma}_{c,\mu},\mu)=c^2(I_\mu-I_0)\boldsymbol{\sigma}''_{c,\mu}=c^2\mu^2\xi''_{\mu,2}\mathbf{e}_2. \tag{4.2}$$

That is to say, $\boldsymbol{\sigma}_{c,\mu}$ solves the first component (1.16) *exactly* and it solves it to $\mathcal{O}(\mu^2)$ in the second component; the residual is thus $\mathcal{O}(\mu^2)$ which is good enough for what follows to work. In our nanopteron solutions, $\boldsymbol{\sigma}_{c,\mu}$ will be the main piece of the localized part.

5. Periodic solutions

In this section we state a theorem about the existence of spatially periodic traveling wave solutions for (1.5). After the statement we will provide an overview of its proof. The proof itself, which is not short, is in Appendix B.

Theorem 5.1. *For all $|c| > c_0$ there exists $\mu_{per} > 0$ and $a_{per} > 0$ such that for all $\mu \in (0, \mu_{per})$ there exist $|\upsilon_\mu| \leq 1$ and maps*

$$\begin{aligned} \omega_\mu^a &: [-a_{per}, a_{per}] \longrightarrow \mathbf{R} \\ \psi_{\mu,1}^a &: [-a_{per}, a_{per}] \longrightarrow C^\infty_{per} \cap \{\textit{even functions}\} \\ \psi_{\mu,2}^a &: [-a_{per}, a_{per}] \longrightarrow C^\infty_{per} \cap \{\textit{odd functions}\} \end{aligned} \tag{5.1}$$

with the following properties.

- *Putting*

$$\mathbf{h}(x) = a\mathbf{p}_\mu^a(x) := \begin{pmatrix} \mu\varphi_{\mu,1}^a(x) \\ \varphi_{\mu,2}^a(x) \end{pmatrix} := a \begin{pmatrix} \mu\upsilon_\mu \cos(\omega_\mu^a x) \\ \sin(\omega_\mu^a x) \end{pmatrix} + a \begin{pmatrix} \mu\psi_{\mu,1}^a(\omega_\mu^a x) \\ \psi_{\mu,2}^a(\omega_\mu^a x) \end{pmatrix} \tag{5.2}$$

  solves (1.16) *for all $|a| \leq a_{per}$. That is*

$$\mathcal{G}(a\mathbf{p}_\mu^a, \mu) = 0. \tag{5.3}$$

- *$\omega_\mu^0 = \omega_\mu$ where $\omega_\mu = \mathcal{O}(\mu^{-1/2})$. The mapping $\mu \mapsto \omega_\mu$ is smooth with respect μ.*
- *$\psi_{\mu,1}^0 = \psi_{\mu,2}^0 = 0$.*
- *For all $s \geq 0$, there exists $C > 0$ such that for all $|a|, |\grave{a}| \leq a_{per}$ we have*

$$\left|\omega_\mu^a - \omega_\mu^{\grave{a}}\right| + \left\|\psi_{\mu,1}^a - \psi_{\mu,1}^{\grave{a}}\right\|_{W^{s,\infty}_{per}} + \left\|\psi_{\mu,2}^a - \psi_{\mu,2}^{\grave{a}}\right\|_{W^{s,\infty}_{per}} \leq C|a - \grave{a}|. \tag{5.4}$$

Here is the formal argument for the proof; it will highlight from where we get the "critical frequency" ω_μ and also the leading order part of $\mathbf{p}_\mu^a$. The key is, of course, linear theory. We want to find small amplitude periodic solutions of $\mathcal{G}(\mathbf{h}, \mu)$ for a fixed value of μ. We make the substitution $\mathbf{h}(x) = a\mathbf{p}(x)$ where we are thinking of $a \neq 0$ as being small and $\mathbf{p}$ as a periodic function of an as yet unspecified frequency but (more or less) unit amplitude. If we put this into (1.16) we find that, after canceling one factor of a from all terms:

$$\underbrace{c^2 I_\mu \mathbf{p}'' + L_\mu \mathbf{p}}_{\mathcal{B}_\mu \mathbf{p}} + aL_\mu Q_\mu(\mathbf{p}, \mathbf{p}) = 0.$$

The obvious starting point is to put $a = 0$; naturally this implies that $\mathcal{B}_\mu \mathbf{p} = 0$. So the question becomes: are there periodic solutions of the linearized problem? The answer is yes, but only for certain frequencies. We make the guess that $\mathbf{p}(x) = \mathbf{u}e^{i\omega x} + \text{c.c.}$ for an undetermined vector $\mathbf{u} \in \mathbf{C}^2$ and frequency ω.

We know that $\partial_x e^{i\omega x} = i\omega e^{i\omega x}$ and $S^d e^{i\omega x} = e^{i\omega d} e^{i\omega x}$. Thus we discover that

$$\mathcal{B}_\mu(\mathbf{u}e^{i\omega x}) = (\widetilde{\mathcal{B}}_\mu(\omega)\mathbf{u})e^{i\omega x}$$

where

$$\widetilde{\mathcal{B}}_\mu(\omega) := -c^2\omega^2 I_\mu + \widetilde{L}_\mu(\omega)$$

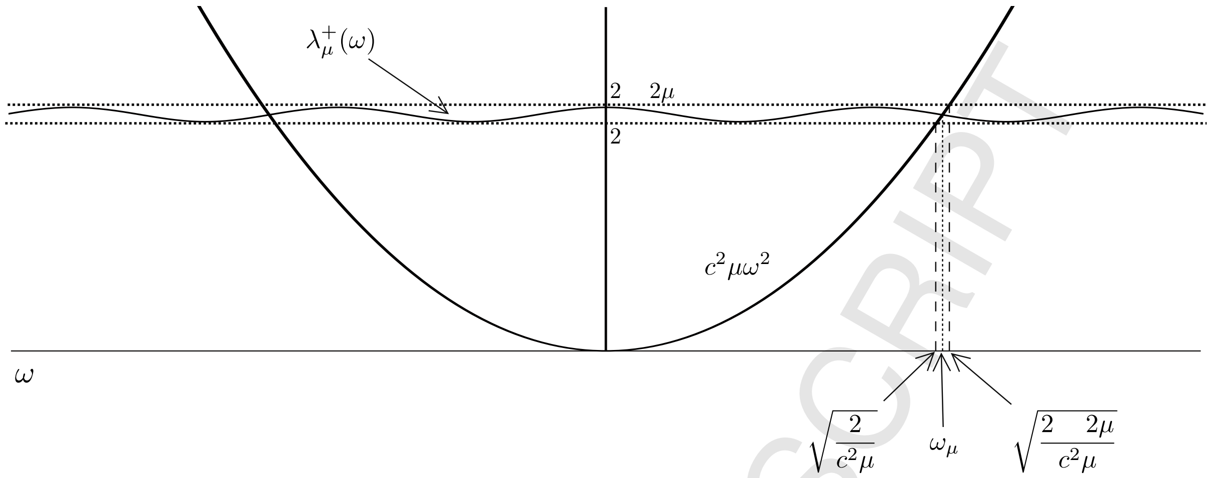


FIGURE 4. *Sketch of the graphs of $\lambda_\mu^+(\omega)$ and $c^2\mu\omega^2$ vs ω; their intersection determines the critical frequency ω_μ.*

and

$$\widetilde{L}_\mu(\omega) := \begin{bmatrix} 2\sin^2(\omega)(1-\mu\cos^2(\omega)) & -\mu i\sin(2\omega)(1-2\cos^2(\omega)-(\mu/4)\sin^2(2\omega)) \\ i\mu\sin(2\omega) & 2\left(1+\mu\cos^2(\omega)+(\mu^2/4)\sin^2(2\omega)\right) \end{bmatrix}. \tag{5.5}$$

Finding nontrivial periodic solutions of the linear part is equivalent to finding ω so that $\widetilde{\mathcal{B}}_\mu(\omega)$ has a nontrivial kernel. Which is to say when

$$\det\left(-c^2\omega^2 I_\mu + \widetilde{L}_\mu(\omega)\right) = 0. \tag{5.6}$$

Some factoring and quite a few trigonometric identities reveal that if

$$c^2\mu\omega^2 = \underbrace{1+\mu+\sqrt{(1+\mu)^2-4\mu\sin^2(\omega)}}_{\lambda_\mu^+(\omega)} \tag{5.7}$$

then we have (5.6).

In Figure 4 we sketch $\lambda_\mu^+(\omega)$ and $c^2\mu\omega^2$ vs ω. This sketch is representative of the general situation when $c > c_0$ and μ sufficiently close to zero. Specifically, we have

Lemma 5.2. *For all $|c| > c_0$ there exists $\mu_\omega \in (0,1)$ such that for all $\mu \in (0,\mu_\omega)$ there is a unique nonnegative number ω_μ for which putting $\omega = \pm\omega_\mu$ solves* (5.7). *Moreover we have*

$$\sqrt{\frac{2}{c^2\mu}} \le \omega_\mu \le \sqrt{\frac{2+2\mu}{c^2\mu}}. \tag{5.8}$$

That is to say, $\omega_\mu = \mathcal{O}(\mu^{-1/2})$. Lastly, the map $\mu \mapsto \omega_\mu$ is C^∞.

One establishes this rigorously using differential calculus, but the picture tells the main story so we omit the proof. It is not so different than the proof of Lemma 2.1 *(vi)* in [9]. It is worth pointing out the the asymptotic frequency, $\widetilde{\omega}_\mu$, of the Jost solutions described in Section 3 is the left hand endpoint of (5.8).

When $\omega = \omega_\mu$, we can find two linearly independent solutions of $\mathcal{B}_\mu(\omega_\mu)\mathbf{p} = 0$ and, more importantly, we can make a linear combination of these that has the even $\times$ odd symmetry we require. It is

$$\mathbf{p}^0_\mu(x) := \boldsymbol{\nu}_\mu(\omega_\mu x) := \begin{pmatrix} \mu\upsilon_\mu \cos(\omega_\mu x) \\ \sin(\omega_\mu x) \end{pmatrix}.$$

Here $\upsilon_\mu \leq \mathcal{O}(1)$ is given by the rather dreadful formula (B.8) below.

In this way we see that $a\mathbf{p}^0_\mu(x)$ is a good first guess for a small amplitude spatially periodic solution of (1.16). To get a rigorous result we employ a Liapunov-Schmidt decomposition which is more or less just a quantitative version of the method of "bifurcation from a simple eigenvalue" developed in [7] and [28]. The main difficulty is not in getting existence of a solution, but rather getting the μ-uniform estimates in Theorem 5. The interested reader can at this point jump to Appendix B to see the gory bits, but for now we return to the search for nanopterons.

6. Nanopteron ansatz and governing equations

We briefly recapitulate where are in our search for traveling wave solutions of (1.5). We have a refined leading order localized solution $\boldsymbol{\sigma}_{c,\mu}$ for which we know that $\|\mathcal{G}(\boldsymbol{\sigma}_{c,\mu}, \mu)\|_{E^s_b \times O^s_b}$ is $\mathcal{O}(\mu^2)$. It is the sum of the $\mathcal{O}(1)$ monatomic solitary wave $\boldsymbol{\sigma}_c$ and an $\mathcal{O}(\mu)$ correction. And we have, for each $\mu > 0$, a family of high frequency periodic solutions parameterized by their amplitude a which we denote by $a\mathbf{p}^a_\mu(x)$. Specifically, $\mathcal{G}(a\mathbf{p}^a_\mu, \mu) = 0$.

The solutions we seek are the sum of $\boldsymbol{\sigma}_{c,\mu}$, $a\mathbf{p}^a_\mu$ and a small, localized remainder. Specifically we make the ansatz

$$\mathbf{g} = \boldsymbol{\sigma}_{c,\mu} + a\mathbf{p}^a_\mu + \boldsymbol{\eta} \tag{6.1}$$

and substitute this into (1.16). To be clear, our unknowns are $\boldsymbol{\eta} \in E^s_b \times O^s_b$ and the amplitude a.

Some algebra leads us from (1.16) to the following system

$$c^2 I_\mu \boldsymbol{\eta}'' + L_\mu \boldsymbol{\eta} + 2L_\mu Q_\mu(\boldsymbol{\sigma}_{c,\mu}, \boldsymbol{\eta}) = J_0 + J_3 + J_4 + J_5$$

with

$$\begin{aligned} J_0 &:= -\mathcal{G}(\boldsymbol{\sigma}_{c,\mu}, \mu) \\ J_3 &:= -2aL_\mu Q_\mu(\boldsymbol{\sigma}_{c,\mu}, \mathbf{p}^a_\mu) \\ J_4 &:= -2aL_\mu Q_\mu(\mathbf{p}^a_\mu, \boldsymbol{\eta}) \\ \text{and} \quad J_5 &:= -L_\mu Q_\mu(\boldsymbol{\eta}, \boldsymbol{\eta}). \end{aligned} \tag{6.2}$$

The defintions for I_μ, L_μ, $L_\mu Q_\mu$ and $\mathcal{G}$ are found in Section 1.

Note that

- J_0 is the residual from the refined leading order limit (see (4.2)) and thus is $\mathcal{O}(\mu^2)$.
- The term J_4 is nonlinear because it consists of terms that look like "$a \cdot \boldsymbol{\eta}$."
- J_5 is quadratic in $\boldsymbol{\eta}$.

Thus J_0, J_4 and J_5 can viewed as being small. The term J_3 encapsulates the interaction between the periodic part and the leading order localized part: it will be at the center of much of our analysis.

It will be advantageous to move the "off-diagonal" parts of the left hand side over to the right.[15] So we denote the off-diagonal part of L_μ as:

$$\mu\Theta_\mu := \begin{bmatrix} 0 & \mu\Theta_{\mu,1} \\ \mu\Theta_{\mu,2} & 0 \end{bmatrix} := \begin{bmatrix} 0 & -2\mu A\delta(1-2A^2+\mu A^2) \\ 2\mu A\delta & 0 \end{bmatrix}.$$

We know from (2.13) that A and δ are $\mathcal{O}(1)$ operators on $W_b^{s,p}$ spaces and thus we have

$$\|\Theta_{\mu,1}\|_{W_b^{s,p}\to W_b^{s,p}} \leq C \quad \text{and} \quad \|\Theta_{\mu,2}\|_{W_b^{s,p}\to W_b^{s,p}} \leq C. \tag{6.3}$$

Which is to say that $\mu\Theta_\mu$ is in fact $\mathcal{O}(\mu)$.

The diagonal part of $2L_\mu Q_\mu(\boldsymbol{\sigma}_{c,\mu},\mathbf{f})$ is

$$\Sigma_\mu \mathbf{f} := \underbrace{[2L_\mu Q_\mu(\boldsymbol{\sigma}_{c,\mu}, f_1\mathbf{e}_1)\cdot\mathbf{e}_1]}_{\Sigma_{\mu,1}f_1}\mathbf{e}_1 + \underbrace{[2L_\mu Q_\mu(\boldsymbol{\sigma}_{c,\mu}, f_2\mathbf{e}_2)\cdot\mathbf{e}_2]}_{\Sigma_{\mu,2}f_2}\mathbf{e}_2. \tag{6.4}$$

The corresponding off-diagonal part is $\mu\Omega_\mu\mathbf{f} := 2L_\mu Q_\mu(\boldsymbol{\sigma}_{c,\mu},\mathbf{f}) - \Sigma_\mu\mathbf{f}$. More explicitly, this is

$$\mu\Omega_\mu \mathbf{f} = \underbrace{[2L_\mu Q_\mu(\boldsymbol{\sigma}_{c,\mu}, f_2\mathbf{e}_2)\cdot\mathbf{e}_1]}_{\mu\Omega_{\mu,1}f_2}\mathbf{e}_1 + \underbrace{[2L_\mu Q_\mu(\boldsymbol{\sigma}_{c,\mu}, f_1\mathbf{e}_1)\cdot\mathbf{e}_2]}_{\mu\Omega_{\mu,2}f_1}\mathbf{e}_2. \tag{6.5}$$

In Lemma 4 we saw that $\boldsymbol{\sigma}_{c,\mu} = \sigma_c\mathbf{e}_1 + \mu\boldsymbol{\xi}_\mu \in E^s_{b_c}\times O^s_{b_c}$. Thus if we use the bilinear estimate (2.20) on $\Sigma_{\mu,1}$ we get

$$\|\Sigma_{\mu,1}f\|_{s,b} \leq C\|\sigma_c + \mu\xi_{\mu,1}\|_{s,b_c}\|f\|_{W^{s,\infty}_{b-b_c}} + C\mu^2\|\xi_{\mu,2}\|_{s,b_c}\|f\|_{W^{s,\infty}_{b-b_c}}.$$

Using the estimates for σ_c in (1.9) and those for $\xi_{\mu,1}$ and $\xi_{\mu,2}$ in (4.1) then gives $\|\Sigma_{\mu,1}f\|_{s,b} \leq C\|f\|_{W^{s,\infty}_{b-b_c}}$. The same sort of reasoning using (2.20)-(2.23) leads to

$$\begin{aligned} &\|\Sigma_{\mu,1}f\|_{s,b} + \|\Sigma_{\mu,2}f\|_{s,b} + \|\Omega_{\mu,1}f\|_{s,b} + \|\Omega_{\mu,2}f\|_{s,b} \leq C\|f\|_{s,b-b_c} \\ \text{and}\quad &\|\Sigma_{\mu,1}f\|_{s,b} + \|\Sigma_{\mu,2}f\|_{s,b} + \|\Omega_{\mu,1}f\|_{s,b} + \|\Omega_{\mu,2}f\|_{s,b} \leq C\|f\|_{W^{s,\infty}_{b-b_c}}. \end{aligned} \tag{6.6}$$

Note that these estimates impute to $\Sigma_{\mu,1}$, $\Sigma_{\mu,2}$, $\Omega_{\mu,1}$ and $\Omega_{\mu,2}$ a "localizing" property: $\Sigma_{\mu,1}f$, $\Sigma_{\mu,2}f$, $\Omega_{\mu,1}f$ and $\Omega_{\mu,2}f$ decay more rapidly at infinity than f does. This is important!

With these definitions we have

$$\underbrace{c^2 I_\mu\boldsymbol{\eta}'' + (L_\mu - \mu\Theta_\mu)\boldsymbol{\eta} + \Sigma_\mu\boldsymbol{\eta}}_{\text{these terms are diagonal}} = J_0 + J_2 + J_3 + J_4 + J_5 \tag{6.7}$$

with

$$J_2 := -\mu\Theta_\mu\boldsymbol{\eta} - \mu\Omega_\mu\boldsymbol{\eta}. \tag{6.8}$$

Written out, the first component of (6.7) reads

$$\underbrace{c^2\eta_1'' - 2\delta^2(1-\mu A^2)\eta_1 + \Sigma_{\mu,1}\eta_1}_{\mathcal{H}_\mu\eta_1} = j_2 + j_3 + j_4 + j_5. \tag{6.9}$$

[15] We did discuss in Section 3 how we wanted keep the $\mathcal{O}(\mu)$ linear parts on the left; for technical reasons these $\mathcal{O}(\mu)$ off-diagonal parts do not create the problems we described there.

Here, we use the notation

$$j_k := J_k \cdot \mathbf{e}_1 \tag{6.10}$$

to denote the first components of the J_k maps. Note that the first component of J_0 is identically zero as can be seen in (4.2). This equation is our "higher order" version of the heavy equation (3.2).

A direct computation shows that $\mathcal{H}_0 = \mathcal{H}$ from (3.2) and the estimates (2.13) and (2.14) give us

$$\|\mathcal{H}_\mu - \mathcal{H}_0\|_{H^s_b \to H^s_b} \leq C\mu.$$

A Neumann series argument allows us to conclude that the results of Proposition 3.1 can be extended to $\mathcal{H}_\mu$. That is to say we have

Lemma 6.1. *For all $|c| \in (c_0, c_1]$ and $b \in (0, b_c]$ there exists $\mu_\mathcal{H}(b) \in (0,1)$ for which $\mu \in (0, \mu_\mathcal{H}(b))$ implies that the map $\mathcal{H}_\mu$ is a homeomorphism of E^{s+2}_b and $E^s_{b,0}$. Moreover we have*

$$\|\mathcal{H}_\mu^{-1}\|_{E^s_{b,0} \to E^{s+2}_b} \leq C_b. \tag{6.11}$$

In addition, $\mu_\mathcal{H}(b)$ is nondecreasing as a function of b.

Noting that j_2, j_3, j_4 and j_5 are even and mean-zero by virtue of the symmetries described at (1.19), we are free to use Lemma 6.1 to rewrite (6.9) as

$$\eta_1 = \underbrace{\mathcal{H}_\mu^{-1}\left(j_2 + j_3 + j_4 + j_5\right)}_{N_1^\mu(\eta_1, \eta_2, a)}. \tag{6.12}$$

The second component of (6.7) reads

$$c^2\mu\eta_2'' + 2(1 + \mu A^2 - \mu^2 A^2 \delta^2)\eta_2 + \Sigma_{\mu,2}\eta_2 = l_0 + l_2 + l_3 + l_4 + l_5. \tag{6.13}$$

Here,

$$l_k := J_k \cdot \mathbf{e}_2 \tag{6.14}$$

are the second components of the J_k maps.

We need to make one more change to the left hand side of this equation. This change is made for a rather subtle reason that will not pay off for quite some time. Let

$$\tau_\mu := \frac{1}{2\mu^2}\left(c^2\mu\omega_\mu^2 - 2 - 2\mu\cos^2(\omega_\mu)\right) \tag{6.15}$$

where ω_μ is the periodic solutions' critical frequency. Remembering that ω_μ is the solution of (5.7), Taylor's theorem shows that

$$|\tau_\mu| \leq C. \tag{6.16}$$

Then we put

$$l_1 := \mu^2(2\tau_\mu + 2A^2\delta^2)\eta_2 \tag{6.17}$$

and see that (6.13) is equivalent to

$$\underbrace{c^2\mu\eta_2'' + 2(1 + \mu A^2 + \mu^2\tau_\mu)\eta_2 + \Sigma_{\mu,2}\eta_2}_{\mathcal{L}_\mu \eta_2} = l_0 + l_1 + l_2 + l_3 + l_4 + l_5. \tag{6.18}$$

This equation is our "higher order" analog of the light equation (3.4).

To summarize, our goal is now to solve

$$\eta_1 = N_1^\mu(\eta_1, \eta_2, a) \quad \text{and} \quad \mathcal{L}_\mu \eta_2 = l_0 + l_1 + l_2 + l_3 + l_4 + l_5. \tag{6.19}$$

This seems like two equations with two unknowns, but recall that buried in the j's and l's are functions which depend on the third variable, the periodic amplitude a. We discuss how to get a third equation in the next section, which is focused on the properties of the light operator $\mathcal{L}_\mu$.

7. The light operator $\mathcal{L}_\mu$ and the final system.

7.1. **Properties of $\mathcal{L}_\mu$.** The first thing to observe is that $\mathcal{L}_\mu$ is a nonlocal perturbation of the Schrödinger map $\mathcal{U}_\mu$ from Section 3. As with its forebear, $\mathcal{L}_\mu$ is injective but not surjective. We describe now the key properties of $\mathcal{L}_\mu$ in a series of lemmas. The proofs of these lemmas are carried out in Appendix C.

The first result we have for $\mathcal{L}_\mu$ is a collection of coercive estimates.

Lemma 7.1. *For all $|c| \in (c_0, c_1]$ and $b \in (0, b_c]$ there exists $\mu_{\mathcal{L}}(b) \in (0,1)$ for which $\mu \in (0, \mu_{\mathcal{L}}(b))$ and $f \in O_b^{s+2}$ imply*

$$\|f\|_{s+k,b} \le C_b \mu^{-(k+1)/2} \|\mathcal{L}_\mu f\|_{s,b}. \tag{7.1}$$

In the above we require $-1 \le k \le 2$ and $s + k \ge 0$. In addition, $\mu_{\mathcal{L}}(b)$ is nondecreasing as a function of b.

This lemma implies that, when viewed as an operator from O_b^{s+2} to O_b^s, $\mathcal{L}_\mu$ is injective, since it tells us that if $\mathcal{L}_\mu g = 0$ and $g \in O_b^{s+2}$, then $g(x) \equiv 0$. The next result tells us that $\mathcal{L}_\mu$ is not surjective.

Lemma 7.2. *For all $|c| \in (c_0, c_1]$ there exists $\mu_\gamma \in (0,1)$ such that $\mu \in (0, \mu_\gamma)$ implies the existence of a nonzero, smooth, bounded, odd function $\gamma_\mu(x)$ for which*

$$f \in O_b^{s+2}, \quad g \in O_b^s \quad \text{and} \quad \mathcal{L}_\mu f = g \implies \underbrace{\int_{\mathbf{R}} g(x)\gamma_\mu(x)dx}_{\iota_\mu[g]} = 0. \tag{7.2}$$

In the above we require $b \in (0, b_c]$. Moreover, for $b \in (0, b_c]$, if $\mu \in (0, \min\{\mu_\gamma, \mu_{\mathcal{L}}(b)\})$ we have the reverse implication:

$$g \in O_b^s \quad \text{and} \quad \iota_\mu[g] = 0 \implies \text{there exists unique } f \in O_b^{s+2} \text{ such that } \mathcal{L}_\mu f = g. \tag{7.3}$$

In this case we write "$f = \mathcal{L}_\mu^{-1} g$."

The coercive estimate (7.1) implies that, when defined, the size of $\mathcal{L}_\mu^{-1}$ (viewed as a map from O_b^s to itself) is $\mathcal{O}_b(\mu^{-1/2})$.

The function $\gamma_\mu(x)$, which defines the range of $\mathcal{L}_\mu$, is utterly central to our analysis. It is very much like the Jost solution $\mathfrak{z}_{\mu,1}(x)$ described in Section 3. For instance, it is asymptotic at infinity to a sinusoid. Importantly, we have quantitative estimates on the amplitude, asymptotic frequency and phase shift of this function.

Lemma 7.3. *For all $|c| \in (c_0, c_1]$ the function $\gamma_\mu(x)$ defined in Lemma 7.2 has the following features.*

- *There are constants $0 < C_1 < C_2$ and a map*

$$\begin{aligned}(0, \mu_\gamma) &\longrightarrow [C_1, C_2] \\ \mu &\longmapsto \vartheta_\mu^\infty\end{aligned} \tag{7.4}$$

  such that

$$\begin{aligned}&\lim_{x\to\infty} |\gamma_\mu(x) - \sin(\omega_\mu(x + \vartheta_\mu^\infty))| \\ =&\lim_{x\to\infty} |\gamma_\mu'(x) - \omega_\mu \cos(\omega_\mu(x + \vartheta_\mu^\infty))| \\ =&0\end{aligned} \tag{7.5}$$

  when $\mu \in (0, \mu_\gamma)$.
- *There exists an open set $M_c \subset \mathbf{R}_+$ for which $0 \in \overline{M}_c$ and*

$$\mu \in M_c \implies |\sin(\omega_\mu \vartheta_\mu^\infty)| > 1/2. \tag{7.6}$$

- *For $s = 0$ or $s = 1$ we have*

$$\|\gamma_\mu^{(s)}\|_{L^\infty} \leq C\mu^{-s/2}. \tag{7.7}$$

- *For all $s \geq 0$ and $b \in (0, b_c]$ we have*

$$|\iota_\mu[g]| \leq C_b \mu^{s/2} \|g\|_{s,b} \tag{7.8}$$

  for all $\mu \in (0, \mu_\gamma)$.

Observe that (7.5) states that $\gamma_\mu(x)$ is asymptotic as $x \to \infty$ to a sinusoid which has frequency ω_μ, the critical frequency for the periodic solutions. It turns out that this is quite an important feature of $\gamma_\mu(x)$ and is why we had to make the "τ_μ" adjustment to the left hand side of the light equation back at (6.15).

7.2. **Amplitude selection and the final system.** Now that we have a better understanding of $\mathcal{L}_\mu$, we can finally close our underdetermined system (6.19). The solvability condition (7.2) in Lemma 7.2 tells us that solving the second equation in (6.19) requires that we have

$$\iota_\mu[l_0 + l_1 + l_2 + l_3 + l_4 + l_5] = 0. \tag{7.9}$$

This is our third equation and in particular we can view it as an equation for the amplitude a.

We could, following the example set in [3] or [1] and desribed in Section 3, use the inverse function theorem to solve (7.9) for a given $\boldsymbol{\eta}$. Success in that venture requires

$$\underbrace{-\partial_a\big|_{\boldsymbol{\eta}=0,a=0} \left(\iota_\mu[l_0 + l_1 + l_2 + l_3 + l_4 + l_5]\right)}_{\kappa_\mu} \neq 0.$$

The negative sign is just there for convenience later. An examination of l_0, l_1, l_2 and l_5 (defined in Section 6) shows that these do not depend on a and thus do not contribute to κ_μ. Likewise, l_4 is, roughly speaking, "$a \cdot \boldsymbol{\eta}$" and so it also does not contribute. The only piece that does is l_3. Carrying out the differentiation shows

$$\kappa_\mu = -\partial_a\big|_{\boldsymbol{\eta}=0,a=0}(\iota_\mu[l_3]) = \iota_\mu[\chi_\mu] \tag{7.10}$$

where

$$\chi_\mu(x) := -\partial_a\big|_{\boldsymbol{\eta}=0,a=0}(l_3) = \Sigma_{\mu,2}\varphi^0_{\mu,2} + \mu^2\Omega_{\mu,2}\varphi^0_{\mu,1}. \tag{7.11}$$

Remarkably, we can find an exact formula for the leading order term of κ_μ. Here is the result, which we verify in Appendix D.

Lemma 7.4. *For all* $|c| \in (c_0, c_1]$ *there exist* $\mu_\kappa \in (0,1)$ *for which*

$$|\kappa_\mu - 2c^2\mu\omega_\mu\sin(\omega_\mu\vartheta^\infty_\mu)| \le C\mu \tag{7.12}$$

holds when $\mu \in (0, \mu_\kappa)$.

This, in combination with (7.6), implies

$$\mu \in M_c \cap (0,\mu_\kappa) \implies C^{-1}\mu^{1/2} \le |\kappa_\mu| \le C\mu^{1/2} \tag{7.13}$$

for some constant $C > 1$ which does not depend on μ. Therefore, when $\mu \in M_c \cap (0, \mu_\kappa)$ we can invoke the inverse function theorem to solve (7.9) for a given $\boldsymbol{\eta}$. Nonetheless we do not directly employ the inverse function theorem here. We require some uniformity with respect to μ in the resulting solution map that we would not find in an "off the shelf" version of that theorem. Instead we proceed along the lines of the proof the inverse function theorem.

Recalling how we defined χ_μ in (7.11), we write l_3 as the part which is linear in a plus a remainder:

$$l_3 = -a\chi_\mu + l_{31}. \tag{7.14}$$

We leave l_{31} implicitly defined. It is small in the sense that it is more or less quadratic in a. Since we know now that κ_μ is non-zero for $\mu \in M_c$ we see that (7.9) can be rewritten as

$$a = \underbrace{\frac{1}{\kappa_\mu}\iota_\mu\left[l_0 + l_1 + l_2 + l_{31} + l_4 + l_5\right]}_{N_3^\mu(\eta_1,\eta_2,a)}. \tag{7.15}$$

This is the "equation for a."

Next we define, for $f \in O^s_b$:

$$\mathcal{P}_\mu f := f - \frac{1}{\kappa_\mu}\iota_\mu[f]\chi_\mu. \tag{7.16}$$

This is a projection onto the range of $\mathcal{L}_\mu$ since, by construction, we have

$$\iota_\mu[\mathcal{P}_\mu f] = 0 \quad \text{and} \quad \mathcal{P}_\mu\chi_\mu = 0. \tag{7.17}$$

Which is to say that an invocation of Lemma 7.2 allows us to solve $\mathcal{L}_\mu f = \mathcal{P}_\mu g$ for f given $g \in O^s_b$. We denote the solution by

$$f = \mathcal{L}_\mu^{-1}\mathcal{P}_\mu g.$$

The following lemma, proved in Appendix D, gives us estimates on the size $\mathcal{L}_\mu^{-1}\mathcal{P}_\mu$.

Lemma 7.5. *For all* $|c| \in (c_0, c_1]$ *and* $b \in (0, b_c]$, *if* $\mu \in (0, \min\{\mu_\kappa, \mu_{\mathcal{L}}(b)\}) \cap M_c$ *we have*

$$\|\mathcal{L}_\mu^{-1}\mathcal{P}_\mu\|_{O^s_b \to O^{s+k}_b} \le C_b\mu^{-(2+k)/2}. \tag{7.18}$$

In the above we require $|k| \le 2$ *and* $s + k \ge 0$.

Putting $k = 0$ in the above gives the norm of $\mathcal{L}_\mu^{-1}\mathcal{P}_\mu$ (viewed as a map from O_b^s to itself) as being $\mathcal{O}_b(\mu^{-1})$. This is a factor of $\mu^{-1/2}$ larger than the size of $\mathcal{L}_\mu^{-1}$ implied by the coercive estimate (7.1). That is to say, the projection $\mathcal{P}_\mu$ is a large operator; this is the same phenomenon descibed in Section 3 wherein "solving for a" renders certain terms much larger than they at first appeared.

Given (7.14) and (7.17), we see that

$$\mathcal{P}_\mu(l_0 + l_1 + l_2 + l_3 + l_4 + l_5) = \mathcal{P}_\mu(l_0 + l_1 + l_2 + l_{31} + l_4 + l_5).$$

Also note that (7.15), since it is derived from (7.9), is equivalent to

$$\mathcal{P}_\mu(l_0 + l_1 + l_2 + l_3 + l_4 + l_5) = l_0 + l_1 + l_2 + l_3 + l_4 + l_5.$$

And so if we revisit the second equation in our system (6.19) we see that

$$\mathcal{L}_\mu \eta_2 = l_0 + l_1 + l_2 + l_3 + l_4 + l_5 = \mathcal{P}_\mu(l_0 + l_1 + l_2 + l_{31} + l_4 + l_5)$$

or rather

$$\eta_2 = \underbrace{\mathcal{L}_\mu^{-1}\mathcal{P}_\mu(l_0 + l_1 + l_2 + l_{31} + l_4 + l_5)}_{N_2^\mu(\eta_1,\eta_2,a)}. \tag{7.19}$$

Summing up: we will have found a solution of (1.16) of the form (6.1), and thus a nanopteron solution of (1.5), if we can find a solution of

$$\eta_1 = N_1^\mu(\eta_1,\eta_2,a), \quad \eta_2 = N_2^\mu(\eta_1,\eta_2,a) \quad \text{and} \quad a = N_3^\mu(\eta_1,\eta_2,a) \tag{7.20}$$

where N_1^μ is given in (6.12), N_2^μ in (7.19) and N_3^μ in (7.15). When convenient we compress the above as

$$(\boldsymbol{\eta}, a) = N^\mu(\boldsymbol{\eta}, a)$$

where $N^\mu := (N_1^\mu, N_2^\mu, N_3^\mu)$.

8. Existence of solutions and properties thereof.

Note that (7.20) has the form of a fixed point equation, and we will look for the solution $(\boldsymbol{\eta}_\mu, a_\mu)$ in

$$X_1 := E_{b_*}^2 \times O_{b_*}^1 \times \mathbf{R}. \tag{8.1}$$

Here we have taken

$$b_* := b_c/2. \tag{8.2}$$

The mismatch in regularity in the defintion of X_1 is done for technical reasons. Ultimately our solutions will be smooth by virtue of a bootstrapping argument. Note that X_1 is a Hilbert space.

We also define

$$X_0 := E_{b_*/2}^2 \times O_{b_*/2}^0 \times \mathbf{R}$$

and, for $s \geq 1$:

$$X_s := E_{b_*}^{s+1} \times O_{b_*}^s \times \mathbf{R}.$$

These are also Hilbert spaces. For ordered triples $\mathfrak{r} = (\mathfrak{r}_1, \mathfrak{r}_2, \mathfrak{r}_3) \in \mathbf{R}^3_+$ we put

$$U^s_{\mu,\mathfrak{r}} := \left\{ (\boldsymbol{\eta}, a) \in X_s : \|\eta_1\|_{s+1,b_*} \leq \mathfrak{r}_1 \mu^3,\ \|\eta_2\|_{s,b_*} \leq \mathfrak{r}_2 \mu^2 \text{ and } |a| \leq \mathfrak{r}_3 \mu^{(6+s)/2} \right\}.$$

These are "balls" in X_s with μ-dependendent radii; note that the larger s is, the smaller is the a-part.

The following lemma contains all the information we need to prove our main result.

Lemma 8.1. *For all $|c| \in (c_0, c_1]$ there exists $\mu_* \in (0,1)$ for which the following hold.*

- *For all $s \geq 1$ and $\mu \in (0, \mu_*) \cap M_c$ we have*

$$(\boldsymbol{\eta}, a) \in E^{s+1}_{b_*} \times O^s_{b_*} \times [-a_{per}, a_{per}] \implies N^\mu(\boldsymbol{\eta}, a) \in E^{s+3}_{b_*} \times O^{s+2}_{b_*} \times \mathbf{R}. \tag{8.3}$$

- *For all $s \geq 1$ and $\mathfrak{r} \in \mathbf{R}^3_+$ (with $\mathfrak{r}_3 < a_{per}$) there exists $\widetilde{\mathfrak{r}} \in \mathbf{R}^3_+$ such that for all $\mu \in (0, \mu_*) \cap M_c$ we have*

$$(\boldsymbol{\eta}, a) \in U^s_{\mu,\mathfrak{r}} \implies N^\mu(\boldsymbol{\eta}, a) \in U^{s+1}_{\mu,\widetilde{\mathfrak{r}}}. \tag{8.4}$$

- *There exists $\mathfrak{r}_* \in \mathbf{R}^3_+$ such that such that for all $\mu \in (0, \mu_*) \cap M_c$ we have*

$$(\boldsymbol{\eta}, a) \in U^1_{\mu,\mathfrak{r}_*} \implies N^\mu(\boldsymbol{\eta}, a) \in U^1_{\mu,\mathfrak{r}_*} \tag{8.5}$$

and

$$(\boldsymbol{\eta}, a), (\grave{\boldsymbol{\eta}}, \grave{a}) \in U^1_{\mu,\mathfrak{r}_*} \implies \|N^\mu(\boldsymbol{\eta}, a) - N^\mu(\grave{\boldsymbol{\eta}}, \grave{a})\|_{X_0} \leq \frac{1}{4} \|(\boldsymbol{\eta}, a) - (\grave{\boldsymbol{\eta}}, \grave{a})\|_{X_0}. \tag{8.6}$$

The proof of this is quite involved and is found in Appendix E. What this lemma is saying is that N^μ is something akin to a contraction on X_1. It proves

Theorem 8.2. *For $c \in (c_0, c_1]$ and $\mu \in (0, \mu_*) \cap M_c$ there exists unique $(\boldsymbol{\eta}_\mu, a_\mu) \in U^1_{\mu,\mathfrak{r}_*} \cap (C^\infty(\mathbf{R}) \times C^\infty(\mathbf{R}) \times \mathbf{R})$ for which $(\boldsymbol{\eta}_\mu, a_\mu) = N^\mu(\boldsymbol{\eta}_\mu, a_\mu)$. Moreover, for all $s \geq 0$ there exists $C_s > 0$ such that*

$$\|\eta_{\mu,1}\|_{s,b_*} \leq C_s \mu^3, \quad \|\eta_{\mu,2}\|_{s,b_*} \leq C_s \mu^2 \quad \text{and} \quad |a_\mu| \leq C_s \mu^s. \tag{8.7}$$

The estimate for $|a_\mu|$ implies that it is small beyond all order of μ.

Given this, we get our main result, Theorem 1.2, by putting

$$\begin{pmatrix} \Upsilon_{c,\mu,1} \\ \Upsilon_{c,\mu,2} \end{pmatrix} := T_\mu \left(\mu \boldsymbol{\xi}_\mu + \boldsymbol{\eta}_\mu \right) \quad \text{and} \quad \begin{pmatrix} \Phi_{c,\mu,1} \\ \Phi_{c,\mu,2} \end{pmatrix} := T_\mu \left(a_\mu \mathbf{p}^{a_\mu}_\mu \right).$$

Since T_μ (defined in (1.13)) is nearly the identity (see (2.14)), the estimates in Theorem 1.2 follow immediately. So let us proceed.

8.1. **The proof of Theorem 8.2.** Fix $\mu \in (0, \mu_*) \cap M_c$. Let $\boldsymbol{\eta}_0 = 0$ and $a_0 = 0$. For $n \geq 0$ put

$$(\boldsymbol{\eta}_{n+1}, a_{n+1}) = N^\mu(\boldsymbol{\eta}_n, a_n). \tag{8.8}$$

Since $(0,0) \in U^1_{\mu,\mathfrak{r}_*}$, (8.5) implies that

$$\{(\boldsymbol{\eta}_n, a_n)\}_{n=0}^\infty \subset U^1_{\mu,\mathfrak{r}_*}.$$

$U^1_{\mu,\mathfrak{r}_*}$ is bounded in X_1, which is a Hilbert space. Therefore there exists a weakly convergent subsequence of $\{(\boldsymbol{\eta}_n, a_n)\}_{n=0}^\infty$. Call the limit $(\boldsymbol{\eta}_\mu, a_\mu) \in X_1$. Since the norm is lower-semicontinuous for weak limits, we have $(\boldsymbol{\eta}_\mu, a_\mu) \in U^1_{\mu,\mathfrak{r}_*}$.

Next, (8.6) implies that $\|(\boldsymbol{\eta}_{n+1}, a_{n+1}) - (\boldsymbol{\eta}_n, a_n)\|_{X_0} \leq 4^{-n}\|(\boldsymbol{\eta}_1, a_1)\|_{X_0}$ which in turn implies that the sequence is Cauchy in X_0 and thus convergent. Since $X_1 \subset X_0$, the sequence's limit in X_0 must agree with the weak limit $(\boldsymbol{\eta}_\mu, a_\mu) \in X_1$.

Moreover, (8.6) implies

$$\|N^\mu(\boldsymbol{\eta}_\mu, a_\mu) - N^\mu(\boldsymbol{\eta}_n, a_n)\|_{X_0} \leq 4^{-1}\|(\boldsymbol{\eta}_\mu, a_\mu) - (\boldsymbol{\eta}_n, a_n)\|_{X_0}.$$

Since the right hand side converges to zero as $n \to \infty$, we see that the iteration (8.8) implies

$$(\boldsymbol{\eta}_\mu, a_\mu) = N^\mu(\boldsymbol{\eta}_\mu, a_\mu)$$

where by " $=$ " we mean equality in X_0. But since the left hand side is in X_1, the equality is in fact equality in this space. Thus we have our solution. Uniqueness within $U^1_{\mu,\mathfrak{r}_*}$ follows from (8.6). The smoothing properties of N^μ in (8.3) immediately imply that $\boldsymbol{\eta}_\mu$, since it is (part of) a fixed point, is smooth. The estimates implicit in (8.4) give the estimates in (8.7).

Appendix A. Proof of Proposition 3.1—properties of $\mathcal{H}$

First we factor $\mathcal{H}$ as

$$\mathcal{H}\xi = (c^2\partial_x^2 - 2\delta^2)\underbrace{\left[\xi - \frac{4\delta^2}{c^2\partial_x^2 - 2\delta^2}(\sigma_c\xi)\right]}_{G_c\xi}$$

where we interpret $4\delta^2/(c^2\partial_x^2 - 2\delta^2)$ as a Fourier multiplier operator, as in [10].

A rescaled version of the operator G_c appears[16] in [10]. Specifically, their operator is equivalent to ours after the following rescaling:

$$c^2 = c_0^2 + \epsilon^2, \quad g(X) = \xi(X/\epsilon) \quad \text{and} \quad \varsigma_\epsilon(X) = \epsilon^{-2}\sigma_c(X/\epsilon). \tag{A.1}$$

Here $\epsilon > 0$ is small. In this case we find that $G_c\xi(x) = P_\epsilon g(X)$ where

$$P_\epsilon g := g - \frac{4\epsilon^2\delta^2[\epsilon]}{(c_0^2 + \epsilon^2)\epsilon^2\partial_X^2 - 2\delta^2[\epsilon]}\left(\varsigma_\epsilon g\right).$$

In the above $\delta[\epsilon] := \dfrac{1}{2}\left(S^\epsilon - S^{-\epsilon}\right)$.

In [10] they show that

$$\left\|\frac{4\epsilon^2\delta^2[\epsilon]}{(c_0^2 + \epsilon^2)\epsilon^2\partial_X^2 - 2\delta^2[\epsilon]} - \frac{\alpha_1}{1 - \beta_1\partial_X^2}\right\|_{H^s \to H^s} \leq C\epsilon^2. \tag{A.2}$$

The constants $\alpha_1 \neq 0$ and $\beta_1 > 0$ can be determined exactly, but are not important for our purposes here. This is shown on the Fourier side by a careful analysis of the associated symbols and their poles. A very similar operator is studied in [9] (specifically in Lemma A.12) and therein the authors show how to take the ideas of [10] and extend estimates like (A.2) from H^s to H^s_q where $q > 0$. Again, it all takes place at the level of symbols. In this

[16] Their operator is called "$(I - P^{(\epsilon)}DN^{(\epsilon)}(\phi))$" and appears at their equation (4.6).

way, the pole analysis on the symbols carried out in [10] is sufficient to show that there exists $q > 0$ such that

$$\left\| \frac{4\epsilon^2\delta^2[\epsilon]}{(c_0^2+\epsilon^2)\epsilon^2\partial_X^2 - 2\delta^2[\epsilon]} - \frac{\alpha_1}{1-\beta_1\partial_X^2} \right\|_{H_q^s \to H_q^s} \le C\epsilon^2. \tag{A.3}$$

Likewise in [10] they show that

$$\|\varsigma_\epsilon(\cdot) - \alpha_2 \operatorname{sech}^2(\beta_2\cdot)\|_{H^1} \le C\epsilon^2.$$

The nonzero constants α_2 and β_2 can be determined exactly, but we do not need them now. One can extract from [10] that this result can be extended to higher regularity and weighted spaces. Specifically we have

$$\|\varsigma_\epsilon(\cdot) - \alpha_2 \operatorname{sech}^2(\beta_2\cdot)\|_{H_q^s} \le C\epsilon^2.$$

All of this together implies that

$$\left\| P_\epsilon - \left[1 - \frac{\alpha_1}{1-\beta_1\partial_X^2}\left(\alpha_2 \operatorname{sech}^2(\beta_2\cdot)\cdot\right)\right]\right\|_{H_q^s\to H_q^s} \le C\epsilon^2.$$

The operator in the square brackets is also analyzed in [10] and is shown to be a homeomorphism of E^1 and itself. Again, this can be extended to E_q^s. And so a Neumann series argument allows us to conclude that P_ϵ is also a homeomorphism of E_q^s and itself. Unraveling the scalings shows that G_c is a homeomorphism of $E_{b_c}^s$ and itself and

$$\|G_c\|_{E_b^s \to E_b^s} \le C.$$

The map $c^2\partial_x^2 - 2\delta^2$ maps E_b^{s+2} to $E_{b,0}^s$. Moreover it is invertible when $b > 0$. Here are the main ideas. If $u \in E_{b,0}^s$ then its Fourier transform $\mathfrak{F}[u](k)$ can be analytically extended from $k \in \mathbf{R}$ to the horizonal strip $\{|\Im k| < b\} \subset \mathbf{C}$. Since u is a mean-zero function we have $\mathfrak{F}[u](0)$. And since since the Fourier transform of an even function is again an even function, we have $\partial_k \mathfrak{F}[u](0) = 0$. Thus $\mathfrak{F}[u](k)$ has a zero of order at least two at $k = 0$. Viewed as a Fourier multiplier operator, $c^2\partial_x^2 - 2\delta^2$ has symbol $\widetilde{w}(k) := -c^2k^2 + 2\sin^2(k)$. When $c > \sqrt{2} = c_0$ one can show (and in fact this is shown in [10]) that the only zero of $\widetilde{w}(k)$ in the horizonal strip $\{|\Im k| < b_c\} \subset \mathbf{C}$ occurs at $k = 0$ and is of order two. Thus $\mathfrak{F}[u](k)/\widetilde{w}(k)$ has a removable singularity at $k = 0$ and is analytic in $\{|\Im k| < b_c\}$. Which implies, by way of the Paley-Wiener theorem (as described in Lemma 3 in [2]) that we have $(c^2\partial_x^2 - 2\delta^2)^{-1}u \in E_b^s$ for $b \in (0, b_c]$ and moreover

$$\|(c^2\partial_x^2 - 2\delta^2)^{-1}\|_{E_{b,0}^s \to E_b^{s+2}} \le C_b.$$

Thus the factorization of $\mathcal{H}$ implies that it is the product of two invertible operators. The usual algebra completes the proof.

Appendix B. Proof of Theorem 5—periodic solutions exist

This proof proceeds along the lines of the proof of the Crandall-Rabinowitz-Zeidler "bifurcation from a simple eigenvalue" theorem. The main difficulty is carrying out the requisite estimates in a way that is uniform in the mass ratio μ.

The first step is to put

$$\mathbf{h}(x) := I^\mu \boldsymbol{\varphi}(\omega x)$$

where $I^\mu := \operatorname{diag}(\mu, 1)$ and $\boldsymbol{\varphi}(X)$ is 2π-periodic. The frequency ω is left undetermined; it is going to become one of our unknowns.

Putting this into (1.16) gives the following equation for $\boldsymbol{\varphi}$:

$$c^2\omega^2\mu\partial_X^2\boldsymbol{\varphi} + L_\mu[\omega]I^\mu\boldsymbol{\varphi} + L_\mu[\omega]Q_\mu[\omega](I^\mu\boldsymbol{\varphi}, I^\mu\boldsymbol{\varphi}) = 0. \tag{B.1}$$

The operators $L_\mu[\omega]$ and $Q_\mu[\omega]$ are formed from L_μ and Q_μ by replacing A and δ in their definitions by

$$A[\omega] := \frac{1}{2}\left(S^\omega + S^{-\omega}\right) \quad \text{and} \quad \delta[\omega] := \frac{1}{2}\left(S^\omega - S^{-\omega}\right)$$

respectively.

We will look for solutions $\boldsymbol{\varphi}$ in

$$\mathcal{X}^s := E^s_{\text{per},0} \times O^s_{\text{per}}$$

The spaces $\mathcal{X}^s$ are Hilbert spaces with the usual inner product and norm, which we abbreviate as $\langle \mathbf{u}, \mathbf{v}\rangle_s := \langle \mathbf{u}, \mathbf{v}\rangle_{\mathcal{X}^s}$ and $\|\mathbf{u}\|_s := \|\mathbf{u}\|_{\mathcal{X}^s}$. We abuse notation and use $\|f\|_s := \|f\|_{H^s_{\text{per}}}$ as well.

The following lemma contain all the information about $L_\mu[\omega]$ we need for the proof.

Lemma B.1. *For all $|c| > c_0$ there exists $\mu_{per} > 0$ such the following hold for all $\mu \in (0, \mu_{per})$.*

- *For all $\omega \in \mathbf{R}$ and $s \geq 0$, $L_\mu[\omega]$ is a bounded operator from $\mathcal{X}^s$ to itself.*
- *For any $\omega_1, \omega_2 \in \mathbf{R}$ we have*

$$\|[(L_\mu[\omega_1] - L_\mu[\omega_2])\mathbf{u}] \cdot \mathbf{e}_1\|_s \leq C|\omega_1 - \omega_2|\|\mathbf{u}\|_{s+1} \tag{B.2}$$

and

$$\|[(L_\mu[\omega_1] - L_\mu[\omega_2])\mathbf{u}] \cdot \mathbf{e}_2\|_s \leq C\mu|\omega_1 - \omega_2|\|\mathbf{u}\|_{s+1}. \tag{B.3}$$

- *There exists ω_μ and $|\upsilon_\mu| \leq 1$ for which we have*

$$\underbrace{c^2\omega_\mu^2\mu\partial_X^2\boldsymbol{\varphi} + L_\mu[\omega_\mu]I^\mu\boldsymbol{\varphi}}_{\Gamma_\mu\boldsymbol{\varphi}} = 0 \quad \textit{and} \quad \boldsymbol{\varphi} \in \mathcal{X}^s$$

if and only if $\boldsymbol{\varphi}(X) = a\boldsymbol{\nu}_\mu(X)$ where

$$\boldsymbol{\nu}_\mu(X) := \begin{bmatrix} \upsilon_\mu \cos(X) \\ \sin(X) \end{bmatrix}$$

and $a \in \mathbf{R}$ is arbitrary. (That is to say $\ker\Gamma_\mu = \operatorname{span}\boldsymbol{\nu}_\mu$.)

- *There exists $|z_\mu| \leq 1$ such that*

$$\Gamma_\mu\boldsymbol{\varphi} = \mathbf{g} \tag{B.4}$$

has a solution $\boldsymbol{\varphi} \in \mathcal{X}^{s+2}$ for a given $\mathbf{g} \in \mathcal{X}^s$ if and only if

$$\langle \mathbf{g}, \boldsymbol{\nu}^*\rangle_0 = 0 \tag{B.5}$$

where

$$\boldsymbol{\nu}^*_\mu(X) := \begin{bmatrix} z_\mu \cos(X) \\ \sin(X) \end{bmatrix}.$$

Also $\Gamma_\mu^\dagger \boldsymbol{\nu}_\mu^* = 0$. *(That is to say* $\operatorname{range}\Gamma_\mu = [\ker \Gamma_\mu^\dagger]^\perp = [\operatorname{span}\boldsymbol{\nu}_\mu^*]^\perp$.*)*

- *Moreover, if* (B.4) *holds and we demand that* $\boldsymbol{\varphi}$ *also meets* (B.5) *then*

$$\|\boldsymbol{\varphi}\|_{s+2} \le C\|\mathbf{g}\|_s \tag{B.6}$$

where the constant $C > 0$ *does not depend on* μ, $\boldsymbol{\varphi}$, $\mathbf{g}$.

Proof. The proof of this lemma is not much more than a lengthy exercise in Fourier series. Specifically, for $u \in L^2_{\text{per}}$ put

$$\widehat{u}(k) := \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ikX}u(X)dX.$$

In which case the Fourier inversion formula reads

$$u(X) = \sum_{k\in Z}\widehat{u}(k)e^{ikX}.$$

Note that since we are working in the space $E^s_{\text{per},0} \times O^s_{\text{per}}$, the $k = 0$ modes are identically zero for all the functions we consider.

From the discussion discussion in Section 5, we have

$$\widehat{L_\mu[\omega]\boldsymbol{\varphi}}(k) = \widetilde{L}_\mu(\omega k)\widehat{\boldsymbol{\varphi}}(k)$$

where $\widetilde{L}_\mu(\omega)$ is given by (5.5). The matrix $\widetilde{L}_\mu(\omega)$ is uniformly bounded in ω and its first row vanishes at $\omega = 0$. Membership in $E^s_{\text{per},0}$ requires that $\widehat{\mathbf{u}}(0)\cdot\mathbf{e}_1 = 0$. In this way we see that L_μ is bounded from $\mathcal{X}^s$ to itself.

The following estimate for the shift map is well-known, and its proof can be found in (for instance) [9]:

$$\|(S^{\omega_1} - S^{\omega_2})u\|_s \le |\omega_1 - \omega_2|\|u\|_{s+1}.$$

This estimate implies similar estimates for $A[\omega]$ and $\delta[\omega]$. With these, proving (B.2) and (B.3) amounts to just keeping track of μ in the definition of $L_\mu[\omega]$.

Now we need to identify the bifurcation frequency so that Γ_μ has the properties ascribed to it. This is more or less exactly the linear calculation carried out in Section 5. Since $\widehat{\partial_X u}(k) = ik\widehat{u}(k)$ we see that $\Gamma_\mu\boldsymbol{\nu} = 0$ if and only if

$$\widetilde{L}_\mu(\omega_\mu k)I^\mu\widehat{\boldsymbol{\nu}}(k) = c^2\mu\omega_\mu^2k^2\widehat{\boldsymbol{\nu}}(k).$$

So we see that we need to take ω_μ and k so that $c^2\mu\omega_\mu^2k^2$ is an eigenvalue of $\widetilde{L}(\omega_\mu k)I^\mu$. For simplicity we take $k = \pm 1$.

The eigenvalues of $\widetilde{L}_\mu(\omega)I^\mu$ are

$$\lambda_\mu^\pm(\omega) := 1 + \mu \pm \sqrt{(1+\mu)^2 - 4\mu\sin^2(\omega)}.$$

We need to understand when we can solve

$$\lambda_\mu^-(\omega) = c^2\mu\omega^2 \quad \text{or} \quad \lambda_\mu^+(\omega) = c^2\mu\omega^2 \tag{B.7}$$

for ω.

We claim that so long $c^2 > 2$ then

$$\lambda_\mu^-(\omega) = c^2\mu\omega^2 \iff \omega = 0.$$

Towards this end, computation of the Maclaurin series of $\lambda_\mu^-(\omega)$ gives

$$\lambda_\mu^-(\omega) = \frac{2\mu}{1+\mu}\omega^2 + \mathcal{O}(\omega^4).$$

For $\mu \geq 0$ and $c^2 > 2$ we have $c^2 > 2/(1+\mu)$ which implies that only $\omega = 0$ solves (B.7) in a neighborhood of the origin. For ω away from the origin, we note that $\lambda_\mu^-(\omega)$ is a uniformly bounded function of ω whereas $c^2\mu\omega^2$ diverges as $\omega \to \infty$. We omit the particulars, but this implies there are no solutions far from the origin. Since we are working in $\mathcal{X}^s = E^s_{\text{per},0} \times O^s_{\text{per}}$, there are no $k = 0$ modes in the Fourier expansion of $\mathbf{u}$. Thus we cannot use λ_μ^- as the eigenvalue in (B.7).

So we will use λ_μ^+. The second equation in (B.7) is precisely (5.7) from Section 5. Lemma 5.2 in that Section tells us what we need to know about its solutions. Specifically there exists a unique nonnegative solution ω_μ and that this solution is $\mathcal{O}(\mu^{-1/2})$.

Thus we now consider the the eigenvector of $\widetilde{L}_\mu(\omega_\mu)I^\mu$ associated with $\lambda_\mu^+(\omega_\mu)$. It is $\mathbf{v}_\mu := \begin{bmatrix} i\upsilon_\mu \\ 1 \end{bmatrix}$ where

$$\upsilon_\mu := -\frac{2\mu\cos(\omega_\mu)\sin(\omega_\mu)(1-2\cos^2(\omega_\mu) - \mu\cos^2(\omega_\mu)\sin^2(\omega_\mu))}{\lambda_\mu^+(\omega_\mu) - 2\mu\sin^2(\omega_\mu)(1-\mu\cos^2(\omega_\mu))}. \tag{B.8}$$

Examination of the formula for $\lambda_\mu^+(\omega)$ shows that it lies in $2 \leq \lambda_\mu^+(\omega) \leq 2+\mu$. (See Figure 4). This tells us that $|\upsilon_\mu| \leq C\mu$ for μ sufficiently close to zero and, in particular, can be made less than one.

All of the above facts are on the "frequency side" but we can reassemble things on the "space side" to find that if we put

$$\boldsymbol{\nu}_\mu(X) := \Im\left(\mathbf{v}_\mu e^{iX}\right) = \begin{bmatrix} \upsilon_\mu\cos(X) \\ \sin(X) \end{bmatrix}$$

then $\boldsymbol{\nu}_\mu \in \mathcal{X}^s$ and $\Gamma_\mu\boldsymbol{\nu}_\mu = 0$. The uniqueness of ω_μ implies that $\boldsymbol{\nu}_\mu$ is the only such solution in $\mathcal{X}^s$, up to scalar multiples. A parallel line of reasoning, together with the usual Fredholm theory, shows that there exists $\boldsymbol{\nu}_\mu^*$, with the stated estimates, such that range $\Gamma_\mu = [\ker\Gamma_\mu^\dagger]^\perp = [\text{span}\,\boldsymbol{\nu}_\mu^*]^\perp$.

Now suppose that $\Gamma_\mu\mathbf{u} = \mathbf{g}$. So, for all $k \in \mathbf{Z}/\{0\}$ we have

$$\left(-c^2\omega_\mu^2\mu k^2 + \widetilde{L}_\mu(\omega_\mu k)I^\mu\right)\widehat{\mathbf{u}}(k) = \widehat{\mathbf{g}}(k).$$

Or rather

$$\left(1 - \frac{1}{c^2\omega_\mu^2\mu k^2}\widetilde{L}_\mu(\omega_\mu k)I^\mu\right)\widehat{\mathbf{u}}(k) = -\frac{1}{c^2\omega_\mu^2\mu k^2}\widehat{\mathbf{g}}(k).$$

For μ sufficiently close to zero, inspection of the entries of $\widetilde{L}_\mu(\omega)$ shows that

$$\|\widetilde{L}_\mu(\omega)I^\mu\| \leq 3.$$

To be clear, we mean the standard matrix norm here. And thus the estimate (5.8) implies

$$\left\|\frac{1}{c^2\omega_\mu^2\mu k^2}\widetilde{L}_\mu(\omega_\mu k)I^\mu\right\| \leq \frac{3}{2k^2}.$$

This quantity is less than one for all $|k| \geq 2$, uniformly in μ. Thus $\left(1 - \frac{1}{c^2\omega_\mu^2\mu k^2}\widetilde{L}_\mu(\omega_\mu k)I^\mu\right)^{-1}$ exists via the Neumann series and has matrix norm uniformly bounded in both k and μ. In this way we see that

$$|k| \geq 2 \implies |\widehat{\mathbf{u}}(k)| \leq Ck^{-2}|\widehat{\mathbf{g}}(k)|.$$

Lastly, for $k = \pm 1$, it is a tedious but mundane exercise in two-by-two matrices to see that $[-c^2\omega_\mu^2\mu + L_\mu(\omega_\mu)I^\mu]\widehat{\mathbf{u}(\pm 1)} = \widehat{\mathbf{g}}(\pm 1)$ plus the condition (B.5) implies $|\widehat{\mathbf{u}}(\pm 1)| \leq C|\widehat{\mathbf{g}}(\pm 1)|$. Thus we use Plancheral's theorem and we see that

$$\|\mathbf{u}\|_{s+2} \leq C\|\mathbf{g}\|_s.$$

This completes the proof of Lemma B.1.

□

Now that we understand L_μ, we put

$$\omega = \omega_\mu + \xi \quad \text{and} \quad \boldsymbol{\varphi} = a\boldsymbol{\nu}_\mu + a\boldsymbol{\psi}$$

where $\boldsymbol{\psi}$ meets condition (B.5). Substituting this into (B.1) returns, after some algebra,

$$\Gamma_\mu\boldsymbol{\psi} = R_0(\xi) + R_1(\xi) + R_2(\xi,\boldsymbol{\psi}) + aR_3(\xi,\boldsymbol{\psi}) \tag{B.9}$$

where

$$\begin{aligned}
R_0(\xi) &:= -2c^2\mu\omega_\mu\xi\partial_X^2\boldsymbol{\nu}_\mu \\
R_1(\xi) &:= -c^2\mu\xi^2\partial_X^2\boldsymbol{\nu}_\mu - \left(L_\mu[\omega_\mu+\xi] - L_\mu[\omega_\mu]\right)I^\mu\boldsymbol{\nu}_\mu \\
R_2(\xi,\boldsymbol{\psi}) &:= -c^2\mu(2\omega_\mu\xi+\xi^2)\partial_X^2\boldsymbol{\psi} - \left(L_\mu[\omega_\mu+\xi] - L_\mu[\omega_\mu]\right)I^\mu\boldsymbol{\psi} \\
R_3(\xi,\boldsymbol{\psi}) &:= -L_\mu[\omega_\mu+\xi]Q_\mu[\omega_\mu+\xi](I^\mu(\boldsymbol{\nu}_\mu+\boldsymbol{\psi}), I^\mu(\boldsymbol{\nu}_\mu+\boldsymbol{\psi})).
\end{aligned}$$

We perform a Liapunov-Schmidt decomposition [29]. Let

$$\pi_\mu\mathbf{u} := \frac{\langle\mathbf{u},\boldsymbol{\nu}_\mu^*\rangle_0}{\langle\boldsymbol{\nu}_\mu,\boldsymbol{\nu}_\mu^*\rangle_0} \quad \text{and} \quad \Pi_\mu\mathbf{u} := [\pi_\mu\mathbf{u}]\boldsymbol{\nu}_\mu.$$

The following properties are easily checked:

$$\pi_\mu\Pi_\mu = \pi_\mu, \quad \Pi_\mu^2 = \Pi_\mu, \quad \Pi_\mu\boldsymbol{\nu}_\mu = \boldsymbol{\nu}_\mu \quad \text{and} \quad \Pi_\mu\Gamma_\mu = \Gamma_\mu\Pi_\mu = 0.$$

Note the the solvability condition (B.5) is equivalent to saying $\pi_\mu\mathbf{g} = 0$. Thus we can always solve equations of the form $\Gamma_\mu\boldsymbol{\varphi} = (1-\Pi_\mu)\mathbf{g}$. Such solutions are not unique since Γ_μ has a non-trivial kernel. By demanding that the solution has $\pi_\mu\boldsymbol{\varphi} = 0$, the solution becomes unique; we denote it by $\boldsymbol{\varphi} =: \Gamma_\mu^{-1}(1-\Pi_\mu)\mathbf{g}$.

Applying π_μ to (B.9) gives, after rearranging terms:

$$\xi = \underbrace{\frac{1}{2c^2\mu\omega_\mu\langle\boldsymbol{\nu}_\mu,\partial_X^2\boldsymbol{\nu}_\mu^*\rangle_0}\pi_\mu\left(R_1(\xi) + R_2(\xi,\boldsymbol{\psi}) + aR_3(\xi,\boldsymbol{\psi})\right)}_{\Xi_\mu(\xi,\boldsymbol{\psi},a)}. \tag{B.10}$$

Applying $1-\Pi_\mu$ to (B.9) gives, after rearranging terms:

$$\boldsymbol{\psi} = \underbrace{\Gamma_\mu^{-1}(1-\Pi_\mu)\left(R_0(\xi) + R_1(\xi) + R_2(\xi,\boldsymbol{\psi}) + aR_3(\xi,\boldsymbol{\psi})\right)}_{\Psi_\mu(\xi,\boldsymbol{\psi},a)}. \tag{B.11}$$

Now we begin to estimate the terms. First we have using the estimates on $\boldsymbol{\nu}_\mu$, $\boldsymbol{\nu}^*_\mu$ and the Cauchy-Schwartz inequality

$$|\pi_\mu \mathbf{u}| \leq C\mu\|\mathbf{u}\cdot\mathbf{e}_1\|_0 + C\|\mathbf{u}\cdot\mathbf{e}_2\|_0.$$

In turn this gives, for any $s \geq 0$:

$$\|\Pi_\mu \mathbf{u}\|_s \leq C\mu\|\mathbf{u}\cdot\mathbf{e}_1\|_0 + C\|\mathbf{u}\cdot\mathbf{e}_2\|_0 \quad \text{and} \quad \|(1-\Pi_\mu)\mathbf{u}\|_s \leq C\|\mathbf{u}\|_s.$$

The latter of these, in tandem with the estimate (B.6) show that

$$\|\Gamma_\mu^{-1}(1-\Pi_\mu)\mathbf{u}\|_{s+2} \leq C\|\mathbf{u}\|_s$$

Also, the estimates for ω_μ, $\boldsymbol{\nu}_\mu$ and $\boldsymbol{\nu}^*_\mu$ imply that

$$\left|\frac{1}{2c^2\mu\omega_\mu\langle\boldsymbol{\nu}_\mu, \partial_X^2\boldsymbol{\nu}^*_\mu\rangle_0}\right| \leq C\mu^{-1/2}. \tag{B.12}$$

We turn our attention to estimating the R_0, R_1 and R_2. The following estimates hold when $s \geq 0$ and when $|\xi|, |\grave{\xi}|, \|\boldsymbol{\psi}\|_s, \|\grave{\boldsymbol{\psi}}\|_s \leq 1$. Each of these estimates follows in routine way from the estimates in Lemma B.1. First, $R_0(0) = 0$ and

$$\|R_0(\xi) - R_0(\grave{\xi})\|_s \leq C\mu^{1/2}|\xi - \grave{\xi}|.$$

Next, $R_1(0) = 0$ and

$$\|R_1(\xi) - R_1(\grave{\xi})\|_s \leq C\mu|\xi - \grave{\xi}|.$$

Then we have $R_2(0,0) = 0$ and

$$\|R_2(\xi,\boldsymbol{\psi}) - R_2(\grave{\xi},\grave{\boldsymbol{\psi}})\|_s \leq C\mu^{1/2}\left(\|\boldsymbol{\psi}\|_{s+2} + \|\grave{\boldsymbol{\psi}}\|_{s+2}\right)|\xi - \grave{\xi}| + C\mu^{1/2}\left(|\xi| + |\grave{\xi}|\right)\|\boldsymbol{\psi} - \grave{\boldsymbol{\psi}}\|_{s+2}.$$

Finally we look at R_3. In some sense this is the most complicated since it has the most terms. On the other hand, the justification of the estimates (2.24) $-$ (2.25) for $L_\mu Q_\mu$ in Section 1 contains the fundamental ideas and there are no subtleties. In the interest of brevity, we merely report the outcome of the estimates. The estimates are

$$\begin{aligned}
\|R_3(0,0)\cdot\mathbf{e}_1\|_s \leq& C\\
\|R_3(0,0)\cdot\mathbf{e}_2\|_s \leq& C\mu\\
\|(R_3(\xi,\boldsymbol{\psi}) - R_3(\grave{\xi},\grave{\boldsymbol{\psi}}))\cdot\mathbf{e}_1\|_s \leq& C\left(1 + \|\boldsymbol{\psi}\|_{s+2} + \|\grave{\boldsymbol{\psi}}\|_{s+2}\right)|\xi - \grave{\xi}|\\
&+C\left(1 + |\xi| + |\grave{\xi}|\right)\|\boldsymbol{\psi} - \grave{\boldsymbol{\psi}}\|_{s+2}\\
\|(R_3(\xi,\boldsymbol{\psi}) - R_3(\grave{\xi},\grave{\boldsymbol{\psi}}))\cdot\mathbf{e}_2\|_s \leq& C\mu\left(1 + \|\boldsymbol{\psi}\|_{s+2} + \|\grave{\boldsymbol{\psi}}\|_{s+2}\right)|\xi - \grave{\xi}|\\
&+C\mu\left(1 + |\xi| + |\grave{\xi}|\right)\|\boldsymbol{\psi} - \grave{\boldsymbol{\psi}}\|_{s+2}.
\end{aligned}$$

The preceding estimates for R_0, R_1, R_2 and R_3 and (B.12) imply

$$|\Xi_\mu(0,0,a)| + \|\Psi_\mu(0,0,a)\|_2 \leq C\mu^{1/2}|a|$$

and

$$\left|\Xi(\xi, \boldsymbol{\psi}, a) - \Xi(\grave{\xi}, \grave{\boldsymbol{\psi}}, a)\right| + \|\Psi_\mu(\xi, \boldsymbol{\psi}, a) - \Psi_\mu(\grave{\xi}, \grave{\boldsymbol{\psi}}, a)\|_2$$
$$\leq C\left(\mu^{1/2} + |\xi| + |\grave{\xi}| + \|\boldsymbol{\psi}\|_2 + \|\grave{\boldsymbol{\psi}}\|_2 + |a|\right)\left(|\xi - \grave{\xi}| + \|\boldsymbol{\psi} - \grave{\boldsymbol{\psi}}\|_2\right).$$

Thus there exists $\rho_* > 0$, $a_{\text{per}} > 0$ and $\mu_{\text{per}} > 0$ such that for all $\mu \in (0, \mu_{\text{per}})$ and $|a| \leq a_{\text{per}}$ the mapping $(\xi, \boldsymbol{\psi}) \to (\Xi_\mu(\xi, \boldsymbol{\psi}, a), \Psi_\mu(\xi, \boldsymbol{\psi}, a))$ take the ball of radius ρ_* in $\mathbf{R} \times \mathcal{X}^2$ to itself and is a contraction on that set with contraction constant less that $1/2$. And so, for each $\mu \in (0, \mu_{\text{per}})$ and $|a| \leq a_{\text{per}}$ there is a unique fixed point $(\xi_\mu^a, \boldsymbol{\psi}_\mu^a)$ of the map (Ξ_μ, Ψ_μ). When $a = 0$ this fixed point is trivial.

Next we notice that, for $|a|, |\grave{a}| \leq a_{\text{per}}$ we have

$$\begin{aligned}
&\|(\xi_\mu^a, \boldsymbol{\psi}_\mu^a) - (\xi_\mu^{\grave{a}}, \boldsymbol{\psi}_\mu^{\grave{a}})\|_{\mathbf{R}\times\mathcal{X}^2} \\
\leq&\|(\Xi_\mu(\xi_\mu^a, \boldsymbol{\psi}_\mu^a, a), \Psi_\mu(\xi_\mu^a, \boldsymbol{\psi}_\mu^a, a)) - (\Xi_\mu(\xi_\mu^a, \boldsymbol{\psi}_\mu^a, \grave{a}), \Psi_\mu(\xi_\mu^a, \boldsymbol{\psi}_\mu^a, \grave{a}))\|_{\mathbf{R}\times\mathcal{X}^2} \\
+&\|(\Xi_\mu(\xi_\mu^a, \boldsymbol{\psi}_\mu^a, \grave{a}), \Psi_\mu(\xi_\mu^a, \boldsymbol{\psi}_\mu^a, \grave{a})) - (\Xi_\mu(\xi_\mu^{\grave{a}}, \boldsymbol{\psi}_\mu^{\grave{a}}, \grave{a}), \Psi_\mu(\xi_\mu^{\grave{a}}, \boldsymbol{\psi}_\mu^{\grave{a}}, \grave{a}))\|_{\mathbf{R}\times\mathcal{X}^2}.
\end{aligned}$$

For the second line in the above we use the contraction estimate and get

$$\begin{aligned}
&\|(\xi_\mu^a, \boldsymbol{\psi}_\mu^a) - (\xi_\mu^{\grave{a}}, \boldsymbol{\psi}_\mu^{\grave{a}})\|_{\mathbf{R}\times\mathcal{X}^2} \\
\leq&2\|(\Xi_\mu(\xi_\mu^a, \boldsymbol{\psi}_\mu^a, a), \Psi_\mu(\xi_\mu^a, \boldsymbol{\psi}_\mu^a, a)) - (\Xi_\mu(\xi_\mu^a, \boldsymbol{\psi}_\mu^a, \grave{a}), \Psi_\mu(\xi_\mu^a, \boldsymbol{\psi}_\mu^a, \grave{a}))\|_{\mathbf{R}\times\mathcal{X}^2}.
\end{aligned}$$

Tracing through the definitions shows that Ξ_μ and Ψ_μ depend on a only through the term R_3 and do so linearly. Thus we can use the estimates for R_3 from above to get

$$\|(\xi_\mu^a, \boldsymbol{\psi}_\mu^a) - (\xi_\mu^{\grave{a}}, \boldsymbol{\psi}_\mu^{\grave{a}})\|_{\mathbf{R}\times\mathcal{X}^2} \leq C|a - \grave{a}|.$$

We can bootstrap this last estimate to

$$\|(\xi_\mu^a, \boldsymbol{\psi}_\mu^a) - (\xi_\mu^{\grave{a}}, \boldsymbol{\psi}_\mu^{\grave{a}})\|_{\mathbf{R}\times\mathcal{X}^s} \leq C|a - \grave{a}|$$

for any $s \geq 0$.

These complete the proof of Theorem 5.1. Summing up, if we put

$$\omega_\mu^a := \omega_\mu + \xi_\mu^a \quad \text{and} \quad \mathbf{p}_\mu^a(x) := \underbrace{I^\mu \boldsymbol{\nu}_\mu(\omega_\mu^a x) + I^\mu \boldsymbol{\psi}_\mu^a(\omega_\mu^a x)}_{\begin{pmatrix} \mu\varphi_{\mu,1}^a(x) \\ \varphi_{\mu,2}^a(x) \end{pmatrix}}$$

then $a\mathbf{p}_\mu^a(x)$ solves (1.16) for all $|a| \leq a_{\text{per}}$ and $\mu \in (0, \mu_{\text{per}})$.

B.1. **Some final estimates.** The estimates in Theorem 5.1 are stated in terms of ω_μ^a, $\boldsymbol{\nu}_\mu(X)$ and $\boldsymbol{\psi}_\mu^a(X)$, but we will be needing estimates on $\mathbf{p}_\mu^a(x)$ and so we close out this appendix with some additional estimates for that function. That is to say estimates for the periodic solutions in the "original coordinates." We use the following lemma:

Lemma B.2. *Suppose that $P(X) \in C^\infty_{per}$. For $|a| \leq a_{per}$ let $P^a(x) := P(\omega_\mu^a x)$. Then, there exists $C > 0$ such that for all $|a| \leq a_{per}$ and $\mu \in (0, \mu_{per})$ we have*

$$\|P^a\|_{W^{s,\infty}} \leq C\|P\|_{C^{s+1}_{per}}\mu^{-s/2}.$$

Also, for all $b<0$ there exists $C_b>0$ (which diverges as $b\to 0^-$) such that
$$\|P^a - P^{\grave{a}}\|_{W_b^{s,\infty}} \le C_b \|P\|_{C_{per}^{s+1}} \mu^{-s/2}|a-\grave{a}|.$$

Proof. The first estimate follows from the chain rule and estimates for ω_μ^a in Theorem 5. We do not provide the details. As for the second estimate, the key quantity to estimate on the left hand side is
$$|(\omega_\mu^a)^s P^{(s)}(\omega_\mu^a x) - (\omega_\mu^{\grave{a}})^s P^{(s)}(\omega_\mu^{\grave{a}} x)|.$$
The triangle inequality give
$$\begin{aligned}|(\omega_\mu^a)^s P^{(s)}(\omega_\mu^a x) &- (\omega_\mu^{\grave{a}})^s P^{(s)}(\omega_\mu^{\grave{a}} x)| \\ &\le |(\omega_\mu^a)^s - (\omega_\mu^{\grave{a}})^s||P^{(s)}(\omega_\mu^a x)| + |\omega_\mu^{\grave{a}}|^s|P^{(s)}(\omega_\mu^a x) - P^{(s)}(\omega_\mu^{\grave{a}} x)|.\end{aligned}$$
Factoring and the estimates for ω_μ^a in Theorem 5 give:
$$|(\omega_\mu^a)^s - (\omega_\mu^{\grave{a}})^s| \le C\mu^{-(s-1)/2}|\omega_\mu^a - \omega_\mu^{\grave{a}}| \le C\mu^{-(s-1)/2}|a-\grave{a}|.$$
Also the Fundametnal Theorem of Calculus (FTOC) tells us
$$|P^{(s)}(\omega_\mu^a x) - P^{(s)}(\omega_\mu^{\grave{a}} x)| \le \|P\|_{C_{\rm per}^{s+1}}|\omega_\mu^a - \omega_\mu^{\grave{a}}||x|.$$
And so
$$|(\omega_\mu^a)^s P^{(s)}(\omega_\mu^a x) - (\omega_\mu^{\grave{a}})^s P^{(s)}(\omega_\mu^{\grave{a}} x)| \le C\mu^{-s/2}|a-\grave{a}|\|P\|_{C_{\rm per}^{s+1}}|x|.$$
Since $\cosh^b(x)|x| \in L^\infty(\mathbf{R})$ so long as $b<0$, we can use these to get the second estimate in the lemma. ∎

Applying this lemma gives us

$$\|\varphi_{\mu,1}^a\|_{W^{s,\infty}} + \|\varphi_{\mu,2}^a\|_{W^{s,\infty}} \le C\mu^{-s/2}. \tag{B.13}$$

And, for $b<0$:

$$\|\varphi_{\mu,1}^a - \varphi_{\mu,1}^{\grave{a}}\|_{W_b^{s,\infty}} + \|\varphi_{\mu,2}^a - \varphi_{\mu,2}^{\grave{a}}\|_{W_b^{s,\infty}} \le C_b\mu^{-s/2}|a-\grave{a}|. \tag{B.14}$$

Appendix C. Proof of Lemmas 7.1-7.3—properties of $\mathcal{L}_\mu$.

Let
$$\mathcal{S}_\mu \eta := c^2\mu\eta'' + c^2\mu\omega_\mu^2\eta + 4\sigma_c(x)\eta.$$
This Schrödinger operator[17] will be the frame on which we build $\mathcal{L}_\mu$. The estimates (2.14), (2.20)-(2.23) and (6.6) can be used to show that

$$\|(\mathcal{S}_\mu - \mathcal{L}_\mu)\eta\|_{s,b} \le C\mu\|\eta\|_{s,b} \tag{C.1}$$

and so we see that $\mathcal{L}_\mu$ is a small perturbation of $\mathcal{S}_\mu$. Nonetheless, observe that we are interested in the approximation when μ is small and as such the singular nature of $\mathcal{S}_\mu$ and $\mathcal{L}_\mu$ create a number of technical difficulties that are not resolvable using standard perturbation methods.

[17]It is almost the same as the Schrödinger operator $\mathcal{U}_\mu$ described in Section 3 at (3.4) and it has the same sort of properties as $\mathcal{U}_\mu$. The only difference is that the constant term "2" in $\mathcal{U}_\mu$ has been replaced by "$c^2\mu\omega_\mu^2$" in $\mathcal{S}_\mu$. The estimates in (5.8) tell us that these two constants are within $\mathcal{O}(\mu)$ of one another, so this is a minor change. The reason we do this is so that the Jost solutions for $\mathcal{S}_\mu$ will be asymptotic as $|x|\to\infty$ to a sinusoid with the critical frequency ω_μ of the periodic solutions from Theorem 5.1.

C.1. **The coercive estimates.** First we prove an *a priori* estimate for solutions of

$$\mathcal{S}_\mu f = c^2\mu f'' + c^2\mu\omega_\mu^2 f + 4\sigma_c(x) f = g \tag{C.2}$$

by means of an energy argument. To begin, we assume f and g are compactly supported, smooth and odd functions.

Multiplying (C.2) by $\sinh(2bx)f'(x)/\left(c^2\mu\omega_\mu^2 + 4\sigma_c(x)\right)$ and integrating on $\mathbf{R}$ returns

$$\int_{\mathbf{R}} \left(\frac{c^2\mu\sinh(2bx)}{c^2\mu\omega_\mu^2 + 4\sigma_c(x)} f''(x)f'(x) + \sinh(2bx)f'(x)f(x)\right)dx = \int_{\mathbf{R}} \frac{\sinh(2bx)}{c^2\mu\omega_\mu^2 + 4\sigma_c(x)} f'(x)g(x)dx.$$

Performing the usual integration by parts sudoku on the left converts this to

$$\begin{aligned}\int_{\mathbf{R}} \frac{c^2\mu\cosh(2bx)}{c^2\mu\omega_\mu^2 + 4\sigma_c(x)}[f'(x)]^2dx - \int_{\mathbf{R}} \frac{2c^2\mu\sinh(2bx)\sigma_c'(x)}{b(c^2\mu\omega_\mu^2 + 4\sigma_c(x))^2}[f'(x)]^2dx \\ + \int_{\mathbf{R}} 2b\cosh(2bx)[f(x)]^2dx = -\frac{1}{b}\int_{\mathbf{R}} \frac{\sinh(2bx)}{c^2\mu\omega_\mu^2 + 4\sigma_c(x)} f'(x)g(x)dx.\end{aligned} \tag{C.3}$$

We also know from Theorem 1.1 that $\sigma_c'(x)$ is negative for $x > 0$ and is positive for $x < 0$. This implies that $\sinh(2bx)\sigma_c'(x) \le 0$ for all x. Which means that the middle term on the left of (C.3) is clearly non-negative. Thus we can omit it from (C.3) to get

$$\begin{aligned}\int_{\mathbf{R}} \left(\frac{c^2\mu\cosh(2bx)}{c^2\mu\omega_\mu^2 + 4\sigma_c(x)}\right)[f'(x)]^2dx \\ + \int_{\mathbf{R}} \cosh(2bx)[f(x)]^2dx \le -\frac{1}{b}\int_{\mathbf{R}} \frac{\sinh(2bx)}{c^2\mu\omega_\mu^2 + 4\sigma_c(x)} f'(x)g(x)dx.\end{aligned} \tag{C.4}$$

Since $\omega_\mu = \mathcal{O}(\mu^{-1/2})$ (from estimate (5.8)), the positivity of σ_c (from Theorem 1.1) implies that

$$C^{-1} \le \frac{1}{c^2\mu\omega_\mu^2 + 4\sigma_c(x)} \le C \tag{C.5}$$

for some $C > 1$ and all $x \in \mathbf{R}$. Also we have a constant $C > 1$ for which

$$C^{-1}\cosh^{2b}(x) \le \cosh(2bx) \le C\cosh^{2b}(x) \quad \text{and} \quad |\sinh(2bx)| \le C\cosh^{2b}(x) \tag{C.6}$$

holds for all $x \in \mathbf{R}$ and $b \in [0,1]$.

Putting the above together with (C.4) and using Cauchy-Schwarz gets us

$$\mu\|f'\|_{0,b}^2 + \|f\|_{0,b}^2 \le Cb^{-1}\|f'\|_{0,b}\|g\|_{0,b} \le \frac{\mu}{2}\|f'\|_{0,b}^2 + \frac{1}{2\mu}C^2b^{-2}\|g\|_{0,b}^2.$$

The last inequality above is just "Cauchy's inequality with parameter." Bringing the first term on the right over to the left and adjusting some constants gives us

$$\mu\|f'\|_{0,b}^2 + \|f\|_{0,b}^2 \le C_b\mu^{-1}\|g\|_{0,b}^2.$$

And so we have shown that $\|f\|_{0,b} \le Cb^{-1}\mu^{-1/2}\|g\|_{0,b}$ and $\|f'\|_{0,b} \le Cb^{-1}\mu^{-1}\|g\|_{0,b}$. And these in combination with the equation (C.2) imply that $\|f''\|_{0,b} \le Cb^{-1}\mu^{-3/2}\|g\|_{0,b}$.

In the above we assumed that f and g were smooth, odd and compactly supported, but a standard density argument implies that the same result holds $f \in O_b^2$ and $g \in O_b^0$. Specifically we have, for $k = 0, 1, 2$

$$f \in O_b^2, \quad g \in O_b^0 \quad \text{and} \quad \mathcal{S}_\mu f = g \implies \|f\|_{k,b} \le C_b \mu^{(k+1)/2} \|g\|_{0,b}. \tag{C.7}$$

Here is an improvement of this estimate that we need. It says that if we are content to meausure f in a less regular space than we measure g in, that we can claw back some powers of μ in the estimate. On the right hand side of (C.4) we integrate by parts to get

$$\mu \|f'\|_{0,b}^2 + \|f\|_{0,b}^2 \le Cb^{-1} \left| \int_{\mathbf{R}} f(x) \frac{d}{dx} \left(\frac{\sinh(bx) g(x)}{c^2 \mu \omega_\mu^2 + 4\sigma_c(x)} \right) dx \right|.$$

On the left we used (C.5) and (C.6) as above. Cauchy-Schwarz on the right, plus (C.5) and (C.6), give us

$$\mu \|f'\|_{0,b}^2 + \|f\|_{0,b}^2 \le Cb^{-1} \|f\|_{0,b} \|g\|_{1,b} \le \frac{1}{2} \|f\|_{0,b}^2 + \frac{1}{2} C^2 b^{-2} \|g\|_{1,b}^2.$$

We have again used Cauchy's inequality in the last step. This implies

$$\mu \|f'\|_{0,b}^2 + \|f\|_{0,b}^2 \le C_b \|g\|_{1,b}^2.$$

And so we get

$$f \in O_b^2, \quad g \in O_b^1 \quad \text{and} \quad \mathcal{S}_\mu f = g \implies \|f\|_{0,b} \le C_b \|g\|_{1,b}. \tag{C.8}$$

Note that unlike (C.7) there are no negative powers of μ on the right hand side of this.

Next we differentiate (C.2) $s-$times and we get

$$\mathcal{S}_\mu f^{(s)} = g^{(s)} + \left(\mathcal{S}_\mu f^{(s)} - (\mathcal{S}_\mu f)^{(s)} \right).$$

It is straightforward to show that $\| \left(\mathcal{S}_\mu f^{(s)} - (\mathcal{S}_\mu f)^{(s)} \right) \|_{0,b} \le C \|f\|_{s-1,b}$. An induction argument takes this last estimate, (C.7) and (C.8) and yields, for $k = -1, 0, 1, 2$ and $s + k \ge 0$:

$$f \in O_b^{s+2}, \quad g \in O_b^s \quad \text{and} \quad \mathcal{S}_\mu f = g \implies \|f\|_{s+k,b} \le C_b \mu^{-(k+1)/2} \|g\|_{s,b}. \tag{C.9}$$

If $\mathcal{S}_\mu f = g_1 + g_2$ then we can deploy different choices for s and k for the different g_j. For instance, if $g_1 \in O_b^s$ and $g_2 \in O_b^{s+k}$ we would have

$$\|f\|_{s+k,b} \le C_b \left(\mu^{-(k+1)/2} \|g_1\|_{s,b} + \mu^{-1/2} \|g_1\|_{s+k,b} \right). \tag{C.10}$$

We can now prove Lemma 7.1.

Proof. (Lemma 7.1) If $\mathcal{L}_\mu f = g$ then $\mathcal{S}_\mu f = g + (\mathcal{S}_\mu - \mathcal{L}_\mu) f$. Using (C.10) tells us that

$$\|f\|_{s+k,b} \le C_b \mu^{-(k+1)/2} \|g\|_{s,b} + C_b \mu^{-1/2} \|(\mathcal{S}_\mu - \mathcal{L}_\mu) f\|_{s+k,b}.$$

Then we use (C.1) on the final term to get

$$\|f\|_{s+k,b} \le C_b \mu^{-(k+1)/2} \|g\|_{s,b} + C_b \mu^{1/2} \|f\|_{s+k,b}.$$

Now fix $b \in (0, b_c]$. By making μ sufficiently close to zero (call the threshold $\mu_{\mathcal{L}}(b)$) we have $C_b \mu^{1/2} \le 1/2$, in which case the last inequality implies $\|f\|_{s+k,b} \le C_b \mu^{-(k+1)/2} \|g\|_{s,b}$. This gives us the estimates in Lemma 7.1. $\square$

C.2. **Jost solutions of $\mathcal{S}_\mu$.** To prove Lemmas 7.2 and 7.3 we first need a good understanding of Jost solutions for the Schrödinger operator $\mathcal{S}_\mu$. This is to say, nontrivial solutions of

$$\mathcal{S}_\mu \zeta = c^2\mu\zeta'' + c^2\mu\omega_\mu^2\zeta + 4\sigma_c(x)\zeta = 0. \tag{C.11}$$

The function $\gamma_\mu(x)$ whose properties are described in Lemmas 7.2 and 7.3 will be shown to be a perturbation of these Jost solutions in the next subsection.

Since (C.11) is a second order linear differential equation, we know there are two linearly independent solutions to this. Since $\sigma_c(x)$ is an even function, we may assume that one of these is odd (which we call $\zeta_{\mu,1}$) and the other which is even (which we call $\zeta_{\mu,0}$). To be clear, let $\zeta_{\mu,0}$ be the solution of $\mathcal{S}_\mu\zeta = 0$ with initial conditions

$$\zeta_{\mu,0}(0) = 1 \quad \text{and} \quad \zeta'_{\mu,0}(0) = 0. \tag{C.12}$$

Let $\zeta_{\mu,1}$ be the solution of $\mathcal{S}_\mu\zeta = 0$ with initial conditions

$$\zeta_{\mu,1}(0) = 0 \quad \text{and} \quad \zeta'_{\mu,1}(0) = \omega_\mu. \tag{C.13}$$

We need some precise information about these functions and their behavior as $\mu \to 0^+$. Most of what happens here is classical, but we do not know of a reference which contains the collection of results we need. Here is that collection:

Lemma C.1. *There exists $\mu_\zeta > 0$ such that for $\mu \in (0, \mu_\zeta)$ and $j = 0, 1$ there exist smooth functions $r_{\mu,j}(x)$ and $\phi_{\mu,j}(x)$ such that putting*

$$\begin{aligned} \zeta_{\mu,j}(x) &= r_{\mu,j}(x)\sin(\omega_\mu(x + \phi_{\mu,j}(x))) \\ \text{and} \quad \zeta'_{\mu,j}(x) &= \omega_\mu r_{\mu,j}(x)\cos(\omega_\mu(x + \phi_{\mu,j}(x))) \end{aligned} \tag{C.14}$$

solves $\mathcal{S}_\mu\zeta_{\mu,j} = 0$ with initial conditions (C.12) *(for $j = 0$) and* (C.13) *(for $j = 1$).*

Moreover $r_{\mu,j}(x)$, $\phi_{\mu,j}(x)$ and $\zeta_{\mu,j}(x)$ have the following properties.

- *We have*

$$\|r_{\mu,j}\|_{W^{s,\infty}} \le C\mu^{-s/2}. \tag{C.15}$$

- *There are constants $0 < C_1 < C_2$ such that*

$$C_1 < r_{\mu,j}(x) < C_2 \tag{C.16}$$

  for all $\mu \in (0, \mu_\zeta)$.
- *There exists $r^\infty_{\mu,j}$ such that*

$$\sup_{x\ge 0} e^{b_c x}|r_{\mu,j}(x) - r^\infty_{\mu,j}| = 0. \tag{C.17}$$

- *We have*

$$\|\phi_{\mu,j}\|_{W^{1,\infty}} \le C \text{ and, for } s \ge 2,\ \|\phi_{\mu,j}\|_{W^{s,\infty}} \le C\mu^{-(s-1)/2}. \tag{C.18}$$

- *There exists $\phi^\infty_{\mu,j}$ such that*

$$\sup_{x\ge 0} e^{b_c x}|\phi_{\mu,j}(x) - \phi^\infty_{\mu,j}| = 0. \tag{C.19}$$

- *There are constants* $0 < C_1 < C_2$ *such that*

$$C_1 < \phi^\infty_{\mu,j} < C_2 \tag{C.20}$$

for all $\mu \in (0, \mu_\zeta)$.
- *The maps* $\mu \mapsto r^\infty_{\mu,j}$ *and* $\mu \mapsto \phi^\infty_{\mu,j}$ *are continuous from* $(0,\mu_\zeta)$ *into* L^∞.
- *We have*

$$\|\zeta^{(s)}_{\mu,j}\|_{L^\infty} \le C\mu^{-s/2}. \tag{C.21}$$

Proof. Note that (C.14) is essentially a polar coordinate decomposition of $\zeta_{\mu,j}(x)$. Putting (C.14) into $\mathcal{S}_\mu \zeta_{\mu,j} = 0$ gives the following system for $r_{\mu,j}(x)$ and $\phi_{\mu,j}(x)$:

$$\begin{aligned} r'_\mu(x) &= -\frac{2}{c^2\mu\omega_\mu}\sigma_c(x) r(x)\sin(2\omega_\mu(x+\phi_{\mu,j}(x))) \\ \phi'_{\mu,j}(x) &= \frac{4}{c^2\mu\omega_\mu^2}\sigma_c(x)\sin^2(\omega_\mu(x+\phi_{\mu,j}(x))). \end{aligned} \tag{C.22}$$

If we are interested in $\zeta_{\mu,1}$ the we have $r_{\mu,1}(0) = 1$ and $\phi_{\mu,1}(0) = 0$. If we are interested in $\zeta_{\mu,0}$ then we put $r_{\mu,0}(0) = 1$ and $\phi_{\mu,0}(0) = \pi/2\omega_\mu$. We will now focus on the what happens for $\zeta_{\mu,1}$; the other case is only different in minor details.

Before getting into the estimates, the exponential decay of $\sigma_c(x)$ (Theoren 1.1) implies that solutions of (C.22) will remain bounded, and in fact converge, as $x \to \infty$. This, together with the fact that solutions of systems of differential equations depend continuously on parameters, is enough to conclude that the maps $\mu \mapsto r_{\mu,1} \in L^\infty$ and $\mu \mapsto \phi_{\mu,1} \in L^\infty$ depend continuously on μ. We spare the details.

Now look at the second equation in (C.22). Since $\sigma_c(x)$ is positive we see immediately that $\phi_{\mu,1}(x)$ is an increasing function of x. This and the exponential decay of $\sigma_c(x)$ imply $\phi_{\mu,1}(x)$ will converge to some positive value $\phi^\infty_{\mu,1}$ as $x \to \infty$. The FTOC then gives us the relation

$$\phi^\infty_{\mu,1} = \frac{4}{c^2\mu\omega_\mu^2}\int_{\mathbf{R}_+}\sigma_c(x)\sin^2(\omega_\mu(x+\phi_{\mu,1}(x)))dx. \tag{C.23}$$

Since $\mu\omega_\mu^2 = \mathcal{O}(1)$ we see that the above implies

$$\phi^\infty_{\mu,1} \le \frac{4}{c^2\mu\omega_\mu^2}\int_{\mathbf{R}_+}\sigma_c(x)dx < C_2$$

where $C_2 > 0$ is independent of μ.

We claim that $\phi^\infty_{\mu,1}$ is also bounded below by a positive constant which is independent of μ. Since $\phi_{\mu,1}(x)$ is increasing with $\phi_{\mu,1}(0) = 0$, it follows that $Y(x) := x + \phi_{\mu,1}(x)$ is an invertible map from $\mathbf{R}_+$ to itself and that $Y(x) \ge x$ for all $x \ge 0$. Thus we can make the change of variables $y = Y(x)$ in (C.23) to get

$$\phi^\infty_{\mu,1} = \frac{4}{c^2\mu\omega_\mu^2}\int_{\mathbf{R}_+}\frac{\sigma_c(X(y))}{1+\phi'_{\mu,1}(X(y))}\sin^2(\omega_\mu y)dy \tag{C.24}$$

where $X(y)$ is the inverse of $Y(x)$.

Since $Y(x) \ge x$ when $x \ge 0$ and $Y(0) = 0$ we deduce that $0 \le X(y) \le y$ for all $y \ge 0$. And since σ_c is a decreasing function on $\mathbf{R}_+$ we have $\sigma_c(X(y)) \ge \sigma_c(y)$. Next note that

(C.22) implies $\phi'_{\mu,1}(x) \leq \frac{4}{c^2\mu\omega_\mu^2}\sigma_c(0)$ for all x and thus $(1+\phi'_{\mu,1}(X(y))) \leq 1 + \frac{4}{c^2\mu\omega_\mu^2}\sigma_c(0)$ for all y. Putting these into (C.24) results in

$$\phi^\infty_{\mu,1} \geq \frac{4}{c^2\mu\omega_\mu^2 + 4\sigma_c(0)} \int_{\mathbf{R}_+} \sigma_c(y)\sin^2(\omega_\mu y)dy. \tag{C.25}$$

Since $\omega_\mu \to \infty$ as $\mu \to 0^+$, the Riemann-Lebesgue lemma implies that

$$\lim_{\mu\to 0^+} \int_{\mathbf{R}} \sigma_c(x)\sin^2(\omega_\mu y)dy = \frac{1}{2}\int_{\mathbf{R}} \sigma_c(x)dy.$$

The right hand side above is bounded below by a constant independent of μ. Thus, for $\mu > 0$ small enough we have $\phi^\infty_{\mu,1} > C_1 > 0$. Therefore $0 < C_1 < \phi^\infty_{\mu,1} < C_2$ for all μ sufficiently close to zero, which is (C.20).

Using (C.23) and the fact that $\mu\omega_\mu^2 = \mathcal{O}(1)$ we see that $|\phi_{\mu,j}(x) - \phi^\infty_{\mu,j}| \leq C\int_x^\infty \sigma_c(x)dx$. We know from Theorem 1.1 that $\sigma_c(x) \leq Ce^{-b_c|x|}$. Therefore $|\phi_{\mu,j}(x) - \phi^\infty_{\mu,j}| \leq Ce^{-b_c x}$ for $x > 0$. This is (C.19). To get (C.18) we use the second equation in (C.22) and bootstrap.

Now we need to estimate $r_{\mu,1}(x)$. We have, from the first equation in (C.22), the formula

$$r_{\mu,1}(x) = \exp\left(-\frac{2}{c^2\mu\omega_\mu}\int_0^x \sigma_c(y)\sin(2\omega_\mu(y+\phi_{\mu,1}(y)))dy\right).$$

The exponential decay of $\sigma_c(x)$ implies that $r_{\mu,1}(x)$ converges to a constant, denoted $r^\infty_{\mu,1}$, as $x \to \infty$. But notice that $1/\mu\omega_\mu = \mathcal{O}(\mu^{-1/2})$ and so it not clear that this constant can be controlled in way which is independent of μ. To get estimates for $r_{\mu,1}(x)$ we instead estimate

$$E_1(x) := c^2\mu[\zeta'_{\mu,1}(x)]^2 + (c^2\mu\omega_\mu^2 + 4\sigma_c(x))[\zeta_{\mu,1}(x)]^2. \tag{C.26}$$

The change of variables (C.14) implies that

$$r^2_{\mu,1}(x) = \omega_\mu^{-2}[\zeta'_{\mu,1}(x)] + [\zeta_{\mu,1}(x)]^2. \tag{C.27}$$

Since $\omega_\mu = \mathcal{O}(\mu^{-1/2})$ and $\sigma_c(x)$ is positive and bounded above it straightforward to conclude that are constants $0 < C_1 < C_2$, independent of μ, such that

$$C_1 r_{\mu,1}(x) < \sqrt{E_1(x)} < C_2 r_{\mu,1}(x) \tag{C.28}$$

holds for all x.

Differentiation of $E_1(x)$ with respect to x followed by using (C.11) gives

$$E'_1 = 4\sigma'_c\zeta^2_{\mu,1}. \tag{C.29}$$

The definition of E_1 and the fact that $\mu\omega_\mu^2 = \mathcal{O}(1)$ implies that there is a (μ-independent) constant $C > 0$ such that

$$\zeta^2_{\mu,1}(x) \leq CE_1.$$

This, together with the fact that $\sigma'_c(x) \leq 0$ for $x \geq 0$ implies, by way of (C.29), that

$$C\sigma'_c E_1 \leq E'_1 \leq 0$$

for all $x \geq 0$. Or rather (since $E_1 \geq 0$)

$$C\sigma'_c \leq E'_1/E_1 \leq 0.$$

Integrating the above from 0 to $x \geq 0$ gives (since $E_1(0) = \mu\omega_\mu^2$):

$$C(\sigma_c(x) - \sigma_c(0)) \leq \ln(E_1(x)/\mu\omega_\mu^2) \leq 0.$$

Thus

$$\mu\omega_\mu^2 e^{C(\sigma_c(x)-\sigma_c(0))} \leq E_1(x) \leq \mu\omega_\mu^2.$$

Since $\mu\omega_\mu^2 = \mathcal{O}(1)$ this implies that, for all $x \in \mathbf{R}_+$, we have constants $0 < C_1 < C_2$ (independent of μ) such that $C_1 < E_1(x) < C_2$. Thus we have $0 < C_1 < r_{\mu,1}(x) < C_2$. This is (C.16).

Using this, the estimate (C.18) and the first equation in (C.22) gives (C.15) by way of bootstrapping. Note that (C.18) and (C.15) yield (C.21).

Now we prove (C.17). If we integrate (C.29) on $\mathbf{R}_+$ we find that $E_1(x)$ converges as $x \to \infty$ (since $\sigma_c'(x)$ decays exponentially) to some limit E_1^∞. And so we have, using the estimate for σ_c in Theorem 1.1,

$$|E_1(x) - E_1^\infty| \leq C \int_x^\infty |\sigma_c'(y)|\, dy \leq Ce^{-b_c x} \tag{C.30}$$

when $x \geq 0$. Next notice that the equation (C.26) and (C.27) tell us that

$$E_1(x) - c^2\mu\omega_\mu^2 r_{\mu,1}^2(x) = 4\sigma_c(x)[\zeta_{\mu,1}(x)]^2.$$

Using (C.21) and the exponential decay of $\sigma_c(x)$ here tell us that

$$|E_1(x) - c^2\mu\omega_\mu^2 r_{\mu,1}^2(x)| \leq Ce^{-b_c x} \tag{C.31}$$

for all $x \geq 0$. This implies

$$E_1^\infty = c^2\mu\omega_\mu^2[r_{\mu,1}^\infty]^2. \tag{C.32}$$

Using (C.16) give

$$|r_{\mu,1}(x) - r_{\mu,1}^\infty| = |r_{\mu,1}(x) + r_{\mu,1}^\infty|^{-1}|r_{\mu,1}^2(x) - [r_{\mu,1}^\infty]^2| \leq C|r_{\mu,1}^2(x) - [r_{\mu,1}^\infty]^2|.$$

Then we use (C.32) and the fact that $\mu\omega_\mu^2 = \mathcal{O}(1)$ to get

$$|r_{\mu,1}(x) - r_{\mu,1}^\infty| \leq C|c^2\mu\omega_\mu^2 r_{\mu,1}^2(x) - E_1^\infty|.$$

The triangle inequality give

$$|r_{\mu,1}(x) - r_{\mu,1}^\infty| \leq C|c^2\mu\omega_\mu^2 r_{\mu,1}^2(x) - E_1(x)| + C|E_1(x) - E_1^\infty|.$$

Then (C.30) and (C.31) give $|r_{\mu,1}(x) - r_{\mu,1}^\infty| < Ce^{-b_c x}$ when $x \geq 0$. This is (C.17).

□

C.3. **Jost solutions of $\mathcal{L}_\mu^*$.** In this subsection we show the existence of a nontrivial, odd, smooth bounded function $\gamma_\mu(x)$ for which $\mathcal{L}_\mu^*\gamma_\mu = 0$. It is this function γ_μ which is described in Lemmas 7.2 and 7.3. By $\mathcal{L}_\mu^*$ we mean the $L^2 \times L^2$ adjoint of $\mathcal{L}_\mu$, specifically[18]

$$\mathcal{L}_\mu^* f = c^2\mu f'' + 2(1+\mu A^2 + \mu^2\tau_\mu)f + \Sigma_{\mu,2}^* f.$$

If we can find such a γ_μ, then for any $f \in O_b^{s+2}$ (with $b>0$) the adjoint property tells us that

$$\mathcal{L}_\mu f = g \implies \int_{\mathbf{R}} \gamma_\mu(x) g(x) dx = 0.$$

This is the first conclusion in Lemma 7.2. It turns out that the function γ_μ we seek is an $\mathcal{O}(\mu^{1/4})$ perturbation[19] of $\zeta_{\mu,1}$ (from the previous subsection) in the L^∞ norm.

C.3.1. *Decomposition of $\mathcal{L}_\mu$.* To show this we first decompose $\mathcal{L}_\mu^*$ as

$$\mathcal{L}_\mu^* = \mathcal{S}_\mu + \mu\Delta_\mu + \mu K_\mu^*$$

where

$$K_\mu^* := \mu^{-1}(\Sigma_{\mu,2}^* - \Sigma_{0,2}^*) \quad \text{and} \quad \mu\Delta_\mu := 2(1+\mu A^2 + \mu^2\tau_\mu) - c^2\mu\omega_\mu^2. \tag{C.33}$$

In the above we used that fact that $\Sigma_{0,2}^* f = 4\sigma_c(x) f$.

The line of reasoning that led to the estimates for $\Sigma_{\mu,2}$ at (6.6) can be repeated to show that

$$\|K_\mu^* f\|_{W_{b+b_c}^{k,p}} \le C\|f\|_{W_b^{k,p}}. \tag{C.34}$$

Note that this estimate implies that K_μ^* is a localizing operator.

Also, the definition of τ_μ at (6.15) and the estimate (2.14) tell us

$$\|\Delta_\mu f\|_{W_b^{k,p}} \le C\|f\|_{W_b^{k,p}}. \tag{C.35}$$

Moreover, the definitions of τ_μ and ω_μ (in (5.7)) imply that

$$\Delta_\mu \cos(\omega_\mu x) = \Delta_\mu \sin(\omega_\mu x) = 0. \tag{C.36}$$

If $\mathcal{L}_\mu^*\gamma_\mu = 0$ then clearly $\mathcal{S}_\mu\gamma_\mu = -\mu\Delta_\mu\gamma_\mu - \mu K_\mu^*\gamma_\mu$. Since γ_μ is odd we may normalize it so that $\gamma_\mu'(0) = \omega_\mu$. In which case the variation of parameters formula tells us that

$$\begin{aligned} \gamma_\mu(x) - \zeta_{\mu,1}(x) = &- \mu\omega_\mu\zeta_{\mu,1}(x)\int_0^x (\Delta_\mu\gamma_\mu(y) + K_\mu^*\gamma_\mu(y))\zeta_{\mu,0}(y)dy \\ &+ \mu\omega_\mu\zeta_{\mu,0}(x)\int_0^x (\Delta_\mu\gamma_\mu(y) + K_\mu^*\gamma_\mu(y))\zeta_{\mu,1}(y)dy. \end{aligned} \tag{C.37}$$

[18] Here is the formula for $\Sigma_{\mu,2}^*$:

$$\Sigma_{\mu,2}^* f = 2T_\mu^* Q_0(T_\mu \boldsymbol{\sigma}_{\mu,c}, (T_\mu^*)^{-1} L_\mu^*(f\mathbf{e}_2)) \cdot \mathbf{e}_2.$$

The $L^2 \times L^2$ adjoints of T_μ and L_μ are easily computed from their definitions in Section 1 and the observation that A is symmetric and δ is anti-symmetric with respect to the L^2 inner-product.

[19] In fact, it is an $\mathcal{O}(\mu^{1/2}|\ln(\mu)|)$ perturbation, but getting this sharper bound is more than we need here and including it would lengthen this already lengthy appendix.

So for functions $f(x)$ put

$$\begin{aligned} V_\mu f(x) := &- \mu\omega_\mu \zeta_{\mu,1}(x) \int_0^x (\Delta_\mu f(y) + K_\mu^* f(y))\zeta_{\mu,0}(y)dy \\ &+ \mu\omega_\mu \zeta_{\mu,0}(x) \int_0^x (\Delta_\mu f(y) + K_\mu^* f(y))\zeta_{\mu,1}(y)dy. \end{aligned} \tag{C.38}$$

Letting $u_\mu := \gamma_\mu - \zeta_{\mu,1}$ we see that (C.37) is equivalent to

$$(1 - V_\mu)u_\mu = V_\mu \zeta_{\mu,1}. \tag{C.39}$$

Our goal will be to show that the operator norm of V_μ is less than one for μ small enough. Thus by the Neumann series $(1 - V_\mu)$ is invertible and we can solve (C.39).

C.3.2. *Asymptotically sinusoidal functions.* However, we will not work in L^∞ but rather in

$$AS_b^s := (W_b^{s,\infty} \cap \{\text{odds}\}) \oplus \text{span}\{\sin(\omega_\mu x)\} \oplus \text{span}\{i_1(x)\cos(\omega_\mu x)\}. \tag{C.40}$$

In the above, $i_1(x)$ is a smooth, odd, non-decreasing function such that

$$i_1(x) = \begin{cases} 0 & \text{when } 0 \le x \le 1/2 \\ 1 & \text{when } x \ge 2. \end{cases} \tag{C.41}$$

The functions in AS_b^s are odd functions which are asymptotically sinusoidal, hence the name of the space. It is obvious that AS_b^s is a subspace of L^∞. So long as $b > 0$, AS_b^s is the direct sum of three subspaces of L^∞ whose intersections are trivial. Thus, if $f \in AS_b^s$ there exist unique $\ell_f \in W_b^{s,\infty} \cap \{\text{odds}\}$ and $\alpha_f, \beta_f \in \mathbf{R}$ such that

$$f(x) = \ell_f(x) + \alpha_f \sin(\omega_\mu x) + \beta_f i_1(x)\cos(\omega_\mu x). \tag{C.42}$$

Moreover, AS_b^s is a Banach space with norm:

$$\|f\|_{AS_b^s} := \|\ell_f\|_{W_b^{s,\infty}} + \omega_\mu^s \sqrt{\alpha_f^2 + \beta_f^2}.$$

The ω_μ^s factor is there to capture the oscillatory part's derivative, though is, strictly speaking, not really needed.

We need a few estimates. First we have the easy embedding estimate

$$\|f\|_{W^{s,\infty}} \le C\|f\|_{AS_b^s}. \tag{C.43}$$

Then we have the "ideal" estimate which states that if $f \in AS_b^s$ and $g \in W_b^{s,\infty}$ then $fg \in W_b^{s,\infty}$ with the estimate

$$\|fg\|_{W_b^{s,\infty}} \le C\|f\|_{AS_b^s}\|g\|_{W_b^{s,\infty}}. \tag{C.44}$$

Here is why:

$$f(x)g(x) = g(x)\ell_f(x) + g(x)\left(\alpha_f \sin(\omega_\mu x) + \beta_f i_1(x)\cos(\omega_\mu x)\right).$$

Both terms on the right hand side are in $W_b^{s,\infty}$ and the rest is bookkeeping.

C.3.3. *The size of* $\zeta_{\mu,1}$. The Jost solution $\zeta_{\mu,1}$ is, unsurprisingly, in AS^s_b. In particular we have this estimate for all $b \in [0, b_c]$:

$$\|\zeta_{\mu,1}\|_{AS^s_b} \leq C\mu^{-s/2}\mu^{-b/2b_c}. \tag{C.45}$$

Note that the larger b is, the larger the right hand side is. This is because it takes a while for $\zeta_{\mu,1}$ to settle into its asymptotic state, and thus the exponential weight takes a greater toll. Here is the computation.

Adding a lot of zeroes to $\zeta_{\mu,1}(x)$ gives us

$$\zeta_{\mu,1}(x) = \ell_1 + \ell_2 + \ell_3 + \ell_3 + \alpha_{\zeta_{\mu,1}} \sin(\omega_\mu x) + \beta_{\zeta_{\mu,1}} i_1(x) \cos(\omega_\mu x) \tag{C.46}$$

where

$$\begin{aligned} \ell_1 :=& (r_{\mu,1}(x) - r^\infty_{\mu,1}) \sin(\omega_\mu(x + \phi_{\mu,1}(x))) \\ \ell_2 :=& r^\infty_{\mu,1} \sin(\omega_\mu x)[\cos(\omega_\mu \phi_{\mu,1}(x)) - \cos(\omega_\mu \phi^\infty_{\mu,1})] \\ \ell_3 :=& r^\infty_{\mu,1} \cos(\omega_\mu x)(1 - |i_1(x)|) \sin(\omega_\mu \phi_{\mu,1}(x)) \\ \ell_4 :=& r^\infty_{\mu,1} \cos(\omega_\mu x) i_1(x)[\sin(\omega_\mu \phi_{\mu,1}(|x|)) - \sin(\omega_\mu \phi^\infty_\mu)] \\ \alpha_{\zeta_{\mu,1}} :=& r^\infty_{\mu,1} \cos(\omega_\mu \phi^\infty_{\mu,1}) \\ \beta_{\zeta_{\mu,1}} :=& r^\infty_{\mu,1} \sin(\omega_\mu \phi^\infty_{\mu,1}). \end{aligned} \tag{C.47}$$

In the above we have used the polar decomposition (C.14), the addition of angles formula and the fact that if $o(x)$ is odd then $o(x) = \operatorname{sgn}(x) o(|x|)$.

The functions ℓ_1, ℓ_2, ℓ_3 and ℓ_4 are in $W^{s,\infty}_{b_c}$, as we show in a moment. If so then we have

$$\|\zeta_{\mu,1}\|_{AS^s_b} \leq \|\ell_1\|_{W^{s,\infty}_b} + \|\ell_2\|_{W^{s,\infty}_b} + \|\ell_3\|_{W^{s,\infty}_b} + \|\ell_4\|_{W^{s,\infty}_b} + \omega^s_\mu r^\infty_{\mu,1}$$

We have used the fact that $\sqrt{\alpha^2_{\zeta_{\mu,1}} + \beta^2_{\zeta_{\mu,1}}} = r^\infty_{\mu,1}$.

The estimates in (C.15) and (C.17) imply that $\|\ell_1\|_{W^{s,\infty}_b} \leq C\mu^{-s/2}$ when $b \in [0, b_c]$. Next, $|i_1(x)| - 1$ is compactly supported and smooth. This, with (C.18), give us $\|\ell_3\|_{W^{s,\infty}_b} \leq C\mu^{-s/2}$ for any $b \in [0, b_c]$.

Now we address ℓ_4; to estimate ℓ_2 is similar. First notice that

$$\|\ell_4\|_{L^\infty} \leq C \tag{C.48}$$

since $|r^\infty_{\mu,1}| = \mathcal{O}(1)$. Next, since $\sin(x)$ is globally Lipschitz, we have

$$|\ell_4(x)| \leq |r^\infty_{\mu,1} \omega_\mu||\phi_{\mu,1}(|x|) - \phi^\infty_{\mu,1}|.$$

Since $\omega_\mu = \mathcal{O}(\mu^{-1/2})$ and $|r^\infty_{\mu,1}| = \mathcal{O}(1)$ we have

$$|\ell_4(x)| \leq C\mu^{-1/2}|\phi_{\mu,1}(x) - \phi^\infty_{\mu,1}|.$$

Then (C.19) tells us that

$$\|\ell_4\|_{L^\infty_{b_c}} \leq C\mu^{-1/2}. \tag{C.49}$$

Interpolating between this estimate and (C.48) implies that

$$\|\ell_4\|_{L^\infty_b} \leq C\mu^{-b/2b_c}. \tag{C.50}$$

when $b \in [0, b_c]$. The same sort of reasoning gets us

$$\|\ell_2\|_{W_b^{s,\infty}} + \|\ell_4\|_{W_b^{s,\infty}} \leq C\mu^{-s/2}\mu^{-b/2b_c}. \tag{C.51}$$

Putting all of these estimates together gives (C.45). A parallel argument shows we have the same sort of estimate for $\zeta_{\mu,0}(x)$.

C.3.4. *Localization.* Next we claim that both Δ_μ and K_μ^* map $AS_b^s \to W_b^{s,\infty}$. Which is to say that these operators localize asymptotically sinusoidal functions. Specifically we have the estimates

$$\|\Delta_\mu f\|_{W_b^{s,\infty}} \leq C\mu^{-s/2}\|f\|_{AS_b^s} \quad \text{and} \quad \|K_\mu^* f\|_{W_b^{s,\infty}} \leq C\mu^{-s/2}\|f\|_{AS_b^s}. \tag{C.52}$$

The estimate for K_μ^* is just a direct consequence of (C.34) and the details are uninteresting.

The estimate for Δ_μ follows from (C.36). Since $\Delta_\mu \cos(\omega_\mu x) = 0$ we have

$$\Delta_\mu(i_1(x)\cos(\omega_\mu x)) = \Delta_\mu(i_1(x)\cos(\omega_\mu x)) - i_1(x)\Delta_\mu(\cos(\omega_\mu x)).$$

Using the definition of Δ_μ at (C.33) converts this to

$$\Delta_\mu(i_1(x)\cos(\omega_\mu x)) = 2A^2(i_1(x)\cos(\omega_\mu x)) - 2i_1(x)A^2(\cos(\omega_\mu x)).$$

Applying the definition of A to this gives

$$\begin{aligned}\Delta_\mu(i_1(x)\cos(\omega_\mu x)) =& \frac{1}{2}(i_1(x+2) - i_1(x))\cos(\omega_\mu(x+2)) \\ &+ \frac{1}{2}(i_1(x-2) - i_1(x))\cos(\omega_\mu(x-2)).\end{aligned}$$

One can check from the definition that $(i_1(x+2) - i_1(x))$ has support in $[-5, 5]$ as does $(i_1(x-2) - i_1(x))$. In L^∞ these functions are no bigger than one. Which is to say that $\Delta_\mu(i_\mu(x)\cos(\omega_\mu(x))$ is compactly supported with $\mathcal{O}(1)$ magnitude. Thus

$$\|\Delta_\mu(i_1(\cdot)\cos(\omega_\mu \cdot))\|_{W_b^{s,\infty}} \leq C\mu^{-s/2}.$$

We also know from (C.35) that $\|\Delta_\mu \ell\|_{W_b^{s,\infty}} \leq \|\ell\|_{W_b^{s,\infty}}$ and from (C.36) that $\Delta_\mu \sin(\omega_\mu x) = 0$. Thus if $f \in AS_b^s$ we have, from (C.42),

$$\Delta_\mu f = \Delta_\mu \ell_f + \beta_f \Delta_\mu(i_1(x)\cos(\omega_\mu x))$$

and the estimate for Δ_μ in (C.52) follows in the obvious way from the preceding estimates.

C.3.5. V_μ *is small.* Now let us estimate $V_\mu f$. Assume $f \in AS^0_{b_c/4}$; we need to estimate $V_\mu f$ in this same space. We show how to estimate the first term in (C.38),

$$V_\mu^1 f(x) := -\mu\omega_\mu\zeta_{\mu,1}(x)\int_0^x (\Delta_\mu f(y) + K_\mu^* f(y))\zeta_{\mu,0}(y)dy.$$

The second term, denoted $V_\mu^2 f(x)$, is no different.

From calculus, we have

$$V_\mu^1 f(x) = -\mu\omega_\mu\zeta_{\mu,1}(x)Z_1^\infty + \mu\omega_\mu\zeta_{\mu,1}(x)Z_1(x) \tag{C.53}$$

where

$$Z_1^\infty = \int_0^\infty (\Delta_\mu f(y) + K_\mu^* f(y))\zeta_{\mu,0}(y)dy \tag{C.54}$$

$$\text{and} \quad Z_1(x) := \int_x^\infty (\Delta_\mu f(y) + K_\mu^* f(y))\zeta_{\mu,0}(y)dy.$$

Recalling from (2.11) that $\|f\|_{L^1} \le C_b \|f\|_{L_b^\infty}$, the localizing property (C.52) tells us

$$|Z_1^\infty| \le \|\Delta_\mu f\|_{L^1} + \|K_\mu^* f\|_{L^1} \le C\|\Delta_\mu f\|_{L^\infty_{b_c/4}} + C\|K_\mu^* f\|_{L^\infty_{b_c/4}} \le C\|f\|_{AS^0_{b_c/4}}.$$

And so using $\omega_\mu = \mathcal{O}(\mu^{-1/2})$ gives us

$$\|\mu\omega_\mu\zeta_{\mu,1}Z_1^\infty\|_{AS^0_{b_c/4}} \le C\mu^{1/2}\|\zeta_{\mu,1}\|_{AS^0_{b_c/4}}|Z_1^\infty| \le C\mu^{1/2}\|\zeta_{\mu,1}\|_{AS^0_{b_c/4}}\|f\|_{AS^0_{b_c/4}}.$$

Then we use (C.45) with $b = b_c/4$ to get

$$\|\mu\omega_\mu\zeta_{\mu,1}Z_1^\infty\|_{AS_{b_c/4}} \le C\mu^{3/8}\|f\|_{AS_{b_c/4}}.$$

Thus the first term in (C.53) is estimated.

Next, if $g \in L_b^\infty$ then one has

$$\int_0^\infty e^{bx}\left|\int_x^\infty g(y)dy\right|dx \le \int_0^\infty e^{bx}\int_x^\infty e^{-by}\|g\|_{L_b^\infty}dydx \le C_b\|g\|_{L_b^\infty}.$$

Using this, together with the boundedness of $\zeta_{\mu,0}$ from (C.18), gives

$$\|Z_1\|_{L^\infty_{b_c/4}} \le C\|\Delta_\mu f\|_{L^\infty_{b_c/4}} + \|K_\mu^* f\|_{L^\infty_{b_c/4}}.$$

And so using (C.44) and $\omega_\mu = \mathcal{O}(\mu^{-1/2})$ gives

$$\|\mu\omega_\mu\zeta_{\mu,1}Z_1\|_{L^\infty_{b_c/4}} \le C\mu^{1/2}\|\zeta_{\mu,1}\|_{AS^0_{b_c/4}}\left(\|\Delta_\mu f\|_{L^\infty_{b_c/4}} + \|K_\mu^* f\|_{L^\infty_{b_c/4}}\right).$$

Using (C.52) and (C.45) with $b = b_c/4$ converts the above to

$$\|\mu\omega_\mu\zeta_{\mu,1}Z_1\|_{L^\infty_{b_c/4}} \le C\mu^{3/8}\|f\|_{AS^0_{b_c/4}}.$$

Thus we have an estimate for the second term in (C.53).

We have therefore shown that $\|V_\mu^1 f\|_{AS^0_{b_c/4}} \le C\mu^{3/8}\|f\|_{AS_{b_c/4}}$. The other term in V_μ is handled in like fashion and we have

$$\|V_\mu f\|_{AS^0_{b_c/4}} \le C\mu^{3/8}\|f\|_{AS^0_{b_c/4}} \tag{C.55}$$

C.3.6. *Inversion and estimates.* For μ small enough, $(1 - V_\mu)$ is invertible on $AS^0_{b_c/4}$ by the Neumann series. And so we have a solution of (C.39)

$$u_\mu = (1 - V_\mu)^{-1}V_\mu\zeta_{\mu,1}.$$

which has the following estimates (from (C.45)):

$$\|u_\mu\|_{AS^0_{b_c/4}} \le C\mu^{3/8}\|\zeta_{\mu,1}\|_{AS^0_{b_c/4}} \le C\mu^{1/4}. \tag{C.56}$$

Also, (C.43) gives

$$\|u_\mu\|_{L^\infty} \le \|u_\mu\|_{AS^0_{b_c/4}} \le C\mu^{1/4}.$$

It is easy enough to conclude that u_μ is a smooth function of x by a bootstrapping argument and in this way get (7.7). This, however, does not imply that $u_\mu \in AS^s_{b_c/4}$ for all $s \geq 0$.[20] So, to get (7.5) in Lemma 7.3, we need to show that $u_\mu \in AS^1_{b_c/4}$.

The FTOC shows that

$$\frac{d}{dx} V_\mu f(x) = \widetilde{V}_\mu f(x)$$

where

$$\begin{aligned}\widetilde{V}_\mu f(x) := & -\mu\omega_\mu \zeta'_{\mu,1}(x) \int_0^x (\Delta_\mu f(y) + K^*_\mu f(y))\zeta_{\mu,0}(y) dy \\ & + \mu\omega_\mu \zeta'_{\mu,0}(x) \int_0^x (\Delta_\mu f(y) + K^*_\mu f(y))\zeta_{\mu,1}(y) dy.\end{aligned}$$

This operator is nearly identical to V_μ, the only difference being that the prefactor functions $\zeta_{\mu,j}(x)$ have been differentiated. Repetition of the the same steps that got us (C.55) gets us

$$\|\widetilde{V}_\mu f\|_{ES^0_{b_c/4}} \leq C\mu^{-1/8} \|f\|_{AS^0_{b_c/4}}. \tag{C.57}$$

Here, ES^s_b is analogous to AS^s_b except it consists of asymptotically sinusoidal *even* functions. We forgo the specifics. Unsurprisingly, if $u \in AS^1_b$ then $u' \in ES^0_b$.

We know that $(1 - V_\mu)u_\mu = V_\mu \zeta_{\mu,1}$ and so

$$u'_\mu = \widetilde{V}_\mu u_\mu + \widetilde{V}_\mu \zeta_{\mu,1}.$$

The estimate for $\widetilde{V}_\mu$ then tells us that $\|u'_\mu\|_{ES^0_{b_c/4}} \leq C\mu^{-1/8} \left(\|u_\mu\|_{AS^0_{b_c/4}} + \|\zeta_{\mu,1}\|_{AS^0_{b_c/4}} \right) \leq C.$

So we know now that $\gamma_\mu = \zeta_{\mu,1} + u_\mu \in AS^1_{b_c/4}$. The estimate in (7.7) follows from those for $\zeta_{\mu,1}$, u_μ and (C.43). Also, we know there exists $\ell_{\gamma_\mu} \in W^{1,\infty}_{b_c/4}$ and constants $\alpha_{\gamma_\mu}, \beta_{\gamma_\mu}$ such that

$$\gamma_\mu(x) = \ell_{\gamma_\mu}(x) + \alpha_{\gamma_\mu} \sin(\omega_\mu x) + \beta_{\gamma_\mu} i_1(x) \cos(\omega_\mu x). \tag{C.58}$$

Which means that

$$\begin{aligned}&\lim_{x\to\infty} |\gamma_\mu(x) - \alpha_{\gamma_\mu} \sin(\omega_\mu x) - \beta_{\gamma_\mu} \cos(\omega_\mu x)| \\ =&\lim_{x\to\infty} |\gamma'_\mu(x) - \omega_\mu \alpha_{\gamma_\mu} \cos(\omega_\mu x) - \omega_\mu \beta_{\gamma_\mu} \sin(\omega_\mu x)| \\ =&0.\end{aligned} \tag{C.59}$$

Furthermore, because of (C.56) and (C.46), we have

$$\sqrt{(\alpha_{\gamma_\mu} - \alpha_{\zeta_{\mu,1}})^2 + (\beta_{\gamma_\mu} - \beta_{\zeta_{\mu,1}})^2} \leq C\mu^{1/4}. \tag{C.60}$$

We know, from (C.17) and (C.19), that $\zeta_{\mu,1}(x)$ converges as $x \to \infty$ to $r^\infty_{\mu,1} \sin(\omega_\mu(x + \phi^\infty_{\mu,1}))$ with $r^\infty_{\mu,1}$ and $\phi^\infty_{\mu,1}$ both $\mathcal{O}(1)$. With this, an exercise in trigonometry shows that

$$\alpha_{\gamma_\mu} \sin(\omega_\mu x) + \beta_{\gamma_\mu} \cos(\omega_\mu x) = \varrho^\infty_\mu \sin(\omega_\mu(x + \vartheta^\infty_\mu))$$

for some constants ϱ^∞_μ and ϑ^∞_μ which satisfy:

$$|\varrho^\infty_\mu - r^\infty_{\mu,1}| \leq C\mu^{1/4}$$

[20]For instance $\sin(\cosh(x))\,\mathrm{sech}(x) \in AS^0_1$ and is smooth, but it is not in AS^1_1.

and

$$|\vartheta_\mu^\infty - \phi_{\mu,1}^\infty| \le C\mu^{3/4}. \tag{C.61}$$

Since $r_{\mu,1}^\infty$ and $\phi_{\mu,1}^\infty$ are $\mathcal{O}(1)$, we see that ϱ_μ^∞ and $\vartheta_{\mu,1}^\infty$ are likewise $\mathcal{O}(1)$. Thus we can renormalize γ_μ so that $\varrho_\mu^\infty = 1$ exactly and not change anything of substance. In this way, (C.59) gives us the map ϑ_μ^∞ described in Lemma 7.3 and the estimates (7.4) and (7.5).

C.4. **The set M_c.** Now we establish the existence of the set M_c described in Lemma 7.3. Let

$$\widetilde{M}_c := \left\{\mu \in (0, \mu_\zeta) : \sin(\omega_\mu \phi_{\mu,1}^\infty) > 3/4\right\}.$$

First, ω_μ and $\phi_{\mu,1}^\infty$ are continuous functions of μ and thus $\sin(\omega_\mu \phi_{\mu,1}^\infty)$ is likewise continuous. Since $\widetilde{M}_c$ is the preimage of an open set it is open. Moreover, we know from (C.20) that $\phi_{\mu,1}^\infty = \mathcal{O}(1)$ and from (5.8) that $\omega_\mu = \mathcal{O}(\mu^{-1/2})$. Thus $\omega_\mu \phi_{\mu,1}^\infty = \mathcal{O}(\mu^{-1/2})$ which means that it diverges to ∞ as $\mu \to 0^+$. This, with continuity, imply that there exists $n_0 \ge 0$ and a sequence $\{\mu_n\}_{n \ge n_0} \subset \mathbf{R}_+$ with $\lim\limits_{n\to\infty} \mu_n = 0$ for which $\omega_{\mu_n} \phi_{\mu_n,1}^\infty = (2n+1)\pi/2$. Thus we have $\sin(\omega_{\mu_n} \phi_{\mu_n,1}^\infty) = 1$ for all n and in this way we see that $0 \in \overline{\widetilde{M}}_c$.

Now, take $\mu \in \widetilde{M}_c$. By the addition of angles formula we have

$$\begin{aligned} &\sin(\omega_\mu \vartheta_\mu^\infty) - \sin(\omega_\mu \phi_{\mu,1}^\infty) \\ &\quad = \sin(\omega_\mu \phi_{\mu,1}^\infty)\left(\cos\left(\omega_\mu\left(\vartheta_\mu^\infty - \phi_{\mu,1}^\infty\right)\right) - 1\right) + \cos(\omega_\mu \phi_{\mu,1}^\infty)\sin\left(\omega_\mu\left(\vartheta_\mu^\infty - \phi_{\mu,1}^\infty\right)\right). \end{aligned} \tag{C.62}$$

From (C.61) we see that $|\omega_\mu\left(\vartheta_\mu^\infty - \phi_{\mu,1}^\infty\right)| \le C\mu^{1/4}$ and thus, for $\mu > 0$ small enough (call the threshold μ_ϑ) we have

$$|\sin\left(\omega_\mu\left(\vartheta_\mu^\infty - \phi_{\mu,1}^\infty\right)\right)| + |\cos\left(\omega_\mu\left(\vartheta_\mu^\infty - \phi_{\mu,1}^\infty\right)\right) - 1| \le 1/4.$$

This, with (C.62) give

$$|\sin(\omega_\mu \vartheta_\mu^\infty) - \sin(\omega_\mu \phi_{\mu,1}^\infty)| \le 1/4.$$

Since $\mu \in \widetilde{M}_c$ we know that $\sin(\omega_\mu \phi_{\mu,1}^\infty) > 3/4$ and therefore if $\mu \in (0, \mu_\vartheta)$ the triangle inequality give

$$\sin(\omega_\mu \vartheta_\mu^\infty) > 1/2.$$

Thus if we put $M_c := \widetilde{M}_c \cap (0, \mu_\vartheta)$ we have (7.6).

C.5. **The oscillatory integral estimate.** As for (7.8), we want to estimate:

$$\iota_\mu[g] := \int_{\mathbf{R}} \gamma_\mu(x) g(x) dx.$$

Since $\mathcal{L}_\mu^* \gamma_\mu = 0$ we can rearrange terms to find

$$\gamma_\mu(x) = \mu \mathcal{R}_\mu \gamma_\mu.$$

where

$$\mathcal{R}_\mu f(x) := -\frac{1}{2 + 4\sigma_c(x)}\left(c^2 f''(x) + A^2 f(x) + \mu\tau_\mu f(x) + K_\mu^* f(x)\right).$$

Our previous estimates tell us that $\|\mathcal{R}f\|_{s,b} \le C\|f\|_{s+2,b}$ and $\|\mathcal{R}^* f\|_{s,b} \le C\|f\|_{s+2,b}$.

Thus we have

$$\iota_\mu[g] = \mu \int_{\mathbf{R}} (\mathcal{R}_\mu \gamma_\mu(x)) g(x) dx = \mu \int_{\mathbf{R}} \gamma_\mu(x) \mathcal{R}^* g(x) dx.$$

Repeated replacement of γ_μ with $\mu \mathcal{R}_\mu \gamma_\mu$ then gives us, for any positive integer n:

$$\iota_\mu[g] := \mu^n \int_{\mathbf{R}} \gamma_\mu(x) (\mathcal{R}^*)^n g(x) dx.$$

This leads to

$$|\iota_\mu[g]| \leq \mu^n \|\gamma_\mu\|_{L^\infty} \|(\mathcal{R}^*)^n g\|_{L^1} \leq C \|(\mathcal{R}^*)^n g\|_{L^1}$$

Then we use the estimate $\|f\|_{L^1} \leq C b^{-1/2} \|f\|_{0,b}$ to get

$$|\iota_\mu[g]| \leq C b^{-1/2} \mu^n \|(\mathcal{R}^*)^n g\|_{0,b} \leq C b^{-1/2} \mu^n \|g\|_{2n,b}.$$

This is the estimate (7.8) for $s = 2n$ and interpolation estimates can be used to get it for odd values of s.

C.6. **Sufficiency.** At this stage we have proven everything in Lemmas 7.1, 7.2 and 7.3 with the exception of the sufficient condition (7.3) in Lemma 7.2. The existence of the bounded function γ_μ for which $\mathcal{L}^*_\mu \gamma_\mu = 0$ implies that codimension of the range of $\mathcal{L}_\mu$ (when viewed as an unbounded map on O^s_b with $b > 0$) is at least one. In this section we prove that the codimension of the range is exactly one, which then implies (7.3).

Recollecting the defintion of $\mathcal{L}_\mu$ at (6.18), we let $\mathcal{L}^0_\mu := c^2 \mu \partial_x^2 + 2(1 + \mu A^2 + \mu^2 \tau_\mu)$ so that we have $\mathcal{L}_\mu = \mathcal{L}^0_\mu + \Sigma_{\mu,2}$. We are viewing both $\mathcal{L}^0_\mu$ and $\mathcal{L}_\mu$ as unbounded operators on O^s_b with domain O^{s+2}_b. We have the compact embedding $O^{s'}_{b'} \subset\subset O^s_b$ when $s' > s$ and $b' > b$. Thus, the localizing property of $\Sigma_{\mu,2}$ described in (6.6) implies that $\Sigma_{\mu,2}$ is compact relative to $\mathcal{L}^0_\mu$. This in turn implies that the Fredholm index[21] of $\mathcal{L}_\mu$ and $\mathcal{L}^0_\mu$ coincide.

The Fredholm index of $\mathcal{L}^0_\mu$ can be computed relatively easily using Fourier methods. We have

$$\mathcal{L}^0_\mu e^{i\omega x} = \underbrace{[-c^2 \mu \omega^2 + 2(1 + \mu \cos^2(\omega) + \mu \tau^2_\mu)]}_{\widetilde{\mathcal{L}}^0_\mu(\omega)} e^{i\omega x}.$$

We can view $\mathcal{L}^0_\mu$ as a Fourier multiplier operator with symbol $\widetilde{\mathcal{L}}^0_\mu(\omega)$. The definition of τ_μ at (6.15) implies that $\widetilde{\mathcal{L}}^0_\mu(\omega) = 0$ if and only $\omega = \pm\omega_\mu$. Lemma 3 in Beale's article [2] tells us how to convert such information into necessary and sufficient conditions for solving the equation $\mathcal{L}^0_\mu f = g$ when f and g are in exponentially weighted spaces like ours. The outcome is that, so long as $b > 0$:

$$\mathcal{L}^0_\mu f = g \in O^s_b \iff \mathfrak{F}[g](\omega_c) = 0. \tag{C.63}$$

This condition tells us that the range of $\mathcal{L}^0_\mu$ is a codimension one subspace of O^s_b. Moreover Beale's lemma implies that $\mathcal{L}^0_\mu$ is injective on O^s_b. Thus the Fredholm index of $\mathcal{L}^0_\mu$ is -1. And so we have proven:

Lemma C.2. *For all $s \geq 0$ and $b > 0$, the Fredholm index of $\mathcal{L}_\mu$, viewed as unbounded operator on O^s_b, is equal to -1.*

[21] We use the defintion in [17] for the Fredholm index of an unbounded operator.

Finally, the coercive estimates in Lemma 7.1 imply that the kernel of $\mathcal{L}_\mu$ is trivial. And since the Fredholm index is the difference of the dimensions of the kernel and the codimension of the range. (C.2) implies that the codimension of the range of $\mathcal{L}_\mu$ is exactly one. And we can move on.

Appendix D. Proof of Lemma 7.4—the computation of κ_μ

By definition

$$\kappa_\mu := \int_{\mathbf{R}} \gamma_\mu(x)\chi_\mu(x)dx = \int_{\mathbf{R}} \gamma_\mu(x)\left(\Sigma_{\mu,2}\varphi^0_{\mu,2}(x) + \mu^2\Omega_{\mu,2}\varphi^0_{\mu,1}(x)\right)dx.$$

Using the localizing properties of $\Omega_{\mu,2}$ and $\Sigma_{\mu,2}$ we know the integral converges even though γ_μ and $\mathbf{p}^0_\mu$ are asymptotically periodic. Moreover, since $\Omega_{\mu,2}$ is a bounded map, we have $|\kappa_\mu - \widetilde{\kappa}_\mu| \le C\mu^2$ where $\widetilde{\kappa}_\mu$ is just the part of κ_μ involving $\Sigma_{\mu,2}$. Using the adjoint property, together with the fact that $\varphi^0_{\mu,2} = \sin(\omega_\mu x)$, gives

$$\widetilde{\kappa}_\mu = \int_{\mathbf{R}} (\Sigma^*_{\mu,2}\gamma_\mu(x))\sin(\omega_\mu x)dx.$$

Since $\mathcal{L}^*_\mu\gamma_\mu = 0$ we can make the substitution

$$\widetilde{\kappa}_\mu = -\int_{\mathbf{R}} (c^2\mu\gamma''_\mu(x) + 2(1+\mu A^2 + \mu^2\tau_\mu)\gamma_\mu(x))\sin(\omega_\mu x)dx.$$

Now the convergence of the integral is less obvious, though of course it must converge. We write (since the integrand is even):

$$\widetilde{\kappa}_\mu = -2\lim_{R\to\infty}\int_0^R (c^2\mu\gamma''_\mu(x) + 2(1+\mu A^2 + \mu^2\tau_\mu)\gamma_\mu(x))\sin(\omega_\mu x)dx.$$

Integration by parts and $u-$substitution tells us that (if f and g are odd) that

$$\int_0^R f''(x)g(x)dx = f'(R)g(R) - f(R)g'(R) + \int_0^R f(x)g''(x)dx$$

and

$$\begin{aligned}\int_0^R (A^2 f(x))g(x)dx = &\int_0^R f(x)(A^2 g(x))dx\\ &+\frac{1}{4}\int_R^{R+2} f(x)S^{-2}g(x)dx - \frac{1}{4}\int_0^2 f(x)S^{-2}g(x)dx\\ &-\frac{1}{4}\int_{R-2}^R f(x)S^2 g(x)dx + \frac{1}{4}\int_{-2}^0 f(x)S^{-2}g(x)dx.\end{aligned}$$

These imply that

$$
\begin{aligned}
\widetilde{\kappa}_\mu =& -2\lim_{R\to\infty}\int_0^R \gamma_\mu(x)(c^2\mu\partial_x^2 + 2(1+\mu A^2+\mu^2\tau_\mu))\sin(\omega_\mu x)dx \\
&- 2c^2\mu\lim_{R\to\infty}\left(\gamma_\mu'(R)\sin(\omega_\mu R) - \omega_\mu\gamma_\mu(R)\cos(\omega_\mu R)\right) \\
&+ \lim_{R\to\infty}\left(\frac{\mu}{2}\int_R^{R+2}\gamma_\mu(x)\sin(\omega_\mu(x-2))dx + \frac{\mu}{2}\int_0^2\gamma_\mu(x)\sin(\omega_\mu(x-2))dx\right. \\
&\left.+\frac{\mu}{2}\int_{R-2}^R\gamma_\mu(x)\sin(\omega_\mu(x+2))dx - \frac{\mu}{2}\int_{-2}^0\gamma_\mu(x)\sin(\omega_\mu(x+2))dx\right).
\end{aligned}
\tag{D.1}
$$

The first line vanishes by virtue of (6.15) and (C.36). From (7.5) we have $|\gamma_\mu(R) - \sin(\omega_\mu(R+\vartheta_\mu^\infty))| \to 0$ and $|\gamma_\mu'(R) - \omega_\mu\cos(\omega_\mu(R+\vartheta_\mu^\infty))| \to 0$ as $R\to\infty$. And so the second line is equal to

$$-2c^2\mu\lim_{R\to\infty}\left(\omega_\mu\cos(\omega_\mu(R+\vartheta_\mu^\infty))\sin(\omega_\mu R) - \omega_\mu\sin(\omega_\mu(R+\vartheta_\mu^\infty))\cos(\omega_\mu R)\right)$$

Trigonometry identies tell us that $\cos(\theta+\theta')\sin(\theta) - \sin(\theta+\theta')\cos(\theta) = -\sin(\theta')$ and so the above is

$$2c^2\mu\omega_\mu\lim_{R\to\infty}\sin(\omega_\mu\vartheta_\mu^\infty) = 2c^2\mu\omega_\mu\sin(\omega_\mu\vartheta_\mu^\infty).$$

As for the third line of (D.1), we could compute it exactly in this same fashion. Note however, that all the integrals are over intervals of fixed length and the integrands are $\mathcal{O}(1)$. The prefactor μ thus means these terms are no bigger than $C\mu$. And so all together we have shown $|\widetilde{\kappa}_\mu - 2c^2\mu\omega_\mu\sin(\omega_\mu\vartheta_\mu^\infty)| \le C\mu$.

D.1. **Proof of Lemma 7.5—estimate of $\mathcal{L}_\mu^{-1}\mathcal{P}_\mu$.** The coercive estimate (7.1) is essentially an estimate for $\mathcal{L}_\mu^{-1}$, so what we need is an estimate for $\mathcal{P}_\mu$. From its definition, (6.6) and (B.13) we have

$$\|\chi_\mu\|_{s,b_c} \le C\mu^{-s/2}.$$

Thus, for $s' \ge s$ and $b\in[0,b_c]$, we have, using (7.12) and (7.8):

$$
\begin{aligned}
\|\mathcal{P}_\mu f\|_{s,b} \le& \|f\|_{s,b} + |\kappa_\mu^{-1}||\iota_\mu[f]|\|\chi_\mu\|_{s,b} \\
\le& \|f\|_{s,b} + C\mu^{(s'-s-1)/2}\|f\|_{s',b} \\
\le& C(1+\mu^{(s'-s-1)/2})\|f\|_{s',b}.
\end{aligned}
\tag{D.2}
$$

Then the coercive estimates (7.1) imply, for $k=-1,0,1,2$

$$\|\mathcal{L}_\mu^{-1}\mathcal{P}_\mu f\|_{s+k,b} \le C_b\mu^{-(k+1)/2}\left(\|f\|_{s,b} + \|\mathcal{P}_\mu f\|_{s,b}\right).$$

Then we use (D.2) to get, for $s'\ge s$:

$$\|\mathcal{L}_\mu^{-1}\mathcal{P}_\mu f\|_{s+k,b} \le C\mu^{-(k+1)/2}(1+\mu^{(s'-s-1)/2})\|f\|_{s',b}.$$

These estimates, together with a little accounting lead to the estimates in Lemma 7.5.

Appendix E. Proof of Lemma 8.1—estimates and more estimates

Now we begin in earnest our proof of Lemma 8.1. Let

$$\mu_\star := \min\left\{\mu_{\mathcal{H}}(b_*/2), \mu_{\mathcal{L}}(b_*/2), \mu_\gamma, \mu_\kappa, \mu_{\text{per}}, \mu_\omega\right\}.$$

We will be working almost entirely in the sets $U^s_{\mu,\mathfrak{r}}$ which have decay rates fixed at $b_* := b_c/2$. We take $\mu \in (0, \mu_\star] \cap M_c$ throughout the remainder of this appendix and thus can use all previous estimates about $\mathcal{H}_\mu$, $\mathcal{L}_\mu$, γ_μ, κ_μ, ω_μ and $a\mathbf{p}^a_\mu$ freely.

The "bootstrapping" estimate (8.3) follows from the estimates for $\mathbf{p}^a_\mu$ in Theorem 5, the fact that H^s_b is an algebra when $s \geq 1$ and $b \geq 0$, and the smoothing properties of $\mathcal{H}^{-1}_\mu$ and $\mathcal{L}^{-1}_\mu \mathcal{P}_\mu$ described in Lemmas 6.1 and 7.1. We leave out the details since most of the key ideas will appear below.

So we turn out attention to establishing (8.4). Fix $|c| \in (c_0, c_1]$, $s \geq 1$ and $\mathfrak{r} \in \mathbf{R}^3_+$ (with $\mathfrak{r}_3 < a_{\text{per}}$). Assume that $(\boldsymbol{\eta}, a) \in U^s_{\mu,\mathfrak{r}}$. We adhere to the following conventions for constants "C" which appear in the remainder of this appendix. First, any unadorned C is positive and is determined only by s and c. Second, a constant denoted $C_{\mathfrak{r}}$ is positive and is determined by s, c and the triple $\mathfrak{r} \in \mathbf{R}^3_+$.

E.1. **First estimates for N^μ_1.** The definition of N^μ_1 is at (6.12) and it is made primarily of the "j" functions. We have $j_k := J_k \cdot \mathbf{e}_1$ where the J_k are defined in Section 6.

Using the definition of j_2 (in (6.8)) together with (6.3) and (6.6) we have

$$\|j_2\|_{s,b_*} \leq \mu\|\Theta_{\mu,1}\eta_2\|_{s,b_*} + \mu\|\Omega_{\mu,1}\eta_2\|_{s,b_*} \leq C\mu\|\eta_2\|_{s,b_*}. \tag{E.1}$$

Then we use the estimates implied by membership in $U^s_{\mu,\mathfrak{r}}$ to get

$$\|j_2\|_{s,b_*} \leq C\mathfrak{r}_2\mu^3.$$

Calling back to the definitions of J_3 at (6.2) as well those of Σ_μ and Ω_μ in (6.4)-(6.5) and $a\mathbf{p}^a_\mu$ in (5.2) we have

$$j_3 = -2aL_\mu Q_\mu(\boldsymbol{\sigma}_{c,\mu}, \mathbf{p}^a_\mu) \cdot \mathbf{e}_1 = -\mu a\Sigma_{\mu,1}\varphi^a_{\mu,1} - \mu a\Omega_{\mu,1}\varphi^a_{\mu,2}. \tag{E.2}$$

Thus

$$\|j_3\|_{s,b_*} \leq \mu|a|\|\Sigma_{\mu,1}\varphi^a_{\mu,1}\|_{s,b_*} + \mu|a|\|\Omega_{\mu,1}\varphi^a_{\mu,2}\|_{s,b_*}.$$

Then applying (6.6) gets us to

$$\|j_3\|_{s,b_*} \leq C\mu|a|\left(\|\varphi^a_{\mu,1}\|_{W^{s,\infty}} + \|\varphi^a_{\mu,2}\|_{W^{s,\infty}}\right).$$

The estimates (B.13) then yield:

$$\|j_3\|_{s,b} \leq C\mu^{(2-s)/2}|a|.$$

Membership in $U^s_{\mu,\mathfrak{r}}$ give

$$\|j_3\|_{s,b} \leq C\mathfrak{r}_3\mu^{(2-s)/2}\mu^{(6+s)/2} \leq C_{\mathfrak{r}}\mu^4.$$

Next, the definition of j_4 in (6.2) give

$$\|j_4\|_{s,b_*} = 2|a|\|L_\mu Q_\mu(\mathbf{p}^a_\mu, \boldsymbol{\eta}) \cdot \mathbf{e}_1\|_{s,b_*}.$$

Then we use (2.20) and (5.2)

$$\|j_4\|_{s,b_*} \leq C|a| \left(\mu\|\varphi^a_{\mu,1}\|_{W^{s,\infty}}\|\eta_1\|_{s,b_*} + \|\varphi^a_{\mu,2}\|_{W^{s,\infty}}\|\eta_2\|_{s,b_*} \right.$$
$$\left. +\mu\|\varphi^a_{\mu,2}\|_{W^{s,\infty}}\|\eta_1\|_{s,b_*} + \mu^2\|\varphi^a_{\mu,1}\|_{W^{s,\infty}}\|\eta_2\|_{s,b_*}\right).$$

Using the estimates (B.13) results in

$$\|j_4\|_{s,b_*} \leq C|a| \left(\mu^{(2-s)/2}\|\eta_1\|_{s,b_*} + \mu^{-s/2}\|\eta_2\|_{s,b_*}\right).$$

Since $(\boldsymbol{\eta}, a) \in U^s_{\mu,\mathfrak{r}}$, the above give

$$\|j_4\|_{s,b_*} \leq C_{\mathfrak{r}}\mu^5.$$

Next we see that the definition of j_5 in (6.2) and the estimates in (2.22) give:

$$\|j_5\|_{s,b_*} \leq C\left(\|\eta_1\|^2_{s,b_*} + \|\eta_2\|^2_{s,b_*}\right).$$

Properties of $U^s_{\mu,\mathfrak{r}}$ convert this to

$$\|j_5\|_{s,b_*} \leq C_{\mathfrak{r}}\mu^4.$$

Now that the j functions are estimated, we can bound N^μ_1. Using the estimate for $\mathcal{H}^{-1}_\mu$ in Proposition 6.1 together with the above estimates gives

$$(\boldsymbol{\eta}, a) \in U^s_{\mu,\mathfrak{r}} \implies \|N^\mu_1(\boldsymbol{\eta}, a)\|_{s+2,b_*} \leq C\mathfrak{r}_2\mu^3 + C_{\mathfrak{r}}\mu^4. \tag{E.3}$$

E.2. **First estimates for N^μ_2 and N^μ_3.** The definitions of N^μ_2 and N^μ_3 are at (7.19) and (7.15) . They are made primarily of the "l" functions. We have $l_k := J_k \cdot \mathbf{e}_2$ where the J_k are defined in Section 6.

Using the definition of l_0 at (6.2), its representation at (4.2) and the estimates in Lemma 4 we have

$$\|l_0\|_{s+10,b_*} = c^2\mu^2\|\xi''_{\mu,2}\|_{s+10,b_*} \leq C\mu^2.$$

From the defintion of l_1 at (6.17) and the estimate for τ_μ at (6.16), we have

$$\|l_1\|_{s,b_*} \leq C\mu^2\|\eta_2\|_{s,b_*}. \tag{E.4}$$

Membership in $U^s_{\mu,\mathfrak{r}}$ then implies

$$\|l_1\|_{s,b_*} \leq C\mathfrak{r}_2\mu^4.$$

Using the definition of l_2 (in (6.8)) together with (6.3) and (6.6) we have

$$\|l_2\|_{s+1,b_*} \leq \mu\|\Theta_{\mu,2}\eta_1\|_{s+1,b_*} + \mu\|\Omega_{\mu,2}\eta_1\|_{s+1,b_*} \leq C\mu\|\eta_1\|_{s+1,b_*} \tag{E.5}$$

And since $(\boldsymbol{\eta}, a) \in U^s_{\mu,\mathfrak{r}}$ this yields:

$$\|l_2\|_{s+1,b_*} \leq C_{\mathfrak{r}}\mu^4.$$

Now we estimate l_{31}. From (7.14) we see $l_{31} = l_3 + a\chi_\mu$. l_3 is derived from J_3, found at (6.2). With the definitions of $\Sigma_{\mu,2}$ and $\Omega_{\mu,2}$ in (6.4)-(6.5) and $a\mathbf{p}^a_\mu$ in (5.2), this give

$$l_3 = -2aL_\mu Q_\mu(\boldsymbol{\sigma}_{c,\mu}, \mathbf{p}^a_\mu) \cdot \mathbf{e}_2 = -a\Sigma_{\mu,2}\varphi^a_{\mu,2} - \mu^2 a\Omega_{\mu,2}\varphi^a_{\mu,1}.$$

Then we refer to the definition of χ_μ at (7.11) to see that

$$l_{31} = -a\Sigma_{\mu,2}(\varphi^a_{\mu,2} - \varphi^0_{\mu,2}) - \mu^2 a\Omega_{\mu,2}(\varphi^a_{\mu,1} - \varphi^0_{\mu,1}). \tag{E.6}$$

Using the estimates in (6.6) and the fact $b_* - b_c = -b_*$ we get

$$\|l_{31}\|_{s,b_*} \le C|a|\|\varphi^a_{\mu,2} - \varphi^0_{\mu,2}\|_{W^{s,\infty}_{-b_*}}.$$

Then we use (B.14):

$$\|l_{31}\|_{s,b_*} \le C\mu^{-s/2}a^2.$$

Then properties of $U^s_{\mu,\mathfrak{r}}$ convert this to

$$\|l_{31}\|_{s,b_*} \le C_{\mathfrak{r}}\mu^{6+s/2} \le C_{\mathfrak{r}}\mu^6.$$

Next, the definition of l_4 in (6.2) give

$$\|l_4\|_{s,b_*} = 2|a|\|L_\mu Q_\mu(\mathbf{p}^a_\mu, \boldsymbol{\eta}) \cdot \mathbf{e}_2\|_{s,b_*}.$$

We use (2.22) and (5.2) on this and get

$$\|l_4\|_{s,b_*} \le C|a| \left(\|\varphi^a_{\mu,2}\|_{W^{s,\infty}}\|\eta_1\|_{s,b_*} + \|\varphi^a_{\mu,1}\|_{W^{s,\infty}}\|\eta_2\|_{s,b_*} + \mu\|\varphi^a_{\mu,1}\|_{W^{s,\infty}}\|\eta_1\|_{s,b_*} + \mu\|\varphi^a_{\mu,2}\|_{W^{s,\infty}}\|\eta_2\|_{s,b_*}\right).$$

Then using estimates in (B.13):

$$\|l_4\|_{s,b_*} \le C\mu^{-s/2}|a| \left(\|\eta_1\|_{s,b_*} + \mu\|\eta_2\|_{s,b_*}\right).$$

The properties of $U^s_{\mu,\mathfrak{r}}$ then give:

$$\|l_4\|_{s,b_*} \le C_{\mathfrak{r}}\mu^6.$$

Estimating l_5 using (2.22) and (2.23) give

$$\|l_5\|_{s,b_*} \le C \left(\|\eta_1\|_{s,b_*}\|\eta_2\|_{s,b_*} + \mu\|\eta_1\|^2_{s,b_*} + \mu\|\eta_2\|^2_{s,b_*}\right).$$

Within $U^s_{\mu,\mathfrak{r}}$, this estimate gives us

$$\|l_5\|_{s,b_*} \le C_{\mathfrak{r}}\mu^5.$$

Now we can use the various estimates for $\mathcal{L}^{-1}_\mu\mathcal{P}_\mu$ in (7.18) to see that

$$\|N^\mu_2(\boldsymbol{\eta}, a)\|_{s+1,b_*} \le C \left(\|l_0\|_{s+3,b_*} + \mu^{-3/2}\|l_1\|_{s,b_*} + \mu^{-1}\|l_2\|_{s+1,b_*} + \mu^{-3/2}\|l_{31}\|_{s,b_*} + \mu^{-3/2}\|l_4\|_{s,b_*} + \mu^{-3/2}\|l_5\|_{s,b_*}\right).$$

Then we use the preceding estimates for the l-functions to get

$$(\boldsymbol{\eta}, a) \in U^s_{\mu,\mathfrak{r}} \implies \|N^\mu_2(\boldsymbol{\eta}, a)\|_{s+1,b_*} \le C\mu^2 + C_{\mathfrak{r}}\mu^{5/2}. \tag{E.7}$$

To estimate N^μ_3 we use the estimate (7.8) for ι_μ together with the estimate that $\kappa_\mu = \mathcal{O}(\mu^{1/2})$ for $\mu \in M_c$ to get

$$|N^\mu_3(\boldsymbol{\eta}, a)| \le C\mu^{-1/2} \left(\mu^{(s+10)/2}\|l_0\|_{s+10,b_*} + \mu^{s/2}\|l_1\|_{s,b_*} + \mu^{(s+1)/2}\|l_2\|_{s+1,b_*} + \mu^{s/2}\|l_{31}\|_{s,b_*} + \mu^{s/2}\|l_4\|_{s,b_*} + \mu^{s/2}\|l_5\|_{s,b_*}\right).$$

Using the above estimates for l_0 through l_5 we get

$$(\boldsymbol{\eta}, a) \in U^s_{\mu,\mathfrak{r}} \implies |N^\mu_3(\boldsymbol{\eta}, a)| \le C\mathfrak{r}_2\mu^{(7+s)/2} + C_{\mathfrak{r}}\mu^{(8+s)/2}. \tag{E.8}$$

E.3. **The estimates** (8.4) **and** (8.5)**.** The estimates (E.3), (E.7) and (E.8) give us (8.4) and (8.5) in Lemma 8.1. For (8.5) put $s = 1$ in those estimates to see that $(\boldsymbol{\eta}, a) \in U^1_{\mu,\mathfrak{r}}$ implies

$$\|N_1^\mu(\boldsymbol{\eta}, a)\|_{2,b_*} \leq C_*\mathfrak{r}_2\mu^3 + C_\mathfrak{r}\mu^4,$$
$$\|N_2^\mu(\boldsymbol{\eta}, a)\|_{1,b_*} \leq C_*\mu^2 + C_\mathfrak{r}\mu^{5/2} \quad \text{and}$$
$$|N_2^\mu(\boldsymbol{\eta}, a)| \leq C_*\mathfrak{r}_2\mu^4 + C_\mathfrak{r}\mu^{9/2}.$$

The constant $C_* > 0$ is determined only by c and is the same across all three inequalities. The inequalities hold for $\mu \in (0, \mu_\star] \cap M_c$. Put $\mathfrak{r}_2 = \mathfrak{r}_{*,2} := 2C_*$, $\mathfrak{r}_1 = \mathfrak{r}_{*,1} = 4C_*^2$ and $\mathfrak{r}_3 = \mathfrak{r}_{*,3} := \mathfrak{r}_{*,2}$ and denote the resulting triple by $\mathfrak{r}_*$. Then there exists $\mu_{\star\star} \in (0, \mu_\star]$ so that $\mu \in (0, \mu_{\star\star}] \cap M_c$ turns the last set of estimates into

$$\|N_1^\mu(\boldsymbol{\eta}, a)\|_{2,b_*} \leq \mathfrak{r}_{*,1}\mu^3, \quad \|N_2^\mu(\boldsymbol{\eta}, a)\|_{1,b_*} \leq \mathfrak{r}_{*,2}\mu^2 \quad \text{and} \quad |N_2^\mu(\boldsymbol{\eta}, a)| \leq \mathfrak{r}_{*,3}\mu^{7/2}.$$

Which is to say that if $(\boldsymbol{\eta}, a) \in U^1_{\mu,\mathfrak{r}_*}$ so is $N^\mu(\boldsymbol{\eta}, a)$. This is (8.5).

As for (8.4), we return our attention to (E.3), (E.7) and (E.8). We put $\widetilde{\mathfrak{r}}_2 := C + C_\mathfrak{r}$, $\widetilde{\mathfrak{r}}_1 := C\widetilde{\mathfrak{r}}_2 + C_\mathfrak{r}$ and $\widetilde{\mathfrak{r}}_3 := C\widetilde{\mathfrak{r}}_2 + C_\mathfrak{r}$ where by C and $C_\mathfrak{r}$ we mean the constants that appear in those estimates at order s once $\mathfrak{r}$ is selected; the implication in (8.4) follows immediately.

E.4. **The contraction estimate.** Now we turn our gaze towards (8.6), which is an estimate for $N(\boldsymbol{\eta}, a) - N(\grave{\boldsymbol{\eta}}, \grave{a})$. Let us assume that $(\boldsymbol{\eta}, a), (\grave{\boldsymbol{\eta}}, \grave{a}) \in U^1_{\mu,\mathfrak{r}_*}$. Let $\grave{j}_n$ and $\grave{l}_n$ to be the same as j_n and l_n but evaluated at $(\grave{\boldsymbol{\eta}}, \grave{a})$. We need to estimate

$$\|\mathcal{H}_\mu^{-1}(j_n - \grave{j}_n)\|_{2,b_*/2}, \quad \|\mathcal{L}_\mu^{-1}\mathcal{P}_\mu(l_n - \grave{l}_n)\|_{0,b_*/2} \quad \text{and} \quad \kappa_\mu^{-1}|\iota_\mu[l_n - \grave{l}_n]|.$$

Revisiting j_2 (at (6.8)) shows that it is linear in η_2. Thus the steps that led to (E.1) lead us to

$$\|\mathcal{H}_\mu^{-1}(j_2 - \grave{j}_2)\|_{2,b_*/2} \leq C\mu\|\eta_2 - \grave{\eta}_2\|_{0,b_*/2}.$$

Using the the formula for j_3 at (E.2), the estimates for $\mathcal{H}_\mu^{-1}$ in Lemma 3.1 and the triangle inequality

$$\begin{aligned}\|\mathcal{H}_\mu^{-1}(j_3 - \grave{j}_3)\|_{2,b_*/2} \leq & C\mu|a - \grave{a}|\|\Sigma_{\mu,1}\varphi^a_{\mu,1}\|_{0,b_*/2} + C\mu|\grave{a}|\|\Sigma_{\mu,1}(\varphi^a_{\mu,1} - \varphi^{\grave{a}}_{\mu,1})\|_{0,b_*/2} \\ & + C\mu|a - \grave{a}|\|\Omega_{\mu,1}\varphi^a_{\mu,1}\|_{0,b_*/2} + C\mu|\grave{a}|\|\Omega_{\mu,1}(\varphi^a_{\mu,1} - \varphi^{\grave{a}}_{\mu,1})\|_{0,b_*/2}.\end{aligned}$$

Using (6.6) and recalling that $b_* = b_c/2$ on the right hand side give

$$\|\mathcal{H}_\mu^{-1}(j_3 - \grave{j}_3)\|_{2,b_*/2} \leq C\mu|a - \grave{a}|\|\varphi^a_{\mu,1}\|_{0,-3b_*/2} + C\mu|\grave{a}|\|\varphi^a_{\mu,1} - \varphi^{\grave{a}}_{\mu,1}\|_{0,-3b_*/2}.$$

The estimates in (B.13) and (B.14) then give:

$$\|\mathcal{H}_\mu^{-1}(j_3 - \grave{j}_3)\|_{2,b_*/2} \leq C\mu|a - \grave{a}|.$$

Using the definition of j_4 in (6.2), boundedness of $\mathcal{H}_\mu^{-1}$ and the triangle inequality gets us

$$\begin{aligned}\|\mathcal{H}_\mu^{-1}(j_4 - \grave{j}_4)\|_{2,b_*/2} \leq & C|a - \grave{a}|\|L_\mu Q_\mu(\mathbf{p}^a_\mu, \boldsymbol{\eta}) \cdot \mathbf{e}_1\|_{0,b_*/2} \\ & + C|\grave{a}|\|L_\mu Q_\mu(\mathbf{p}^a_\mu - \mathbf{p}^{\grave{a}}_\mu, \boldsymbol{\eta}) \cdot \mathbf{e}_1\|_{0,b_*/2} \\ & + C|\grave{a}|\|L_\mu Q_\mu(\mathbf{p}^{\grave{a}}_\mu, \boldsymbol{\eta} - \grave{\boldsymbol{\eta}}) \cdot \mathbf{e}_1\|_{0,b_*/2}.\end{aligned} \tag{E.9}$$

Let us focus on the middle term. Using (2.20) with $s=0$, $b=b_*/2$ and $b'=b''=b'''=b''''=-b_*$ give

$$\begin{aligned}
&C|\grave{a}|\|L_\mu Q_\mu(\mathbf{p}_\mu^a-\mathbf{p}_\mu^{\grave{a}},\boldsymbol{\eta})\cdot\mathbf{e}_1\|_{0,b_*/2}\\
\le&C|\grave{a}|\left(\mu\|\varphi_{\mu,1}^a-\varphi_{\mu,1}^{\grave{a}}\|_{W^{0,\infty}_{-b_*/2}}\|\eta_1\|_{0,b_*}+\|\varphi_{\mu,2}^a-\varphi_{\mu,2}^{\grave{a}}\|_{W^{0,\infty}_{-b_*/2}}\|\eta_2\|_{0,b_*}\right)\\
&+C|\grave{a}|\mu\left(\mu\|\varphi_{\mu,1}^a-\varphi_{\mu,1}^{\grave{a}}\|_{W^{0,\infty}_{-b_*/2}}\|\eta_2\|_{0,b_*}+\|\varphi_{\mu,2}^a-\varphi_{\mu,2}^{\grave{a}}\|_{W^{0,\infty}_{-b_*/2}}\|\eta_1\|_{0,b_*}\right).
\end{aligned}$$

Then (B.14) gives

$$C|\grave{a}|\|L_\mu Q_\mu(\mathbf{p}_\mu^a-\mathbf{p}_\mu^{\grave{a}},\boldsymbol{\eta})\cdot\mathbf{e}_1\|_{0,b_*/2}\le C|\grave{a}|\left(\|\eta_1\|_{0,b_*}+\|\eta_2\|_{0,b_*}\right)|a-\grave{a}|$$

and since we are working in $U^1_{\mu,\mathfrak{r}_*}$:

$$C|\grave{a}|\|L_\mu Q_\mu(\mathbf{p}_\mu^a-\mathbf{p}_\mu^{\grave{a}},\boldsymbol{\eta})\cdot\mathbf{e}_1\|_{L^2}\le C\mu^{11/2}|a-\grave{a}|.$$

The same sort of reasoning on the other two lines in (E.9) leads us to

$$\|\mathcal{H}_\mu^{-1}(j_3-\grave{j}_3)\|_{2,b_*/2}\le C\mu^2|a-\grave{a}|+C\mu^{7/2}\|\eta_1-\grave{\eta}_1\|_{0,b_*/2}+C\mu^{7/2}\|\eta_2-\grave{\eta}_2\|_{0,b_*/2}.$$

The definition of j_5 and boundedness of $\mathcal{H}_\mu^{-1}$ yield:

$$\|\mathcal{H}_\mu^{-1}(j_5-\grave{j}_5)\|_{2,b_*/2}\le\|L_\mu Q_\mu(\boldsymbol{\eta}+\grave{\boldsymbol{\eta}},\boldsymbol{\eta}-\grave{\boldsymbol{\eta}})\cdot\mathbf{e}_1\|_{0,b_*/2}\tag{E.10}$$

Using (2.20) converts this to

$$\begin{aligned}
\|\mathcal{H}_\mu^{-1}(j_5-\grave{j}_5)\|_{2,b_*/2}\le&C\left(\|\eta_{\mu,1}+\grave{\eta}_{\mu,1}\|_{L^\infty}\|\eta_{\mu,1}-\grave{\eta}_{\mu,1}\|_{0,b_*/2}+\|\eta_{\mu,2}+\grave{\eta}_{\mu,2}\|_{L^\infty}\|\eta_{\mu,2}-\grave{\eta}_{\mu,2}\|_{0,b_*/2}\right)\\
&+C\mu\left(\|\eta_{\mu,2}+\grave{\eta}_{\mu,2}\|_{L^\infty}\|\eta_{\mu,1}-\grave{\eta}_{\mu,1}\|_{0,b_*/2}+\|\eta_{\mu,2}+\grave{\eta}_{\mu,2}\|_{L^\infty}\|\eta_{\mu,1}-\grave{\eta}_{\mu,1}\|_{0,b_*/2}\right)
\end{aligned}$$

Then using Sobolev embedding and the properties of $U^1_{\mu,\mathfrak{r}_*}$ give

$$\|\mathcal{H}_\mu^{-1}(j_5-\grave{j}_5)\|_{2,b_*/2}\le C\mu^3\|\eta_{\mu,1}-\grave{\eta}_{\mu,1}\|_{0,b_*/2}+C\mu^2\|\eta_{\mu,2}-\grave{\eta}_{\mu,2}\|_{0,b_*/2}.$$

With this, we have

$$(\boldsymbol{\eta},a),(\grave{\boldsymbol{\eta}},\grave{a})\in U^1_{\mu,\mathfrak{r}_*}\implies\|N_1^\mu(\boldsymbol{\eta},a)-N_1^\mu(\grave{\boldsymbol{\eta}},\grave{a})\|_{2,b_*/2}\le C\mu\|(\boldsymbol{\eta},a)-(\grave{\boldsymbol{\eta}},\grave{a})\|_{X_0}.$$

Now we work on the l functions. First, l_0 (defined at (6.2) and (4.2)) depends on neither a nor $\boldsymbol{\eta}$ and so $l_0-\grave{l}_0=0$. Second, l_1 and l_2 (see (6.8) and (6.17)) are linear in $\boldsymbol{\eta}$ and thus we have, using the same ideas that gave us (E.5) and (E.4):

$$\|l_1-\grave{l}_1\|_{0,b_*/2}\le C\mu^2\|\eta_2-\grave{\eta}_2\|_{0,b_*/2}\quad\text{and}\quad\|l_2-\grave{l}_2\|_{2,b_*/2}\le C\mu\|\eta_1-\grave{\eta}_1\|_{2,b_*/2}.\tag{E.11}$$

For l_{31} we use the formula (E.6) and see, by way of the triangle inequality, that

$$\begin{aligned}
\|l_{31}-\grave{l}_{31}\|_{0,b_*/2}\le&2|a-\grave{a}|\|\Sigma_{\mu,2}(\varphi_{\mu,2}^a-\varphi_{\mu,2}^0)\|_{0,b_*/2}+2|\grave{a}|\|\Sigma_{\mu,2}(\varphi_{\mu,2}^a-\varphi_{\mu,2}^{\grave{a}})\|_{0,b_*/2}\\
&+2\mu^2|a-\grave{a}|\|\Omega_{\mu,2}(\varphi_{\mu,1}^a-\varphi_{\mu,1}^0)\|_{0,b_*/2}+2\mu^2|\grave{a}|\|\Omega_{\mu,2}(\varphi_{\mu,1}^a-\varphi_{\mu,1}^{\grave{a}})\|_{0,b_*/2}.
\end{aligned}\tag{E.12}$$

Using (6.6) and the fact that $b_*=b_c/2$ turns this into

$$\begin{aligned}
\|l_{31}-\grave{l}_{31}\|_{0,b_*/2}\le&C|a-\grave{a}|\|\varphi_{\mu,2}^a-\varphi_{\mu,2}^0\|_{0,-3b_*/4}+C|\grave{a}|\|\varphi_{\mu,2}^a-\varphi_{\mu,2}^{\grave{a}}\|_{0,-3b_*/4}\\
&+C\mu^2|a-\grave{a}|\|\varphi_{\mu,1}^a-\varphi_{\mu,1}^0\|_{0,-3b_*/4}+C\mu^2|\grave{a}|\|\varphi_{\mu,1}^a-\varphi_{\mu,1}^{\grave{a}}\|_{0,-3b_*/4}.
\end{aligned}$$

Then we (B.13) and (B.14) to get

$$\|l_{31} - \grave{l}_{31}\|_{0,b_*/2} \leq C(|a| + |\grave{a}|)|a - \grave{a}|.$$

And since we are in $U^1_{\mu,\mathfrak{r}_*}$:

$$\|l_{31} - \grave{l}_{31}\|_{0,b_*/2} \leq C\mu^{7/2}|a - \grave{a}|.$$

Using the definition of l_4 in (6.2) and the triangle inequality gives us

$$\begin{aligned}\|l_4 - \grave{l}_4\|_{0,b_*/2} \leq & C|a - \grave{a}|\|L_\mu Q_\mu(\mathbf{p}^a_\mu, \boldsymbol{\eta}) \cdot \mathbf{e}_2\|_{0,b_*/2} \\ & + C|\grave{a}|\|L_\mu Q_\mu(\mathbf{p}^a_\mu - \mathbf{p}^{\grave{a}}_\mu, \boldsymbol{\eta}) \cdot \mathbf{e}_2\|_{0,b_*/2} \\ & + C|\grave{a}|\|L_\mu Q_\mu(\mathbf{p}^{\grave{a}}_\mu, \boldsymbol{\eta} - \grave{\boldsymbol{\eta}}) \cdot \mathbf{e}_2\|_{0,b_*/2}.\end{aligned} \tag{E.13}$$

Focus on the middle line. We use (2.21) to get

$$\begin{aligned}& C|\grave{a}|\|L_\mu Q_\mu(\mathbf{p}^a_\mu - \mathbf{p}^{\grave{a}}_\mu, \boldsymbol{\eta}) \cdot \mathbf{e}_2\|_{0,b_*/2} \\ \leq & C|\grave{a}| \left(\mu\|\varphi^a_{\mu,1} - \varphi^{\grave{a}}_{\mu,1}\|_{W^{0,\infty}_{-b_*/2}}\|\eta_2\|_{0,b_*} + \|\varphi^a_{\mu,2} - \varphi^{\grave{a}}_{\mu,2}\|_{W^{0,\infty}_{-b_*/2}}\|\eta_1\|_{0,b_*}\right) \\ & + C|\grave{a}|\mu \left(\mu\|\varphi^a_{\mu,1} - \varphi^{\grave{a}}_{\mu,1}\|_{W^{0,\infty}_{-b_*/2}}\|\eta_1\|_{0,b_*} + \|\varphi^a_{\mu,2} - \varphi^{\grave{a}}_{\mu,2}\|_{W^{0,\infty}_{-b_*/2}}\|\eta_2\|_{0,b_*}\right).\end{aligned}$$

Then (B.14) give

$$C|\grave{a}|\|L_\mu Q_\mu(\mathbf{p}^a_\mu - \mathbf{p}^{\grave{a}}_\mu, \boldsymbol{\eta}) \cdot \mathbf{e}_2\|_{0,b_*/2} \leq C|\grave{a}| \left(\|\eta_1\|_{b_*} + \mu\|\eta_2\|_{b_*}\right)|a - \grave{a}|.$$

And then membership in $U^1_{\mu,\mathfrak{r}_*}$ give

$$C|\grave{a}|\|L_\mu Q_\mu(\mathbf{p}^a_\mu - \mathbf{p}^{\grave{a}}_\mu, \boldsymbol{\eta}) \cdot \mathbf{e}_2\|_{0,b_*/2} \leq C\mu^{13/2}|a - \grave{a}|.$$

The remaining terms in (E.13) are handled similarly and we find

$$\|l_4 - \grave{l}_4\|_{0,b_*/2} \leq C\mu^4|a - \grave{a}| + C\mu^{7/2}\|\eta_1 - \grave{\eta}_1\|_{0,b_*/2} + C\mu^{7/2}\|\eta_2 - \grave{\eta}_2\|_{0,b_*/2}.$$

The definition of l_5 at (6.2) give

$$\|l_5 - \grave{l}_5\|_{0,b_*/2} \leq \|L_\mu Q_\mu(\boldsymbol{\eta} + \grave{\boldsymbol{\eta}}, \boldsymbol{\eta} - \grave{\boldsymbol{\eta}}) \cdot \mathbf{e}_2\|_{0,b_*/2} \tag{E.14}$$

Using (2.21) converts this to

$$\begin{aligned}\|l_5 - \grave{l}_5\|_{0,b_*/2} \leq & C\left(\|\eta_{\mu,2} + \grave{\eta}_{\mu,2}\|_{L^\infty}\|\eta_{\mu,1} - \grave{\eta}_{\mu,1}\|_{0,b_*/2} + \|\eta_{\mu,1} + \grave{\eta}_{\mu,1}\|_{L^\infty}\|\eta_{\mu,2} - \grave{\eta}_{\mu,2}\|_{0,b_*/2}\right) \\ & + C\mu\left(\|\eta_{\mu,1} + \grave{\eta}_{\mu,1}\|_{L^\infty}\|\eta_{\mu,1} - \grave{\eta}_{\mu,1}\|_{0,b_*/2} + \|\eta_{\mu,2} + \grave{\eta}_{\mu,2}\|_{L^\infty}\|\eta_{\mu,2} - \grave{\eta}_{\mu,2}\|_{0,b_*/2}\right)\end{aligned}$$

Then using Sobolev embedding and the properties of $U^1_{\mu,\mathfrak{r}_*}$ give

$$\|l_5 - \grave{l}_5\|_{0,b_*/2} \leq C\mu^2\|\eta_{\mu,1} - \grave{\eta}_{\mu,1}\|_{0,b_*/2} + C\mu^3\|\eta_{\mu,2} - \grave{\eta}_{\mu,2}\|_{0,b_*/2}.$$

Then we can use the various estimates for $\mathcal{L}^{-1}_\mu \mathcal{P}_\mu$ in (7.18) to get

$$\begin{aligned}\|N^\mu_2(\boldsymbol{\eta}, a) - N^\mu_2(\grave{\boldsymbol{\eta}}, \grave{a})\|_{0,b_*/2} \leq C \Big(& \mu^{-1}\|l_1 - \grave{l}_1\|_{0,b_*/2} + \|l_2 - \grave{l}_2\|_{2,b_*/2} \\ & + \mu^{-1}\|l_{31} - \grave{l}_{31}\|_{0,b_*/2} + \mu^{-1}\|l_4 - \grave{l}_4\|_{0,b_*/2} + \mu^{-1}\|l_5 - \grave{l}_5\|_{0,b_*/2}\Big).\end{aligned}$$

Using the preceding estimates together with this gives us

$$(\boldsymbol{\eta}, a), (\grave{\boldsymbol{\eta}}, \grave{a}) \in U^1_{\mu,\mathfrak{r}_*} \implies \|N^\mu_2(\boldsymbol{\eta}, a) - N^\mu_2(\grave{\boldsymbol{\eta}}, \grave{a})\|_{0,b_*/2} \leq C\mu\|(\boldsymbol{\eta}, a) - (\grave{\boldsymbol{\eta}}, \grave{a})\|_{X_0}.$$

Similarly using (7.8) shows that

$$|N_3^\mu(\boldsymbol{\eta}, a) - N_3^\mu(\grave{\boldsymbol{\eta}}, \grave{a})| \le C\mu^{-1/2} \Big(\|l_1 - \grave{l}_1\|_{0,b_*/2} + \mu\|l_2 - \grave{l}_2\|_{2,b_*/2} \\ + \|l_{31} - \grave{l}_{31}\|_{0,b_*/2} + \|l_4 - \grave{l}_4\|_{0,b_*/2} + \|l_5 - \grave{l}_5\|_{0,b_*/2} \Big).$$

Using the estimates above give

$$(\boldsymbol{\eta}, a), (\grave{\boldsymbol{\eta}}, \grave{a}) \in U^1_{\mu,\mathfrak{r}_*} \implies |N_3^\mu(\boldsymbol{\eta}, a) - N_2^\mu(\grave{\boldsymbol{\eta}}, \grave{a})| \le C\mu^{3/2}\|(\boldsymbol{\eta}, a) - (\grave{\boldsymbol{\eta}}, \grave{a})\|_{X_0}.$$

Thus all together we have (8.6) so long as we take $\mu \in M_c$ sufficiently close to zero; we call the threshold μ_*. This concludes the proof of Lemma 8.1 and also this paper.

Conflict of Interest: The authors declare that they have no conflict of interest.

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Highlights:

-The existence of traveling waves in diatomic FPUT is considered.

-They are studied in the limit of small mass ratio.

-The solutions constructed are nanopterons.

-Nanopterons are the superposition of a solitary wave and a periodic wave.

-The periodic wave's amplitude is small beyond all algebraic orders of the mass ratio.