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Stochastic least-action principle for the incompressible Navier–Stokes equation

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We dedicate our paper with respect and affection to K.R. Sreenivasan, on the occasion of his 60th birthday.

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ABSTRACT

We formulate a stochastic least-action principle for solutions of the incompressible Navier–Stokes equation, which formally reduces to Hamilton's principle for the incompressible Euler solutions in the case of zero viscosity. We use this principle to give a new derivation of a stochastic Kelvin Theorem for the Navier–Stokes equation, recently established by Constantin and Iyer, which shows that this stochastic conservation law arises from particle-relabelling symmetry of the action. We discuss issues of irreversibility, energy dissipation, and the inviscid limit of Navier–Stokes solutions in the framework of the stochastic variational principle. In particular, we discuss the connection of the stochastic Kelvin Theorem with our previous "martingale hypothesis" for fluid circulations in turbulent solutions of the incompressible Euler equations.

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1. Introduction

Alternative formulations of standard equations can be very illuminating and can cast new light on old problems. As just one example, consider how Feynman's path-integral solution of the Schrödinger equation enabled intuitive new approaches to difficult problems with many degrees-of-freedom, such as quantum electrodynamics and superfluid helium. In this same spirit, many different mathematical formulations have been developed for the equations of classical hydrodynamics, both ideal and non-ideal. Recently, Constantin and Iyer [1] have presented a very interesting representation of solutions of the incompressible Navier–Stokes equation by averaging over stochastic Lagrangian trajectories in the Weber formula [2] for incompressible Euler solutions. Their formulation is a nontrivial application of the method of stochastic characteristics, well known in pure mathematics [3] (Chapter 6), in theoretical physics [4,5] and in engineering modeling [6,7]. The characterization of the Navier–Stokes solutions in [1] is through a nonlinear fixed-point problem, since the velocity field that results from the average over stochastic trajectories must be the same as that which advects the fluid particles. Constantin and Iyer have shown that their stochastic representation implies remarkable properties of Navier–Stokes solutions in close analogy to those of ideal Euler solutions, such as a stochastic Kelvin Theorem for fluid circulations and a stochastic Cauchy formula for the vorticity field.

In this paper, we point out some further remarkable features of the stochastic Lagrangian formulation of [1]. Most importantly, we show that the nonlinear fixed-point problem that characterizes the Navier–Stokes solution is, in fact, a variational problem which generalizes the well-known Hamilton–Maupertuis least-action principle for incompressible Euler solutions [8]. We shall demonstrate this result by a formally exact calculation, at the level of rigor of theoretical physics. A more careful mathematical proof, with the set-up of relevant function spaces, precise definitions of variational derivatives, etc. shall be given elsewhere. Closely related stochastic variational formulations of incompressible Navier–Stokes solutions have been developed recently by others [9–11] and a detailed comparison with these approaches will also be made in future work.

Our variational formulation sheds some new light on a basic proposition of [1], the stochastic Kelvin Theorem which was established there for smooth Navier–Stokes solutions at any finite Reynolds number. We show that this result is a consequence of particle-relabelling symmetry of our stochastic action functional for Navier–Stokes solutions, in the same manner as the usual Kelvin Theorem arises from particle-relabelling symmetry of the standard action functional for Euler solutions [8]. This result strengthens the conjecture made by us in earlier work [12,13] that a "martingale property" of circulations should hold for generalized solutions of the incompressible Euler equations obtained in the zero-viscosity limit. Indeed, the stochastic variational principle for Navier–Stokes solutions considered in the present work is very closely similar to a stochastic least-action principle for generalized solutions of incompressible Euler equations that was developed by Brenier [14–16]. One of the arguments advanced for the "martingale property" in [12] was particle-relabelling symmetry

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in a Brenier-type variational formulation of generalized Euler solutions. That argument, however, did not distinguish an arrow of time, so that fluid circulations might satisfy the martingale property either forward or backward in time. It was subsequently argued in [13] that the backward-martingale property is the correct one, consistent with time-irreversibility in the limit of vanishing viscosity. The present work shows that a small but positive viscosity indeed selects the backward martingale property, as expected for a causal solution.

2. The action principle

The action principle formulated here for Navier–Stokes solutions involves *stochastic flows* [3]. The relevant flows are those which solve a *backward Ito equation*:

$$\begin{cases} \hat{d}_t \mathbf{x}^\varpi(\mathbf{a},t) = \mathbf{u}^\varpi(\mathbf{x}^\varpi(\mathbf{a},t),t)\mathrm{d}t + \sqrt{2\nu}\hat{d}\mathbf{W}^\varpi(t), \quad t < t_f \\ \mathbf{x}^\varpi(\mathbf{a},t_f) = \mathbf{a}. \end{cases} \tag{1}$$

Here $\mathbf{W}^\varpi(t)$, $t \in [t_0, t_f]$ is a d-dimensional Brownian motion on a probability space $(\Omega, P, \mathcal{F})$ which is adapted to a two-parameter filtration $\mathcal{F}_t^{t'}$ of sub-σ-fields of $\mathcal{F}$, with $t_0 \le t < t' \le t_f$. Thus, $\mathbf{W}^\varpi(s) - \mathbf{W}^\varpi(s')$ is $\mathcal{F}_t^{t'}$-measurable for all $t \le s < s' \le t'$. The constant ν that appears in the amplitude of the white-noise term in the SDE (1) will turn out to be the kinematic viscosity in the Navier–Stokes equation. Note that, for such an additive noise as appears in (1), the (backward) Ito and Stratonovich equations are equivalent.

In order to describe the space of flow maps which appear in the action principle, we must make a few slightly technical, preliminary remarks. The random velocity field $\mathbf{u}^\varpi(\mathbf{r},t)$ in Eq. (1) is assumed to be smooth and, in particular, continuous in time, as well as adapted to the filtration $\mathcal{F}_t^{t_f}$, $t < t_f$ backward in time. It then follows from standard theorems (e.g. see Corollary 4.6.6 of [3]) that the solution $\mathbf{x}^\varpi(\mathbf{a},t)$ of (1) is a backward semi-martingale of flows of diffeomorphisms. Conversely, any backward semi-martingale of flows of diffeomorphisms has a backward Stratonovich random infinitesimal generator $\mathring{\hat{\mathbf{F}}}^\varpi(\mathbf{r},t)$ which is a spatially-smooth backward semi-martingale (e.g. see Theorem 4.4.4 of [3]). The class of such flows for which the martingale part of the generator is $\sqrt{2\nu}\mathbf{W}^\varpi(t)$ and for which the bounded-variation part of the generator is absolutely-continuous with respect to dt coincides with the class of solutions of equations of form (1), for all possible choices of $\mathbf{u}^\varpi(\mathbf{r},t)$. Clearly, the random fields $\mathbf{u}^\varpi(\mathbf{r},t)$ and $\mathbf{x}^\varpi(\mathbf{a},t)$ uniquely determine each other. We consider here the incompressible case, where $\mathbf{u}^\varpi(\mathbf{r},t)$ is divergence-free and $\mathbf{x}^\varpi(\mathbf{a},t)$ is volume-preserving a.s.

The action is defined as a functional of the backward-adapted random velocity fields $\mathbf{u}^\varpi(\mathbf{r},t)$—or, equivalently, of the random flow maps $\mathbf{x}^\varpi(\mathbf{a},t)$—by the formula

$$S[\mathbf{x}] = \int P(\mathrm{d}\varpi)\int_{t_0}^{t_f}\mathrm{d}t\int \mathrm{d}^d r\,\frac{1}{2}|\mathbf{u}^\varpi(\mathbf{r},t)|^2 \tag{2}$$

when this is well defined and as $+\infty$ otherwise. The *variational problem* (VP) is to find a stationary point of this action such that $\mathbf{x}^\varpi(\mathbf{a},t_f) = \mathbf{a}$ and $\mathbf{x}^\varpi(\mathbf{a},t_0) = \boldsymbol{\varphi}^\varpi(\mathbf{a})$ for $P - a.e.\ \varpi$, where $\boldsymbol{\varphi}^\varpi(\mathbf{a})$ is a given random field of volume-preserving diffeomorphisms of the flow domain. It is interesting that this problem is very similar to that considered by Brenier [14–16] for generalized Euler solutions. The above problem leads instead to the incompressible Navier–Stokes equation, in the following precise sense:

Proposition 1. *A stochastic flow $\mathbf{x}^\varpi(\mathbf{a},t)$ which satisfies both the initial and final conditions is a solution of the above variational problem if and only if $\mathbf{u}^\varpi(\mathbf{r},t)$ solves the incompressible Navier–Stokes equation with viscosity $\nu > 0$*

$$\partial_t \mathbf{u}^\varpi + (\mathbf{u}^\varpi\cdot\nabla)\mathbf{u}^\varpi = -\nabla p^\varpi + \nu\Delta\mathbf{u}^\varpi, \quad P\text{-}a.s. \tag{3}$$

where kinematic pressure p^ϖ is chosen so that $\nabla\cdot\mathbf{u}^\varpi = 0$.

Proof. Making a variation $\delta\mathbf{u}^\varpi(\mathbf{r},t)$ in the random velocity field, the Eq. (1) becomes

$$\begin{cases} \hat{d}_t\delta\mathbf{x}^\varpi(\mathbf{a},t) \\ \quad = [\delta\mathbf{x}^\varpi(\mathbf{a},t)\cdot\nabla_r\mathbf{u}^\varpi(\mathbf{x}^\varpi,t) + \delta\mathbf{u}^\varpi(\mathbf{x}^\varpi,t)]\mathrm{d}t, \quad t < t_f \\ \delta\mathbf{x}^\varpi(\mathbf{a},t_f) = \mathbf{0}. \end{cases} \tag{4}$$

Since the VP requires that $\mathbf{x}^\varpi(\mathbf{a},t_0) = \boldsymbol{\varphi}^\varpi(\mathbf{a})$, one can only consider variations such that, also, $\delta\mathbf{x}^\varpi(\mathbf{a},t_0) = \mathbf{0}$. (We shall consider below an alternative approach with a Lagrange multiplier that permits unconstrained variations.) This equation may also be written as

$$\hat{d}_t\delta\mathbf{x}^\varpi(\mathbf{a},t) - (\nabla_r\mathbf{u}^\varpi(\mathbf{x}^\varpi,t))^\top\delta\mathbf{x}^\varpi(\mathbf{a},t)\mathrm{d}t = \delta\mathbf{u}^\varpi(\mathbf{x}^\varpi,t)\mathrm{d}t \tag{5}$$

for $t < t_f$ and then easily solved by Duhamel's formula (backward in time) to give $\delta\mathbf{x}^\varpi(\mathbf{a},t)$ in terms of $\delta\mathbf{u}^\varpi(\mathbf{r},t)$. Since the martingale term vanished under variation, the process $\delta\mathbf{x}^\varpi(\mathbf{a},t)$ is of bounded variation and clearly adapted to the backward filtration $\mathcal{F}_t^{t_f}$, $t < t_f$. Conversely, any such flow variation will determine the corresponding velocity variation $\delta\mathbf{u}^\varpi(\mathbf{r},t)$ by Eq. (5) directly. Lastly, note that the volume-preserving condition $\det(\nabla_a\mathbf{x}^\varpi(\mathbf{a},t)) = 1$ becomes

$$\nabla_r\cdot\delta\mathbf{x}^\varpi(\mathbf{a}^\varpi,t) = 0 \tag{6}$$

under variation, where $\mathbf{a}^\varpi(\mathbf{r},t)$ is the "back-to-labels" map inverse to the flow map $\mathbf{x}^\varpi(\mathbf{a},t)$. Because these maps are diffeomorphisms, we see that the Eulerian variation of the flow map, $\delta\bar{\mathbf{x}}^\varpi(\mathbf{r},t) \equiv \delta\mathbf{x}^\varpi(\mathbf{a}^\varpi(\mathbf{r},t),t)$, is an arbitrary divergence-free field.

With these preparations, we obtain for the variation of the action (2):

$$\begin{aligned} \delta S[\mathbf{x}] &= \int P(\mathrm{d}\varpi)\int_{t_0}^{t_f}\mathrm{d}t\int \mathrm{d}^d r\,\mathbf{u}^\varpi(\mathbf{r},t)\cdot\delta\mathbf{u}^\varpi(\mathbf{r},t) \\ &= \int P(\mathrm{d}\varpi)\int d^d a\int_{t_0}^{t_f}\mathbf{u}^\varpi(\mathbf{x}^\varpi(\mathbf{a},t),t) \\ &\quad\cdot\left[\hat{d}_t\delta\mathbf{x}^\varpi(\mathbf{a},t) - \delta\mathbf{x}^\varpi(\mathbf{a},t)\cdot\nabla_r\mathbf{u}^\varpi(\mathbf{x}^\varpi,t)\mathrm{d}t\right] \\ &= -\int P(\mathrm{d}\varpi)\int d^d a\int_{t_0}^{t_f}\left[\hat{d}_t\mathbf{u}^\varpi(\mathbf{x}^\varpi,t)\right. \\ &\quad\left. + \nabla_r\left(\frac{1}{2}|\mathbf{u}^\varpi|^2\right)\Big|_{\mathbf{x}^\varpi}\mathrm{d}t\right]\cdot\delta\mathbf{x}^\varpi(\mathbf{a},t). \end{aligned} \tag{7}$$

In the second line we employed (5). In the third line we integrated by parts, using the facts that $\delta\mathbf{x}^\varpi(\mathbf{a},t_f) = \delta\mathbf{x}^\varpi(\mathbf{a},t_0) = 0$ and that $\delta\mathbf{x}^\varpi(\mathbf{a},t)$ is a bounded-variation process, so that the quadratic variation vanishes: $\hat{d}_t\langle\mathbf{u}^\varpi(\mathbf{x}^\varpi,t),\delta\mathbf{x}^\varpi(\mathbf{a},t)\rangle = 0$. We note that the final gradient term vanishes, because $\delta\bar{\mathbf{x}}^\varpi(\mathbf{r},t)$ is divergence-free. We can evaluate the remaining term using the chain rule

$$\hat{d}_t\mathbf{u}^\varpi(\mathbf{x}^\varpi,t) = \partial_t\mathbf{u}^\varpi(\mathbf{x}^\varpi,t)\mathrm{d}t + (\mathbf{x}^\varpi(\mathbf{a},\circ\hat{\mathrm{d}}t)\cdot\nabla)\mathbf{u}^\varpi(\mathbf{x}^\varpi,t), \tag{8}$$

in terms of the backward Stratonovich differential. This result can also be written using Ito calculus. Calculating from (1) and (8) the quadratic variation

$$\sqrt{2\nu}\hat{d}_t\langle W_j^\varpi(t),\partial_{x_j}\mathbf{u}^\varpi(\mathbf{x}^\varpi,t)\rangle = 2\nu\Delta\mathbf{u}^\varpi(\mathbf{x}^\varpi,t)\mathrm{d}t,$$

one obtains the backward Ito equation

$$\begin{aligned} \hat{d}_t\mathbf{u}^\varpi(\mathbf{x}^\varpi,t) &= [\partial_t\mathbf{u}^\varpi + (\mathbf{u}^\varpi\cdot\nabla)\mathbf{u}^\varpi - \nu\Delta\mathbf{u}^\varpi](\mathbf{x}^\varpi,t)\mathrm{d}t \\ &\quad + \sqrt{2\nu}(\hat{d}\mathbf{W}^\varpi(t)\cdot\nabla)\mathbf{u}^\varpi(\mathbf{x}^\varpi,t). \end{aligned} \tag{9}$$

A crucial point is that the martingale part of (9) vanishes when the expression is substituted back into (7), since both $(\nabla\mathbf{u}^{\varpi})(\mathbf{x}^{\varpi},t)$ and $\delta\mathbf{x}^{\varpi}(\mathbf{a},t)$ are adapted to the backward filtration $\mathcal{F}_t^{t_f}$, $t<t_f$. Thus, the final result is

$$\delta S[\mathbf{x}] = \int P(d\varpi)\int_{t_0}^{t_f} dt \int d^d r\,[\partial_t\mathbf{u}^{\varpi} + (\mathbf{u}^{\varpi}\cdot\nabla)\mathbf{u}^{\varpi} - \nu\Delta\mathbf{u}^{\varpi}](\mathbf{r},t)\cdot\delta\bar{\mathbf{x}}^{\varpi}(\mathbf{r},t). \tag{10}$$

Since the integrands are smooth in space and continuous in time and since the flow variation is an arbitrary divergence-free field, the theorem statement follows. □

There are alternative formulations of the VP which should be mentioned. Rather than performing the variation with the constraint $\mathbf{x}^{\varpi}(\mathbf{a},t_0)=\boldsymbol{\varphi}^{\varpi}(\mathbf{a})$, one can instead modify the action with a Lagrange multiplier term:

$$S'[\mathbf{x},\mathbf{v}_0] = S[\mathbf{x}] + \int P(d\varpi)\int d^d a\,\mathbf{v}_0^{\varpi}(\mathbf{a})\cdot[\mathbf{x}^{\varpi}(\mathbf{a},t_0)-\boldsymbol{\varphi}^{\varpi}(\mathbf{a})]. \tag{11}$$

Varying with respect to the Lagrange multiplier $\mathbf{v}_0^{\varpi}$ yields the constraint, whereas an unconstrained variation with respect to $\mathbf{x}^{\varpi}$ yields

$$\delta S'[\mathbf{x},\mathbf{v}_0] = \int P(d\varpi)\int_{t_0}^{t_f} dt\int d^d r\,[\partial_t\mathbf{u}^{\varpi}+(\mathbf{u}^{\varpi}\cdot\nabla)\mathbf{u}^{\varpi} - \nu\Delta\mathbf{u}^{\varpi}](\mathbf{r},t)\cdot\delta\bar{\mathbf{x}}^{\varpi}(\mathbf{r},t) + \int P(d\varpi) \times \int d^d a\,[\mathbf{v}_0^{\varpi}(\mathbf{a}) - \mathbf{u}^{\varpi}(\mathbf{x}^{\varpi}(\mathbf{a},t_0),t_0)]\cdot\delta\mathbf{x}^{\varpi}(\mathbf{a},t_0). \tag{12}$$

The second term on the righthand side arises partly from the Lagrange multiplier term and partly from integration-by-parts in time. It follows that $\mathbf{v}_0^{\varpi}(\mathbf{a})$ can be identified as the Lagrangian fluid velocity at the initial time t_0 and, likewise, $\mathbf{u}_0^{\varpi}(\mathbf{r}) = \mathbf{v}_0^{\varpi}((\boldsymbol{\varphi}^{\varpi})^{-1}(\mathbf{r}))$ is the Eulerian fluid velocity at time t_0. Another alternative is to add a Lagrange multiplier term $\int P(d\varpi)\int_{t_0}^{t_f} dt\int d^d a\,\pi^{\varpi}(\mathbf{a},t)\ln\det(\nabla_a\mathbf{x}^{\varpi}(\mathbf{a},t))$ for the incompressibility constraint and to allow variations over flows which are not volume-preserving. In that case $\pi^{\varpi}(\mathbf{a},t)$ is the Lagrangian pressure field and $p^{\varpi}(\mathbf{r},t) = \pi^{\varpi}(\mathbf{a}^{\varpi}(\mathbf{r},t),t)$ is the Eulerian pressure.

There are several mathematical questions that deserve to be pursued. Some technical issues remain, e.g. the precise degree of smoothness of solutions required to make the above argument fully rigorous, etc. It would also be very interesting to know under what conditions the solution of the VP corresponds to a minimum of the action and not just a stationary point. Although we have characterized the solutions of the VP, we have not proved either their existence or their uniqueness. We just remark on the latter point that a unique stationary point certainly exists if the initial velocity $\mathbf{u}_0(\mathbf{r})$ is deterministic and if the Navier–Stokes equation has a unique solution $\mathbf{u}(\mathbf{r},t)$ over the time interval $[t_0,t_f]$ for that initial datum. This will be the case, for example, if the initial velocity is smooth enough and the Reynolds number $Re = UL/\nu$ is low enough. In that case, the solution of the VP is also deterministic and is given by the corresponding Navier–Stokes solution.

3. The Stochastic Kelvin Theorem

We now mention a closely related result of [1]:

Proposition 2 (*Constantin & Iyer [1]*). *The following two properties for a divergence-free velocity field* $\mathbf{u}(\mathbf{r},t)$ *are equivalent:* (i) *For all closed, rectifiable loops C and for any pair of times* $t_0\le t<t'\le t_f$,

$$\oint_C \mathbf{u}(\mathbf{a},t')\cdot d\mathbf{a} = \int P(d\varpi)\left[\oint_{\mathbf{x}_{t',t}^{\varpi}(C)}\mathbf{u}(\mathbf{r},t)\cdot d\mathbf{r}\right], \tag{13}$$

where $\mathbf{x}_{t',t}^{\varpi}(\mathbf{a})$ *are the stochastic backward flows which solve Eq.* (1) *with velocity* $\mathbf{u}(\mathbf{r},t)$ *for times* $t<t'$ *with final condition* $\mathbf{x}_{t',t'}^{\varpi}(\mathbf{a})=\mathbf{a}$*; and,* (ii) *the velocity* $\mathbf{u}(\mathbf{r},t)$ *satisfies the incompressible Navier–Stokes equation over the time interval* $[t_0,t_f]$.

This is just a slight restatement of Theorem 2.2 and Proposition 2.9 of [1]. The result (13) is a stochastic version of the Kelvin Theorem on conservation of circulations for incompressible Euler solutions. Although circulations are not conserved for Navier–Stokes solutions in the usual sense, (13) states that circulations on loops advected by the stochastic Lagrangian flow are a martingale backward in time. This property of the Navier–Stokes solutions is closely related to the "martingale conjecture" of Eyink [12] for generalized Euler solutions obtained in the limit $\nu\to 0$. This connection will be discussed in detail in the next section.

It is well known that the Kelvin Theorem for incompressible Euler equations can be derived by the least-action principle as a consequence of an infinite-dimensional symmetry [8], called "particle-relabelling symmetry" and corresponding to the group of volume-preserving diffeomorphisms of the flow domain. This may be done by applying the general Noether Theorem relating symmetries and conservation laws. In this section, we shall show that the result of Proposition 2 can be similarly derived from the stochastic action principle of Section 1 as a consequence of particle-relabelling symmetry. We shall not make use of the Noether Theorem but, following Salmon [8], shall instead employ a more direct method of Lanczos [17] based on time-dependent symmetry transformations.

Suppose given a smooth 1-parameter family $\{\boldsymbol{\varphi}(\mathbf{a},t),\ t\in[t_0,t_f]\}$ of volume-preserving diffeomorphisms satisfying $\boldsymbol{\varphi}(\mathbf{a},t_f) = \boldsymbol{\varphi}(\mathbf{a},t_0) = \mathbf{a}$. Then any incompressible flow $\mathbf{x}(\mathbf{a},t)$ may be deformed into another such flow

$$\mathbf{x}_{\varphi}(\mathbf{a},t)\equiv\mathbf{x}(\boldsymbol{\varphi}(\mathbf{a},t),t) \tag{14}$$

with initial and final values the same. It follows furthermore from (14) by chain rule that

$$\hat{d}_t\mathbf{x}_{\varphi}(\mathbf{a},t) = \hat{d}_t\mathbf{x}(\bar{\mathbf{a}},t) + (\dot{\boldsymbol{\varphi}}(\mathbf{a},t)\cdot\nabla_{\bar{a}})\mathbf{x}(\bar{\mathbf{a}},t)dt, \tag{15}$$

for $\bar{\mathbf{a}}=\boldsymbol{\varphi}(\mathbf{a},t)$. If $\mathbf{x}^{\varpi}(\mathbf{a},t)$ is the solution of the stochastic equation (1), then (15) implies that $\mathbf{x}_{\phi}^{\varpi}(\mathbf{a},t)$ also satisfies (1) for the modified velocity field

$$\mathbf{u}_{\phi}^{\varpi}(\mathbf{r},t) = \mathbf{u}^{\varpi}(\mathbf{r},t) + (\dot{\boldsymbol{\varphi}}(\mathbf{a}^{\varpi},t)\cdot\nabla_{\bar{a}})\mathbf{x}^{\varpi}(\bar{\mathbf{a}}^{\varpi},t), \tag{16}$$

where we employ the shorthands $\mathbf{a}^{\varpi}=\mathbf{a}^{\varpi}(\mathbf{r},t)$ and $\bar{\mathbf{a}}^{\varpi} = \boldsymbol{\varphi}(\mathbf{a}^{\varpi}(\mathbf{r},t),t)$. It is easy to see from (16) that $\mathbf{u}_{\phi}^{\varpi}(\mathbf{r},t)$ is adapted to the backward filtration $\mathcal{F}_t^{t_f}$, $t<t_f$ whenever the original velocity $\mathbf{u}^{\varpi}(\mathbf{r},t)$ is adapted.

In infinitesimal form, $\boldsymbol{\varphi}(\mathbf{a},t) = \mathbf{a}+\varepsilon\mathbf{g}(\mathbf{a},t)+O(\varepsilon^2)$ with $\nabla_a\cdot\mathbf{g}(\mathbf{a},t)=0$ and $\mathbf{g}(\mathbf{a},t_f)=\mathbf{g}(\mathbf{a},t_0)=\mathbf{0}$. Formula (16) then yields the velocity variation

$$\delta\mathbf{u}^{\varpi}(\mathbf{r},t) = \varepsilon(\dot{\mathbf{g}}(\mathbf{a}^{\varpi},t)\cdot\nabla_a)\mathbf{x}^{\varpi}(\mathbf{a}^{\varpi},t)+O(\varepsilon^2). \tag{17}$$

The corresponding variation of the action is thus [see the first line of (7)]:

$$0=\delta S[\mathbf{x}] = \varepsilon\int P(d\varpi)\int_{t_0}^{t_f}dt\int d^d a\,\dot{\mathbf{g}}(\mathbf{a},t)\cdot\mathbf{w}^{\varpi}(\mathbf{a},t)+O(\varepsilon^2), \tag{18}$$

where $\mathbf{w}^{\varpi}(\mathbf{a},t)$ is the *stochastic Weber velocity* [1,2,8]

$$\mathbf{w}^{\varpi}(\mathbf{a},t) = \nabla_a\mathbf{x}^{\varpi}(\mathbf{a},t)\cdot\mathbf{u}^{\varpi}(\mathbf{x}^{\varpi}(\mathbf{a},t),t). \tag{19}$$

We can conclude that the $O(\epsilon)$ variation in (18) must vanish for any smooth divergence-free function $\mathbf{g}(\mathbf{a},t)$ with $\mathbf{g}(\mathbf{a},t_f)=\mathbf{g}(\mathbf{a},t_0)=\mathbf{0}$. Taking limits of such functions, one may approximate a divergence-free distribution of the form

$$\mathbf{g}_{C,t,t'}(\mathbf{a},\tau)=\chi_{[t',t]}(\tau)\oint_C \delta^d(\mathbf{a}-\mathbf{a}')\,\mathrm{d}\mathbf{a}' \tag{20}$$

for any closed, rectifiable loop C and any $t_f>t'>t>t_0$. Here $\chi_{[t,t']}(\tau)$ is the characteristic function of the interval $[t,t']$ and $\delta^d(\mathbf{a}-\mathbf{a}')$ is the Dirac delta-distribution. If we use the property of the Weber velocity that

$$\oint_C \mathbf{w}^\varpi(\mathbf{a}',t)\boldsymbol{\cdot}\mathrm{d}\mathbf{a}'=\oint_{\mathbf{x}^\varpi(C,t)}\mathbf{u}^\varpi(\mathbf{r},t)\boldsymbol{\cdot}\mathrm{d}\mathbf{r}, \tag{21}$$

it then follows that

$$\int P(\mathrm{d}\varpi)\oint_{\mathbf{x}^\varpi(C,t')}\mathbf{u}^\varpi(\mathbf{r},t')\boldsymbol{\cdot}\mathrm{d}\mathbf{r} = \int P(\mathrm{d}\varpi)\oint_{\mathbf{x}^\varpi(C,t)}\mathbf{u}^\varpi(\mathbf{r},t)\boldsymbol{\cdot}\mathrm{d}\mathbf{r}, \tag{22}$$

for any $t_f>t'>t>t_0$. Taking the limit $t'\to t_f$ allows us to identify the above constant average as $\oint_C \overline{\mathbf{u}}(\mathbf{r},t_f)\boldsymbol{\cdot}\mathrm{d}\mathbf{r}$ with $\overline{\mathbf{u}}(\mathbf{r},t)=\int P(\mathrm{d}\varpi)\mathbf{u}^\varpi(\mathbf{r},t)$, since $\mathbf{x}^\varpi(\mathbf{a},t_f)=\mathbf{a}$ *P-a.s.* When the velocity field that solves the VP is deterministic, i.e. corresponds to a unique Navier–Stokes solution $\mathbf{u}(\mathbf{r},t)$, then this result gives the statement (13) of Proposition 2 for the special case where $t'=t_f$. However, the VP may be applied not only over the entire interval $[t_0,t_f]$, but over any subinterval $[t,t']$ as well and this yields the general case.

4. Irreversibility and the zero-viscosity limit

At first sight, it is strange to obtain the dissipative Navier–Stokes equation from a principle of least-action, which ordinarily leads to time-reversible equations. There is no paradox, however, since an "arrow-of-time" is built into the stochastic action principle of Section 1. We may say that this is a *causal* variational principle, since labels are assigned at the final time and variations are over prior histories. The VP may be recast instead to be anti-causal, with fluid particle labels assigned at the initial time t_0 and with flow maps solving a forward Ito equation:

$$\begin{cases} d_t\mathbf{x}^\varpi(\mathbf{a},t)=\mathbf{u}^\varpi(\mathbf{x}^\varpi(\mathbf{a},t),t)\mathrm{d}t+\sqrt{2\nu}\,d\mathbf{W}^\varpi(t), & t>t_0\\ \mathbf{x}^\varpi(\mathbf{a},t_0)=\mathbf{a}. \end{cases} \tag{23}$$

The random velocity field $\mathbf{u}^\varpi(\mathbf{r},t)$ must now be adapted to the forward filtration $\mathcal{F}_{t_0}^t$, $t>t_0$. An exact analogue of Proposition 1 holds, but with the conclusion that the velocity must satisfy

$$\partial_t\mathbf{u}^\varpi+(\mathbf{u}^\varpi\boldsymbol{\cdot}\nabla)\mathbf{u}^\varpi=-\nabla p^\varpi-\nu\Delta\mathbf{u}^\varpi,\quad P\text{-}a.s. \tag{24}$$

or the *negative-viscosity* Navier–Stokes equation. An analogue of Proposition 2 also holds, in the form

$$\oint_C \mathbf{u}(\mathbf{a},t')\boldsymbol{\cdot}\mathrm{d}\mathbf{a}=\int P(\mathrm{d}\varpi)\left[\oint_{\mathbf{x}^\varpi_{t',t}(C)}\mathbf{u}(\mathbf{r},t)\boldsymbol{\cdot}\mathrm{d}\mathbf{r}\right],\quad t_0\le t'<t\le t_f, \tag{25}$$

with circulations at the present time given anti-causally as averages over future values. The process of circulations in this case is a forward martingale. The proofs of all of the above statements follow by straightforward modifications of the previous arguments for the causal case.

Conservation of energy is another property of Hamiltonian systems derived from the least-action principle, as a consequence of time-translation invariance. For example, consider a standard incompressible Euler fluid, with the action functional

$$S[\mathbf{x}]=\frac{1}{2}\int_{t_0}^{t_f}\mathrm{d}t\int \mathrm{d}^d r\,|\mathbf{u}(\mathbf{r},t)|^2. \tag{26}$$

Following the procedure of Lanczos [17], one considers an arbitrary increasing function $\tau(t)$ on the interval $[t_0,t_f]$, with $\tau(t_0)=t_0$ and $\tau(t_f)=t_f$, and defines a modified flow

$$\mathbf{x}_\tau(\mathbf{a},t)=\mathbf{x}(\mathbf{a},\tau(t)). \tag{27}$$

When $\dot{\mathbf{x}}(\mathbf{a},t)=\mathbf{u}(\mathbf{x}(\mathbf{a},t),t)$, then $\mathbf{x}_\tau(\mathbf{a},t)$ satisfies the analogous equation with

$$\mathbf{u}_\tau(\mathbf{r},t)=\dot{\tau}(t)\mathbf{u}(\mathbf{r},\tau(t)). \tag{28}$$

In an infinitesimal form, $\tau(t)=t+\varepsilon\delta(t)+O(\varepsilon^2)$ with $\delta(t_f)=\delta(t_0)=0$, corresponding to a time-translation by a time-dependent shift. The variation in the velocity resulting from (28) is $\delta\mathbf{u}(\mathbf{r},t)=(\mathrm{d}/\mathrm{d}t)[\delta(t)\mathbf{u}(\mathbf{r},t)]$, which implies a variation of the action

$$\delta S[\mathbf{x}]=\int_{t_0}^{t_f}\mathrm{d}t\,\dot{\delta}(t)\int \mathrm{d}^d r\,\frac{1}{2}|\mathbf{u}(\mathbf{r},t)|^2. \tag{29}$$

From the stationarity of the action it follows that kinetic energy $E(t)=\frac{1}{2}\int \mathrm{d}^d r\,|\mathbf{u}(\mathbf{r},t)|^2$ is conserved.

This argument is not valid, however, for the stochastic action principle of Section 1. Indeed, if $\mathbf{x}^\varpi(\mathbf{a},t)$ solves the stochastic equation (1) for backward Lagrangian trajectories, then the time-reparameterized flow (27) satisfies

$$\hat{d}_t\mathbf{x}^\varpi_\tau(\mathbf{a},t)=\mathbf{u}^\varpi_\tau(\mathbf{x}^\varpi_\tau(\mathbf{a},t),t)\mathrm{d}t+\sqrt{2\nu\dot{\tau}(t)}\,\hat{d}\widetilde{\mathbf{W}}^\varpi(t), \tag{30}$$

where $\mathbf{u}^\varpi_\tau(\mathbf{r},t)$ is given by the analogue of (28) and $\widetilde{\mathbf{W}}^\varpi(t)$ is a Brownian motion defined on the same probability space as $\mathbf{W}^\varpi(t)$. This is a consequence of standard results on time-change in stochastic differential equations (e.g. see [18], Ch.IV, Section 7). We thus see that the reparameterization (27) leads to a flow map which is outside the class obeying an equation of the form (1) and for which the stochastic action (2) is formally $+\infty$. Thus the argument leading to energy-conservation based on time-translation invariance of the action is no longer valid.

One interest of the characterization of Navier–Stokes solutions via an action principle is that it may give some hint as to the character of their zero-viscosity limit. It was long ago conjectured by Onsager [19] that singular solutions of the Euler equations may result from that limit, relevant to the description of turbulent energy dissipation at high Reynolds numbers. For recent reviews, see [20,21]. The variational principle formulated for the Navier–Stokes solutions in the present work is similar to that of Brenier [16] for generalized Euler solutions, in which deterministic Lagrangian trajectories are also replaced by distributions over histories. There is other evidence to suggest that this may be a physical feature of the zero-viscosity limits of Navier–Stokes solutions, based upon recent results in a simpler problem, the Kraichnan model of random advection by a rough velocity field that is white-noise in time [22,5]. Unlike the smooth-velocity case considered in [3], it has been shown for the case of rough velocities that the solutions of the stochastic equations (1) and (23) for backward and forward Lagrangian trajectories do *not* become deterministic in the limit as $\nu\to 0$ [23–29]. Instead, there are unique and nontrivial probability distributions on Lagrangian histories in the limit, a property referred to as "spontaneous stochasticity". This property is a direct consequence of Richardson's law of 2-particle turbulent diffusion [30] and thus extends very plausibly to Navier–Stokes turbulence in the limit of large Reynolds numbers.

The results discussed above helped to motivate the conjecture in [12,13] that a martingale property of circulations should hold for Euler solutions obtained in the limit $\nu \to 0$. The derivation of the stochastic Kelvin theorem in the present paper based on particle-relabelling symmetry is closely related to a similar argument in [12] for the martingale property of circulations in generalized Euler solutions (see Section 4 there). Note, however, that it was incorrectly proposed in [12] that circulations for such Euler solutions should be martingales forward in time, and this conjecture was only later emended to a backward-martingale property in [13]. The present work shows that the backward-martingale property is indeed the natural one, which could be expected to hold for dissipative Euler solutions obtained as the zero-viscosity limit of Navier–Stokes solutions.

5. Final remarks

The stochastic Lagrangian representation of Constantin and Iyer [1] and our closely related variational formulation should clearly extend to a wide class of Hamiltonian fluid-mechanical models with added Laplacian dissipation. In a forthcoming paper [31] we prove the analogous results for several non-ideal (resistive and viscous) plasma models, including the two-fluid model of electron–ion plasmas, Hall magnetohydrodynamics (MHD), and standard MHD. As we shall show there, those models possess two stochastic Lagrangian conservation laws, one corresponding to the Alfvén Theorem [32] on conservation of magnetic flux and another corresponding to a generalized Kelvin Theorem [33,34]. In following work we shall apply these results to important physical problems of magnetic reconnection and magnetic dynamo, especially in turbulent MHD regimes.

As noted earlier, similar stochastic least-action principles have recently been proposed for the incompressible Navier–Stokes equations [9–11]. A detailed discussion of the relation of these different variational principles to ours, as well as a rigorous treatment of the latter, will be the subject of future work. It is worth remarking that there is another variational principle for fluid equations, Onsager's principle of least dissipation [35,36], which determines the probability of molecular fluctuations away from hydrodynamic behavior in terms of the dissipation required to produce them. A modern formulation is presented in [37], and [38] gives a rigorous derivation of Onsager's principle for incompressible Navier–Stokes in a microscopic lattice-gas model. It would be interesting to know if any relation exists between the least-action and least-dissipation principles.

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