

# Embedded solitons in Lagrangian and semi-Lagrangian systems

D.J. Kaup<sup>a</sup>, Boris A. Malomed<sup>b,\*</sup>

<sup>a</sup> Department of Mathematics, University of Central Florida, Orlando, FL 32816, USA

<sup>b</sup> Department of Interdisciplinary Studies, Faculty of Engineering, Tel Aviv University, Ramat Aviv 69978, Israel

## Abstract

We develop the technique of the variational approximation (VA) for solitons in two directions. First, one may have a physical model which does not admit the usual Lagrangian representation, as some terms were discarded for various reasons. For instance, the second-harmonic-generation (SHG) model considered here, which includes the Kerr nonlinearity, lacks the usual Lagrangian representation if one ignores the Kerr nonlinearity of the second-harmonic, as compared to that of the fundamental. However, we show that, with a natural modification, one may still apply the VA to those seemingly flawed systems as efficiently as it applies to their fully Lagrangian counterparts. We call such models, that do not admit the usual Lagrangian representation, *semi-Lagrangian* systems. Second, we show that, upon adding an infinitesimal tail that does not vanish at infinity, to a usual soliton ansatz, one can obtain an analytical criterion which (within the framework of VA) gives a condition for finding *embedded solitons* (ESs), i.e., isolated truly localized solutions existing inside the continuous spectrum of the radiation modes. The criterion takes a form of orthogonality of the radiation mode in the infinite tail to the soliton core. To test the criterion, we have applied it to both the semi-Lagrangian truncated version of the SHG model and to the same model in its full form. In the former case, the criterion (combined with VA for the soliton proper) yields an *exact* solution for the ES. In the latter case, the criterion selects the ES with a relative error  $\approx 1\%$ .

© 2003 Elsevier B.V. All rights reserved.

**Keywords:** Solitons; Lagrangian; Second-harmonic-generation; Variational approximation

## 1. Introduction

One of the many pioneer contributions by A.C. Newell has been the study of solitary waves (which we will simply call “solitons” here, in line with currently adopted terminology). These occur not only in integrable models, but also in many non-integrable nonlinear-wave systems [1,2]. It is toward an improved understanding of solitons in non-integrable systems, that we present this work, dedicated to Prof. Newell.

In non-integrable models, more complex forms of solitons are found as one considers higher-order systems. An issue of fundamental importance is to find where, in the space of the soliton parameters, such solutions could exist. Where this could be is dominantly determined by the model’s linear dispersion relation,  $\omega(k)$ . Typically, dispersion curves have gaps or partial gaps, which are intervals of the frequency of  $\omega$  in which the wavenumber  $k$  would take imaginary or complex values. For example, for the nonlinear Schrödinger (NLS) equation, one has  $\omega = Dk^2$ , where

\* Corresponding author. Tel.: +972-3-640-6399.

E-mail addresses: [kaup@ucf.edu](mailto:kaup@ucf.edu) (D.J. Kaup), [malomed@eng.tau.ac.il](mailto:malomed@eng.tau.ac.il), [malomed@post.tau.ac.il](mailto:malomed@post.tau.ac.il) (B.A. Malomed).

$D$  is the dispersion coefficient, hence  $k$  must be imaginary if  $\omega/D < 0$ . This is precisely the region where one finds solitons of the NLS equation.

More complex systems, with several branches of the dispersion relation, may have regions of  $\omega$  where one branch gives imaginary  $k$ , while another one gives it real. Inside such a “partial gap”, one typically encounters “delocalized solitons” (see the book [3] and relevant examples from nonlinear-optical models in Ref. [4]), which are quasi-solitary waves with non-vanishing oscillatory tails. Such objects, obviously, have an infinite energy, and therefore are unphysical, except, possibly, in finite-size systems. However, it may occur that the amplitude of the tail vanishes at some special values of  $\omega$ . Then, one has a truly localized object at an isolated (discrete) value of  $\omega$  at which a real value of  $k$  does exist. Since this value of  $\omega$  lies inside a continuous part of the spectrum, these objects are called embedded solitons (ESs). A number of physically meaningful models which support ESs are now known (see examples of such solitons found in hydrodynamic models in Refs. [5,6], and a short review in Ref. [7]), their stability (which turns out to be *semi-stability*) being quite distinct from the stability of ordinary solitary waves [8].

Currently, the only method for locating ESs is to search for them numerically (in a model of a different type, based on the fifth-order Korteweg–de Vries equation, an approach which makes it possible to identify ES by means of a perturbation theory was developed in Ref. [6]). One does know that they may be found inside a partial gap of the dispersion relation, but otherwise one has no analytical tool for locating them (in exceptional cases, exact ES solutions can be found by guess [8]). The first objective of this paper is to develop an approximate analytical method for locating ESs. The method is based on the variational approximation (VA; see a recent review of the application of this technique to solitons in Ref. [9]), which incorporates an amplitude of the infinitesimal tail of the related delocalized solitons as a key variational parameter. The approach will be tested on two versions of an ES-generating model introduced in Ref. [8], which combines the second-harmonic-generation (SHG) through quadratic  $[\chi^{(2)}]$  nonlinearity, and the usual cubic  $[\chi^{(3)}]$  nonlinear terms. The difference between the two versions of the model is that one is a full model (it has the usual Lagrangian representation), while the other one is a truncated model, wherein, upon assuming that the fundamental-harmonic (FH) field is much stronger than the second-harmonic (SH), some of the SH  $\chi^{(3)}$  terms are omitted. The truncated model does *not* admit the usual Lagrangian representation. However, it was this truncated version of the  $\chi^{(2)} : \chi^{(3)}$  model in which the above-mentioned exact analytic ES solution was found in Ref. [8]. On the other hand, in the full version of the model, ES solutions could only be found by means of numerical methods. In this work, we will obtain a natural variational criterion which makes it possible to distinguish ESs from delocalized solitons in each of these systems. Furthermore, we will demonstrate that, in the case of the truncated model, this method yields an exact result, and in the full model, a relative error is  $\approx 1\%$  in the prediction of the location of ES (in comparison with numerical results).

As it was mentioned above, due to a missing (omitted) term, the truncated model does not have a “complete” Lagrangian representation. Namely, it can be obtained from a Lagrangian, but only if one term in it is not subjected to the variation with respect to the FH field, when deriving the system of the FH and SH equations. Accordingly, the truncated model does not conserve any Hamiltonian. But it is, nevertheless, a conservative system, as it conserves the norm of the solution (which is usually called “energy” in nonlinear optics and is different from the Hamiltonian).

As the truncated model does not admit the full Lagrangian representation, it is necessary to work out a special version of the VA for it, which is another objective of this paper. In fact, this can be done in a very simple way: after inserting the variational ansatz into the Lagrangian and performing the integration in order to obtain the corresponding effective Lagrangian (the one which is an explicit function of variational parameters, rather than a functional depending on the field variables), the above-mentioned special term in the effective Lagrangian should not be varied with respect to variational parameters that belong to the FH component of the ansatz. Thus, models similar to the above-mentioned truncated one, even though they cannot be represented in the usual Lagrangian form, can still be handled by means of VA. Since one cannot freely vary all the fields in the Lagrangian of these systems, we call them *semi-Lagrangian* systems.

The rest of the paper is organized as follows. In [Section 2](#), we introduce the full and truncated models and the Lagrangian representations of each. In [Section 3](#), the VA for the semi-Lagrangian case is developed. The general VA-based analytical condition for identifying ESs (in an approximate form) is obtained in [Section 4](#). [Section 5](#) concludes the paper.

## 2. The full and truncated models

Following Refs. [\[8,10\]](#), we first introduce the full  $\chi^{(2)} : \chi^{(3)}$  model:

$$iu_z + \frac{1}{2}u_{tt} + u^*v + \gamma_1|u|^2u + 4\gamma_2|v|^2u = 0, \quad (1)$$

$$iv_z - \frac{1}{2}\delta v_{tt} + qv + \frac{1}{2}u^2 + 2\gamma_2(|v|^2 + 2|u|^2)v = 0, \quad (2)$$

which is written in the usual “optical” notation, so that  $z$  and  $t$  are the propagation distance and reduced time,  $u$  and  $v$  the FH and SH fields,  $-\delta$  is the relative dispersion coefficient at SH,  $q$  the SHG mismatch, and  $\gamma_{1,2}$  are the Kerr coefficients. In fact, the ratio of the self-phase-modulation (SPM) and cross-phase-modulation (XPM) coefficients in [Eq. \(2\)](#) is not necessarily 1:2, but this feature of the model is not a crucially important one.

The form of [Eqs. \(1\) and \(2\)](#) implies that the group-velocity dispersion is anomalous at FH, while at SH it may be both normal, if  $\delta > 0$ , and anomalous, if  $\delta < 0$  (both cases are physically possible). As for the Kerr coefficients  $\gamma_1$  and  $\gamma_2$ , they always have one sign. Most typically, they are positive (corresponding to the self-focusing nonlinearity), but may be negative too, see a detailed discussion of this point in Ref. [\[11\]](#).

In many cases, the SH field is much weaker than the FH field—for instance, if the mismatch is large. Then, assuming that  $|v|^2 \ll |u|^2$ , one may neglect the XPM term in comparison with the SPM one in [Eq. \(1\)](#), and the SPM term in comparison with its XPM counterpart in [Eq. \(1\)](#), which leads to the *truncated model*,

$$iu_z + \frac{1}{2}u_{tt} + u^*v + \gamma_1|u|^2u = 0, \quad (3)$$

$$iv_z - \frac{1}{2}\delta v_{tt} + qv + \frac{1}{2}u^2 + 4\gamma_2|u|^2v = 0. \quad (4)$$

In this work, we are interested in stationary fundamental-soliton solutions, which are looked in the form

$$u(z, t) = e^{ikz}U(t), \quad v(z, t) = e^{2ikz}V(t), \quad (5)$$

where  $k$  is the FH wavenumber, and real even functions  $U(t)$  and  $V(t)$  with a single maximum at  $t = 0$  exponentially decay at  $t \rightarrow \infty$ . Note that ordinary (non-embedded) solitons may exist in the regions

$$0 < k < \frac{1}{2}q \quad \text{if } \delta > 0, \quad k > \max\{0, \frac{1}{2}q\} \quad \text{if } \delta < 0 \quad (6)$$

(which implies that  $q$  must be positive for the existence of ordinary solitons if  $\delta$  is positive). On the other hand, ESs may exist in a range of  $k$  which does not overlap with the continuous spectrum in the FH equation, [\(1\)](#) or [\(3\)](#), but falls into the continuous spectrum of the SH equation, [\(2\)](#) or [\(4\)](#). Thus, ES may exist in the regions (cf. [Eq. \(6\)](#))

$$k > \max\{0, \frac{1}{2}q\} \quad \text{if } \delta > 0, \quad 0 < k < \frac{1}{2}q \quad \text{if } \delta < 0. \quad (7)$$

This implies that, in the case  $\delta < 0$ , the existence of ESs makes it necessary to have positive  $q$ .

Substituting the expressions [\(5\)](#) into [Eqs. \(1\) and \(2\)](#), we arrive at a system of ordinary differential equations,

$$-kU + \frac{1}{2}U'' + UV + \gamma_1U^2 + \gamma_1U^3 + 4\gamma_2V^2U = 0, \quad (8)$$

$$-2kV - \frac{1}{2}\delta V'' + qV + \frac{1}{2}U^2 + 2\gamma_2(V^2 + 2U^2)V = 0, \quad (9)$$

where the prime stands for  $d/dt$ . Accordingly, the system of stationary equations corresponding to the truncated model is

$$-kU + \frac{1}{2}U'' + UV + \gamma_1 U^3 = 0, \quad (10)$$

$$-2kV - \frac{1}{2}\delta V'' + qV + \frac{1}{2}U^2 + 4\gamma_2 U^2 V = 0. \quad (11)$$

It is obvious that Eqs. (8) and (9) can be derived from the Lagrangian

$$L = \frac{1}{2} \int_{-\infty}^{+\infty} \left[ -kU^2 - (2k + q)V^2 - \frac{1}{2}(U')^2 + \frac{\delta}{2}(V')^2 + U^2 V + \frac{\gamma_1}{2}U^4 + 4\gamma_2 U^2 V^2 + \gamma_2 V^4 \right] dt. \quad (12)$$

In the case of the truncated system of stationary equations, (10) and (11), one may still obtain this system from the variational principle, provided we do the following. First, the last term in the integrand in Eq. (12) should be dropped. Second, the next to the last term should only be subjected to varying in  $V$ , but not in the  $U$  field.

### 3. The VA for the semi-Lagrangian system

In this section, we focus on the application of VA to the semi-Lagrangian (truncated) system (10) and (11). The most natural variational ansatz to search for ESs proper (without the tail, which will be considered in the next section) in this system is based on the following expressions:

$$U = A \operatorname{sech}(\sqrt{2k}x), \quad V = B \operatorname{sech}^2(\sqrt{2k}x), \quad (13)$$

where the amplitudes  $A$  and  $B$  are variational parameters, while the inverse width  $\sqrt{2k}$ , which is *not* to be varied, has been fixed to match to the linearized form of Eqs. (8) and (9) or (10) and (11) at  $t \rightarrow \infty$ . Substituting the ansatz (13) into the Lagrangian (12), dropping the last term, and then performing the integration, we find the effective Lagrangian of the truncated system:

$$3\sqrt{2k}L_{\text{eff}} = -4kA^2 - 2 \left[ 2 \left( 1 - \frac{2\delta}{5} \right) k - q \right] B^2 + 2A^2 B + \gamma_1 A^4 + \frac{35}{5} \gamma_2 A^2 B^2. \quad (14)$$

As it was said above, in order to go from the full system to its truncated counterpart, one not only has to omit the last term in Eq. (12), but also must avoid varying the term  $4\gamma_2 U^2 V^2$  with respect to  $U$ . In terms of the effective Lagrangian (14), this means that, when deriving the variational equations for  $A$  and  $B$ , one should not vary the last term of the effective Lagrangian with respect to  $A$ . In this case, the variation with respect to  $A$  yields a simple equation which allows one to eliminate  $B$ ,

$$B = -\gamma_1 A^2 + 2k. \quad (15)$$

Using this result, the equation produced by varying the Lagrangian with respect to  $B$  can be cast in the final form of a biquadratic equation for  $A$ ,

$$\frac{32}{5} \gamma_1 \gamma_2 A^4 - \left[ 1 + \frac{64}{5} k \gamma_2 + 2\gamma_1 \left( 2 \left( 1 - \frac{2\delta}{5} \right) k - q \right) \right] A^2 + 4k \left[ 2 \left( 1 - \frac{2\delta}{5} \right) k - q \right] = 0. \quad (16)$$

Thus, depending on values of the parameters, Eq. (16) may give up to two different physical solutions. The simplest non-trivial case (which still allows an ES to exist) is with  $\gamma_1 = 0$ , but  $\gamma_2 \neq 0$ . Then, Eq. (16) yields a single solution,

$$A^2 = \frac{4k[2(5 - 2\delta)k - 5q]}{5 + 64k\gamma_2}, \quad (17)$$

which is physical if it gives  $A^2 > 0$ ; the amplitude  $B$  can then be obtained from Eq. (15).

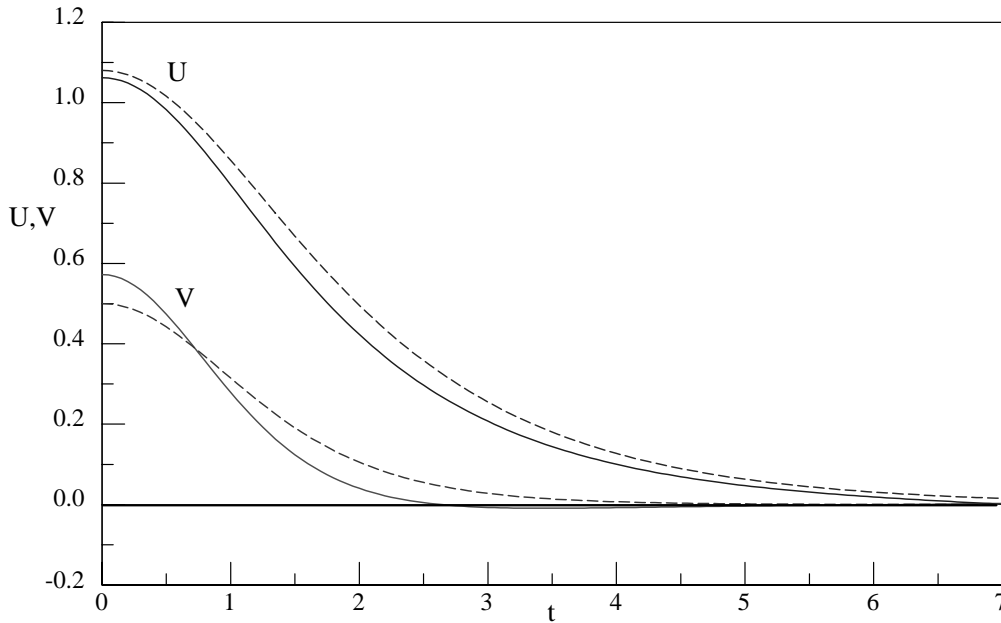


Fig. 1. A comparison between the numerical shape (solid curves) of an ordinary (non-embedded) soliton of the semi-Lagrangian (truncated) system of equations (10) and (11), as found by the shooting method, and the result (dashed curves) obtained from the modified VA based on Eqs. (13), (15) and (16). The parameters are  $q = 1$ ,  $\delta = 1$ ,  $\gamma_1 = 0$ , and  $\gamma_2 = -1/4$ , and both the numerical and analytical solutions are taken for  $k = 1/4$ .

It should be stressed that, as it was demonstrated in Ref. [8], the ansatz (13) yields exact soliton solutions to Eqs. (10) and (11) at some uniquely selected values of  $k$  (an expression for it is given in Eq. (25)). Depending on the value of the mismatch parameter  $q$ , this exact solution may be either ES or an ordinary soliton. Comparing Eqs. (15) and (16) with that solution, one can easily verify that expressions (15) and (16) are precisely parts of the exact solution.

While the exact solution for the soliton in the truncated model is available at a single value of  $k$ , in the region (6), where a *continuous* family of ordinary solitons is expected to exist, no general exact solution is known. So, to illustrate the accuracy and reliability of the modified VA for producing approximate solutions to the truncated equations (10) and (11), in Fig. 1 we display a typical example of a numerically found ordinary soliton in the region  $0 < k < q/2$  (in this example,  $\delta = 1$ ), together with the analytical approximation generated by Eqs. (13), (15) and (16).

Looking at the numerical solution in Fig. 1, one observes that the SH component goes slightly negative along the shoulder of the soliton, and then appears to oscillate as it decays. Note that in the region where this behavior of the SH component is observed,  $U$  is not small, and, according to Eq. (4), the solution for  $V$  in this region should oscillate indeed. Thus, ordinary solitons in the truncated system may have fine features which are not found in the ESs.

#### 4. An analytical criterion to identify ESs

##### 4.1. General analysis

In the case when the wavenumber  $k$  falls into the regions (7), ESs can exist in both the full and truncated models, i.e., (1)–(4) [8]. However, except for using the exact ES solutions in the truncated model to guess where ESs might

exist in its full counterpart, the only known method for locating ESs in the full model was to search for them by means of direct numerical computations.

Thus, there is a need for an analytical approach to the quest for ESs. Such an approach can be based on VA, if one assumes that, at values of  $k$  close to  $k_{\text{ES}}$ , at which an ES exists, there also exists a family of delocalized solitons, with small-amplitude oscillating tails in the SH component that vanish when  $k$  become exactly equal to  $k_{\text{ES}}$ . As it immediately follows from Eq. (9) or (11), the free oscillating tail of a delocalized soliton, which has an infinitesimal amplitude  $b$  and arbitrary phase shift  $\psi$ , is given by the expression

$$V_{\text{tail}} = b \cos \left( \sqrt{\frac{2}{\delta}} (2k - q)t + \psi \right), \quad (18)$$

that neglects the nonlinear terms (hence  $b$  is an arbitrary small amplitude). Now, one may add this tail to the ansatz (13), to produce a more general tailed ansatz,

$$V(t) = V_{\text{sol}}(t) + b \cos \left( \sqrt{\frac{2}{\delta}} (2k - q)t + \psi \right), \quad (19)$$

where  $V_{\text{sol}}(t)$  corresponds to the ansatz for the core of the delocalized soliton, and  $b$  is considered as an extra variational parameter.

Thus, a new variational equation,

$$\frac{\partial L_{\text{eff}}}{\partial b} = 0, \quad (20)$$

must be added to the set of equations obtained by varying the effective Lagrangian with respect to other free parameters, which pertain to the soliton's core (irrespective of the fact if the system is complete Lagrangian or semi-Lagrangian). As we are interested in the location of the ES which, by itself, has  $b = 0$  (no tail), one should set  $b = 0$  *after* completing the differentiation in Eq. (20). This means that, prior to varying in  $b$ , one should only keep terms in  $L_{\text{eff}}$  which are linear in  $b$ , hence Eq. (20) takes the general form

$$\int_{-\infty}^{+\infty} \left[ \left( \frac{\delta L}{\delta V} \right) \Big|_{U=U_{\text{sol}}(t), V=V_{\text{sol}}(t)} \right] \cos \left( \sqrt{\frac{2}{\delta}} (2k - q)t + \psi \right) dt = 0 \quad (21)$$

with  $\delta/\delta V$  standing for the variational derivative of the underlying Lagrangian (the one given by Eq. (12)).

It is relevant to make some observations here. First, we stress again that Eq. (21) applies equally well to the full system and to the truncated one, since only the variation in the SH field is involved, while the differences between the two types of the systems is solely in the variation with respect to  $U$ . Second, since the solitons sought for are even, the expression  $\delta L/\delta V$  with  $V$  substituted by  $V_{\text{sol}}(t)$  is also even. Hence Eq. (21) amounts to an *orthogonality* condition between the infinitesimal tail and the soliton,

$$\int_{-\infty}^{+\infty} \left[ \left( \frac{\delta L}{\delta V} \right) \Big|_{U=U_{\text{sol}}(t), V=V_{\text{sol}}(t)} \right] \cos \left( \sqrt{\frac{2}{\delta}} (2k - q)t \right) dt = 0. \quad (22)$$

This consideration also shows that the phase parameter  $\psi$  in the expression (18) is not important in the limit of  $b \rightarrow 0$ .

We notice that the above derivation actually circumvents the formal problem of the divergence of the integral expression (12) for the Lagrangian, when there is a tail which does not vanish as  $|t| \rightarrow \infty$ . The divergence did not appear since the tail was designed as a solution to the linearized version of Eq. (9) or (11). One can readily verify

that the tail-induced divergence can also be eliminated in the fully nonlinear case, provided one suitably adjusts the nonlinear shift of the tail's frequency.

As concerns the latter issue, the divergence of the integrated Lagrangian is a rather general problem, which occurs in other contexts, for instance, in the application of the variational method to dark solitons. In those cases, regularization procedures were proposed that are based on properly justified subtraction of a divergence-generating term in the Lagrangian density [12].

Thus, Eq. (22) is a general criterion that can be used to locate ES solutions within the framework of VA. In Eq. (22), we recognize that the variational derivative  $(\delta L / \delta V)|_{U=U_{\text{sol}}, V=V_{\text{sol}}(t)}$  is just the left-hand side (l.h.s.) of the stationary equation for SH, with  $U$  and  $V$  taken as per the chosen ansatz for the core of the delocalized soliton. Another point is that upon applying integration by parts to the second-derivative term in Eq. (11), one sees that all contributions from the linear terms cancel in Eq. (22), leaving only the nonlinear terms in Eq. (11) to determine this condition.

#### 4.2. ES in the truncated system

To test the efficiency of the criterion (22), we first apply it to the truncated system. Then, according to what was said above, the variational derivative in Eq. (22) should be replaced by the nonlinear part of l.h.s. of Eq. (11), which yields

$$\int_{-\infty}^{+\infty} \left[ \frac{1}{2} U^2(t) + 4\gamma_2 U^2(t) V(t) \right] \cos \left( \sqrt{\frac{2}{\delta}} (2k - q)t \right) dt = 0. \quad (23)$$

Using the ansatz (13), it is easy to explicitly perform the integration in Eq. (23), which finally yields a simple result (note that the FH amplitude drops out),

$$4\gamma_2 B = -\frac{3\delta \cdot k}{2k(1 + 2\delta) - q}. \quad (24)$$

Now, combining the above results (15) and (16), which were obtained by means of the VA, with the relation (24) that locates where the ES must be (and is also based on VA), one can easily verify that this set of three relations is *precisely tantamount* to the exact analytical solution for the ES which was found by guess in Ref. [8]. In particular, an eventual expression for the wavenumber of ES is

$$k_{\text{ES}} = \frac{1}{2}(1 + 2\delta)^{-1} [q - \frac{3}{2}\delta(4\gamma_2 + 3\delta\gamma_1)^{-1}]. \quad (25)$$

The fact that, in the case of the semi-Lagrangian truncated system, VA reproduces the *exact* ES is remarkable, although the reason for this occurrence is not fully understood (the fact can be understood easily if one knows a priori that the exact solution is expressed in terms of the first and second powers of sech, as in Eq. (13), as in this case the number of variational equations is equal to the number of free parameters in the corresponding analytical expressions). We also note that this exact soliton is not always an embedded one, as the wavenumber (25) is not necessarily restricted to the region (7) in which ES may exist: the wavenumber may instead fall into the region (6), in which ordinary solitons are to be found. A criterion showing whether the exact soliton is embedded or ordinary has already been given, for this model, in Ref. [8].

#### 4.3. ES in the full system

The next step is to apply the general ES-selecting criterion (22) to the full system which is based on Eqs. (8) and (9). To this end, we assume that the soliton proper may again be approximated by the ansatz (13). Then,



substituting the nonlinear part of l.h.s. of Eq. (9) into Eq. (22), we arrive, instead of Eq. (24), at a more complicated relation,

$$4\gamma_2 B \left[ 1 + \frac{B^2}{A^2} \frac{2(8\delta + 1)k_{\text{ES}} - q}{40\delta \cdot k_{\text{ES}}} \right] = -\frac{3\delta \cdot k_{\text{ES}}}{2k_{\text{ES}}(1 + 2\delta) - q}. \quad (26)$$

We tested the validity of this relation in the following way: take a particular example of the ES in the full system that was found in a numerical form in Ref. [8], for which  $\delta = 1$ ,  $q = 1$ , and  $\gamma_1 = \gamma_2 = -0.05$ . Borrowing values of the amplitudes  $A$  and  $B$  directly from the numerical data, we find that  $A = 3.794$  and  $B = 2.735$ . Substituting these values into Eq. (26) yields  $k_{\text{ES}} = 0.688$ , while the numerical value found in Ref. [8] was  $k_{\text{ES}} = 0.696$ . Thus, the relative error of the criterion (22) for this case is 1.1%.

## 5. Conclusion

In this work, we have put forward two modifications to the technique of the VA for solitons. First, it may happen that a physical model does not admit the full Lagrangian representation, as some terms may be missing due to various reasons. In the case of the  $\chi^{(2)} : \chi^{(3)}$  model considered in this work, this means that one term in the Lagrangian should not be varied when deriving the equation for the fundamental wave. We demonstrate that the VA can be applied to such *semi-Lagrangian* systems as efficiently as to their full Lagrangian counterparts. Second, we have shown that, by the addition of an infinitesimal tail, which does not vanish at infinity, to the usual soliton ansatz, and demanding, after performing the variation, that the amplitude of the tail be zero, we obtain an approximate analytical criterion for locating ESs inside a family of delocalized ones, i.e., isolated truly localized solutions existing inside the continuous spectrum of radiation modes. The criterion takes the form of orthogonality between the radiation mode contained in the infinitesimal tail and the core of the delocalized soliton. To test the criterion, we have applied it to both the semi-Lagrangian truncated version of the  $\chi^{(2)} : \chi^{(3)}$  model and to the same model in its full form. In the former model, the criterion, combined with the VA for the soliton proper, yields a result which completely coincides with the previously found exact solution for the ES. In the latter model, the criterion predicts the wavenumber corresponding to the ES with a relative error  $\approx 1\%$ .

## Acknowledgements

BAM appreciates hospitality of the Department of Mathematics at the University of Central Florida. This research has been supported in part by NSF grant #DMS0129714.

## References

- [1] A.C. Newell, J. Math. Phys. 18 (1977) 922;  
A.C. Newell, J. Math. Phys. 19 (1978) 1126;  
A.C. Newell, Rocky Mount. J. Math. 8 (1979) 25.
- [2] D.J. Kaup, A.C. Newell, Phys. Rev. B 18 (1978) 5162;  
D.J. Kaup, A.C. Newell, Proc. R. Soc. London A 361 (1978) 413.
- [3] J.P. Boyd, Weakly Nonlocal Solitary Waves and Beyond-all-orders Asymptotics, Kluwer Academic Publishers, Dordrecht, 1998.
- [4] D.J. Kaup, T.I. Lakoba, B.A. Malomed, J. Opt. Soc. Am. B 14 (1997) 1199;  
D.J. Kaup, B.A. Malomed, J. Opt. Soc. Am. B 15 (1998) 2838.
- [5] R. Grimshaw, P. Cook, in: A.T. Chang, J.H. Lee, D.Y.C. Leung (Eds.), Hydrodynamics, Balkema, Rotterdam, 1996;  
A.R. Champneys, M.D. Groves, J. Fluid Mech. 342 (1997) 199.



- [6] J. Yang, *Stud. Appl. Math.* 106 (2001) 337.
- [7] A.R. Champneys, B.A. Malomed, J. Yang, D.J. Kaup, *Physica D* 152–153 (2001) 340.
- [8] J. Yang, B.A. Malomed, D.J. Kaup, *Phys. Rev. Lett.* 83 (1999) 1958.
- [9] B.A. Malomed, *Prog. Opt.* 43 (2002) 71.
- [10] S. Trillo, A.V. Buryak, Y.S. Kivshar, *Opt. Commun.* 122 (1996) 200;  
O. Bang, Y.S. Kivshar, A.V. Buryak, *Opt. Lett.* 22 (1997) 1680.
- [11] L. Torner, *IEEE Phot. Tech. Lett.* 11 (1999) 1268;  
O. Bang, C. Clausen, P. Christiansen, L. Torner, *Opt. Lett.* 24 (1999) 1413.
- [12] Y.S. Kivshar, W. Krolikowski, *Opt. Commun.* 114 (1995) 353.