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Liouville operator approach to symplecticity-preserving renormalization group method

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Abstract

We present a method to construct symplecticity-preserving renormalization group maps by using the Liouville operator. The resultant RG maps accurately reproduce the long-time behavior of the original symplectic maps even when a resonant island chain appears.

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1. Introduction

There has been a long history to study the asymptotic behavior of Hamiltonian flows by means of singular perturbation methods such as the averaging method and the method of multiple time-scales. A Hamiltonian flow can be reduced to a symplectic discrete map called the Poincaré map, which has a lower dimension than the original flow and is, therefore, extensively studied [1,2].

The perturbative renormalization group (RG) method developed recently may be a useful tool to tackle asymptotic behaviors of discrete maps as well as flows. The original RG method is an asymptotic singular perturbation technique developed for differential equations [3]. Secular or divergent terms of perturbation solutions of differential equations are removed by renormalizing integral constants of the lowest order solution. The RG method is reformulated on the basis of a naive renormalization transformation and the Lie group [4]. This reformulated RG method based on the Lie group is easy to apply to discrete systems, by which asymptotic expansions of unstable manifolds of some chaotic discrete systems are obtained [5]. The extension of the RG method to discrete symplectic systems is not trivial because the symplectic structures are not preserved in naive RG equations (maps) as shown in Ref. [6], while

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the application of the RG method to Hamiltonian flows does not cause such a problem as the destroyed symplectic symmetry except for a special case [7].

The application of the RG method to some non-symplectic discrete systems has been attempted in the framework of the envelope method [8]. However, the method, if applied to a symplectic map, would give only a naive RG map which destroys the symplectic symmetry. To recover the destroyed symmetry not only the paper [6] has introduced a process of the “exponentiation” which gives us analytical expressions of some physical quantities [6,9], but also another paper [10] has used a symplectic integrator by taking the time-continuous limit of the time-step parameter. Furthermore, Tzenov and Davidson have shown that the exponentiation procedure can be successfully applied in the Hénon map [11]. They also have found a general way to obtain the symplectic RG method including the exponentiation method. However, these procedures may seem somewhat artificial, and resonant islands have never been studied by the RG method.

The main purpose of the present paper is to present a general RG procedure to preserve the symplectic structure in RG maps and to obtain correctly reduced symplectic RG maps. In this paper, this procedure is called a symplecticity-preserving RG procedure, which consists of the following two steps. First, using the reformulated RG method [4], we get a naive RG map near an elliptic fixed point of a symplectic discrete system. The naive RG map preserves the symplectic symmetry only approximately and fails to describe a long-time behavior of the original map. Second, in order to recover the symplectic symmetry by means of taking account of higher orders in the naive perturbation result, we equate the naive RG map to the appropriately discretized Hamiltonian flow, which is done by using the Liouville operator. In this procedure we identify the small parameter ε with the time-step, which yields a canonical equation in the limit of $\varepsilon \to 0$. This process and a symplectic RG map are called, respectively, a symplecticity-preserving procedure and a symplecticity-preserving RG map. A reduction of the Liouville equation in a non-integrable Hamiltonian flow has been developed in Ref. [12]. It should be pointed out that Dragt and Finn [13] have mathematically discussed the relation between symplectic maps and invariant functions. We practically show how to construct invariant functions, which correspond to Hamiltonians, in the course of obtaining symplectic RG maps.

In Section 2, a long-time behavior of a simple linear map is analyzed in order to elucidate both the broken symplectic symmetry in a naive RG map and our process. In Section 3, a symplecticity-preserving RG map is obtained near elliptic fixed points of a two-dimensional nonlinear symplectic map even when a resonant island chain appears. In Section 4, we mention the advantages of our newly developed RG method as the summary of this paper.

2. Linear symplectic map

It may be instructive to analyze a linear symplectic map, which is exactly solvable, $(x^n, y^n) \mapsto (x^{n+1}, y^{n+1})$:

$$x^{n+1} = x^n + y^{n+1}, \qquad y^{n+1} = y^n - ax^n + 2\varepsilon Jx^n.$$

This map can be rewritten as

$$L_\theta x^n = \varepsilon 2Jx^n, \tag{1}$$

where ε is the small parameter, J is the real parameter, θ and $L_\theta x^n$ are defined as

$$\cos\theta \equiv \left(1 - \frac{a}{2}\right), \tag{2}$$

$$L_\theta x^n \equiv x^{n+1} - 2x^n \cos\theta + x^{n-1}, \tag{3}$$

respectively. We assume that the map (1) has an elliptic fixed point at the origin (0, 0). The linear map (1) has the following exact solution x_E^n:

$$x_E^n = A\exp(\mathrm{i}\arccos(\cos\theta + \varepsilon J)n) + \text{c.c.},$$
$$x_E^n = A\exp\left[\mathrm{i}\left(\theta + \varepsilon\frac{-J}{\sin\theta} + \varepsilon^2\frac{-\cos\theta}{2\sin\theta}\left(\frac{J}{\sin\theta}\right)^2 + \cdots\right)n\right] + \text{c.c.}, \tag{4}$$

where $A(\in \boldsymbol{C})$ is the complex "integration" constant and c.c. stands for the complex conjugate of the preceding terms.

Let us derive an asymptotic solution of the map (1) for small ε by means of the RG method. Substituting the expansion

$$x^n = x^{(0)n} + \varepsilon x^{(1)n} + \varepsilon^2 x^{(2)n} + \mathcal{O}(\varepsilon^3), \tag{5}$$

into Eq. (1), we have

$$L_\theta x^{(0)n} = 0, \qquad L_\theta x^{(1)n} = 2Jx^{(0)n}, \qquad L_\theta x^{(2)n} = 2Jx_n^{(1)n},$$

and

$$x^{(0)n} = A\exp(\mathrm{i}\theta n) + \text{c.c.}, \qquad x^{(1)n} = \frac{-\mathrm{i}JA}{\sin\theta}n\exp(\mathrm{i}\theta n) + \text{c.c.},$$
$$x^{(2)n} = \frac{-J^2A}{2\sin^2\theta}\left(n^2 + \mathrm{i}\frac{\cos\theta}{\sin\theta}n\right)\exp(\mathrm{i}\theta n) + \text{c.c.},$$

where $A(\in \boldsymbol{C})$ is the integration constant. To remove secular terms ($\propto n, n^2$), we introduce the renormalization transformation $A \mapsto A^n$ [4]:

$$A^n \equiv A + \varepsilon\frac{-\mathrm{i}JA}{\sin\theta}n + \varepsilon^2\frac{-J^2A}{2\sin^2\theta}\left(n^2 + \mathrm{i}\frac{\cos\theta}{\sin\theta}n\right) + \mathcal{O}(\varepsilon^3). \tag{6}$$

A discrete version of the RG equation is just the first order difference equation of A^n, whose local solution is given by Eq. (6). From Eq. (6), we have

$$A^{n+1} - A^n = \left(-\mathrm{i}\varepsilon\frac{J}{\sin\theta} - \varepsilon^2\frac{J^2}{2\sin^2\theta}\left(2n + 1 + \mathrm{i}\frac{\cos\theta}{\sin\theta}\right)\right)A + \mathcal{O}(\varepsilon^3), \tag{7}$$

where A should be expressed in terms of A^n. This is done by taking the inversion of the renormalization transformation (6) iteratively:

$$A = \left(1 + \mathrm{i}\varepsilon\frac{Jn}{\sin\theta} + \mathcal{O}(\varepsilon^2)\right)A^n. \tag{8}$$

Substituting (8) into (7), we obtain the following RG equation (RG map) up to $\mathcal{O}(\varepsilon^2)$

$$A^{n+1} = \left(1 + \frac{-\mathrm{i}\varepsilon J}{\sin\theta} + \frac{1}{2!}\left(\frac{-\mathrm{i}\varepsilon J}{\sin\theta}\right)^2 - \mathrm{i}\varepsilon^2\frac{J^2\cos\theta}{2\sin^3\theta}\right)A^n + \mathcal{O}(\varepsilon^3), \tag{9}$$

of which the solution is

$$A^n = \left(1 + \frac{-\mathrm{i}\varepsilon J}{\sin\theta} + \frac{1}{2!}\left(\frac{-\mathrm{i}\varepsilon J}{\sin\theta}\right)^2 - \mathrm{i}\varepsilon^2\frac{J^2\cos\theta}{2\sin^3\theta} + \mathcal{O}(\varepsilon^3)\right)^n A^0. \tag{10}$$

On the other hand, from Eq. (4), we have A^n exactly as

$$A^n = A^0 \exp\left[\mathrm{i}\left(\varepsilon\frac{-J}{\sin\theta} - \varepsilon^2\frac{\cos\theta}{2\sin\theta}\left(\frac{J}{\sin\theta}\right)^2 + \cdots\right)n\right]. \tag{11}$$

Notice that $|A^n|^2$ is an exact constant of motion while it is merely an approximate conserved quantity of the (truncated) RG map (9). The symplectic structure is also not exactly preserved in the RG map, that is, for the truncated RG map (9) up to $\mathcal{O}(\varepsilon^k)$, we have

$$\mathrm{d}A^{n+1} \wedge \mathrm{d}A^{*n+1} - \mathrm{d}A^n \wedge \mathrm{d}A^{*n} = \mathcal{O}(\varepsilon^{(k+1)}) \neq 0,$$

where $k = 1, 2, \ldots$; A^* is complex conjugate to A^n and should also be a canonical conjugate to A^n.[1] However, this fault of the RG map vanishes in the limit of $\varepsilon \to 0$.

In order to remedy a fault like this, we take advantage of the crucial observation that Hamiltonian flows satisfy the following relation:

$$Z(t+\mu) = \left(1 + \mu\mathcal{L}_H + \frac{\mu^2}{2!}\mathcal{L}_H^2 + \cdots\right)Z(t) = \exp(\mu\mathcal{L}_H)Z(t), \tag{12}$$

where t is the time variable, μ is a real number, H is a Hamiltonian, Z is a canonical variable $(q_1, q_2, \ldots, q_N, p_1, p_2, \ldots, p_N)$ and

$$\mathcal{L}_H Z \equiv \{Z, H\} \equiv \sum_{j=1}^{N}\left(\frac{\partial Z}{\partial q_j}\frac{\partial H}{\partial p_j} - \frac{\partial Z}{\partial p_j}\frac{\partial H}{\partial q_j}\right), \qquad \mathcal{L}_H^2 Z = \mathcal{L}_H(\mathcal{L}_H Z) = \{\{Z, H, \}, H\}, \tag{13}$$

where $\mathcal{L}_H$ is the Liouville operator defined by the Poisson bracket. We can identify the relation (12) as the map by defining $Z^{n+1} \equiv Z(t+\mu)$, and $Z^n \equiv Z(t)$:

$$Z^{n+1} = \Psi(Z^n; \mu), \qquad \Psi(Z^n; \mu) \equiv \exp(\mu\mathcal{L}_H)Z(t)|_{Z(t)\equiv Z^n}. \tag{14}$$

This is the symplectic mapping associated with the Hamiltonian system. Identifying $\mu = \varepsilon$ in (14), we can obtain both the Hamiltonian system and the associated symplectic mapping in the following steps:

1. Expanding a Hamiltonian H in (12) in powers of ε, $H = H^{(1)} + \varepsilon H^{(2)} + \cdots$, we obtain the following relation:

$$Z(t+\varepsilon) = \left(1 + \varepsilon\mathcal{L}_H + \frac{\varepsilon^2}{2!}\mathcal{L}_H^2 + \mathcal{O}(\varepsilon^3)\right)Z(t) = \left\{1 + \varepsilon\mathcal{L}_{H^{(1)}} + \varepsilon^2\left(\frac{\mathcal{L}_{H^{(1)}}^2}{2!} + \mathcal{L}_{H^{(2)}}\right) + \mathcal{O}(\varepsilon^3)\right\}Z(t). \tag{15}$$

2. $H^{(1)}$ can be found by taking the limit $(A^{n+1} - A^n)/\varepsilon \to \mathrm{d}A/\mathrm{d}t$ $(\varepsilon \to 0)$ in the naive RG map by comparing the naive RG map with (15). Using $H^{(1)}$ and equating the naive RG map with the Liouville operator relation (15), we obtain the Hamiltonian $H = H^{(1)} + \varepsilon H^{(2)}$. Similarly, we can obtain the higher order Hamiltonian $H^{(3)}$, $H^{(4)}, \ldots,$ order by order in ε. This procedure yields the approximate Hamiltonian.
3. To obtain the symplecticity-preserving RG mapping, we discretize the continuous-time system obtained in the second step. The time-step should be chosen to be ε.

[1] For the corresponding Hamiltonian flows, the truncated RG equation exactly preserves the symplectic structure with canonical variables (A, A^*) order by order in the small parameter.

In the present case, we take the limit $\varepsilon \to 0$ in (9), which gives

$$\frac{\mathrm{d}A}{\mathrm{d}t} = -\frac{\mathrm{i}J}{\sin\theta}A = \frac{\partial H}{\partial A^*} = \mathcal{L}_{H^{(1)}}A, \qquad \frac{\mathrm{d}A^*}{\mathrm{d}t} = -\frac{\partial H}{\partial A^*} = \mathcal{L}_{H^{(1)}}A^*.$$

Here $\mathrm{d}A/\mathrm{d}t$ is from $(A^{n+1} - A^n)/\varepsilon$, and so on. Then we have

$$H^{(1)} = -\mathrm{i}\frac{J|A|^2}{\sin\theta} \quad \text{and} \quad \mathcal{L}^2_{H^{(1)}}A = \{\{A, H^{(1)}\}, H^{(1)}\} = \frac{-J^2A}{\sin^2\theta}.$$

At this stage, the Liouville operator relation is

$$\left\{1 + \varepsilon\mathcal{L}_{H^{(1)}} + \varepsilon^2\left(\frac{\mathcal{L}^2_{H^{(1)}}}{2} + \mathcal{L}_{H^{(2)}}\right)\right\}A = A + \varepsilon\frac{-\mathrm{i}JA}{\sin\theta} + \varepsilon^2\left(\frac{-J^2A}{2\sin^2\theta} + \frac{\partial H^{(2)}}{\partial A^*}\right).$$

Equating this to the right-hand side of the naive RG map (9), we have

$$H^{(2)} = \frac{-\mathrm{i}J^2\cos\theta|A|^2}{2\sin^3\theta}.$$

The Hamiltonian $H = H^{(1)} + \varepsilon H^{(2)}$ leads to the following equation:

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{-\mathrm{i}J}{\sin\theta}A + \varepsilon\frac{-\mathrm{i}J^2\cos\theta}{2\sin^3\theta}A = \frac{\partial H}{\partial A}, \qquad \frac{\mathrm{d}A}{\mathrm{d}t} = -\frac{\partial H}{\partial A^*}.$$

The solution of the continuous-time system is

$$A(t) = A(0)\exp\left\{\mathrm{i}\left(\frac{-J}{\sin\theta} + \varepsilon\frac{-J^2\cos\theta}{2\sin\theta^3}\right)t\right\},$$

which yields the symplecticity-preserving RG map:

$$A^{n+1} = A^n\exp\left\{\mathrm{i}\varepsilon\left(\frac{-J}{\sin\theta} + \varepsilon\frac{-J^2\cos\theta}{2\sin^3\theta}\right)\right\}.$$

Here the time-step has been chosen to be ε. Then we recover the exact solution of (1), is (11), in terms of the original variable x^n.

Finally, the present method is summarized in the following diagram.

A symplecticity-preserving RG method for symplectic maps:

$$\begin{tikzcd}
\text{[discrete-time]} & & \text{[continuous-time]} \\
\text{Symp. maps} \arrow[d, "\text{naive RG}"'] & & \\
\begin{array}{c}\text{naive RG maps}\\ \text{(dissipative maps)}\end{array} \arrow[r, "\text{Liouville operator}"] & \begin{array}{c}\text{RG Eqs.}\\ \text{(canonical Eqs.)}\end{array} \arrow[dl, "\text{Symp.-Pres. discretizations}"] & \\
\text{Symp.-Pres. RG maps} & &
\end{tikzcd}$$

3. Two-dimensional nonlinear symplectic map

3.1. Non-resonant case

Let us analyze the weakly-nonlinear symplectic map $(x^n, y^n) \mapsto (x^{n+1}, y^{n+1})$:

$$x^{n+1} = x^n + y^{n+1}, \qquad y^{n+1} = y^n - ax^n + 2\varepsilon J (x^n)^3,$$

or

$$L_\theta x^n = \varepsilon 2 J (x^n)^3, \tag{16}$$

where ε is the small parameter, J is the real parameter, θ and $L_\theta x^n$ are defined in (2) and (3) respectively. Expanding x^n as a power series of ε

$$x^n = x^{(0)n} + \varepsilon x^{(1)n} + \varepsilon^2 x^{(2)n} + \mathcal{O}(\varepsilon^3),$$

we have

$$L_\theta x^{(0)n} = 0, \qquad L_\theta x^{(1)n} = 2J(x^{(0)n})^3, \qquad L_\theta x^{(2)n} = 6J(x^{(0)n})^2 x^{(1)n}, \tag{17}$$

and the solutions of the perturbed equations are given by

$$x^{(0)n} = A\,\mathrm{e}^{\mathrm{i}\theta n} + \mathrm{c.c.}, \tag{18}$$

$$x^{(1)n} = \frac{-3\mathrm{i}|A|^2 AJ}{\sin\theta} n\,\mathrm{e}^{\mathrm{i}\theta n} + \frac{JA^3}{\cos 3\theta - \cos\theta}\mathrm{e}^{3\mathrm{i}\theta n} + \mathrm{c.c.}, \tag{19}$$

$$\begin{aligned} x^{(2)n} = {} & \left\{ \frac{-9}{2}\frac{J^2|A|^4 A}{\sin^2\theta} n^2 - \mathrm{i}\frac{J^2|A|^4 A}{\sin\theta}\left(\frac{3}{\cos 3\theta - \cos\theta} + \frac{9\cos\theta}{2\sin^2\theta}\right) n \right\} \mathrm{e}^{\mathrm{i}\theta n} \\ & + \left\{ \frac{-9\mathrm{i}J^2|A|^2A^3}{(\cos 3\theta - \cos\theta)\sin\theta} n + \frac{J^2|A|^2A^3}{2(\cos 3\theta - \cos\theta)^2}\left\{12 - 18\frac{\sin 3\theta}{\sin\theta}\right\}\right\} \mathrm{e}^{3\mathrm{i}\theta n} \\ & + \left\{ \frac{3JA^5}{(\cos 5\theta - \cos\theta)(\cos 3\theta - \cos\theta)} \right\} \mathrm{e}^{5\mathrm{i}\theta n} + \mathrm{c.c.} \end{aligned} \tag{20}$$

Here $A (\in \boldsymbol{C})$ is the integration constant. To avoid resonant secular terms, we assume in addition that

$$\cos\theta \neq \cos 3\theta, \qquad \cos\theta \neq \cos 5\theta.$$

The construction of the reduced map in the case of near resonance is considered in Section 3.2. In order to remove the secular terms in the coefficient of the fundamental harmonic ($\exp(\mathrm{i}\theta n)$), we introduce the renormalization transformation $A \mapsto A^n$:

$$A^n \equiv A + \varepsilon\frac{-3\mathrm{i}|A|^2AJ}{\sin\theta} n + \varepsilon^2\left\{\frac{-9}{2}\frac{J^2|A|^4A}{\sin^2\theta}n^2 - \mathrm{i}\frac{J^2|A|^4A}{\sin\theta}\left(\frac{3}{\cos 3\theta - \cos\theta} + \frac{9\cos\theta}{2\sin^2\theta}\right)n\right\}. \tag{21}$$

Following the same procedure as that in the preceding section, we derive the naive RG map from (21):

$$\begin{aligned} A^{n+1} = {} & A^n + \varepsilon\frac{-3\mathrm{i}J}{\sin\theta}|A^n|^2 A^n \\ & + \varepsilon^2\left\{\frac{1}{2!}\left(\frac{-3\mathrm{i}J}{\sin\theta}|A^n|^2\right)^2 A^n - \left(\frac{9\mathrm{i}\cos\theta}{2\sin^3\theta}J^2 + \frac{3\mathrm{i}J^2}{\sin\theta(\cos 3\theta - \cos\theta)}\right)|A^n|^4 A^n\right\}, \end{aligned} \tag{22}$$

which destroys symplectic symmetry. This RG map should recover the symplectic symmetry. The naive RG map can be written in terms of the real variables A_1^n, A_2^n ($A^n = A_1^n + \mathrm{i}A_2^n$):

$$A_1^{n+1} = A_1^n + \varepsilon\frac{3J}{\sin\theta}(A_1^{n2} + A_2^{n2})A_2^n + \varepsilon^2\left[\frac{-1}{2!}\left\{\frac{3J}{\sin\theta}(A_1^{n2} + A_2^{n2})\right\}^2 A_1^n + \left\{\frac{9\cos\theta}{2\sin^3\theta} + \frac{3}{\sin\theta(\cos 3\theta - \cos\theta)}\right\} J^2(A_1^{n2} + A_2^{n2})^2 A_2^n\right], \tag{23}$$

$$A_2^{n+1} = A_2^n + \varepsilon\frac{-3J}{\sin\theta}(A_1^{n2} + A_2^{n2})A_1^n + \varepsilon^2\left[\frac{-1}{2!}\left\{\frac{3J}{\sin\theta}(A_1^{n2} + A_2^{n2})\right\}^2 A_2^n - \left\{\frac{9\cos\theta}{2\sin^3\theta} + \frac{3}{\sin\theta(\cos 3\theta - \cos\theta)}\right\} J^2(A_1^{n2} + A_2^{n2})^2 A_1^n\right]. \tag{24}$$

The ($\varepsilon \to 0$)-limit yields

$$\frac{\mathrm{d}A_1}{\mathrm{d}t} = \frac{3J}{\sin\theta}(A_1^2 + A_2^2)A_2 = \frac{\partial H}{\partial A_2} = \mathcal{L}_{H^{(1)}}A_1, \qquad \frac{\mathrm{d}A_2}{\mathrm{d}t} = -\frac{3J}{\sin\theta}(A_1^2 + A_2^2)A_1 = -\frac{\partial H}{\partial A_1} = \mathcal{L}_{H^{(1)}}A_2,$$

where

$$H^{(1)} = \frac{3J(A_1^2 + A_2^2)^2}{4\sin\theta}.$$

This Hamiltonian generates the relations:

$$\left\{1 + \varepsilon\mathcal{L}_{H^{(1)}} + \varepsilon^2\left(\frac{\mathcal{L}_{H^{(1)}}^2}{2!} + \mathcal{L}_{H^{(2)}}\right)\right\} A_1(t) = A_1 + \varepsilon\frac{3J}{\sin\theta}(A_1^2 + A_2^2)A_2 + \varepsilon^2\left\{\frac{-1}{2!}\left(\frac{3J}{\sin\theta}\right)^2\left(A_1^2 + A_2^2\right)^2 A_1 + \frac{\partial H^{(2)}}{\partial A_2}\right\}, \tag{25}$$

$$\left\{1 + \varepsilon\mathcal{L}_{H^{(1)}} + \varepsilon^2\left(\frac{\mathcal{L}_{H^{(1)}}^2}{2!} + \mathcal{L}_{H^{(2)}}\right)\right\} A_2(t) = A_2 + \varepsilon\frac{-3J}{\sin\theta}(A_1^2 + A_2^2)A_1 + \varepsilon^2\left\{\frac{-1}{2!}\left(\frac{3J}{\sin\theta}\right)^2(A_1^2 + A_2^2)^2 A_2 - \frac{\partial H^{(2)}}{\partial A_1}\right\}. \tag{26}$$

According to the general procedure, $H^{(2)}$ is obtained by equating these to (23) and (24). That is,

$$H^{(2)} = \left\{\frac{9\cos\theta}{2\sin^3\theta} + \frac{3}{\sin\theta(\cos 3\theta - \cos\theta)}\right\}\frac{J^2(A_1^2 + A_2^2)^3}{6}.$$

Then we obtain the approximate Hamiltonian $H = H^{(1)} + \varepsilon H^{(2)}$:

$$H = \alpha(A_1^2 + A_2^2)^2 + \beta(A_1^2 + A_2^2)^3, \quad \alpha \equiv \frac{3J}{4\sin\theta}, \;\; \beta \equiv \varepsilon\left\{\frac{9\cos\theta}{2\sin^3\theta} + \frac{3}{\sin\theta(\cos 3\theta - \cos\theta)}\right\}\frac{J^2}{6}.$$

The canonical transformation $\mathrm{d}A_1 \wedge \mathrm{d}A_2 = \mathrm{d}\Theta \wedge \mathrm{d}I$:

$$A_1 = \sqrt{2I}\sin\Theta, \qquad A_2 = \sqrt{2I}\cos\Theta,$$

gives us the simple canonical equations:

$$\frac{\mathrm{d}\Theta}{\mathrm{d}t} = \frac{\partial H}{\partial I} = 8\alpha I + 24\beta I^2, \qquad \frac{\mathrm{d}I}{\mathrm{d}t} = -\frac{\partial H}{\partial \Theta} = 0.$$

In addition to this expression, there exists the exponential form

$$\begin{aligned} A &= A_1 + \mathrm{i}A_2 = \sqrt{2I(0)}(\sin\Theta + \mathrm{i}\cos\Theta) \\ &= \sqrt{2I(0)}\mathrm{i}(\cos\Theta - \mathrm{i}\sin\Theta) = \sqrt{2I(0)}\exp\left(-\mathrm{i}(8\alpha I(0) + 24\beta I(0)^2)t - \mathrm{i}\theta(0) + \frac{\mathrm{i}\pi}{2}\right). \end{aligned}$$

Therefore, the solution of ($\varepsilon \to 0$)-system which is written in the form of an exponential function is

$$\begin{aligned} A(t) &= A(0)\exp\left\{-\mathrm{i}t\left(8\frac{3J}{4\sin\theta}\frac{|A(0)|^2}{2} + 24\varepsilon\left\{\frac{9\cos\theta}{2\sin^3\theta} + \frac{3}{\sin\theta(\cos 3\theta - \cos\theta)}\right\}\frac{J^2}{6}\frac{|A(0)|^4}{4}\right)\right\} \\ &= A(0)\exp\left\{-\mathrm{i}t\left(\frac{3J}{\sin\theta}|A(0)|^2 + \varepsilon\left\{\frac{9\cos\theta}{2\sin^3\theta} + \frac{3}{\sin\theta(\cos 3\theta - \cos\theta)}\right\}J^2|A(0)|^4\right)\right\}, \end{aligned}$$

where we have used the relation $\sqrt{2I(t)} = |A(t)| = \text{const}$. To construct the reduced map, we discretize the Hamiltonian flow. Identifying

$$A^{n+1} \equiv A(t+\varepsilon), \qquad A^n \equiv A(t),$$

yields a symplecticity-preserving RG map

$$A^{n+1} = A^n\exp\left[\mathrm{i}\varepsilon\left\{\frac{-3J|A^n|^2}{\sin\theta} + \varepsilon J^2|A^n|^4\left(-\frac{9\cos\theta}{2\sin^3\theta} - \frac{3}{\sin\theta(\cos 3\theta - \cos\theta)}\right)\right\}\right]. \tag{27}$$

This map can also be obtained by the exponentiation RG method [6].

3.2. Resonant case

Let us consider a case of resonance in the mapping (16). The solution to the perturbation equation (17) is obtained in the form (18)–(20) assuming that the parameter θ is far from the resonance with $\cos\theta = \cos 3\theta$. Although resonant islands are important structures in symplectic maps, they have never been analyzed by the RG method. Here we demonstrate how our symplecticity-preserving RG method works near a resonant island.

Let us expand θ near the resonance

$$\theta = \tfrac{1}{2}\pi + \varepsilon\theta^{(1)} + \varepsilon^2\theta^{(2)} + \mathcal{O}(\varepsilon^3).$$

Eq. (16) can then be expressed in the form

$$L_{\pi/2}x^n = \varepsilon(2J(x^n)^3 - 2\theta^{(1)}x^n) - 2\varepsilon^2\theta^{(2)}x^n,$$

where $L_{\pi/2}x^n$ is defined by

$$L_{\pi/2}x^n \equiv x^{n+1} + x^{n-1}.$$

The perturbation expansion yields

$$\begin{aligned} &L_{\pi/2}x^{(0)n} = 0, \qquad L_{\pi/2}x^{(1)n} = 2J(x^{(0)n})^3 - 2\theta^{(1)}x^{(0)n}, \\ &L_{\pi/2}x^{(2)n} = 6J(x^{(0)n})^2x^{(1)n} - 2\theta^{(1)}x^{(1)n} - 2\theta^{(2)}x^{(0)n}. \end{aligned}$$

These equations can be solved, yielding the result

$$x^{(0)n} = A\mathrm{i}^n + \text{c.c.}, \qquad x^{(1)n} = (-\mathrm{i})\mathrm{i}^n n[J(A^{*3} + 3|A|^2 A) - \theta^{(1)} A] + \text{c.c.},$$
$$x^{(2)n} = \mathrm{i}^n n^2[\tfrac{3}{2}J^2(-2|A|^4 A + |A|^2 A^{*3} + A^5) + J\theta^{(1)}(3|A|^2 A - A^{*3}) - \tfrac{1}{2}(\theta^{(1)2})A] + \mathrm{i}^n n\mathrm{i}\theta^{(2)} A + \text{c.c.}$$

Here $A(\in \boldsymbol{C})$ is the integration constant.

As before, we define the renormalization transformation by

$$A^n \equiv A + \varepsilon(-\mathrm{i})n\{J(A^{*3} + 3|A|^2 A) - \theta^{(1)} A\} + \varepsilon^2 n^2\{\tfrac{3}{2}J^2(-2|A|^4 A + |A|^2 A^{*3} + A^5) + J\theta^{(1)}(3|A|^2 A - A^{*3}) - \tfrac{1}{2}(\theta^{(1)2})A\} + \varepsilon^2 n\mathrm{i}\theta^{(2)} A + \text{c.c.}$$

Taking into account the expression

$$A = A^n + \varepsilon\mathrm{i}n\{J((A^{*n})^3 + 3|A^n|^2 A^n) - \theta^{(1)} A^n\} + \mathcal{O}(\varepsilon^3)$$

which relates the amplitude A to the renormalization variable A^n, we obtain the naive RG map

$$A_1^{n+1} = A_1^n + \varepsilon(4J(A_2^n)^3 - \theta^{(1)} A_2^n) + \varepsilon^2\{-24J^2(A_1^n)^3(A_2^n)^2 + 2J\theta^{(1)}((A_1^n)^3 + 3A_1^n(A_2^n)^2) - \tfrac{1}{2}\theta^{(1)2} A_1^n - \theta^{(2)} A_2^n\}, \tag{28}$$

$$A_2^{n+1} = A_2^n + \varepsilon(-4J(A_2^n)^3 + \theta^{(1)} A_1^n) + \varepsilon^2\{-24J^2(A_1^n)^2(A_2^n)^3 + 2J\theta^{(1)}((A_1^n)^3 + 3(A_1^n)^2 A_2^n) - \tfrac{1}{2}\theta^{(1)2} A_2^n - \theta^{(2)} A_1^n\}. \tag{29}$$

Here $A^n = A_1^n + \mathrm{i}A_2^n$, and A_1^n, A_2^n are real variables. $H^{(1)}$ can be determined by taking the limit $\varepsilon \to 0$

$$\frac{\mathrm{d}A_1}{\mathrm{d}t} = 4JA_2^3 - \theta^{(1)} A_2 = \frac{\partial H^{(1)}}{\partial A_2},$$
$$\frac{\mathrm{d}A_2}{\mathrm{d}t} = -4JA_1^3 + \theta^{(1)} A_1 = -\frac{\partial H^{(1)}}{\partial A_1}, \quad H^{(1)}(A_1, A_2) = \left(JA_1^4 - \theta^{(1)}\frac{A_1^2}{2}\right) + \left(JA_2^4 - \theta^{(1)}\frac{A_2^2}{2}\right).$$

The symplecticity-preserving RG procedure gives the expression for $H^{(2)}$. Identifying A^{n+1} in (28) and (29) as

$$A(t) + \varepsilon\mathcal{L}_H A(t) + \frac{\varepsilon^2\mathcal{L}_H^2}{2!}A(t) = A(t) + \varepsilon\{A(t), H^{(1)}\} + \varepsilon^2\left(\{A(t), H^{(2)}\} + \frac{1}{2!}\{\{A(t), H^{(1)}\}, H^{(1)}\}\right),$$

then the truncated Liouville operator relations are

$$A_1(t+\varepsilon) = A_1(t) + \varepsilon(4JA_2^3 - \theta^{(1)} A_2) + \varepsilon^2\left[-24J^2 A_1^3 A_2^2 + 2J\theta^{(1)}(A_1^3 + 3A_1 A_2^2) - \frac{\theta^{(1)2}}{2}A_1 - \theta^{(2)} A_2 + \frac{\partial H^{(2)}}{\partial A_2}\right], \tag{30}$$

$$A_2(t+\varepsilon) = A_2 + \varepsilon[-4JA_2^3 + \theta^{(1)} A_1] + \varepsilon^2\left[-24J^2 A_1^2 A_2^3 + 2J\theta^{(1)}(A_1^3 + 3A_1^2 A_2) - \frac{\theta^{(1)2}}{2}A_2 - \theta^{(2)} A_1 - \frac{\partial H^{(2)}}{\partial A_1}\right]. \tag{31}$$

Comparing (30) and (31) with (28) and (29), we have the Hamiltonian whose trajectory can approximately interpolate the trajectory of the naive RG map,

$$H = H^{(1)} + \varepsilon H^{(2)} = J(A_1^4 + A_2^4) - \tfrac{1}{2}(\theta^{(1)} + \varepsilon\theta^{(2)})(A_1^2 + A_2^2).$$

This Hamiltonian system is integrable because of 1 degree of freedom. However, the solution may be too complicated to write the analytic expression, which yields the difficulty of finding a symplecticity-preserving RG map. Instead of using the analytical expression to the Hamiltonian flow, we use a symplectic integrator, which is known to be a discretization method designed for preserving the symplecticity for Hamiltonian flows [14].

We split the Hamiltonian as $H_1 + H_2$, not $H^{(1)} + \varepsilon H^{(2)}$, so that we use a symplectic integrator,

$$H = H_1 + H_2 = (JA_1^4 - \tfrac{1}{2}\theta'^{(1)} A_1^2) + (JA_2^4 - \tfrac{1}{2}\theta'^{(1)} A_2^2), \quad \theta'^{(1)} \equiv \theta^{(1)} + \varepsilon\theta^{(2)}.$$

The Hamiltonian H_1 provides the mapping

$$e^{\tau D_{H_1}}: \quad A_1(t+\tau) = A_1(t), \qquad A_2(t+\tau) = A_2(t) + (-4JA_1^3(t) + \theta'^{(1)} A_1(t))\tau.$$

Here τ is a real number. Similarly, H_2 provides

$$e^{\tau D_{H_2}}: \quad A_1(t+\tau) = A_1(t) + (4JA_2^3(t) - \theta'^{(1)} A_2(t))\tau, \qquad A_2(t+\tau) = A_2(t).$$

Therefore we can obtain the reduced symplectic map

$$A_1^{n+1} = A_1^n + \varepsilon[4J\{A_2^n + \tfrac{1}{2}\varepsilon(-4JA_1^{n3} + \theta'^{(1)} A_1^n)\}^3 - \theta'^{(1)}\{A_2^n + \tfrac{1}{2}\varepsilon(-4JA_1^{n3} + \theta'^{(1)} A_1^n)\}], \tag{32}$$

$$A_2^{n+1} = A_2^n + \tfrac{1}{2}\varepsilon(-4JA_1^{n3} + \theta'^{(1)} A_1^n) + \tfrac{1}{2}\varepsilon(-4JA_1^{(n+1)3} + \theta'^{(1)} A_1^{n+1}). \tag{33}$$

Here we have taken the following symplectic integrator:

$$e^{\varepsilon D_H} = \exp(\tfrac{1}{2}\varepsilon D_{H_1}) \exp(\varepsilon D_{H_2}) \exp(\tfrac{1}{2}\varepsilon D_{H_1}) + \mathcal{O}(\varepsilon^3).$$

3.3. Numerical results in case of a resonance

In this section we present illustrative numerical results for our symplecticity-preserving RG method in the case of near resonance with $\cos\theta = \cos 3\theta$. To show that our RG method can successfully be applied to study a resonant structure for the two-dimensional symplectic map (16), we use both the reduced maps (27), (32) and (33) up to $\mathcal{O}(\varepsilon)$.

In Figs. 1 and 2 the phase portraits to the map near the resonance with $\cos\theta = \cos 3\theta$, are depicted. Although the exponentiated RG map agrees well with the exact numerical result for near the origin in the phase space, it derivates considerably from the exact result near the resonance point. In contrast, the map obtained by the Liouville operator

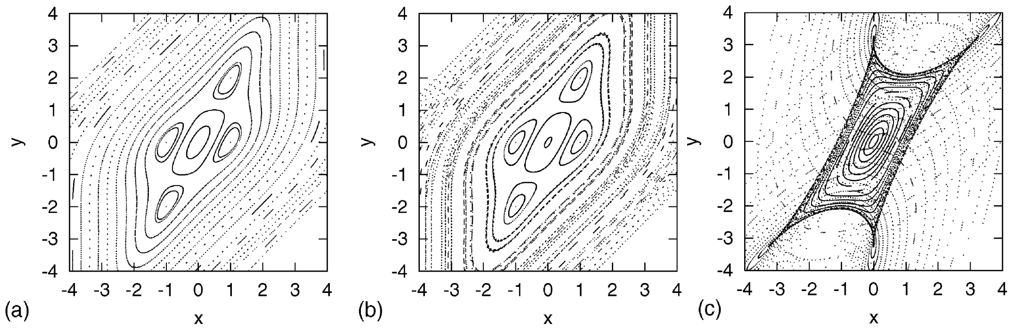

Fig. 1. Phase portraits of the two-dimensional symplectic map model with the parameters are $\varepsilon = 0.01$, $J = 1.0$, and $\theta^{(1)} = 1.0$: (a) the original map (Eq. (16)), (b) the Liouville operator approach to the RG method (Eqs. (32) and (33) up to $\mathcal{O}(\varepsilon)$), (c) the exponentiated RG method (Eq. (27) up to $\mathcal{O}(\varepsilon)$).

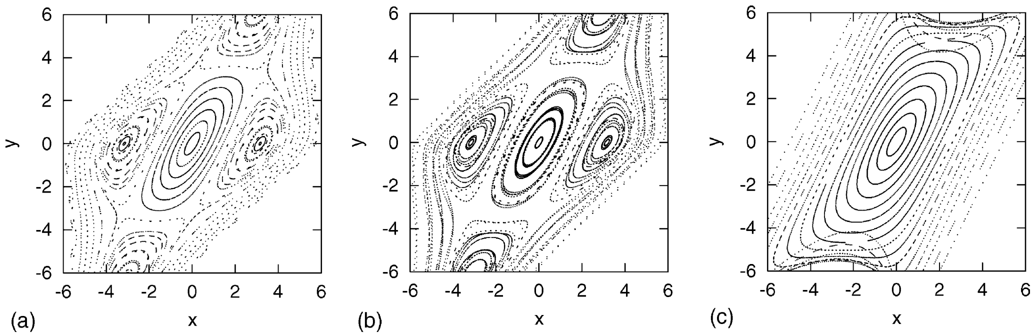

Fig. 2. Phase portraits of the two-dimensional symplectic map model, with the parameters are $\varepsilon = 0.01$, $J = 1.0$, and $\theta^{(1)} = 10.0$: (a) the original map (Eq. (16)), (b) the Liouville operator approach to the RG method (Eqs. (32) and (33) up to $\mathcal{O}(\varepsilon)$), (c) the exponentiated RG method (Eq. (27) up to $\mathcal{O}(\varepsilon)$).

approach to a symplecticity-preserving RG method can globally give a good approximation even in the case of the first RG approximation. Detailed studies of such resonant island structure by means of the present RG method will be published elsewhere for other maps including the Hénon map.

4. Conclusions

We have presented the Liouville operator approach to the RG method to preserve symplectic structures in RG maps near elliptic fixed points of symplectic discrete systems. The symplecticity-preserving procedure is accomplished by comparing naive RG maps with the Liouville operator relation order by order in the small parameter and gives symplectic maps, which successfully describes the long-time asymptotic behavior of the original systems. Although the exponentiation procedure has never been successfully applied to study resonant islands, the Liouville operator approach to symplecticity preservation has given correctly reduced maps. Furthermore, the advantages of this new method are not only that a time-step parameter is not needed, but also a reduced map can be explicit, which is done by choosing an explicit symplectic integrator.

It is easy to see that the present symplecticity-preserving method is also applicable to general weakly-nonlinear symplectic maps. Other symplecticity-preserving RG maps are to be studied in future.

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