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On collapse of wave maps

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ABSTRACT

We derive the universal collapse law of degree 1 equivariant wave maps (solutions of the sigma model) from the 2 + 1 Minkowski space–time, to the 2-sphere. To this end, we introduce a nonlinear transformation from original variables to blowup ones. Our formal derivations are confirmed by numerical simulations.

1. Introduction

In this paper, we investigate the phenomenon of collapse of degree 1 equivariant wave maps from the 2 + 1 Minkowski space–time, $\mathbb{M}^{2+1}$, to the 2-sphere, S^2. Besides purely mathematical interest and relation to the σ-model of particle physics, the study of the blowup phenomena for such maps is motivated by the recent efforts to understand the singularity formation in general relativity [1].

A wave map, Φ, from a $(d+1)$-dimensional Minkowski space–time, $\mathbb{M}^{d+1}$, with the Minkowski metric $\eta = \mathrm{diag}(1, -1, \ldots, -1)$, to a Riemannian manifold, N, with a metric (g_{ab}), is a critical point of the action functional, given as

$$S(\Phi) := \frac{1}{2}\int_{\mathbb{M}^{d+1}} \langle \partial_\mu \Phi, \partial^\mu \Phi\rangle,$$

and known as the σ-model. Here $\langle \partial_\mu \Phi, \partial^\mu \Phi\rangle$ is the Riemann scalar product in N, which, in local coordinates, is $g_{ab}\partial_\mu \Phi^a \partial^\mu \Phi^b$, as usual, $\partial_\mu = \frac{\partial}{\partial x^\mu}$ and $\partial^\mu := \eta^{\mu\nu}\partial_\nu$, and we assume the summation over repeated indices $\mu = 1, \ldots, d+1$, $a, b, c = 1, \ldots, \dim(N)$, $i, j = 1, \ldots, d$. Critical points of $S(\Phi)$ satisfy the Euler–Lagrange equation

$$\partial_\mu \partial^\mu \Phi^a + \Gamma^a_{bc}(\Phi)\partial_\mu \Phi^b \partial^\mu \Phi^c = 0,$$

where $\Gamma^a_{bc}(\Phi)$ is the Christoffel symbols on N. This system of nonlinear PDEs is Hamiltonian, and in particular has conserved energy, $\mathcal{E}(\Phi) := \frac{1}{2}\int_{\mathbb{R}^d} g_{ab}\partial_i \Phi^a \partial^i \Phi^b$, and is scale invariant in the sense that if $\Phi(x)$ is a solution then so is $\Phi(\lambda x)$. The energy, $\mathcal{E}(\Phi)$, is transformed under scaling as

$$\mathcal{E}(\Phi_\lambda) = \lambda^{2-d}\mathcal{E}(\Phi),$$

where $\Phi_\lambda(x) = \Phi(\lambda x)$. Thus the case $d = 2$ of interest for us is the energy critical.

For a map, Φ, to have finite energy, it should converge to a constant at infinity. In this case for each moment of time, t, Φ can be extended to a continuous map from S^d to N taking the point at infinity to the limit of $\Phi(x)$ at the spatial infinity. Then one can define the degree, $\deg \Phi$, as the homotopy class of Φ as a map from S^d to N. This degree is conserved under the dynamics generated by the Euler–Lagrange equations above.

In the most important case $N = G/H$, where G is a compact Lie group and H is its subgroup, specifically, $G = \mathrm{SO}(n+1)$ and $H = \mathrm{SO}(n)$, so that $N = S^n$. This is exactly our case, with $d = n = 2$, i.e. $\Phi : \mathbb{M}^{2+1} \to S^2$, and consequently the degree of Φ is an integer (the degree for maps from S^2 to S^2).

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If we consider S^2 to be embedded in $\mathbb{R}^3$ in the standard way, then Φ can be thought of as a map from $\mathbb{R}^{2+1}$ to $\mathbb{R}^3$, satisfying $|\Phi| = 1$, and the action can be written as $S(\Phi) = \frac{1}{2}\int\left(\int_{\mathbb{R}^2}|\partial_0\Phi|^2 \mathrm{d}^2x - \mathcal{E}(\Phi)\right)\mathrm{d}x_0$, where

$$\mathcal{E}(\Phi) = \frac{1}{2}\int_{\mathbb{R}^2}\partial_i\Phi\cdot\partial_i\Phi \mathrm{d}^2x$$

is the energy. ($a\cdot b$ denotes the dot product in $\mathbb{R}^3$) Moreover, the degree is given by $\deg\Phi = \frac{1}{8\pi}\int\epsilon^{ij}\Phi\cdot(\partial_i\Phi\wedge\partial_j\Phi)\mathrm{d}^2x$, where ϵ^{ij} is the Levi-Civita antisymmetric symbol with $\epsilon^{12} = -\epsilon^{21} = 1$, $\epsilon^{11} = \epsilon^{22} = 0$, and one has the Bogomolnyi-type identity

$$\mathcal{E}(\Phi) = \pm 4\pi\deg\Phi + \frac{1}{4}\int|\partial_i\Phi\pm\epsilon_{ij}\Phi\wedge\partial_j\Phi|^2\mathrm{d}^2x,$$

which implies that the minimizers in every homotopy class satisfy self-dual/anti-dual equations, $\partial_i\Phi\pm\epsilon^{ij}\Phi\wedge\partial_j\Phi = 0$ (as well as the Bogomolnyi-type inequality $\mathcal{E}(\Phi)\geq 4\pi|\deg\Phi|$). For any $\deg\Phi = k$, these equations have explicit solutions (harmonic or anti-harmonic maps), Φ_k^{stat}, given by $\Phi_k^{\mathrm{stat}}(\rho,\phi,t) = (U_k(\rho,t),k\phi)$, where (ρ,ϕ) are the polar coordinates in $\mathbb{R}^2$ and (φ,θ) are the spherical coordinates in S^2 and $U_k(\rho) = 2\arctan\rho^k$.

Among the maps of the degree k the simplest, most symmetric maps are the 'radially symmetric' or equivariant maps which are of the form $\Phi_k(\rho,\phi,t) = (u_k(\rho,t),k\phi)$, where, to repeat, (ρ,ϕ) are the polar coordinates in $\mathbb{R}^2$ and (φ,θ) are the spherical coordinates in S^2. Then the Euler–Lagrange equation for Φ reduces to the equation:

$$\ddot{u} = \Delta u - \frac{k^2}{2\rho^2}\sin(2u) \tag{1}$$

for u_k. Here Δ is the $2D$ spherical Laplacian. This equation inherits the key properties of the original equation for Φ, mentioned above: scaling invariance, and existence of the static solutions, $U_k(\rho) = 2\arctan\rho^k$, minimizing the static energy. Moreover, $\deg\Phi_k = Q(u_k) = k$, where

$$Q(u) := \frac{1}{\pi}(u(\infty)-u(0)).$$

Numerical studies of Eq. (1) led to a conjecture that large energy, degree-one initial data develop singularities in finite time and the singularity formation has the universal form of adiabatic shrinking of the degree-one harmonic map from $\mathbb{R}^2$ to S^2 [2]. Later, it was shown by Struwe [3] that the existence of a nontrivial harmonic map is in fact the necessary condition for blowup for 2 + 1 equivariant wave maps. In this paper, we address the question of the dynamics of the blowup process. We show that there is $0 < t_* < \infty$ such that, as $t\to t_*$, we have on bounded domains in $\mathbb{R}^2$

$$u(\rho,t)\approx U(\rho/\lambda(t)),$$

where $U(\rho) = U_1(\rho)$ is the profile of the degree 1 equivariant, static (and in particular harmonic) map, minimizing static energy (see below), and the scaling parameter $\lambda(t)$, satisfies the following second order ODE:

$$\lambda\ddot{\lambda} = \frac{\dot{\lambda}^2}{\ln\left(\frac{a}{\lambda\ddot{\lambda}}\right)},\quad\text{with } a = (1.04)^2\mathrm{e}^{-2}\approx 0.146. \tag{2}$$

We expect that, proceeding as in [4], we can show that the error term in the above relation is $O(\dot{\lambda}^2)$.

Note that Eq. (2) shows that if $\dot{\lambda}|_{t=0} < 0$, then $\dot{\lambda} < 0$, $\ddot{\lambda} > 0$ for $t > 0$ and therefore λ and $|\dot{\lambda}|$ decrease as $t\to t_*$. Since $\dot{\lambda}^2$ is the small parameter in our analysis (adiabatic regime), our approximation improves as $t\to t_*$.

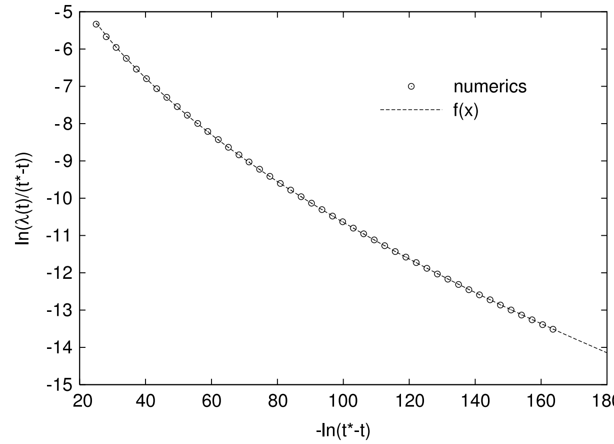

Fig. 1. For a numerical solution that blows up at time t^* we plot $y = \ln\frac{\lambda(t)}{t^*-t}$ as a function of $x = -\ln(t^*-t)$ (circles) and compare it with the analytic formula $y = f(x) = \frac{1}{2}\ln(a) - \sqrt{x+b}$, where $a = 0.146$ and b is a free (non-universal) parameter. Fitting b we get an excellent agreement between numerical and analytical results.

An approximate solution of Eq. (2) with two free parameters (constants of integration), t^* and c, is (see Section 7 below)

$$\sqrt{a}(t^*-t) = \lambda\mathrm{e}^{\ln^{1/2}\left(\frac{c}{\lambda}\right)} + c\frac{\sqrt{\pi}}{2}\mathrm{e}^{1/4}\times\left[1-\Phi\left(-1/2+\ln^{1/2}\left(\frac{c}{\lambda}\right)\right)\right], \tag{3}$$

where $\Phi(x)\equiv\mathrm{erf}(x)$ is the Fresnel integral [5]. An exact solution of Eq. (2) is obtained in Section 7 (see Eq. (80)). A comparison of the leading term of this solution with a numerical solution of Eq. (1) is given in Fig. 1. (The initial datum for the solution shown in the plot was $u(0,\rho) = A\rho^3\mathrm{e}^{-(\rho-2)^2}$, $\frac{\partial}{\partial t}u(0,\rho) = 0$. The scaling factor $\lambda(t)$ was read off from the formula $\frac{\partial}{\partial\rho}u(t,0) = 2/\lambda(t)$. The blowup time was simply taken as the last moment of time before the code crashed.) This figure shows that the two resulting curves are indistinguishable for times sufficiently close to the blowup time.

Observe that like Eqs. (1) and (2) is a Hamiltonian equation. Its Lagrangian is

$$L := h(\dot{\lambda}) - \ln\lambda, \tag{4}$$

where the function h is defined by $h''(x) = -1/(af^{-1}(x^2/a))$ with $f(x) = x\ln(1/x)$ (see Section 8).

The local well-posedness for the wave map equations in Sobolev spaces was proven in [6–8], while the global well-posedness for small initial conditions, in [9–14] (see also [15–22,10,23,24]). The research on the problem of blowup for the wave maps started with numerical work [2,25–27]. (We do not review here related works for nonlinear wave equations.)

The first numerical evidence for singularity formation for $2+1$ equivariant wave maps to the 2-sphere was given in [2]. In this paper, (concerned only with $k = 1$ homotopy) the authors showed that blowup has the form of adiabatic shrinking of the harmonic map and formulated conjectures about blowup for large energy, blowup profile and energy concentration and that $\lambda(t)/(t^*-t)$ must go to zero. As was already mentioned, it was shown rigorously in [28] that the existence of a stationary solution is a necessary condition for the blowup to take place. The blowup scenarios were further numerically investigated in [29,30] (see references therein for additional works).

The first rigorous results on the blowup rate and profile were obtained in [31,32]. In particular, [31] has obtained the lower bound on the contraction rate $\lambda(t)$ for $k\geq 4$ wave maps. As it turned out this lower bound conforms exactly to the dynamical law derived for the $4+1$ Yang–Mills $k=1$ equivariant solutions in [4] using a formal but careful analysis, explained below in this introduction, justified by numerical computations. (Earlier numerical analysis for the latter model was announced in [33] and described more completely in the survey [34].) (It was noticed in [31] (see below), that the $k \geq 2$ wave map equation are similar to the $4+1$ Yang–Mills one for $k=1$.) Finally, for each $b > 1/2$, [32] has constructed special solutions of the $k=1$ equivariant wave map equation, Eq. (1) with $k=1$, which blow up at the rate $\lambda(t) \sim (t^*-t)^{1+b}$.

Eq. (1) belongs to a general class of semilinear wave equations in $\mathbb{R}^{2+1}$ of the form

$$\ddot{u} = \Delta u - \frac{k^2}{\rho^2} f(u). \tag{5}$$

In the case of $f(u) = \frac{1}{2}\sin(2u)$, Eq. (5) is, as was already mentioned, the equation for the profile of the equivariant wave map from the $2+1$ Minkowski space–time of degree k, $\mathbb{R}^{2+1}$, to the 2-sphere, S^2. More generally, (5) is satisfied by equivariant maps for the case when N is the surface of revolution with the metric $g := \mathrm{d}u^2 + g^2(u)\mathrm{d}\varphi^2$, where $g(u)$ is related to $f(u)$ as $f(u) = g(u)g'(u)$.

In the case of $k=2$ and $f(u) = \frac{1}{2}u(u^2-1)$ the corresponding equation,

$$\ddot{u} = \Delta u - \frac{2}{\rho^2}(u^2-1)u, \tag{6}$$

is related to the equation for equivariant Yang–Mills fields of degree 1 in the $4+1$ dimensions.

Note that

(i) Eq. (5) is invariant with respect to the scaling transformation,

$$u(\rho, t) \to u\left(\frac{\rho}{\lambda}, \frac{t}{\lambda}\right);$$

(ii) Eq. (5) can be presented as a Hamiltonian system with the standard symplectic form and the Hamiltonian

$$H(u, v) := \int_0^\infty \left(\frac{1}{2}v^2 + \frac{1}{2}|\nabla u|^2 + \frac{k^2}{\rho^2}F(u)\right)\rho \mathrm{d}\rho, \tag{7}$$

with $F'(u) = f(u)$. The scaling properties of the Hamiltonian $H(u, v)$ imply that the dimension $d=2$ is the critical dimension for Eq. (5). This is the dimension treated in this paper.

We assume now that $f(u)$ is a derivative of a *double-well potential* $F(u)$, i.e. $F(u)$ is nonnegative and has at least two global minima, say at a and b for some $b > a$, with $F(a) = F(b) = 0$, and no minima between a and b ($F(u) = \frac{1}{2}\sin^2 u$ and $a=0, b=\pi$ in the case of $f(u) = \frac{1}{2}\sin(2u)$ and $F(u) = \frac{1}{2}(u^2-1)^2$ and $a = -1, b = 1$ in the case of $k=2$ and $f(u) = \frac{1}{2}u(u^2-1)$). In this case, Eq. (5) has the following features:

(A) For each $k \in \mathbb{N}$, Eq. (5) has static solutions, $U_k(\rho)$ and $U_{-k}(\rho) = -U_k(\rho)$; they have topological degrees $Q(U_k) = k$ and $Q(U_{-k}) = -k$.
(B) For $k=1$, the solution $U_1(\rho)$ is monotonically increasing from a to b, while $U_{-1}(\rho)$ is monotonically decreasing from a to b.
(C) The solution, $U_k(\rho)$, is a minimizer of the static energy functional $E(u)$ under the constrain, $Q(u) = k$, on the topological charge.
(D) Eq. (5) conserves the topological charge $Q(u) := \frac{1}{b-a}(u(\infty) - u(0))$.

Existence of the solutions $U_k(\rho)$ follows from the Bogomolnyi argument; see above. The solutions $U_1(\rho)$ and $U_{-1}(\rho)$ are called the kink solution and anti-kink solution, or simply *kink* and *anti-kink*, respectively. Since the analysis for $k < 0$ can be obtained from the analysis for the case $k > 0$ by simply flipping the signs, in what follows we assume that $k > 0$. Note that though Eq. (5) is scale invariant, its static kink solution $U_k(\rho)$ are not. Hence Eq. (5) has an entire family, $U_k\left(\frac{\rho}{\lambda}\right)$, of kink solutions (symmetry breaking).

From now on *we concentrate on the kink solution*, $U_1(\rho)$ and omit the subindex 1: $U_1(\rho) \equiv U(\rho)$.

There is a feature of Eq. (5) which is not apparent at the first sight but which plays an important role in our analysis of the collapse. The fact that the kink, $U(\rho)$, breaks scale invariance manifests itself in appearance of the dilation zero mode

$$\zeta(\rho) := \frac{1}{2}\rho\partial_\rho U(\rho).$$

This is a zero eigenfunction, $L_\rho \zeta = 0$, for the linearization operator

$$L_\rho = -\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2}f'(U(\rho)) \tag{8}$$

(negative Fréchet derivative) for the r.h.s. of (5) around $U(\rho)$. This zero mode presents an obstruction to solving Eq. (5) perturbatively, starting with $U(\rho)$, which can be resolved by a modulation theory, provided ζ is an L^2 function, i.e. one can use a Hilbert space spectral theory.

Thus equations of the form (5) can be organized in two classes according to which of the following two properties takes place

(i) ζ is in L^2
(ii) ζ is not in L^2.

The Yang–Mills equation, (6), belongs to the first class while the wave map equation, (1), belong to the second. Indeed, the kink solution for (6) is $U(\rho) = \frac{1-\rho^2}{1+\rho^2}$ and the corresponding zero mode is $\zeta(\rho) = \frac{4\rho^2}{(1+\rho^2)^2}$ (see [34,4]). For Eq. (1) with $k=1$ the kink solution is

$$U(\rho) = 2\arctan\rho,$$

while the scaling zero mode is

$$\zeta(\rho) := \frac{1}{2}\rho\partial_\rho U(\rho) = \frac{\rho}{1+\rho^2}.$$

Clearly, ζ is L^2 in the former case and is not L^2 in the latter case. (This is possible due to the fact that the operator L has no spectral gap: $\sigma(L) = \sigma_{\mathrm{cont}}(L) = [0, \infty)$ (see below). The fact that there is a problem with the modulation approach due to the nonintegrability of the zero mode was pointed out by Bizoń in 2001, [35].)

Note that $U_k(\rho) = 2\arctan(\rho^k)$ and the corresponding zero mode is square-integrable for $k > 1$. Thus, in this case, we expect that at least the formal analysis of [4] of the Yang–Mills equation should go through. (Higher degree equivariant, static solutions for the Yang–Mills equations in 4+1 dimensions, known as instantons, can be found in [36,37].)

We are interested in solution with initial conditions near the kink manifold

$$M_{\mathrm{kink}} := \{U(\rho/\lambda) | \lambda > 0\}.$$

Let $S(u) = \int \left\{\frac{1}{2}\|\dot{u}\|^2 - V(u)\right\}\mathrm{d}t$, where $V(u) := \int\left(\frac{1}{2}|\nabla u|^2 + \frac{1}{\rho^2}F(u)\right)$, with $F'(u) = f(u)$, be the action for the Eq. (5). The effective action $S(U_\lambda)$ on the family $U_\lambda(\rho) := U(\rho/\lambda)$ (the 'effective action' of λ) is equal to

$$S(U_\lambda) = \int \{2\dot{\lambda}^2\|\zeta\|^2 - V(U)\}\mathrm{d}t.$$

Here and in what follows $\dot\lambda = \frac{\partial\lambda}{\partial t}$. The fact that the square integrability of the zero mode ζ plays an important role in analysis of such solutions can be gleaned from the observation that the effective action $S(U_\lambda)$ diverges, if $\zeta \notin L^2$. (For a connection to the geodesic hypothesis see [38,39].)

We present heuristic arguments motivating our approach. It is natural to guess that for an initial condition close to the manifold M_{kink} the solution evolves along this manifold. Let $U\left(\frac{\rho}{\lambda(t)}\right)$ be the projection of the solution on this manifold. If for this projection $\lambda(t) \to 0$ as $t \to t_*$ for some t_*, then the solution collapses at the time t_*. With this in mind, we look for solutions to Eq. (5) in the form

$$u(\rho,t) = U(x) + w(x,t), \tag{9}$$

where $x = \rho/\lambda$, a blowup variable, with λ a slowly varying function of time t (we do not pass to the blowup time variable). Note that while in a standard approach the scaling, λ, is fixed at the very beginning (with corrections at certain scales possibly considered later on) we leave it free and we look for a differential equation for λ which guarantees that $|w| \ll 1$. However, this simple procedure which works in the case of the Yang–Mills equation mentioned above (see [4,31]) does not work in the present case as we explain below.

Note that if $\zeta \in L^2$, then $\lambda(t)$ is uniquely determined by the orthogonality condition

$$\langle \zeta, w\rangle = 0. \tag{10}$$

If $\zeta \notin L^2$, then this condition is not well defined unless we assume w belongs to a space of sufficiently fast decaying functions.

Substituting decomposition (9) into (5) leads to the equation for the function w and parameter λ:

$$L_x w + F(w) = -\lambda^2\frac{\partial^2 U}{\partial t^2}, \tag{11}$$

where $F(w)$ absorbs higher order terms $\left(F(w) = \lambda^2\frac{\partial^2 w}{\partial t^2} + N(w),\ N(w) = \text{nonlinearity in } w\right)$ and L_x is the linearization operator for the r.h.s. of (5) around $U(x)$ given by (8). The operator L_x is self-adjoint. The scaling zero mode, ζ, is a zero mode of this operator: $L_x\zeta = 0$. Since ζ is positive and not L^2 we conclude by the Perron–Frobenius theory that $\sigma(L_x) = [0,\infty)$ and 0 is not an eigenvalue of L_x.

We compute explicitly

$$\begin{aligned}\lambda^2\frac{\partial^2 U}{\partial t^2} &= \lambda^2[-2\partial_t(\dot\lambda\lambda^{-1})\zeta + 2(\dot\lambda\lambda^{-1})^2 x\partial_x\zeta] \\ &= -2\ddot\lambda\lambda\zeta + 2\dot\lambda^2(\zeta + x\partial_x\zeta).\end{aligned} \tag{12}$$

We multiply Eq. (11) scalarly by $\zeta(x)$. If ζ is not L^2 one can show using a limiting procedure that $\langle\zeta, L_x w\rangle = 0$, provided $w = o(x)$ and $\partial_x w = o(1)$ at ∞. Thus we obtain

$$\lambda^2\left\langle \zeta, \frac{\partial^2 U}{\partial t^2}\right\rangle + \langle \zeta, F(w)\rangle = 0. \tag{13}$$

Following [4] we try to develop a perturbation theory in the small parameter $\dot\lambda^2$ assuming that term $\lambda\ddot\lambda$ is of the order $o(\dot\lambda^2)$ (and $\dot\lambda < 0$) and similarly for higher order time derivatives of λ, e.g. $\partial_t(\lambda\ddot\lambda) = O(\dot\lambda^3)$, etc. Furthermore, if our assumption that $|w| \ll 1$ is correct and the integral in $\langle\zeta, F(w)\rangle$ is convergent, then we can drop the term $\langle\zeta, F(w)\rangle$ in (13). Hence we obtain in the leading order $O(\dot\lambda^2)$

$$\dot\lambda^2\langle\zeta, \zeta + x\partial_x\zeta\rangle = 0. \tag{14}$$

Considering the integral on the l.h.s. over a bounded domain and integrating by parts one shows that the inner product on the l.h.s. is

$$1/2\lim_{x\to\infty}(x^2\zeta^2(x)). \tag{15}$$

For Eq. (6), this is 0 so we can solve Eq. (11) in the leading order, $w = -\dot\lambda^2 L^{-1}(\zeta + x\partial_x\zeta)$. Plugging this result into Eq. (13) and keeping only the terms up to the order $O(\dot\lambda^4)$, we obtain the equation for scaling dynamics,

$$\lambda\ddot\lambda = \frac{3}{4}\dot\lambda^4, \tag{16}$$

in the leading order $O(\dot\lambda^4)$ (see [4,31]). Next, in order to obtain a correction to this equation, we use (13) at the order $O(\dot\lambda^6)$ to solve Eq. (11) to the order $O(\dot\lambda^4)$ and plug the result to (13). However, at this step we run into logarithmically divergent terms. To overcome this problem we use a multiscale expansion, by introducing an additional scale at infinity (see [4]).

For Eq. (1) with $k = 1$ we have $\lim_{x\to\infty}(x^2\zeta^2(x)) = 1$ and so we go to the next term, $-2\ddot\lambda\lambda\|\zeta\|^2$, and discover that it diverges logarithmically. Thus for Eq. (1) with $k = 1$ one runs into a problem right away. This shows that decomposition (9) is incompatible with the condition $|w| \ll 1$.

The problem for Eq. (1) with $k = 1$ mentioned above can be also seen in a different but related way. Let us try to solve Eq. (11) by perturbation theory. In the leading order we drop the term $F(w)$ to obtain the leading order approximation to the solution: $w = L^{-1}\varphi$, where $\varphi = -\lambda^2\frac{\partial^2 U}{\partial t^2}$ and L^{-1} is understood as the Green function of the equation $Lw = \varphi$ (see Section 3). It is easy to check, using Eqs. (37)–(38) of Section 3, that if $\ddot\lambda \neq 0$ then the function $L^{-1}\varphi$ grows at infinity as $x\ln x$, and a straightforward perturbation theory fails. (Not only the correction w is large at ∞, its energy is infinite.)

The point here is that the function $U(\rho/\lambda)$ is not a good adiabatic solution to Eq. (1) with $k = 1$:

$$\begin{aligned}&\lambda^2\left(\partial_t^2 U(x) - \Delta_\rho U(x) + \frac{1}{2\rho^2}\sin(2U(x))\right) \\ &\quad = -2\ddot\lambda\lambda\zeta(x) + \frac{4\dot\lambda^2 x}{(1+x^2)^2},\end{aligned} \tag{17}$$

where $x := \rho/\lambda$ and where we used (12) and the relation $\zeta + x\partial_x\zeta = 2x(1+x^2)^{-2}$. The r.h.s. is not L^2. The problematic term is $2\ddot\lambda\lambda\zeta(x)$. In particular, it leads to the logarithmically divergent term, $2\ddot\lambda\lambda\|\zeta\|^2$ in the orthogonality condition. Hence, one has to find a better leading term.

We deal with the problem above by introducing instead of the linear, one-parameter transformation, $\rho \to \rho/\lambda$, a nonlinear, three-parameter transformation, $\rho \to f(\rho,\lambda,\alpha,\beta)$, chosen so that $U(f(\rho,\lambda,\alpha,\beta))$ becomes a better approximate solution to Eq. (1) with $k = 1$ than $U(\rho/\lambda)$. In particular, the problematic term $2\ddot\lambda\lambda\zeta$ entering the r.h.s. of Eq. (17) is canceled and therefore the large ρ divergence in Eq. (13) mentioned above is eliminated.

Thus, instead of (9), we look for solutions of Eq. (1) in the form

$$u(\rho,t) = U(y) + w(y,t). \tag{18}$$

We consider initial conditions close to $U(y) \equiv U(f(\rho,\lambda,\alpha,\beta))$ (we do not specify the norm, the latter must be determined by a rigorous analysis; see e.g. [31]). After this we proceed as above with Eq. (9). The conditions $|w| \ll 1$ and $w \to 0$ at $\rho \to \infty$ and constraints on the energy (7) and its fluctuations imply the differential equation (2) on the parameter $\lambda = \lambda(t)$. We expect that proceeding as in [4] one can obtain corrections to Eq. (2).

The paper is organized as follows. In Section 2 we introduce a change of variables, $\rho \to f(\rho,\lambda,\alpha,\beta)$, depending nonlinearly on the original variable ρ and on the scaling parameter λ^{-1} (and depending on additional parameters α,β). This is our main new idea. In Section 3 we derive, modulo some technical details which are provided in Appendices B and C, an approximate solution to Eq. (1) with $k = 1$. In Sections 4 and 5 we use an orthogonality condition of the type of (10), the smallness condition on energy fluctuations and the minimum condition on the energy of the approximate solution in order to find our main equation on the scaling parameter λ, Eq. (2). In Section 6 we find exact and approximate solutions of Eq. (2), and in Section 7 we show that this equation is Hamiltonian. In Appendices A–E we provide technical calculations used in the main text and explanations of the numerical approaches.

2. Nonlinear blowup variables

In this section we introduce a nonlinear, three-parameter (scaling) transformation of the independent spatial variable ρ : $\rho \to y = y(x, \lambda\ddot{\lambda}, \alpha, \beta)$, where $x := \rho/\lambda$ and α and β are free parameters to be chosen. This replaces the standard, linear, one-parameter transformation, $\rho \to \rho/\lambda$. Write

$$v(y,t) = u(\rho,t), \quad \text{where } y = y(x, \lambda\ddot{\lambda}, \alpha, \beta). \tag{19}$$

In the new variables, Eq. (1) with $k = 1$ becomes

$$-\frac{\partial^2 v}{\partial y^2} - \frac{1}{y}\frac{\partial v}{\partial y} + \frac{\sin(2v)}{2y^2} = \Psi(v), \tag{20}$$

where

$$\Psi(v) := \frac{x^2}{y^2}\left\{\left(\frac{2y}{x}\chi + \chi^2\right)\frac{\partial^2 v}{\partial y^2} + \left(\frac{2\chi}{x} + \frac{\partial\chi}{\partial x}\right)\frac{\partial v}{\partial y} - \lambda^2\frac{\partial^2 v}{\partial t^2}\right\}. \tag{21}$$

In the last expression, $\partial^2/\partial t^2$ is the total derivative in t (i.e. taking into account that y depends on t) and the function χ is defined according to the equation

$$\frac{\partial y}{\partial x} = \frac{y}{x} + \chi. \tag{22}$$

Eq. (20) is our transformed equation. Initial conditions for (20) are chosen to be close, in an appropriate norm, to $U(y) \equiv U(y(x, \lambda\ddot{\lambda}, \alpha, \beta))$, where, recall, $U(\rho)$ is the static – kink – solution to Eq. (1). To simplify the exposition we take the initial condition to be just $U(y(x, \lambda\ddot{\lambda}, \alpha, \beta))$.

We look for solutions of Eq. (20) in the form

$$v(y,t) = U(y) + w(y,t), \tag{23}$$

where w is a small correction. We plug this decomposition into Eq. (20) to obtain

$$L_y w + N(w) = \Psi(U + w), \tag{24}$$

where operator L_y is defined in Eq. (8), $N(w)$ is the nonlinear in w term defined by this equation and the function $\Psi(v)$ is defined in (21).

To find an approximate solution of the latter equation we drop the nonlinearity, $N(w)$, and the term w in $\Psi(U + w)$ to obtain the leading order equation

$$L_y w = \psi, \tag{25}$$

where $\psi(y,t) := \Psi(U(y))$. The latter function is given explicitly by

$$\psi(y,t) := \frac{2x^2}{y^2(1+y^2)}\left\{\frac{4\chi}{x(1+y^2)} + \left(\frac{\partial\chi}{\partial x} - \frac{2\chi}{x}\right) - \frac{2y\chi^2}{1+y^2} + \frac{2\lambda^2 y}{1+y^2}\left(\frac{\partial y}{\partial t}\right)^2 - \lambda^2\frac{\partial^2 y}{\partial t^2}\right\}. \tag{26}$$

Above we omitted the term $\lambda^2\frac{\partial^2 w}{\partial t^2}$. To justify this we show in Section 6, (65), that treating this term as a perturbation yields $\lambda^2\frac{\partial^2 w}{\partial t^2} = O\left(\dot{\lambda}^4\frac{1}{y} + \dot{\lambda}^2\lambda\ddot{\lambda}\frac{\ln y}{y} + \dot{\lambda}^4\lambda\ddot{\lambda}y^3\ln(\sqrt{\lambda\ddot{\lambda}}y)\right)$, for $1 \le y \le y_{cr}$, and similarly for other terms in $\Psi(w)$, which we dropped.

The last two terms in Eq. (26) come from $\lambda^2\frac{\partial^2 U(y)}{\partial t^2}$ and they contain a slowly decaying at infinity term; see (12). To compute these terms in the leading order we replace y by $x = \rho/\lambda$ to obtain

$$\begin{aligned}
&\frac{2\lambda^2 x}{1+x^2}\left(\frac{\partial x}{\partial t}\right)^2 - \lambda^2\frac{\partial^2 x}{\partial t^2}\\
&= \frac{2\lambda^2 x}{1+x^2}(\lambda^{-1}\dot{\lambda}x)^2 + \lambda^2\left(\lambda^{-1}\ddot{\lambda}x - 2(\lambda^{-1}\dot{\lambda})^2 x\right)\\
&= -\dot{\lambda}^2\frac{2x}{1+x^2} + \lambda\ddot{\lambda}x.
\end{aligned}$$

We choose χ so that $\partial\chi/\partial x - 2\chi/x$ removes the undesirable term, $x\lambda\ddot{\lambda}$. In other words, we would like χ to solve the equation

$$\frac{\partial\chi}{\partial x} - 2\frac{\chi}{x} = -x\lambda\ddot{\lambda}. \tag{27}$$

The solution of this equation is $\chi = -\lambda\ddot{\lambda}x^2\ln\left(\frac{\sqrt{\lambda\ddot{\lambda}}}{\beta'}x\right)$, for any constant β'. This and the definition of χ, (22), gives the equation

$$\frac{\partial y}{\partial x} - \frac{y}{x} = -\lambda\ddot{\lambda}x^2\ln\left(\frac{\sqrt{\lambda\ddot{\lambda}}}{\beta}x\right), \tag{28}$$

which can be solved approximately to give the transformation $y = x - \frac{\lambda\ddot{\lambda}}{2}x^3\ln\left(\frac{\sqrt{\lambda\ddot{\lambda}}}{\beta}x\right)$ in the leading approximation, say for $x \ll (\lambda\ddot{\lambda})^{-1}$. We extend this transformation to a two-parameter family by introducing an extra parameter α, which will be used to determine an appropriate value for the parameter β by minimizing the energy under obtained constraints.

We define $y = y(x, \lambda\ddot{\lambda}, \alpha, \beta)$ as the solution of the equation

$$\begin{cases} y = x - \dfrac{\lambda\ddot{\lambda}}{2}x^3\ln\left(\dfrac{\sqrt{\lambda\ddot{\lambda}}}{\beta}y^\alpha x^{1-\alpha}\right) & \text{if } x \le x_{cr},\\ y = 2y_{cr} - x + \dfrac{\lambda\ddot{\lambda}}{2}x^3\ln\left(\dfrac{\sqrt{\lambda\ddot{\lambda}}}{\beta}y^\alpha x^{1-\alpha}\right) & \text{if } x > x_{cr}, \end{cases} \tag{29}$$

where $0 \le \alpha \le 1$, $\beta > 0$, $x_{cr} = x_{cr}(\lambda, \alpha, \beta)$ and $y_{cr} = y_{cr}(\lambda, \alpha, \beta)$ solve the equations

$$\begin{aligned}
&\frac{\partial}{\partial x}\left(x - \frac{\lambda\ddot{\lambda}}{2}x^3\ln\left(\frac{\sqrt{\lambda\ddot{\lambda}}}{\beta}y^\alpha x^{1-\alpha}\right)\right) = 0,\\
&y = x - \frac{\lambda\ddot{\lambda}}{2}x^3\ln\left(\frac{\sqrt{\lambda\ddot{\lambda}}}{\beta}y^\alpha x^{1-\alpha}\right).
\end{aligned} \tag{30}$$

(In the first equation y is fixed.)

First we note that Eqs. (30) have a unique solution, (x_{cr}, y_{cr}). Indeed, we define $\gamma = \gamma(\alpha, \beta) = \frac{\sqrt{\lambda\ddot{\lambda}}}{\beta}y_{cr}^\alpha x_{cr}^{1-\alpha}$. Then (30) can be rewritten as $1 - 3\frac{\lambda\ddot{\lambda}}{2}x_{cr}^2\ln\gamma - (1-\alpha)\frac{\lambda\ddot{\lambda}}{2}x_{cr}^2 = 0$ and $y_{cr} = x_{cr}\left(1 - \frac{\lambda\ddot{\lambda}}{2}x_{cr}^2\ln\gamma\right)$. Solving the first of these equations for x_{cr} and substituting the result into the second equation gives

$$x_{cr} = \left(\frac{2}{\lambda\ddot{\lambda}}\right)^{1/2}(3\ln\gamma + 1 - \alpha)^{-1/2} \tag{31}$$

and

$$y_{cr} = \left(\frac{2}{\lambda\ddot{\lambda}}\right)^{1/2}\frac{2\ln\gamma + 1 - \alpha}{(3\ln\gamma + 1 - \alpha)^{3/2}}. \tag{32}$$

The last two equations imply that $\gamma = \gamma(\alpha, \beta) = \frac{\sqrt{\lambda\ddot{\lambda}}}{\beta} y_{cr}^{\alpha} x_{cr}^{1-\alpha}$ satisfies the equation

$$\frac{\gamma}{\sqrt{2}} \frac{(3\ln\gamma + 1 - \alpha)^{1/2+\alpha}}{(2\ln\gamma + 1 - \alpha)^{\alpha}} = \beta^{-1}. \tag{33}$$

This equation has a unique solution for each $\alpha \in [0, 1]$, $\beta > 0$ (note that the l.h.s. is monotonically increasing in $\tau := \ln\gamma$ from 0 at $\tau = -\frac{1-\alpha}{3}$ to ∞ at $\tau = \infty$) and therefore so does (30).

Finally, we show that, for $\ln\gamma + \frac{1}{2} > \alpha$, where γ is the solution of Eq. (33), Eq. (29) has at least one solution and, in the case it has more than one solution, make a choice among the solutions. For each fixed $x \le x_{cr}$, the r.h.s. of (29) is decreasing in $y \ge 0$ from ∞ to $-\infty$. Hence Eq. (29) has a unique solution for $y > 0$ for any fixed $x \le x_{cr}$. This solution is monotonically increasing in x and reaches y_{cr} at x_{cr}.

Consider $x \ge x_{cr}$ and let $f(x, y) :=$ the r.h.s. of (29) for $x \ge x_{cr}$. For $x \ge x_{cr}$ fixed, $y - f(x, y) \to \infty$, as $y \to 0$ or $y \to \infty$. Hence it has a minimum. Computing the derivatives $\partial_y f(x, y)$ and $\partial_y^2 f(x, y)$, we conclude that $y - f(x, y)$ has a unique critical point in y at $y = \alpha\frac{\lambda\ddot{\lambda}}{2}x^3$ and this point is a minimum point. Therefore, $y = f(x, y)$ has at most two solutions for y for each fixed $x \ge x_{cr}$. Since $y = f(x_{cr}, y)$ has a solution $y = y_{cr}$ and $\partial_y f(x_{cr}, y_{cr}) - 1 = -2(\ln\gamma + \frac{1}{2} - \alpha)(2\ln\gamma + 1 - \alpha)^{-1} < 0$, and since the minimum value of $y - f(x, y)$ (in y) decreases with x, we see that $y = f(x, y)$, i.e. (29), has exactly two solutions for each fixed $x \ge x_{cr}$, with y_{cr} being the larger solution for $x = x_{cr}$. Of these two solutions we choose the one, $y = y(x)$, satisfying $y_{cr} = y(x_{cr})$, i.e. the larger one. A simple computation shows that among the solutions, $y = y(x)$, of $y = f(x, y)$, larger one increases while smaller one decreases.

The solution $y = y(x, \lambda\ddot{\lambda}, \alpha, \beta)$, obtained above, increases monotonically in x for $x \ge 0$. Indeed, $\frac{\partial y}{\partial x} > 0$ for x sufficiently small and sufficiently large and has a single zero at $x = x_{cr}$ and therefore $\frac{\partial y}{\partial x} > 0$ for all $x \ge 0$, $x \ne x_{cr}$.

With the transformation (29), we show in Appendix B that

$$\psi(y, t) = O\left(\dot{\lambda}^2 \left(\ln(\sqrt{\lambda\ddot{\lambda}}y)\lambda\ddot{\lambda}y^2\right)^{-1} y^{-1}\right) \quad \text{for } y \ge y_{cr}, \tag{34}$$

and, in particular, it decays sufficiently fast at infinity.

Following [4] we develop a perturbation theory in the small parameter $\dot{\lambda}^2$ assuming that term $\lambda\ddot{\lambda}$ is of the order $o(\dot{\lambda}^2)$ (and $\dot{\lambda} < 0$) and similarly for higher order time derivatives of λ, e.g. $\partial_t(\lambda\ddot{\lambda}) = o(\dot{\lambda}^3)$, etc.

3. Approximate solution of Eq. (25)

In this section we find an approximate solution to Eq. (25). Recall that y is a function of $x := \rho/\lambda$ given in (29) and the operator L_y entering (25) has a zero mode:

$$L_y\zeta = 0. \tag{35}$$

Multiplying Eq. (25) scalarly by ζ and using the self-adjointness of the operator L (and some elementary limiting procedure) and (35), we obtain

$$\int_0^{\infty} dyy\,\zeta\psi = 0. \tag{36}$$

This is a (necessary) solvability condition for Eq. (25). It gives an equation on the parameters λ, α and β. (Note that it is an approximate solvability condition for the exact Eq. (24).)

So far we obtained one equation for the three parameters λ, α and β. To derive another equation we analyze the solution $W(y, t) = L^{-1}\psi$ to (25), with ψ satisfying (36), which is an approximate solution to (24). (We denote the solution to (25) by the capital W to distinguish it from solutions to (24).) Our goal in the rest of this section is to isolate the leading contribution to W. This will be used in the next section to derive the second equation for the parameters.

To find $L^{-1}\psi$ we compute the Green function for the operator L. Two linearly independent solutions of the homogeneous equation $Lw = 0$ are

$$w_1(y) = \frac{y}{1 + y^2} \quad \text{and} \quad w_2(y) = \frac{y}{2} - \frac{1}{2y} + \frac{2y\ln y}{1 + y^2} \tag{37}$$

(the first of these solutions is just the scaling zero mode, ζ, the second solution is found in Appendix A). Hence, by the ODE theory, the general solution, $W = L^{-1}\psi$, of Eq. (25) decreasing at infinity is of the form

$$W(y, t) = c_1 w_1(y) + w_1(y)\int_0^y w_2(s)\psi(s, t)sds - w_2(y)\int_0^y w_1(s)\psi(s, t)sds, \tag{38}$$

where c is chosen to guarantee solvability of the equation to the second order correction term or by minimizing the energy. Since the function w_2 is singular at $y = 0$, we use $-w_2(y)\int_0^y w_1(s)\psi(s, t)sds$ in (38), rather than $+w_2(y)\int_y^{\infty} w_1(s)\psi(s, t)sds$. Since the function w_2 grows at infinity, W is bounded only if the condition (36) is satisfied.

We find the leading contribution to the solution $W = L^{-1}\psi$ of Eq. (25). In what follows, we use results of Appendix B, where we used the assumptions

$$0 < \lambda\ddot{\lambda} \ll \dot{\lambda}^2 \ll 1, \qquad \lambda\partial_t(\lambda\ddot{\lambda}) = o(\dot{\lambda}^3),$$
$$\lambda\partial_t(\dot{\lambda}^2) = o(\dot{\lambda}^3), \quad \beta = O(1), \tag{39}$$

and

time derivatives of α, β are small and can be neglected. (40)

Consider first the region $y \le y_{cr} \Leftrightarrow x \le x_{cr}$. It is shown in Appendix B that in this region the function ψ can be written as

$$\psi := \psi_1 + \psi_2, \tag{41}$$

where the functions ψ_1 and ψ_2 satisfy for $y \ll y_{cr} = O\left(\frac{1}{\sqrt{\lambda\ddot{\lambda}}}\right) \gg 1$ the estimates (see Eqs. (105) and (106) and the text that follows)

$$\psi_1(y, t) = \frac{-4y\dot{\lambda}^2}{(1 + y^2)^2}\left(1 + O\left(\frac{\lambda\ddot{\lambda}}{\dot{\lambda}^2}\right)\right) \quad \text{and}$$
$$\psi_2(y, t) = O(\dot{\lambda}^2\lambda\ddot{\lambda}y\ln(\sqrt{\lambda\ddot{\lambda}}y)) \tag{42}$$

(as indicated by these estimates, the functions $\psi_1(y, t)$ and $\psi_2(y, t)$ are localized on the scales $y \sim 1$, and $y \sim y_{cr} = O\left(\frac{1}{\sqrt{\lambda\ddot{\lambda}}}\right) \gg 1$, respectively).

Using that for $y \ge 1$, $w_1(y) = O(1/y)$ (see (37)) this and using (42), which implies in particular that ψ satisfies the estimate $\psi(y, t) = O\left(\frac{\dot{\lambda}^2}{y^3} + \dot{\lambda}^2\lambda\ddot{\lambda}y\ln(\sqrt{\lambda\ddot{\lambda}}y)\right)$, we find that for $1 \le y \ll y_{cr}$,

$$w_1(y)\int_0^y w_2(s)\psi(s, t)sds = O\left(\dot{\lambda}^2\frac{\ln y}{y} + \dot{\lambda}^2\lambda\ddot{\lambda}y^3\ln(\sqrt{\lambda\ddot{\lambda}}y)\right).$$

Next, using that for $y \ge 1$, $w_2(y) = y/2 + O(\ln y/y)$ and using the estimates (42) for ψ_1 and ψ_2 in the domain $y \le y_{cr}$, we find that for $1 \le y \ll y_{cr}$,

$$-w_2(y)\int_0^y w_1(s)\psi(s, t)sds$$
$$= -w_2(y)\int_0^y w_1(s)\psi_1(s, t)sds + O\left(\dot{\lambda}^2\lambda\ddot{\lambda}y^3\ln(\sqrt{\lambda\ddot{\lambda}}y)\right)$$
$$= -\frac{y}{2}\int_0^{\infty} w_1(s)\psi_1(s, t)sds + O\left(\frac{\dot{\lambda}^2}{y} + \dot{\lambda}^2\lambda\ddot{\lambda}y^3\ln(\sqrt{\lambda\ddot{\lambda}}y)\right),$$

where, in the second equality, we used that $\psi_1(y,t) = (\dot\lambda^2 y^{-3})$ for $y \geq 1$ and therefore $\int_y^\infty w_1(s)\psi_1(s,t)s\mathrm{d}s = O\left(\frac{\dot\lambda^2}{y^2}\right)$. Hence, due to (38), we have that for $y \ll y_{\rm cr}$

$$W(y,t) = -\frac{y}{2}\int_0^\infty w_1(s)\psi_1(s,t)s\mathrm{d}s + O\left(\dot\lambda^2\frac{\ln(y)}{y} + \dot\lambda^2\lambda\ddot\lambda y^3\ln(\sqrt{\lambda\ddot\lambda}y)\right). \tag{43}$$

From the estimate (42), it is easy to see that

$$\int_0^\infty w_1(s)\psi_1(s,t)s\mathrm{d}s \asymp \dot\lambda^2, \tag{44}$$

i.e. the first expression on the r.h.s. of (43) is the leading term for $1 \ll y \ll y_{\rm cr} = O\left(\frac{1}{\sqrt{\lambda\ddot\lambda}}\right)$.

4. Energy of the approximate solution and the equation on λ

We compute the energy of our approximate solution $u(\rho,t) = U_{\lambda,\alpha,\beta}(\rho) + W(y)$, where $U_{\lambda,\alpha,\beta}(\rho) := U(y,t)$, with $y = y(x, \lambda\ddot\lambda, \alpha, \beta)$ and U defined in the Introduction, and where $W = L^{-1}\psi$, the solution to Eq. (25) (see the previous section). Due to (7), the energy functional is

$$E(u) = \int_0^\infty \left(\frac{1}{2}\dot u^2 + \frac{1}{2}|\nabla u|^2 + \frac{1}{2\rho^2}\sin^2 u\right)\rho\mathrm{d}\rho. \tag{45}$$

Inserting the approximate solution (43) for W into this expression, we obtain that $E(U_{\lambda,\alpha,\beta} + W) = E(U_{\lambda,\alpha,\beta}) + \delta E_1$, with

$$\delta E_1 = O(\dot\lambda^2 y_{\rm cr}^4)\left(\int_0^\infty w_1(s)\psi_1(s,t)s\mathrm{d}s\right)^2 + O(\dot\lambda^2). \tag{46}$$

Furthermore, we have that

$$E(U_{\lambda,\alpha,\beta}) = E(U) + \delta E_0 \quad \text{with } \delta E_0 = O(\dot\lambda^2\ln(1/\dot\lambda^2)). \tag{47}$$

We require that the energy correction due to the fluctuation, W, be much smaller than the one due to the modulation:

$$|\delta E_1| \ll |\delta E_0|. \tag{48}$$

Since $\int_0^\infty w_1(s)\psi_1(s,t)s\mathrm{d}s \asymp \dot\lambda^2$ and $y_{\rm cr} = O\left(\frac{1}{\sqrt{\lambda\ddot\lambda}}\right) \gg O\left(\frac{1}{|\dot\lambda|}\right)$, this implies that the integral in the leading term in the above expression for δE_1 must vanish:

$$\int_0^\infty \mathrm{d}y y\zeta\psi_1 = 0, \tag{49}$$

where, recall, ψ_1 is given in Appendix B, Eq. (105). This gives an implicit equation on the parameters λ, α, β.

In the leading order, we can replace y by $x = \rho/\lambda$ (see the first equation in (29)) and use (105), so that Eq. (49) becomes

$$\int_0^\infty \mathrm{d}x x\frac{x}{1+x^2}\left\{\frac{8\lambda\ddot\lambda x}{(1+x^2)^2}\left[\ln\left(\frac{\sqrt{\lambda\ddot\lambda}}{\beta}x\right) + 1/2\right] + \frac{4\dot\lambda^2 x}{(1+x^2)^2}\right\} = 0. \tag{50}$$

Computing the integrals in (50) (see Appendix B for detailed computations), we obtain

$$\dot\lambda^2 + 2\lambda\ddot\lambda\left[\ln\left(\frac{\sqrt{\lambda\ddot\lambda}}{\beta}\right) + 1\right] = 0. \tag{51}$$

This is our explicit equation for the parameter λ. It depends on the parameter β whose value we still have to determine. Since in the leading approximation ($y \to x$) the first equation on the r.h.s. of (29) is independent of α, then so are the resulting Eqs. (50) and (51). Eqs. (51) and (2) coincide, provided

$$a = \beta^2 e^{-2}. \tag{52}$$

Clearly, solutions of Eq. (51) have the property (39) assumed above. Moreover, if $\lambda(0) > 0$, $\dot\lambda(0) < 0$, then, by Eq. (51), $\lambda(t) > 0$, $\dot\lambda(t) < 0$, $\ddot\lambda(t) > 0$ and therefore $\dot\lambda(t)^2 \leq \dot\lambda(0)^2$ for $t > 0$. As $t \to t_*$, $|\dot\lambda|$ decreases so that our approximation improves as $t \to t_*$.

Thus, it remains to find the value of the parameter β. This is addressed in the next section.

5. Values of the parameters α and β

In this section we find the values of the parameters α and β. We are interested in an adiabatic (very slow) evolution close to the collapse time. This means that the energy has smallest possible value. (The system lowers its energy by radiating excess energy to infinity.) Hence we require that the leading approximation, $U_{\lambda,\alpha,\beta}(\rho) := U(y(x,\lambda\ddot\lambda,\alpha,\beta))$, to the solution of our equation minimizes the energy under existing constraints. To find the constraint we use the Eqs. (36) and (49) to derive an equation, $I(\alpha,\beta) = 0$, on the parameters α and β (see (61) below). Then we find these parameters by minimizing the energy $E(\alpha,\beta) := E(U_{\lambda,\alpha,\beta})$ under the constraint $I(\alpha,\beta) = 0$.

First we rewrite Eq. (36) as $\int_0^{y_{\rm cr}}\zeta\psi y\mathrm{d}y + \int_{y_{\rm cr}}^\infty \zeta\psi y\mathrm{d}y = 0$. Next, using (49) and the estimate in (42) on ψ_1, we obtain $\int_0^{y_{\rm cr}}\zeta\psi_1 y\mathrm{d}y = -\int_{y_{\rm cr}}^\infty\zeta\psi_1 y\mathrm{d}y = O(\dot\lambda^2 y_{\rm cr}^{-2}) = O(\dot\lambda^2\lambda\ddot\lambda)$. The last two equations and decomposition (41) imply that, modulo $O(\dot\lambda^2\lambda\ddot\lambda)$,

$$\int_0^{y_{\rm cr}}\psi_2\zeta y\mathrm{d}y + \int_{y_{\rm cr}}^\infty \psi\zeta y\mathrm{d}y = 0. \tag{53}$$

To find the leading approximation to the l.h.s. in this equation, we use the expressions for the functions ψ and ψ_2 given in (105)–(107) in Appendix B. A little contemplation of these expressions shows that the most important region in the above integral is where y is of the order of $y_{\rm cr}$. Since $y_{\rm cr} = O\left(\frac{1}{\sqrt{\lambda\ddot\lambda}}\right) \gg 1$ (see Eq. (32)), it is natural to pass in the integrals in (53) to the new variables

$$x' = \sqrt{\lambda\ddot\lambda}x, \qquad y' = \sqrt{\lambda\ddot\lambda}y, \tag{54}$$

where, recall, x and y are connected by (29). Note that (29) implies that x' and y' are related by

$$\begin{cases} y' = x' - \frac{1}{2}x'^3\ln\left(\frac{1}{\beta}y'^{\alpha}x'^{1-\alpha}\right) & \text{if } x' \leq x'_{\rm cr}, \\ y' = 2y'_{\rm cr} - x' + \frac{1}{2}x'^3\ln\left(\frac{1}{\beta}y'^{\alpha}x'^{1-\alpha}\right) & \text{if } x' > x'_{\rm cr}, \end{cases} \tag{55}$$

where $x'_{\rm cr} := \sqrt{\lambda\ddot\lambda}x_{\rm cr}$ and $y'_{\rm cr} := \sqrt{\lambda\ddot\lambda}y_{\rm cr}$ are just numbers,

$$x'_{\rm cr} = \sqrt{2}(3\ln\gamma + 1 - \alpha)^{-1/2} \tag{56}$$

and

$$y'_{\rm cr} = \sqrt{2}\frac{2\ln\gamma + 1 - \alpha}{(3\ln\gamma + 1 - \alpha)^{3/2}}, \tag{57}$$

with the number γ being the solution to Eq. (33). Using this change of variables it is easy to show that

$$\int_0^{y_{\rm cr}}\psi_2\zeta y\mathrm{d}y + \int_{y_{\rm cr}}^\infty\psi\zeta y\mathrm{d}y = \dot\lambda^2 I(\alpha,\beta) + O(\lambda\ddot\lambda), \tag{58}$$

where the function $I(\alpha,\beta)$, given in (59), is, in fact, independent of $\dot\lambda^2$ and $\lambda\ddot\lambda$ and is given by the integral

$$I(\alpha,\beta)=2\int_0^{y'_{cr}}dy'\frac{x'^6}{y'^5\left(1+\frac{\alpha x'^3}{2y'}\right)}\left\{\frac{\omega'-\alpha}{\left(1+\frac{\alpha x'^3}{2y'}\right)}-\frac{(1+\sigma')(3\omega'+1-\alpha)}{\left(1+\frac{\alpha x'^3}{2y'}\right)}-\frac{\alpha x'}{2y'}\frac{\sigma'^2}{\left(1+\frac{\alpha x'^3}{2y'}\right)^2}+\frac{3\alpha}{\left(1+\frac{\alpha x'^3}{2y'}\right)}+\frac{3y'(1-\alpha)}{2x'}+\frac{2(3\omega'+1-\alpha)y'\left(1-\alpha\frac{x'^3}{4y'}\right)}{\left(1+\frac{\alpha x'^3}{2y'}\right)x'}\right\}$$
$$+2\int_{y'_{cr}}^{\infty}dy'\frac{x'^4}{\left(1-\frac{\alpha x'^3}{2y'}\right)y'^4}\left\{\frac{2\sigma'^2}{y'\left(1-\frac{\alpha x'^3}{2y'}\right)}+\frac{2\sigma'}{x'}+\frac{x'}{2}\left[\frac{\alpha\sigma'^2x'^2}{\left(1-\frac{\alpha x'^3}{2y'}\right)^2y'^2}+\frac{6\alpha x'}{y'\left(1-\frac{\alpha x'^3}{2y}\right)}-3(1-\alpha)-\frac{2\left(1+\frac{\alpha x'^3}{y'}\right)(3\omega'+1-\alpha)}{\left(1-\frac{\alpha x'^3}{2y'}\right)}\right]\right\}. \quad (59)$$

Here

$$\omega':=\ln\left(\frac{1}{\beta}y'^{\alpha}x'^{1-\alpha}\right),\qquad \sigma'=1-\frac{x'^2}{2}\left(3\omega'+1-\alpha\right). \quad (60)$$

This reduces Eq. (53), in the leading approximation in $1/\ln(1/\lambda\ddot\lambda)$, to the equation

$$I(\alpha,\beta)=0. \quad (61)$$

The resulting integral can be computed numerically. In particular, one can show that for $\alpha=0$, $I(\alpha=0,\beta)=1$, independently of the value of β. Thus we cannot take $\alpha=0$ in our transformation (29).

We chose the parameters α and β which minimize the energy $E(\alpha,\beta):=E(U_{\lambda,\alpha,\beta})$, where, recall, $U_{\lambda,\alpha,\beta}(\rho):=U(y(x,\lambda\ddot\lambda,\alpha,\beta))$, given that Eq. (61), $I(\alpha,\beta)=0$, holds. To find these minimizers, we use Eqs. (45) and $U(\rho)=2\arctan\rho$ to rewrite the energy $E(\alpha,\beta)$ as

$$E(\alpha,\beta)=2\int_0^\infty d\rho\,\rho\frac{1}{(1+y^2)^2}\left\{\left(\frac{\partial y}{\partial t}\right)^2+\left(\frac{\partial y}{\partial\rho}\right)^2+\frac{y^2}{\rho^2}\right\}. \quad (62)$$

We find numerically (see Appendix E for the analytical part) that the energy $E(\alpha,\beta)$ is minimized on the curve $I(\alpha,\beta)=0$ at the point

$$\beta_0=1.04\quad\text{and}\quad\alpha_0=0.65436. \quad (63)$$

Note that the solvability condition $\ln\gamma+\frac12-\alpha>0$ (see the paragraph after (33) is satisfied at this point: $\gamma_0:=\gamma(\alpha_0,\beta_0)\approx 1.21$ and $\ln\gamma_0+\frac12-\alpha_0\approx 0.193-0.154>0$.

This is a special point for the curve $I(\alpha,\beta)=0$. Our numerics show that while the functions $\alpha=\alpha(\beta)$ and $\beta=\beta(\alpha)$ determined by the equation $I(\alpha,\beta)=0$ are double-valued, their branches originate exactly at this point (and form a wedge there). So the equation $I(\alpha,\beta)=0$ has a unique solution only for $\beta=\beta_0$ or for $\alpha=\alpha_0$ and has no solutions for $\beta>\beta_0$ or for $\alpha<\alpha_0$.

Substituting $\beta=\beta_0=1.04$ into Eq. (51), we obtain the following value for the parameter a:

$$a=0.146.$$

This proves Eq. (2) with $a=0.146$.

6. Justification of Eq. (25)

To justify Eq. (25), i.e. that the term $\lambda^2\frac{\partial^2 w}{\partial t^2}$ can be dropped from the r.h.s. of (24), we estimate this term in the leading order, i.e. we estimate $\lambda^2\frac{\partial^2 W}{\partial t^2}$. Eqs. (38), (41) and (49) imply the following expression for the general solution, $W=L^{-1}\psi$, of Eq. (25), which decreases at infinity:

$$W(y,t)=c_1w_1(y)+w_1(y)\int_0^y w_2(s)\psi(s,t)sds+w_2(y)\left(\int_y^\infty w_1(s)\psi_1(s,t)sds-\int_0^y w_1(s)\psi_2(s,t)sds\right), \quad (64)$$

where, recall, c_1 is chosen to guarantee solvability of the equation to the second order correction term or by minimizing the energy. Using (64), (39) and the estimates (97)–(100) of Appendix B on $\lambda\frac{\partial y}{\partial t}$ and $\lambda^2\frac{\partial^2 y}{\partial t^2}$, we obtain after lengthy computations

$$\lambda^2\frac{\partial^2 W}{\partial t^2}=O\left(\dot\lambda^4\frac1y+\dot\lambda^2\lambda\ddot\lambda\frac{\ln y}{y}+\dot\lambda^4\lambda\ddot\lambda y^3\ln(\sqrt{\lambda\ddot\lambda}y)\right). \quad (65)$$

We demonstrate these computations on estimating, say, the term $\lambda^2\frac{\partial^2}{\partial t^2}\left(w_1(y)\int_0^y w_2(s)\psi_1(s,t)sds\right)$ for $1\ll y\ll y_{cr}$. We have

$$\lambda^2\frac{\partial^2}{\partial t^2}\left(w_1(y)\int_0^y w_2(s)\psi_1(s,t)sds\right)=\lambda^2\frac{\partial^2}{\partial t^2}(w_1(y))\int_0^y w_2(s)\psi_1(s,t)sds+2\lambda\frac{\partial}{\partial t}(w_1(y))\lambda\frac{\partial y}{\partial t}w_2(y)\psi_1(y,t)+w_1(y)\left[\lambda^2\frac{\partial^2 y}{\partial t^2}w_2(y)\psi_1(y,t)+2\lambda\frac{\partial y}{\partial t}w_2(y)\lambda\frac{\partial}{\partial t}\psi_1(y,t)+\int_0^y w_2(s)\lambda^2\frac{\partial^2}{\partial t^2}\psi_1(s,t)sds\right]. \quad (66)$$

Using definitions (37) and (105) of w_1, w_2 and ψ_1 and the assumptions (39) on time derivatives of λ, we compute, for $y\gg1$,

$$\lambda\frac{\partial}{\partial t}w_1(y)=O\left(\frac{1}{y^2}\lambda\frac{\partial y}{\partial t}\right),\qquad \lambda\frac{\partial}{\partial t}w_2(y)=O\left(\lambda\frac{\partial y}{\partial t}\right),$$
$$\lambda^2\frac{\partial^2}{\partial t^2}w_1(y)=O\left(\frac1y\left|2\left(\frac1y\lambda\frac{\partial y}{\partial t}\right)^2-\frac1y\lambda^2\frac{\partial^2 y}{\partial t^2}\right|\right)$$

and

$$\lambda\frac{\partial}{\partial t}\psi_1(y,t)=O\left(\frac{\dot\lambda^2}{y^4}\left|\lambda\frac{\partial y}{\partial t}\right|+|\dot\lambda^3\lambda\ddot\lambda|y\ln(\sqrt{\lambda\ddot\lambda}y)\right).$$

Next, using the estimates (97)–(100) of Appendix B on $\lambda\frac{\partial y}{\partial t}$ and $\lambda^2\frac{\partial^2 y}{\partial t^2}$, we find for $1\ll y\ll y_{cr}$

$$\frac1y\lambda\frac{\partial y}{\partial t}=O(\dot\lambda)\quad\text{and}\quad 2\left(\frac1y\lambda\frac{\partial y}{\partial t}\right)^2-\frac1y\lambda^2\frac{\partial^2 y}{\partial t^2}=O(\lambda\ddot\lambda).$$

Similarly one estimates $\lambda^2\frac{\partial^2}{\partial t^2}\psi_1(y,t)$. Finally, we note that definition (37) and (105) of Appendix B imply that $w_1=O\left(\frac1y\right)$, $w_2=O(y)$. Inserting all these estimates into (66) and using estimate (42) for ψ_1, we arrive, for $1\ll y\ll y_{cr}$, at

$$\lambda^2\frac{\partial^2}{\partial t^2}\left(w_1(y)\int_0^y w_2(s)\psi_1(s,t)sds\right)=O\left(\frac{\dot\lambda^4}{y}+\dot\lambda^2\lambda\ddot\lambda\frac{\ln y}{y}+\dot\lambda^4|\lambda\ddot\lambda|y^3\ln(\sqrt{\lambda\ddot\lambda}y)\right). \quad (67)$$

(To check ourselves, we see that in the domain $1 \ll y \ll y_{cr}$, $w_1 \sim y^{-1} \sim x^{-1} = \lambda\rho^{-1}$ and therefore, e.g. $\lambda\partial_t w_1 \sim \lambda\dot{\lambda}\rho^{-1} = \dot{\lambda}x^{-1} \sim \dot{\lambda}y^{-1}$.)

To obtain the next term in the perturbation theory for Eq. (24) we write $w = W + \theta$, plug this into (24) to obtain in the leading order

$$L_y\theta = -N(W) - \frac{x^2}{y^2}\lambda^2\frac{\partial^2 W}{\partial t^2},$$

where, recall, the operator L is given in (8). Using the expression for the inverse of L, given on the r.h.s. of (38), and estimates (43), with (49) and (65), one can easily estimate θ and show that it is of a higher order than W.

7. Investigation of Eq. (2)

In this section we find an approximate solution to Eq. (2) (which is, up to a redefinition of the parameters, Eq. (51)). Iterating this equation, we find, in the leading approximation, the following equation

$$\frac{\ddot{\lambda}}{\dot{\lambda}} = \frac{\dot{\lambda}}{\lambda\ln(a/\dot{\lambda}^2)}. \tag{68}$$

Solution of the Eq. (68) with two free parameters of integration, $c > 0$ and t^*, is

$$\sqrt{a}(t^* - t) = \int_0^\lambda dx e^{\ln^{1/2}\left(\frac{c}{x}\right)}. \tag{69}$$

Changing the variable of integration as $\ln\left(\frac{c}{x}\right) = z^2$, we reduce Eq. (69) for the parameter λ to the form Eq. (3) given in Introduction.

Now we derive an exact expression for a general solution of (2). We introduce the function $f(x) := x\ln(1/x)$. For $0 < x < e^{-1}$ this function has the inverse, $f^{-1}(x)$. Using this inverse we rewrite Eq. (2) as

$$\frac{\lambda\ddot{\lambda}}{a} = f^{-1}\left(\frac{\dot{\lambda}^2}{a}\right). \tag{70}$$

(Note that for $x \to 0$, $f^{-1}(x) = \frac{-x}{\ln x} + \cdots$, so in the leading approximation of (70) gives (68).) Integrating equation (70) gives

$$\ln\lambda = F(\dot{\lambda}) \quad \text{where } F(y) = \frac{1}{2}\int^{\frac{y^2}{a}} \frac{dz}{z}g(z), \tag{71}$$

with $g(z) := z/f^{-1}(z)$. Using the equation $f(f^{-1}(y)) = y$, or, more explicitly, $f^{-1}(y)\ln(1/f^{-1}(y)) = y$, we find that the function $g(z)$ satisfies the equation

$$g(z) = \ln\left(\frac{g(z)}{z}\right). \tag{72}$$

Differentiating the latter equation, we find

$$g'(z) = -\frac{g(z)}{z(g(z)-1)}. \tag{73}$$

Using this equation we integrate

$$\int^x \frac{dz}{z}g(z) = -\int^x dz g'(z)(g(z)-1) = -\frac{1}{2}(g(x)-1)^2 + \text{const}. \tag{74}$$

This gives

$$F(y) = -\frac{1}{4}(g(y^2/a)-1)^2 + \text{const}, \tag{75}$$

which together with Eq. (71) yields

$$g\left(\frac{\dot{\lambda}^2}{a}\right) = 1 + 2\sqrt{\ln\left(\frac{c}{\lambda}\right)} \tag{76}$$

for some constant c. The latter equation can be integrated as follows

$$\sqrt{a}(t^* - t) = \int_0^\lambda \frac{dx}{\left[g^{-1}\left(1 + 2\sqrt{\ln\left(\frac{c}{x}\right)}\right)\right]^{1/2}}. \tag{77}$$

Next we find the function $g^{-1}(x)$. The definition of the function $f(x)$ implies $f(e^{-x}) = xe^{-x}$, which yields

$$\frac{xe^{-x}}{f^{-1}(xe^{-x})} = x, \tag{78}$$

which, in turn, leads to $g(xe^{-x}) = x$, which finally gives the expression

$$g^{-1}(x) = xe^{-x}. \tag{79}$$

Now Eqs. (77) and (79) imply

$$\sqrt{a}(t^* - t) = \int_0^\lambda dx \frac{e^{1/2 + \sqrt{\ln\left(\frac{c}{x}\right)}}}{\sqrt{1 + 2\sqrt{\ln\left(\frac{c}{x}\right)}}}. \tag{80}$$

Our goal is to expand the r.h.s. by perturbation theory in powers of $(\ln(c/\lambda))^{-1/2} \ll 1$ to the second order. To this end, we change the variable of integration as $\ln(c/x) = z^2$, to find

$$\sqrt{a}(t^* - t) = \sqrt{2}ce^{1/2}\int_{\sqrt{\ln(c/\lambda)}}^\infty dz \frac{z}{\sqrt{z + 1/2}}e^{z - z^2}. \tag{81}$$

Next, we change the integration variable on the r.h.s. as $z = \beta + y$, where $\beta := \sqrt{\ln(c/\lambda)}$ and expand the integrand as

$$\begin{aligned}
&\frac{\beta + y}{\sqrt{\beta + y + 1/2}}e^{\beta + y - (\beta + y)^2} \\
&= \frac{e^{\beta - \beta^2}\beta}{\sqrt{\beta + 1/2}}\frac{1 + \frac{y}{\beta}}{\sqrt{1 + \frac{y}{\beta + 1/2}}}e^{-(2\beta - 1)y - y^2} \\
&= \frac{e^{\beta - \beta^2}\beta}{\sqrt{\beta + 1/2}}\left(1 + \frac{y}{\beta}\right)\left(1 - \frac{1}{2}\frac{y}{\beta + 1/2}\right. \\
&\quad \left. + O\left(\left(\frac{y}{\beta + 1/2}\right)^2\right)\right)(1 - y^2 + O(y^4))e^{-(2\beta - 1)y}.
\end{aligned}$$

Note that $e^{-\beta^2} = \lambda/c$. Hence for $\beta = \sqrt{\ln(c/\lambda)} \gg 1$ this gives modulo $O((\ln(c/\lambda))^{-2})$ the equation

$$\begin{aligned}
\sqrt{a}(t^* - t) &= \frac{\lambda}{\sqrt{2}[\ln(c/\lambda)]^{1/4}}e^{1/2 + \sqrt{\ln(c/\lambda)}} \\
&\quad \times \left[1 + \frac{1}{4\sqrt{\ln(c/\lambda)}} - \frac{1}{32}\frac{1}{\ln(c/\lambda)}\right].
\end{aligned} \tag{82}$$

Eq. (3) is an approximation for this exact expression, it differs from the latter by a slowly varying factor which can be found in the next approximation to (3).

8. Hamiltonian formulation

Eq. (2) is a Hamiltonian system. Indeed, it can be obtained from the Lagrangian

$$L = h(\dot{\lambda}) - \ln\lambda \tag{83}$$

where the function h is defined by

$$h''(x) = -\frac{1}{af^{-1}(x^2/a)} \tag{84}$$

with $f(x) = x\ln(1/x)$ (see Section 5). Now the generalized momentum, Hamiltonian and energy can be found in the standard way. In particular, the energy is given by

$$E = \dot\lambda\frac{\partial L}{\partial\dot\lambda} - L = \dot\lambda h'(\dot\lambda) - h(\dot\lambda) + \ln\lambda. \tag{85}$$

This is the energy conservation law. On the other hand, differentiating Eq. (85) w.r.t. t, we obtain the equation of motion (2).

9. Conclusion

We presented detailed arguments that for initial conditions close to the degree 1 equivariant, static wave map, the solutions of the wave map equation (σ-model) collapse in a finite time. Near the collapse point the solutions have a universal profile given by the modified degree 1 equivariant, static wave map depending on a time-dependent parameter λ. This parameter describes the rate of compression (scaling) of the collapse profile. We derived a second order Hamiltonian dynamical equation for the scaling parameter, λ. We also found approximate solutions of this equation. These solutions are of a rather complex form. They are in excellent agreement with direct numerical simulations of the wave map equation. We expect that the set of initial conditions for which our analysis works is open.

Note added in the proofs

A rigorous proof of Eq. (3), in the leading order and without specifying the multiplicative constant, was achieved in e-print [40], posted soon after the posting of this manuscript.

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Appendix A

In this appendix we find two linearly independent solutions of linear homogeneous equation

$$L_x w = 0, \tag{86}$$

which are used in order to find the Green function of the equation $Lw = g$, and then we find the general solution of the latter equation. The first solution of $L_x w = 0$ is the scaling zero mode ζ

$$w_1 = \zeta = \frac{x}{1+x^2}. \tag{87}$$

The second solution w_2 satisfies the inhomogeneous equation of first order:

$$w_1 w_2' - w_2 w_1' = \frac{1}{x}. \tag{88}$$

The standard solution of this equation is

$$w_2 = w_1 z; \qquad z' = x + \frac{2}{x} + \frac{1}{x^3}; \qquad z = C + \frac{x^2}{2} + 2\ln x - \frac{1}{2x^2}. \tag{89}$$

Setting $C = 0$, we obtain

$$w_2 = \frac{x}{2} + \frac{2x\ln x}{1+x^2} - \frac{1}{2x}. \tag{90}$$

To obtain a general solution of the equation $L_x w = g$, we rewrite it as a first order ODE

$$\frac{\partial}{\partial x}\begin{pmatrix} w \\ v\end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{1}{x^2}\left(1 - \frac{8x^2}{(1+x^2)^2}\right) & -\frac{1}{x}\end{pmatrix}\begin{pmatrix} w \\ v\end{pmatrix} - g\begin{pmatrix} 0 \\ 1\end{pmatrix}. \tag{91}$$

Two linear independent solutions of (91) are

$$\begin{pmatrix} w_1 \\ w_1'\end{pmatrix}, \qquad \begin{pmatrix} w_2 \\ w_2'\end{pmatrix}. \tag{92}$$

By the method of variation of constants we look for a general solution of inhomogeneous equation (91) in the form

$$\begin{pmatrix} w \\ v\end{pmatrix} = c_1\begin{pmatrix} w_1 \\ w_1'\end{pmatrix} + c_2\begin{pmatrix} w_2 \\ w_2'\end{pmatrix} \tag{93}$$

where c_1 and c_2 are functions of x. Inserting (93) into Eq. (91), we find

$$\frac{\partial c_1}{\partial x} = x w_2 g; \qquad \frac{\partial c_2}{\partial x} = -x w_1 g, \tag{94}$$

which together with (93) give the general solution to the equation $L_x w = g$.

Appendix B

In this appendix we will find an explicit expression for the inhomogeneous term ψ and use this expression to show Eq. (34). In what follows we assume (39) and (40).

We consider separately two domains $\{y \le y_{cr}\} \equiv \{\rho \le \rho_{cr}\}$ and $\{y \ge y_{cr}\} \equiv \{\rho \ge \rho_{cr}\}$. First, we compute $\frac{\partial y}{\partial t}$ and $\frac{\partial^2 y}{\partial t^2}$. Recall the notation $x := \rho/\lambda$ and

$$\omega := \ln\left(\frac{\sqrt{\lambda\ddot\lambda}}{\beta} y^\alpha x^{1-\alpha}\right), \qquad \sigma := 1 - \frac{\lambda\ddot\lambda x^2}{2}(3\omega + 1 - \alpha), \tag{95}$$

and let

$$\mu := 1 + \frac{\alpha\lambda\ddot\lambda x^3}{2y} \quad\text{and}\quad \nu := 1 - \frac{\alpha\lambda\ddot\lambda x^3}{2y}. \tag{96}$$

Now, we take time derivatives of Eq. (29) to obtain expressions for $\lambda\frac{\partial y}{\partial t}$ and $\lambda^2\frac{\partial^2 y}{\partial t^2}$. We explain at the end only a derivation of one of these expressions, with the remaining expressions obtained in a similar way. Taking time derivatives of Eq. (29) in the domain $y \le y_{cr}$, we obtain

$$\begin{cases} \lambda\dfrac{\partial y}{\partial t} = -\dot\lambda x\dfrac{\sigma}{\mu} + O\left(\omega x^3\lambda\dfrac{\partial}{\partial t}(\lambda\ddot\lambda)\right), \\ \lambda^2\dfrac{\partial^2 y}{\partial t^2} = (2\dot\lambda^2 - \ddot\lambda\lambda)x\dfrac{\sigma}{\mu} + \dfrac{\dot\lambda x^3}{2}\lambda^3\dfrac{\partial}{\partial t} \\ \qquad\times\left[\dfrac{\lambda\ddot\lambda}{\lambda^2}\dfrac{3\omega + 1 - \alpha + \frac{\alpha x}{y}}{\mu}\right] + O\left(x^3\dot\lambda\lambda\dfrac{\partial}{\partial t}(\lambda\ddot\lambda)\right). \end{cases} \tag{97}$$

Using (29) and (39), we compute

$$\frac{\partial}{\partial t}\left[\frac{\lambda\ddot\lambda}{\lambda^2}\frac{3\omega + 1 - \alpha + \frac{\alpha x}{y}}{\mu}\right] = \frac{\dot\lambda\lambda\ddot\lambda}{\lambda^3}\left\{\frac{\alpha x^2}{y^2}\frac{\sigma^2}{\mu^3} - \frac{6\alpha x}{y\mu^2} - \frac{3(1-\alpha)}{\mu} - \frac{2(2\nu - 1)(3\omega + 1 - \alpha)}{\mu^2}\right\}. \tag{98}$$

In the domain $y \geq y_{cr}$, again, using (29), (32) and (39), we compute

$$\begin{cases} \lambda \dfrac{\partial y}{\partial t} = \dot{\lambda} x \dfrac{\sigma}{\nu} + O\left(\dfrac{\omega x^3}{\nu} \lambda \dfrac{\partial}{\partial t}(\lambda \ddot{\lambda})\right), \\ \lambda^2 \dfrac{\partial^2 y}{\partial t^2} = -(2\dot{\lambda}^2 x - \lambda \ddot{\lambda} x)\dfrac{\sigma}{\nu} - \dfrac{\dot{\lambda} x^3}{2} \lambda^3 \dfrac{\partial}{\partial t} \\ \quad \times \left[\dfrac{\lambda \ddot{\lambda}}{\lambda^2} \dfrac{3\omega + 1 - \alpha - \frac{\alpha x}{y}}{\nu}\right] + O\left(x^3 \lambda \dot{\lambda} \dfrac{\partial}{\partial t}(\lambda \ddot{\lambda})\right), \end{cases} \tag{99}$$

and

$$\frac{\partial}{\partial t}\left[\frac{\lambda \ddot{\lambda}}{\lambda^2} \frac{3\omega + 1 - \alpha - \frac{\alpha x}{y}}{\nu}\right]$$
$$= \frac{\dot{\lambda} \lambda \ddot{\lambda}}{\lambda^3} \left\{ \frac{\alpha x^2}{y^2} \frac{\sigma^2}{\nu^3} + \frac{6\alpha x}{y \nu^2} - \frac{3(1-\alpha)}{\nu} - \frac{2(2\mu - 1)(3\omega + 1 - \alpha)}{\nu^2} \right\}. \tag{100}$$

We explain the derivation of the first relation in (97). Differentiating Eq. (29) with respect to t in the domain $y \leq y_{cr}$, gives

$$\lambda \frac{\partial y}{\partial t} = \lambda \frac{\partial x}{\partial t} - \lambda \frac{\partial}{\partial t}(\lambda \ddot{\lambda}) \frac{1}{2} x^3 \omega - \frac{\lambda \ddot{\lambda}}{2} 3x^2 \lambda \frac{\partial x}{\partial t} \omega - \frac{\lambda \ddot{\lambda}}{2} x^3 \left(\frac{\lambda \frac{\partial}{\partial t}(\lambda \ddot{\lambda})}{2\lambda \ddot{\lambda}} + \alpha \frac{\lambda \frac{\partial y}{\partial t}}{y} + (1-\alpha) \frac{\lambda \frac{\partial x}{\partial t}}{x} \right).$$

Solving this for $\lambda \frac{\partial y}{\partial t}$ and using the notation (96) and the relation $\lambda \frac{\partial x}{\partial t} = -\dot{\lambda} x$, we arrive at $\lambda \frac{\partial y}{\partial t} \mu = -\dot{\lambda} x + \frac{\lambda \ddot{\lambda}}{2} 3\dot{\lambda} x^3 \omega + \frac{\lambda \ddot{\lambda}}{2} \dot{\lambda} x^3 (1 - \alpha) - \lambda \frac{\partial}{\partial t}(\lambda \ddot{\lambda}) \frac{1}{2} x^3 \omega - \frac{1}{2} x^3 \lambda \frac{\partial}{\partial t}(\lambda \ddot{\lambda})$. Finally, from Eqs. (51) and (31) (see also (39)) we obtain easily

$$\frac{1}{\lambda \ddot{\lambda}} \lambda \frac{\partial(\lambda \ddot{\lambda})}{\partial t} = 2\dot{\lambda} \ln^{-1}\left(\frac{1}{\lambda \ddot{\lambda}}\right) \left[1 + O\left(1/\ln\left(\frac{1}{\lambda \ddot{\lambda}}\right)\right)\right]$$

and

$$\frac{1}{x_{cr}} \lambda \frac{\partial x_{cr}}{\partial t} = -\dot{\lambda} \ln^{-1}\left(\frac{1}{\lambda \ddot{\lambda}}\right) \left[1 + O\left(1/\ln\left(\frac{1}{\lambda \ddot{\lambda}}\right)\right)\right]. \tag{101}$$

Collecting the above estimates gives the first relation in (97).

Now we present an explicit form of the function χ entering the definition of ψ, (26), and introduced in (22). Due to Eq. (29) we have

$$\chi = \begin{cases} -\lambda \ddot{\lambda} x^2 (\omega + 1/2) \mu^{-1}, & x < x_{cr} \\ \left(-\dfrac{2y_{cr}}{x} + \lambda \ddot{\lambda} x^2 (\omega + 1/2)\right) \nu^{-1}, & x > x_{cr}. \end{cases} \tag{102}$$

Next, we give here an explicit expression for the expression $\partial \chi / \partial x - 2\chi / x$. We compute

$$\frac{\partial \chi}{\partial x} - \frac{2\chi}{x} = -\frac{\lambda \ddot{\lambda} x^2}{\mu} \left\{ \frac{1-\alpha}{x} + \frac{\alpha \sigma}{y \mu} - \frac{\alpha \lambda \ddot{\lambda} x^2}{2y\mu} \times (\omega + 1/2) \left(3 - \frac{x}{y} \frac{\sigma}{\mu}\right) \right\}, \tag{103}$$

for $x < x_{cr}$, and

$$\frac{\partial \chi}{\partial x} - \frac{2\chi}{x} = \frac{6y_{cr}}{x^2 \nu} + \frac{\lambda \ddot{\lambda} x^2}{\nu} \left(\frac{1-\alpha}{x} - \frac{\alpha}{y} \frac{\sigma}{\nu} \right) + \frac{\alpha \lambda \ddot{\lambda} x}{2y\nu^2} \times \left(-2y_{cr} + \lambda \ddot{\lambda} x^3 (\omega + 1/2)\right) \left(3 + \frac{x}{y} \frac{\sigma}{\nu}\right), \tag{104}$$

for $x > x_{cr}$.

Note that the function y, Eq. (29), is chosen so as to cancel the term $-2\lambda \ddot{\lambda} x / (\lambda^2 (1 + y^2))$ arising from the last term in expression (26) (see the first term on the r.h.s. of (97) and the first term on the r.h.s. of (17)). Using Eqs. (26), (103) and (97) and using the relation

$$\frac{y}{x} - \frac{\sigma}{\mu} = \frac{\lambda \ddot{\lambda} x^2}{2} \left[\frac{\frac{\alpha x}{y} - 3\omega + 1 - \alpha}{\mu} - \omega \right],$$

we obtain the following expression for the function ψ in the domain $x \leq x_{cr}$:

$\psi = \psi_1 + \psi_2$, with

$$\psi_1(y, t) := -\left[4\dot{\lambda}^2 + 8\lambda \ddot{\lambda} \left(\omega + \frac{1}{2}\right)\right] \frac{x^3}{\mu y^2 (1 + y^2)^2}, \tag{105}$$

and

$$\psi_2 = \frac{\lambda \ddot{\lambda} x^5}{y^2 (1 + y^2)} \left\{ \frac{\dot{\lambda}^2}{\mu} \left[\frac{2xy(\omega - \alpha)}{(1 + y^2)\mu} - \frac{\alpha x^2}{y^2} \frac{\sigma^2}{\mu^2} + \frac{6\alpha x}{\mu y} + 3(1-\alpha) - 2(3\omega + 1 - \alpha) \left(\frac{(1+\sigma) yx}{\mu(1 + y^2)} - 1 - \frac{2\nu - 1}{\mu} \right) \right] \right.$$
$$\left. + \frac{\lambda \ddot{\lambda}}{\mu} \left[\frac{\alpha x}{y} \left(\frac{\frac{\alpha x}{y} - 3\omega + 1 - \alpha}{\mu} - \omega + \frac{\omega + 1/2}{\mu} \left(3 - \frac{x}{y} \frac{\sigma}{\mu}\right) \right) - \frac{4(\omega + 1/2)^2 xy}{\mu(1 + y^2)} - (3\omega + 1 - \alpha) \right] \right\}. \tag{106}$$

Note that in the region $1 \leq y \ll y_{cr} = O\left(\frac{1}{\sqrt{\lambda \ddot{\lambda}}}\right)$, $\omega, \sigma, \mu, \nu = O(1)$ and $x = O(y)$. Using this we conclude that in the region $0 < y \ll y_{cr}$, the estimates (42) hold.

Now, using Eqs. (102), (26), (103) and (99), we find expression for function ψ in the domain $y \geq y_{cr}$. In fact, to obtain the equation on the parameter λ we need to know only the part of ψ in $\{y \geq y_{cr}\}$, proportional to $\dot{\lambda}^2$. For this reason we write out only this part:

$$\psi = 2\dot{\lambda}^2 \frac{x^3}{y^4} \left\{ \frac{2x}{y} \frac{\sigma^2}{\nu^2} + \frac{2\sigma}{\nu} + \frac{\lambda \ddot{\lambda} x^2}{2\nu} \left(\frac{\alpha x^2}{y^2} \frac{\sigma^2}{\nu^2} + \frac{6\alpha x}{y\nu} - 3(1-\alpha) - \frac{2(2\mu - 1)(3\omega + 1 - \alpha)}{\nu} \right) \right\}$$
$$+ \text{ term proportional to } \lambda \ddot{\lambda}. \tag{107}$$

Now, we show Eq. (34) which was stated in Section 3. Indeed, the definitions of σ and ν and the second equation in (29) imply that for $y \geq y_{cr}$

$$\frac{\sigma}{\nu} \sim \frac{3\lambda \ddot{\lambda} x^2}{2} \ln(\lambda \ddot{\lambda} x^2), \qquad \sigma \sim \lambda \ddot{\lambda} x^2 \ln(\lambda \ddot{\lambda} x^2),$$
$$y \sim \lambda \ddot{\lambda} x^3 \ln(\lambda \ddot{\lambda} x^2).$$

Using these relations and Eq. (107), we arrive at the desired relation (34).

Appendix C

In this appendix we derive Eq. (51) from Eq. (50). To this end, we calculate two simple integrals. For the first one, we have

$$\int_0^\infty \frac{dy y^3}{(1 + y^2)^3} = \frac{1}{2} \int_0^\infty \frac{dx x}{(1 + x)^3} = \frac{1}{4}. \tag{108}$$

For the second integral, we compute

$$\int_0^\infty \frac{\mathrm{d}y y^3 \ln y}{(1+y^2)^3} = \frac{1}{4}\int_0^\infty \frac{\mathrm{d}x x \ln x}{(1+x)^3}$$
$$= -\lim_{\varepsilon\to 0}\frac{1}{4}\int_\varepsilon^\infty \ln x \mathrm{d}\left(\frac{1}{x+1} - \frac{1}{2}\frac{1}{(x+1)^2}\right)$$
$$= \lim_{\varepsilon\to 0}\left(\frac{1}{8}\ln\varepsilon + \frac{1}{4}\int_\varepsilon^\infty \frac{\mathrm{d}x}{x}\left(\frac{1}{x+1} - \frac{1}{2}\frac{1}{(x+1)^2}\right)\right)$$
$$= \lim_{\varepsilon\to 0}\left(\frac{1}{8}\ln\varepsilon + \frac{1}{8}\int_\varepsilon^\infty \mathrm{d}x\left[\frac{1}{x} - \frac{1}{x+1} + \frac{1}{(x+1)^2}\right]\right)$$
$$= \frac{1}{8}. \tag{109}$$

Using the values of these two integrals, we obtain Eq. (51) from Eq. (50).

Appendix D

In this appendix we compute the partial derivatives of energy $E = E(\alpha,\beta)$ w.r.t. parameters α, β. Using expression (62), we obtain

$$\frac{\partial E}{\partial\beta} = 4\int_0^\infty \mathrm{d}\rho \frac{\rho}{(1+y^2)^2}\left\{\frac{\partial y}{\partial t}\frac{\partial}{\partial\beta}\left(\frac{\partial y}{\partial t}\right) + \frac{\partial y}{\partial\rho}\frac{\partial}{\partial\beta}\left(\frac{\partial y}{\partial\rho}\right) + \frac{y}{\rho^2}\frac{\partial y}{\partial\beta} - \frac{2y}{1+y^2}\frac{\partial y}{\partial\beta}\left(\left(\frac{\partial y}{\partial t}\right)^2 + \left(\frac{\partial y}{\partial\rho}\right)^2 + \frac{y^2}{\rho^2}\right)\right\} \tag{110}$$

and

$$\frac{\partial E}{\partial\alpha} = 4\int_0^\infty \mathrm{d}\rho \frac{\rho}{(1+y^2)^2}\left\{\frac{\partial y}{\partial t}\frac{\partial}{\partial\alpha}\left(\frac{\partial y}{\partial t}\right) + \frac{\partial y}{\partial\rho}\frac{\partial}{\partial\alpha}\left(\frac{\partial y}{\partial\rho}\right) + \frac{y}{\rho^2}\frac{\partial y}{\partial\alpha} - \frac{2y}{1+y^2}\frac{\partial y}{\partial\alpha}\left(\left(\frac{\partial y}{\partial t}\right)^2 + \left(\frac{\partial y}{\partial\rho}\right)^2 + \frac{y^2}{\rho^2}\right)\right\}. \tag{111}$$

Recall the notation $\sigma := 1 - \frac{\lambda\ddot\lambda x^2}{2}\left(3\ln\left(\frac{\sqrt{\lambda\ddot\lambda}}{\beta}y^\alpha x^{1-\alpha}\right) + 1 - \alpha\right)$,

$$\omega := \ln\left(\frac{\sqrt{\lambda\ddot\lambda}}{\beta}y^\alpha x^{1-\alpha}\right), \qquad \mu := 1 + \frac{\alpha\lambda\ddot\lambda x^3}{2y}, \quad \text{and}$$
$$\nu := 1 - \frac{\alpha\lambda\ddot\lambda x^3}{2y}, \tag{112}$$

and let

$$\mu_{\mathrm{cr}} := 1 + \frac{\alpha\lambda\ddot\lambda x_{\mathrm{cr}}^3}{2y_{\mathrm{cr}}} \quad \text{and} \quad \nu_{\mathrm{cr}} := 1 - \frac{\alpha\lambda\ddot\lambda x_{\mathrm{cr}}^3}{2y_{\mathrm{cr}}}. \tag{113}$$

From Eq. (29) we find

$$\begin{cases} \dfrac{\partial y}{\partial\beta} = \dfrac{\lambda\ddot\lambda x^3}{2\beta}\dfrac{1}{\mu}, & x < x_{\mathrm{cr}}, \\ \dfrac{\partial y}{\partial\beta} = \dfrac{\lambda\ddot\lambda}{2\beta}\dfrac{1}{\nu}\left[\dfrac{2x_{\mathrm{cr}}^3}{\mu_{\mathrm{cr}}} - x^3\right], & x > x_{\mathrm{cr}}, \end{cases} \tag{114}$$

and

$$\frac{\partial y_{\mathrm{cr}}}{\partial\beta} = \frac{\lambda\ddot\lambda x_{\mathrm{cr}}^3}{2\beta}\frac{1}{\mu_{\mathrm{cr}}}.$$

Using Eq. (114) and assumptions (39) (or differentiating (51)) and (40), we obtain readily, in the leading approximation in $1/\ln\left(\frac{1}{\lambda\ddot\lambda}\right)$, the time derivative of $\frac{\partial y}{\partial\beta}$:

$$\begin{cases} \dfrac{\partial}{\partial t}\left(\dfrac{\partial y}{\partial\beta}\right) = -\dfrac{\dot\lambda\lambda\ddot\lambda x^3}{2\beta\lambda\nu^2}\left[3 + \dfrac{\alpha\lambda\ddot\lambda x^4}{2y^2}\dfrac{\sigma}{\mu}\right], & x < x_{\mathrm{cr}}, \\ \dfrac{\partial}{\partial t}\left(\dfrac{\partial y}{\partial\beta}\right) = -\dfrac{\dot\lambda\lambda\ddot\lambda}{2\beta\lambda\nu}\left\{\dfrac{3 + \frac{\alpha\lambda\ddot\lambda x^4}{2y^2}\frac{\sigma}{\nu}}{\nu}\left(\dfrac{2x_{\mathrm{cr}}^3}{\mu_{\mathrm{cr}}} - x^3\right) - \dfrac{6x_{\mathrm{cr}}^3}{\mu_{\mathrm{cr}}}\right\}, & x > x_{\mathrm{cr}}. \end{cases} \tag{115}$$

In a similar way, we find the derivative of y w.r.t. α:

$$\begin{cases} \dfrac{\partial y}{\partial\alpha} = -\dfrac{\lambda\ddot\lambda x^3}{2}\dfrac{\ln\left(\frac{y}{x}\right)}{\mu}, & x < x_{\mathrm{cr}}, \\ \dfrac{\partial y}{\partial\alpha} = \dfrac{\lambda\ddot\lambda}{2}\dfrac{1}{\nu}\left[x^3\ln\left(\dfrac{y}{x}\right) - 2x_{\mathrm{cr}}^3\dfrac{\ln\left(\frac{y_{\mathrm{cr}}}{x_{\mathrm{cr}}}\right)}{\mu_{\mathrm{cr}}}\right], & x > x_{\mathrm{cr}}, \end{cases} \tag{116}$$

and

$$\frac{\partial y_{\mathrm{cr}}}{\partial\alpha} = -\frac{\lambda\ddot\lambda x_{\mathrm{cr}}^3}{2}\frac{\ln\left(\frac{y_{\mathrm{cr}}}{x_{\mathrm{cr}}}\right)}{\mu_{\mathrm{cr}}}.$$

Taking the time derivative of Eq. (116), we obtain

$$\frac{\partial}{\partial t}\left(\frac{\partial y}{\partial\alpha}\right) = -\frac{\dot\lambda\lambda\ddot\lambda x^3}{2\lambda\mu^2}\left\{\frac{\lambda\ddot\lambda x^3}{y}\left(\omega + \frac{1}{2}\right) - \ln\left(\frac{y}{x}\right)\left(3 + \frac{\alpha\lambda\ddot\lambda x^4}{2y^2}\frac{\sigma}{\mu}\right)\right\}, \quad x < x_{\mathrm{cr}}, \tag{117}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial y}{\partial\alpha}\right) = -\frac{\dot\lambda\lambda\ddot\lambda}{2\lambda\nu}\left\{-\frac{3 + \frac{\alpha\lambda\ddot\lambda x^4}{2y}\frac{\sigma}{\nu}}{\nu}\left[2x_{\mathrm{cr}}^3\frac{\ln\left(\frac{y_{\mathrm{cr}}}{x_{\mathrm{cr}}}\right)}{\mu_{\mathrm{cr}}} - x^3\ln\left(\frac{y}{x}\right)\right] + \frac{6x_{\mathrm{cr}}^3\ln\left(\frac{y_{\mathrm{cr}}}{x_{\mathrm{cr}}}\right)}{\mu_{\mathrm{cr}}} - \frac{x^3}{y\nu}\left(2y_{\mathrm{cr}} - \lambda\ddot\lambda x^3\left(\omega + \frac{1}{2}\right)\right)\right\},$$
$$x > x_{\mathrm{cr}}. \tag{118}$$

Note that the main contribution to the partial derivatives (110) and (111) comes from the domain $x \sim x_{\mathrm{cr}}$. Both derivatives are sums of terms proportional to $\dot\lambda^2$ and to $\lambda\ddot\lambda$. The coefficients for these terms are of the order of 1. As result, since we assumed that $|\lambda\ddot\lambda| \ll \dot\lambda^2$, we have to find in the expressions for (110) and (111) only the terms proportional to $\dot\lambda^2$.

Using Eqs. (114)–(118) we can write the r.h.s. of (110) and (111) in a more explicit form

$$\frac{1}{4}\frac{\partial E}{\partial\beta} = \dot\lambda^2\int_0^{x_{\mathrm{cr}}} \frac{\mathrm{d}x x^5\lambda\ddot\lambda}{2\beta y^4}\frac{\sigma}{\left(1 + \frac{\alpha\lambda\ddot\lambda x^3}{2y}\right)^3} \times \left[3 + \frac{\alpha\lambda\ddot\lambda x^4}{2y^2}\frac{\sigma}{1 + \frac{\alpha\lambda\ddot\lambda x^3}{2y}} - \frac{2x\sigma}{y}\right]$$
$$- \dot\lambda^2\int_{x_{\mathrm{cr}}}^\infty \frac{\mathrm{d}x x^2\lambda\ddot\lambda}{2\beta y^4}\frac{\sigma}{\left(1 - \frac{\alpha\lambda\ddot\lambda x^3}{2y}\right)^2} \times \left\{\left[\frac{1}{1 - \frac{\alpha\lambda\ddot\lambda x^3}{2y}}\left(3 + \frac{\alpha\lambda\ddot\lambda x^4}{2y^2}\frac{\sigma}{1 - \frac{\alpha\lambda\ddot\lambda x^3}{2y}}\right)\right.\right.$$

$$\times\left(\frac{2x_{cr}^3}{1+\frac{\alpha\lambda\ddot{\lambda}x_{cr}^3}{2y_{cr}}}-x^3\right)-\frac{6x_{cr}^3}{1+\frac{\alpha\lambda\ddot{\lambda}x_{cr}^3}{2y_{cr}}}\Bigg]$$
$$+\frac{2x}{y}\frac{\sigma}{1-\frac{\alpha\lambda\ddot{\lambda}x^3}{2y}}\left(\frac{2x_{cr}^3}{1+\frac{\alpha\lambda\ddot{\lambda}x_{cr}^3}{2y_{cr}}}-x^3\right)\Bigg\}, \tag{119}$$

$$\frac{1}{4}\frac{\partial E}{\partial\alpha}=\dot{\lambda}^2\int_0^{x_{cr}}\frac{\mathrm{d}x x^5\lambda\ddot{\lambda}\sigma}{2y^4\left(1+\frac{\alpha\lambda\ddot{\lambda}x^3}{2y}\right)^3}$$
$$\times\left\{\frac{\lambda\ddot{\lambda}x^3}{y}\left(\ln\left(\frac{\sqrt{\lambda\ddot{\lambda}}}{\beta}y^\alpha x^{1-\alpha}\right)+1/2\right)\right.$$
$$\left.-\ln\left(\frac{y}{x}\right)\left(3+\frac{\alpha\lambda\ddot{\lambda}x^4}{2y^2}\frac{\sigma}{1+\frac{\alpha\lambda\ddot{\lambda}x^3}{2y}}\right)+\frac{2x}{y}\ln\left(\frac{y}{x}\right)\sigma\right\}$$
$$+\dot{\lambda}^2\int_{x_{cr}}^{\infty}\frac{\mathrm{d}x x^2\lambda\ddot{\lambda}\sigma}{2y^4\left(1-\frac{\alpha\lambda\ddot{\lambda}x^3}{2y}\right)^2}$$
$$\times\left\{\frac{1}{1-\frac{\alpha\lambda\ddot{\lambda}x^3}{2y}}\left(3+\frac{\alpha\lambda\ddot{\lambda}x^4}{2y^2}\frac{\sigma}{1-\frac{\alpha\lambda\ddot{\lambda}x^3}{2y}}\right)\right.$$
$$\times\left(\frac{2x_{cr}^3\ln\left(\frac{y_{cr}}{x_{cr}}\right)}{1+\frac{\alpha\lambda\ddot{\lambda}x_{cr}^3}{2y_{cr}}}-x^3\ln\left(\frac{y}{x}\right)\right)$$
$$-\frac{6x_{cr}^3\ln\left(\frac{y_{cr}}{x_{cr}}\right)}{1+\frac{\alpha\lambda\ddot{\lambda}x_{cr}^3}{2y_{cr}}}+\frac{x^3}{y\left(1-\frac{\alpha\lambda\ddot{\lambda}x^3}{2y}\right)}$$
$$\times\left(2y_{cr}-\lambda\ddot{\lambda}x^3\left(\ln\left(\frac{\sqrt{\lambda\ddot{\lambda}}}{\beta}y^\alpha x^{1-\alpha}\right)+1/2\right)\right)$$
$$\left.+\frac{2x}{y}\frac{\sigma}{1-\frac{\alpha\lambda\ddot{\lambda}x^3}{2y}}\left(\frac{2x_{cr}^3\ln\left(\frac{y_{cr}}{x_{cr}}\right)}{1+\frac{\alpha\lambda\ddot{\lambda}x_{cr}^3}{2y_{cr}}}-x^3\ln\left(\frac{y}{x}\right)\right)\right\}. \tag{120}$$

Let $\alpha=\alpha(\beta)$ be a solution of Eq. (61). We find numerically (see Appendix E) that β changes on the interval $(0,\beta_0]$, where $\beta_0=1.0405$ (the corresponding value of α is $\alpha_0=0.65436$). Using expressions for $\frac{\partial E}{\partial\alpha}$ and $\frac{\partial E}{\partial\beta}$, derived above, we show numerically that the function

$$\Phi:=\frac{\partial E}{\partial\beta}+\frac{\partial E}{\partial\alpha}\frac{\partial\alpha}{\partial\beta} \tag{121}$$

is negative for $\beta=\beta_0$ and for $\beta\to 0$, with $E(\alpha,\beta)$ having absolute minimum at $\beta=\beta_0$.

Appendix E

Numerical calculations with the help of Eq. (59) show that there is a point (α_0,β_0),

$$\beta_0=1.0405,\qquad \alpha_0=0.6543626,$$

so that Eq. (61), $I(\alpha,\beta)=0$, has no solution for $\alpha<\alpha_0$ and for $\beta>\beta_0$. Moreover, the solution of the equation $I(\alpha,\beta)=0$ for β determines a double-valued function $\beta=\beta(\alpha)$, whose branches coalesce at $\alpha=\alpha_0$ and have different derivatives there (see Eqs. (122) and (123)). Moreover, $I(\alpha,\beta)=0$ has the unique solution β_0 at $\alpha=\alpha_0$. Hence the solution of the equation $I(\alpha,\beta)=0$ for α also leads to a double-valued function $\alpha=\alpha(\beta)$.

Numerical calculations give the following expansions for the lowest branch,

$$\beta=\beta_0-\beta_1(\alpha-\alpha_0)-\beta_2(\alpha-\alpha_0)^2, \tag{122}$$

$\alpha>\alpha_0$, and for the distance, Δ, between the branches along the α-axis,

$$\Delta=\gamma_1(\beta_0-\beta)-\gamma_2(\beta_0-\beta)^2, \tag{123}$$

where

$$\beta_1=2.54732,\qquad \beta_2=13.8297, \tag{124}$$
$$\gamma_1=0.08029,\qquad \gamma_2=0.42736. \tag{125}$$

(Solving (122) for α gives the lower branch of the function $\alpha=\alpha(\beta)$. Adding (123) to this solution gives the upper branch of $\alpha=\alpha(\beta)$.)

To find the second "end" point on the α-interval we check the point $\alpha=1$ where the dependence of y on x in Eq. (29) can be found in an explicit form. To do this, we note that (32) and (33) with $\alpha=1$ imply that

$$\gamma=\frac{\sqrt{\lambda\ddot{\lambda}}y_{cr}}{\beta},\quad \beta^2=\frac{8}{27\gamma^2\ln\gamma}\quad\text{and}\quad \lambda\ddot{\lambda}{y_{cr}}^2=\frac{8}{27\ln\gamma}. \tag{126}$$

We also have $y_{cr}/x_{cr}=2/3$. For $\alpha=1$ solvability condition of Eq. (29) is

$$\gamma>\mathrm{e}^{1/2}. \tag{127}$$

Indeed, set

$$y=y_{cr}z,\qquad z=1+\delta,\qquad \frac{x}{y_{cr}}=\frac{3}{2}+\tau. \tag{128}$$

In the range $0<\delta\ll 1$ we have

$$\frac{2}{3}\tau^2=\delta\left(1-\frac{1}{2\ln\gamma}\right). \tag{129}$$

From this equation we see that β should satisfy the inequality given in Eq. (127).

Now we set $y=y_{cr}z$. For $z<1$ we obtain from the first equation in (29), with $\alpha=1$, and from (126) the following cubic equation for the ratio $\frac{x}{y_{cr}}$

$$\frac{4}{27}\left(\frac{x}{y_{cr}}\right)^3\frac{\ln(\gamma z)}{\ln\gamma}-\frac{x}{y_{cr}}+z=0. \tag{130}$$

Solution of Eq. (130) in the range $z<1$ is

$$\frac{x}{y_{cr}}=3\left(\frac{\ln\gamma}{\ln(1/(\gamma z))}\right)^{1/2}$$
$$\times\sinh\left[\frac{1}{3}\ln\left(z\sqrt{\frac{\ln(1/(\gamma z))}{\ln\gamma}}+\sqrt{1+z^2\frac{\ln(1/(\gamma z))}{\ln\gamma}}\right)\right] \tag{131}$$

for $\gamma z<1$ and

$$\frac{x}{y_{cr}}=3\sqrt{\frac{\ln\gamma}{\ln(\gamma z)}}\sin\left[\frac{1}{3}\arctan\frac{z\sqrt{\ln(\gamma z)/\ln\gamma}}{\sqrt{1-z^2\ln(\gamma z)/\ln\gamma}}\right] \tag{132}$$

for $\gamma z>1$.

In the range $z\geq 1$ the ratio x/y_{cr} solves the following cubic equation (see the second equation in (29))

$$\frac{4}{27}\frac{\ln(\gamma z)}{\ln\gamma}\left(\frac{x}{y_{cr}}\right)^3-\frac{x}{y_{cr}}+2-z=0. \tag{133}$$

Let z_0 be the solution of equation

$$1=(z_0-2)^2\frac{\ln(\gamma z_0)}{\ln\gamma}. \tag{134}$$

We split the semi-interval $z > 1$ into two sub-intervals. In the interval $1 < z < z_0$ we have

$$\frac{x}{y_{\rm cr}} = 3\sqrt{\frac{\ln\gamma}{\ln(\gamma z)}}\sin\phi, \tag{135}$$

where

$$\begin{aligned}\phi &= \frac{\pi}{6} + \frac{1}{3}\arctan\left(\frac{\sqrt{1-(2-z)^2\ln(\gamma z)/\ln\gamma}}{(2-z)\sqrt{\ln(\gamma z)/\ln\gamma}}\right),\\ &\quad 1 < z < 2,\\ \phi &= \frac{\pi}{3} + \frac{1}{3}\arctan\left(\frac{(z-2)\sqrt{\ln(\gamma z)/\ln\gamma}}{\sqrt{1-(z-2)^2\ln(\gamma z)/\ln\gamma}}\right),\\ &\quad 2 < z < z_0.\end{aligned} \tag{136}$$

In the range $z > z_0$ we have

$$\frac{x}{y_{\rm cr}} = \frac{3}{2}\left(Q^{1/3} + \frac{\ln\gamma}{\ln(\gamma z)}Q^{-1/3}\right) \tag{137}$$

where

$$Q = (z-2)\frac{\ln\gamma}{\ln(\gamma z)} + \sqrt{\left((z-2)\frac{\ln\gamma}{\ln(\gamma z)}\right)^2 - \left(\frac{\ln\gamma}{\ln(\gamma z)}\right)^3}. \tag{138}$$

Using Eqs. (131) and (138) we obtain with the help of numerical calculations, that Eq. (61) at $\alpha = 1$ has a solution only as β goes to zero. This means that $\alpha = 1$ is the second end point of the α-interval.

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